Contributions to the Theory of Nonmonotonic Inference Systems

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Abstract

In the present paper an approach to nonmonotonic reasoning is discussed which is inspired by the paradigm of abstract model theory. The aim is to analyse nonmonotonic reasoning within the framework of inference operations based on model operators. This is done by using the concept of a deductive frame and its semantical counterpart, a semantical frame. These are considered as a basis for nonmonotonic model theory. Then several representation theorems are proved and compactness properties are analysed in detail. These concepts are applied to classical nonmonotonic systems, in particular to minimal reasoning in propositional calculus and predicate calculus, and to propositional Poole systems.

keywords : nonmonotonic inference, compactness, model theory

1 Introduction

Given a language L a logic can be described on an abstract level by a functor $C : 2^L \to 2^L$ such that for every set $X \subseteq L$, $C(X)$ is understood to be the set of all the consequences of the set X of premises. If C is monotonic and compact then C is uniquely determined by its restriction to finite sets $X \subseteq L$. This implies that the finitary part of a deductive operation captures already its relevant properties. The situation changes if the monotonicity requirement is weakened. Nowadays, systems of nonmonotonic inference have won acceptance, particularly in the area of artificial intelligence, for their ability to deal with several forms of commensense reasoning.

Many of the investigated systems are introduced on a syntactical level and have no sufficient broad semantical framework. Semantics can be studied using the notions and concepts of model theory which is a branch of mathematical logic dealing with the relation between a formal language and its interpretations. This relation is achieved by introducing a basic truth definition, which connects the formal language with its interpretations by means of models. The truth relation can be used to define an operator $\mathrm{Mod} : 2^L \to 2^{Int}$ assigning to every set $X \subseteq L$ a class $\mathrm{Mod}(X)$ of interpretations satisfying the formulas in X; these interpretations are called models of X. We attain *nonmonotonic model theory* if we add an operator $\Phi : 2^L \to 2^{Int}$ satisfying the condition $\Phi(X) \subseteq \mathrm{Mod}(X)$. Elementary properties of such model operators were studied in [He91]; similar approaches were presented in [Bl91], [Li91], and in [Th89]. Thes area of research needs further development, for example are the interrelations between finitary inference operations and their infinitary extensions not yet sufficiently understood. Here is a need to find new versions of compactness which allow to create tools for constructing new nonmonotonic logics and to characterize known nonmonotonic systems. The present paper is devoted to these topics. Our investigation takes a view which was inspired by the paradigm of abstract model theory [Ba85] and the ideas in [KL90]. The paper continues the investigation of [He91], [He94], and is intended as a contribution to the emerging field of nonmonotonic model theory.

Section 2 contains the background concerning pure conditions of nonmonotonic inference operations and notions from monotonic systems, including predicate calculus. In section 3 we propose a general framework for studying relevant aspects of nonmonotonic inference. The main notions are *deductive frames* and *semantical frames*. Also, the notion of a predicative semantical frame is used which gives a natural generalization of classical abstract model theory to nonmonotonic abstract model theory. In the section 4 the representation problem is investigated, and in section 5 compactness properties are studied. Several new versions of compactness are presented and analysed. Section 6 is devoted to classical nonmonotonic systems in propositional logic and predicate logic. In section 7 we give a short summary and discuss similar approaches. Finally, in section 8 problems for further research are outlined.

2 Background and Preliminaries

Let L be a nonempty language whose elements are denoted by ϕ, ψ, χ. An operation $C : 2^L \to 2^L$ is called an *inference operation*, and the pair (L, C) is said to be an *inference system*. The operation C represents the notion of logical inference on its most abstract level. An inference system (L, C_L) is a *closure system* and C_L a closure operation if it satisfies the following conditions:

$$\begin{array}{ll} X \subseteq C_L(X) & \text{(inclusion)} \\ C_L(C_L(X)) = C_L(X) & \text{(idempotence)} \\ X \subseteq Y \Rightarrow C_L(X) \subseteq C_L(Y) & \text{(monotony)} \end{array}$$

An inference operation C_L satisfies *compactness* if $\phi \in C_L(X)$ implies the existence of a finite subset $Y \subseteq X$ such that $\phi \in C_L(Y)$. A closure system (L, C_L) is a *deductive system* if C_L satisfies compactness; then C_L is said to be a *deductive operation.* A set $X \subseteq L$ is closed under C_L if $C_L(X) = X$. (L, Cn) denotes an inference system based on classical logic; the natural example for such a system is classical propositional logic with standard semantics. An inference system (L, C) is said to be *partial* if C is not necessarily defined for all $X \subseteq L$; let $dom(C)$ be the domain of C, and $Fin(L) = \{X : X \subseteq L \text{ and } X \text{ is finite }\}$. A partial inference operation C is said to be *finitary* iff $dom(C) = Fin(L)$. $F \downarrow X$ denotes the restriction of a function F to the set $X \subseteq dom(F)$; define $C_f = C \downarrow Fin(L)$. The partial relation $\leq$ between inference operations C_1, C_2 is defined as follows: $C_1 \leq C_2 \Leftrightarrow \forall X(C_1(X) \subseteq C_2(X))$.

A semantics for a closure system (L, C_L) can be defined by a *model-theoretic system.* A *model-theoretic system* $(L, M, \models)$ is determined by a language L, a set (or class) M whose elements are called *worlds* and a *relation of satisfaction* $\models \;\subseteq M \times L$ between worlds and formulas. Given a logical system $(L, M, \models)$, we introduce the following notions. Let $X \subseteq L$, $\mathrm{Mod}^{\models}(X) = \{m : m \in M$ and $m \models X\}$, where $m \models X$ iff for every $\phi \in X : m \models \phi$. Let $K \subseteq M$, then $Th^{\models}(K) = \{\phi : \phi \in L$ and $K \models \phi\}$, where $K \models \phi$ iff for all $m \in K : m \models \phi$. For $K \subseteq M$ let $\overline{K} = Mod^{\models}(Th^{\models}(K))$. $C^{\models}(X) = \{\phi : Mod^{\models}(X) \subseteq Mod^{\models}(\phi)\}$, and $X \models \phi$ iff $\phi \in C^{\models}(X)$. Obviously, $(L, C^{\models})$ is a closure system and if $C^{\models}(X) = X$ then $Th^{\models}(Mod^{\models}(X)) = X$. The model-theoretic system $(L, M, \models)$ *has negation* if there is an unary functor $\neg$ in the language L such that $m \not\models \phi \Leftrightarrow m \models \neg\phi$ for every $m \in M$; it *has contradiction* if there is a formula $\phi \in L$ such that $\mathrm{Mod}^{\models}(\phi) = \emptyset$. $(L, M, \models)$ is said to be compact if the closure operation $C^{\models}$ is compact. The inference system (L, C_L) is *correct* (*complete*) with respect to the model-theoretic system $(L, M, \models)$ iff $C_L(X) \subseteq C^{\models}(X)$ $(C_L(X) = C^{\models}(X))$. In case of completeness we say also that $(L, M, \models)$ *represents* (L, C_L).

The study of the general properties of inference operations $C : 2^L \rightarrow 2^L$ that do not satisfy monotony is well-established. In [Ma93] this field is reviewed and several nonmonotonic inference operations are investigated that have been proposed in the literature. A condition on inference operations is said to be *pure* if it concerns the operation alone without regard to its interrelations to classical consequence operation and truth-functional connectives. The most important pure conditions are the following.

$$
\begin{array}{ll}
X \subseteq Y \subseteq C(X) \Rightarrow C(Y) \subseteq C(X) & \text{(Cut)} \\
X \subseteq Y \subseteq C(X) \Rightarrow C(X) \subseteq C(Y) & \text{(Cautious Monotony)} \\
X \subseteq Y \subseteq C(X) \Rightarrow C(X) = C(Y) & \text{(Cumulativity)} \\
X \subseteq C(Y) \text{ and } Y \subseteq C(X) \Rightarrow C(X) = C(Y) & \text{(Reciprocity)}
\end{array}
$$

There exist certain formal dependencies between these conditions. Given inclusion, idempotence is a special case of cut. An inference operation C is cumulative iff C satisfies inclusion, cut and cautious monotony. Given inclusion, cumulativity is equivalent to reciprocity.

We close this section with an overview about notions from predicate calculus that are used in the following. ω denotes the set of natural numbers. A predicative *signature* σ is a set of relational, functional and constant symbols. Let L be a first order language of a certain signature, $Const(L)$ the set of the constant symbols, $Rel(L)$ the set of the relational symbols and $Fun(L)$ the set of the function symbols of L. $Fm(L)$ is the set of formulas of L and a set $X \subseteq Fm(L)$ of formulas is called a *theory.* A prenex formula is said to be *universal* if it contains no existential quantifier, an universal theory is a set of universal formulas.

The *signature* of a theory P, $sign(P)$, is the set containing the relational, functional and constant symbols appearing in P. $L(P)$ is the first order language associated to P. $B(P)$ denotes the *Herbrand base* of P, i.e. the set of all ground atoms of the signature of P. $U(P)$ is the set of all variable-free terms of $sign(P)$. We use the notion of an *interpretation* for P as a relational structure $\mathcal{A}$ which associates a declarative meaning to the symbols of $sign(P)$. $U(\mathcal{A})$ denotes the universe of the structure $\mathcal{A}$. A structure $\mathcal{A}$

is a *model* for P if every formula ϕ in P is true in $\mathcal{A}$, denoted by $\mathcal{A} \models \phi$. Let $\mathrm{Mod}(P)$ be the class of all models of P in the language $L(P)$. Then $X \models \phi$ iff $\mathrm{Mod}(X) \subseteq \mathrm{Mod}(\phi)$ and $X^{\models} = \{\phi : X \models \phi\}$. A *Herbrand structure* for P in the language L is one for which the universe equals $U(P), L(P) \subseteq L$, and the function symbols have their canonical interpretation. A *Herbrand model* of P is a Herbrand structure in the language $L(P)$ being a model of P. Herbrand structures of the signature $sign(P)$ are determined by subsets $I \subseteq B(P)$. We identify Herbrand structures $\mathcal{A}$ with their associated set $I_{\mathcal{A}} \subseteq B(P)$. For a class of $\mathcal{K}$ of structures let $Th(\mathcal{K}) = \{F : F \in L(\mathcal{K}) \text{ and } \mathcal{K} \models F\}$. A Herbrand model I is said to be *minimal* for S if I is a model of S and no proper subsets of I represents a model of S.

3 Model Operators and Frames

Besides the three conditions of cut, cautious monotony and cumulativity *Makinson* [Ma93] emphasizes several mixed conditions of inference: *supraclassicality*, *distributivity*, and *rationality*. C is said to be supraclassical if it extends the usual consequence operation Cn of classical logic, ie. $Cn(X) \subseteq C(X)$ for all $X \subseteq L$. He claims that if C behaves logically it is supraclassical and satisfies certain additional properties, as left absorption and congruence. Obviously, these mixed conditions do not depend on classical logic and may be formulated for every logic represented by a deductive system. For this reason we use the following definition [He94].

Definition 1 *1. A system $\mathcal{IF} = (L, C_L, C)$ is said to be an inference frame iff the following conditions are satisfied:*

(a) L is a language.

(b) C_L is a deductive inference operation on L, i.e. C_L satisfies inclusion , idempotence, monotony and compactness.

(c) C satisfies $C_L(X) \subseteq C(X)$ (supradeductivity) [1]

(L, C_L) is said to be a basis for C.

2. C satisfies left absorption iff $C_L(C(X)) = C(X)$; and C satisfies congruence or right absorption iff $C_L(X) = C_L(Y) \Rightarrow C(X) = C(Y)$. C satisfies full absorption if C satisfies left absorption and congruence.

3. An inference frame $\mathcal{DF} = (L, C_L, C)$ is said to be a deductive frame if it satisfies full absorption. In this case C is said to be logical over C_L, and (L, C_L) is called a deductive basis for C.

If C_L is not assumed to be compact then the system (L, C_L, C) is called a *weak inference frame*, and analogously, a *weak deductive frame*.

[1] *Pearce*[Pe93] discussed nonmonotonic extensions of the constructive inference relation $\vdash_N$, and introduced the notion of *supraconstructivity*. Our notion of *supradeductivity* is more general because it is not connected to a particular logic, and thus seems to cover all nonmonotonic systems having a logical interpretation. A similar view is also presented in [FL94].

Definition 2 *$\mathcal{SF} = (L, M, \models, \Phi)$ is a semantical frame iff*

1. *$(L, M, \models)$ is a model-theoretic system.*
2. *$\Phi : 2^L \to 2^M$ is a functor such that $\Phi(X) \subseteq Mod^{\models}(X)$. Let $C_\Phi(X) = Th^{\models}(\Phi(X))$. Φ is called model operator.*
3. *The operator Φ is said to be invariant if $\forall X \subseteq L(\Phi(X) = \Phi(C_L(X)))$.*
4. *$\mathcal{SF}$ is said to be compact if $C^{\models}$ satisfies compactness; it is called invariant if the model operator Φ is invariant.*

C_Φ satisfies extension and left absorption, and hence $(L, C^{\models}, C_\Phi)$ is an inference frame. The following class of model operators is of importance. Similar as in [Li91] we use the following notions.

Definition 3 *Let $\mathcal{L} = (L, M, \models)$ be a model-theoretic system and $cl(M) = \{K : K \subseteq M$ and $K = Mod^{\models}(X)$ for a certain set $X \subseteq L\}$; $cl(M)$ is the set of closed subsets or elementary classes of M. A function $\Sigma : cl(M) \to 2^M$ is a selection function if $\Sigma(K) \subseteq K$. The selection function Σ is closed if $\Sigma(K) \in cl(M)$ for every $K \in cl(M)$. A model operator $\Phi : 2^L \to 2^M$ is said to be a model selector if there is a selection function $\Sigma : cl(M) \to 2^M$ such that $\Phi(X) = \Sigma(Mod^{\models}(X))$. The system $(L, M, \models, \Sigma)$ is said to be a selection frame. Let $C_\Sigma(X) = Th(\Sigma(Mod(X)))$.*

The following straightforward proposition states that selection frames and invariant frames coincide.

Proposition 1
1. *Let $\mathcal{F} = (L, M, \models, \Sigma)$ be a selection frame. Then the model operator $\Phi_\Sigma(X) := \Sigma(Mod^{\models}(X))$ is invariant.*
2. *Let $\mathcal{F} = (L, M, \models, \Phi)$ be a invariant semantical frame. Then $\Sigma(K) = \Phi(Th^{\models}(K))$ is a selection function.*
3. *Let $\mathcal{S} = (L, M, \models)$ be a model-theoretic system and Σ a selection function for $\mathcal{S}$. Then there exists a closed selection function Δ such that for every $X \subset L$ $C_\Sigma(X) = C_\Delta(X)$.*

We recall further properties of inference systems and frames. Let $\mathcal{F} = (L, C_L, C)$ be a deductive frame. $\mathcal{F}$ is called cumulative if C satisfies cumulativity. C satisfies *loop* if for any $n \in \omega$: $X_1 \subseteq C(X_2)$, $X_2 \subseteq C(X_3)$, $\ldots$ $X_{n-1} \subseteq C(X_n)$, $X_n \subseteq C(X_1)$ implies $C(X_1) = C(X_n)$. C is *distributive* iff C is cumulative and satisfies the condition $C(X) \cap C(Y) \subseteq C(C_L(X) \cap C_L(Y))$. C satisfies *conditionalization* iff C is cumulative and satisfies the condition $C(X \cup Y) \subseteq C_L(X \cup C(Y))$. C is *rational* iff C satisfies conditionalization, cumulativity and the following condition: $X \subseteq Y$ and $C_L(C(X) \cup Y) \neq L \Rightarrow C(X) \subseteq C(Y)$.

In the following we consider two concrete types of semantical frames. A propositional signature $\sigma = (\{f_i : i \in I\}, r)$ is given by functional symbols $f_i, i \in I$, called propositional connectives, and an arity-function $r : I \to \omega$ fixing an arity $r(i)$ for every of the functions f_i. Let V be a countably infinite set whose elements are called variables. The set $PFm(\sigma)$ of propositional formulas of signature σ is the smallest set containing the set V and closed with respect to the following condition: If $t_1, \ldots, t_{r(i)} \subseteq PFm(\sigma)$ then $f_i(t_1, \ldots, t_{r(i)}) \in PFm(\sigma)$.

A partially ordered *truth-algebra* $\mathcal{N} = (N, \{\overline{f}_i\}_{i \in I}, T, \leq)$ of signature σ is defined by:

1. a nonempty set N (the truth values);
2. a subset $T \subseteq N$ (the designated truth-values);
3. functions $\overline{f}_i : N^{r(i)} \to N$ (interpretation of the connective f_i);
4. a partial ordering $\leq$ on N.

A *propositional base* $\mathcal{P} = (\sigma, \mathcal{N})$ is determined by a propositional signature σ and a truth-algebra $\mathcal{N}$ of signature σ. Let $\mathcal{P} = (\sigma, \mathcal{N})$ a propositional base, N the set of truth-values. $M(\mathcal{P}) = N^V$ is the set of $\mathcal{P}$-valuations. Every valuation $v \in N^V$ can be extended to a function $\overline{v}$ defined on $PFm(\sigma)$ in the following canonical way:

(1) $\overline{v}(\phi) = v(\phi)$ if $\phi \in V$;
(2) $\overline{v}(f_i(t_1, \ldots, f_{r(i)})) = \overline{f}_i(\overline{v}(t_1), \ldots, \overline{v}(t_{r(i)}))$.

v is a model of ϕ, denoted by $v \models_{\mathcal{P}} \phi$, iff $\overline{v}(\phi) \in T$. Let be $Mod_{\mathcal{P}}(X) = \{v : v \models_{\mathcal{P}} \phi$ for every $\phi \in X\}$. A *generalized propositional base* $\mathcal{P} = (\sigma, (\mathcal{N})_{i \in I})$ is defined by a propositional signature and a family $(\mathcal{N})_{i \in I}$ of truth-algebras of signature σ.

Definition 4 *$\mathcal{F} = (L, M, \models, \Phi)$ is a propositional semantical frame iff there is a propositional base $\mathcal{P} = (\sigma, \mathcal{N})$ such that:*

1. *L equals $PFm(\sigma)$;*
2. *$\models$ equals $\models_{\mathcal{P}}$;*
3. *M equals $M(\mathcal{P})$;*
4. *Φ is a functor defined on 2^L such that $\Phi(X) \subseteq Mod_{\mathcal{P}}(X)$.*

The language of a *predicative semantical frame* is constructed from predicative signatures. Signatures are denotes by σ, τ. The class of τ-structures is denoted by $Str(\tau)$. A set $\mathcal{T}$ of signatures is *closed* if $\sigma \subseteq \tau, \tau \in \mathcal{T}$ implies $\sigma \in \mathcal{T}$ and $\sigma, \tau \in \mathcal{T}$ implies $\tau \cup \sigma \in \mathcal{T}$. For structures $\mathcal{A}, \mathcal{B} \in Str(\tau)$ let $\mathcal{A} \cong \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Definition 5 *A predicative semantical frame $\mathcal{LF} = (L, \mathcal{T}, \models_L, \Phi)$ is defined by following conditions:*

1. *$\mathcal{T}$ is a closed set of signatures.*
2. *L is a mapping defined on $\mathcal{T}$, $L(\tau)$ is a class of formulas.*
3. *$\models_{L(\tau)}$ is a relation $\subseteq L(\tau) \times Str(\tau)$, $\models_L = \bigcup_{\tau \in \mathcal{T}} \models_{L(\tau)}$.*

4. $\tau \subseteq \sigma \Rightarrow L(\tau) \subseteq L(\sigma)$.

5. *Isomorphism property: Let* $\mathcal{A}, \mathcal{B} \in Str(\tau)$, $\phi \in L(\tau)$, *then* $\mathcal{A} \models_L \phi$ *and* $\mathcal{A} \simeq \mathcal{B}$ *implies* $\mathcal{B} \models_L \phi$.

6. *Reduct property:* $\mathcal{A} \in Str(\sigma), \phi \in L(\tau), \tau \subseteq \sigma$ *and* $\mathcal{A} \models_L \phi$ *then* $\mathcal{A} \downarrow \tau \models_L \phi$. *Furthermore, if* $\mathcal{A} \models_L \phi$ *then* $\phi \in L(\tau_{\mathcal{A}})$.

7. *Renaming property. Let* $F : \tau \to \sigma$ *a renaming. Then for each* $\phi \in L(\tau)$ *there is a sentence* ϕ^f *from* $L(\tau)$ *such that* $\mathcal{A} \models_L \phi$ *iff* $\mathcal{A}^f \models_L \phi^f$.

8. Φ *is a functor defined on* $\mathcal{T}$ *such that for* $\tau \in \mathcal{T}$ $\Phi(\tau)$ *is a function from* $2^{L(\tau)}$ *in* $2^{Str(\tau)}$ *satisfying the condition* $\Phi(\tau)(X) \subseteq Mod(X)$ *for every* $X \subseteq L(\tau)$.

Let $\mathcal{K}$ be a function defined on $\mathcal{T}$ such that for every $\tau \in \mathcal{T}$ the condition $\mathcal{K}(\tau) \subseteq Str(\tau)$ is satisfied. $\mathcal{K}$ is said to be *reductive* if for $\sigma \subseteq \tau$ the condition $\mathcal{K}(\sigma) = \{\mathcal{A} \downarrow \sigma : \mathcal{A} \in \mathcal{K}(\tau)\}$ is fulfilled. Φ is said to be *contextually independent* if there is a reductive function $\mathcal{K}$ such that for every set $X \subseteq L(\tau)$: $\Phi(X) = \mathrm{Mod}(X) \cap \mathcal{K}(\tau)$. A predicative semantical frame is said to be an *abstract logic* if the functor Φ is contextually independent. This notion coincides with the concept of *abstract logic* introduced in [KV93]. The following proposition is obvious.

Proposition 2 *Let* $\mathcal{F} = (L, \mathcal{T}, \models_L, \Phi)$ *a predicative semantical frame such that* Φ *is contextually independent. Then* C_Φ *is monotonic.*

Proof: Assume $X \subseteq Y$, $X \subseteq L(\sigma)$, $Y \subseteq L(\tau)$; then $\sigma \subseteq \tau$. $X \subseteq Y$ implies $\mathrm{Mod}(Y) \downarrow \sigma \subseteq \mathrm{Mod}(X)$. Since $\mathcal{K}$ is reductive it follows $(\mathrm{Mod}(Y) \cap \mathcal{K}(\tau)) \downarrow \sigma = \mathrm{Mod}(Y) \downarrow \sigma \cap \mathcal{K}(\sigma)$. Hence $\Phi(Y) \downarrow \sigma \subseteq \Phi(X)$. This implies $C_\Phi(X) \subseteq C_\Phi(Y)$. □

Finite model theory is an example of a predicative semantical frames. It is defined by the model operator $\Phi(X) = \{\mathcal{A} : \mathcal{A} \text{ is a finite model of } X\}$. Since this operator Φ is contextually independent, C_Φ is monotonic.

4 Representation Theorems

The syntactical notion of an inference frame and the notion of a semantical frame can be connected by using a standard *Lindenbaum-Tarski*-construction. Let (L, C_L) be a deductive system. A set $X \subseteq L$ is *maximally consistent* if $C_L(X) \neq L$ and for every proper superset Y of X holds $C_L(Y) = L$. X is said to be *relatively maximal consistent* (abbreviated by the term *r-maximal*), iff there is a formula $\phi \in L$ such that $\phi \notin C_L(X)$ and for every proper superset Y of X the condition $\phi \in C_L(Y)$ is satisfied. Let $rmax(L)$ the set of all r-maximal subsets of L, $max(L)$ the set of all maximal consistent subsets, and $dc(L) = \{X : C_L(X) = X\}$ the set of all deductivel closed subsets. Obviously, $max(L) \subseteq rmax(L) \subseteq dc(L)$. The system (L, C_L) is said to be *absolute* if $max(L) = rmax(L)$. A semantics $(L, M, \models)$ for (L, C_L) can be defined as follows. Let $M \subseteq dc(L)$ and define for $\phi \in L : m \models \phi$ iff $\phi \in m$. Then $C^{\models} = C_L$

iff $rmax(L) \subseteq M$. The model-theoretic system $(L, M, \models)$ is called *Lindenbaum-Tarski-semantics* (abbreviated *LT-semantics*) for (L, C_L) if $M = rmax(L)$. Let $(L, M, \models)$ a model-theoretic system; then the set $cl(M)$ of elementary classes of M is closed with respect to arbitrary intersections. If $cl(M)$ is closed with respect to finite unions then $(M, cl(M))$ represents a topological space.

Definition 6 *1. A deductive system (L, C_L) is said to be auto-distributive iff for arbitrary deductively closed subsets $X, Y, Z \subseteq L$ holds: $C_L(X \cup (Y \cap Z)) \supseteq C_L(X \cup Y) \cap C_L(X \cup Z)$.*

2. A model-theoretic system $(L, M, \models)$ is said to be topological iff $cl(M)$ is closed with respect to set-theoretical union.

Proposition 3 *Let (L, C_L) be a deductive system and $(L, M, \models)$ its Lindenbaum-Tarski-semantics.*

1. *If $(L, M, \models)$ is topological then (L, C_L) is auto-distributive.*

2. *If (L, C_L) is auto-distributive and absolute then $(L, M, \models)$ is topological.*

Proof: (1) Let $(L, M, \models)$ be topological and $M_1 = Mod(X_1), M_2 = Mod(X_2)$ elementary classes.Then there exists a set $Z \subseteq L$ such that $M_1 \cup M_2 = Mod(Z)$. Because of $Mod(X) = Mod(C_L(X)$ we may assume that the sets X_1, X_2, Z are deductively closed. Then, we get $Z = X_1 \cap X_2$. To prove auto-distributivity we assume $\phi \in C_L(X_1 \cup Y) \cap C_L(X_2 \cup Y)$ and show that $\phi \in C_L(Y \cup (X_1 \cap X_2))$. Since $Z = X_\cap X_2$ it remains to show that $\phi \in C_Z(Y \cup Z)$. The assumption $\phi \in C_L(X_1 \cup Y) \cap C_L(X_2 \cup Y)$ implies that $Mod(Y \cup X_1) \cup Mod(Y \cup X_2) \subseteq Mod(\phi)$. Now, we have $Mod(Y \cup Z) = Mod(Y) \cap Mod(Z)$, and by the property of Z it is $Mod(Y) \cap Mod(Z) = Mod(Y) \cap (Mod(X_1) \cup Mod(X_2))$. This implies $Mod(Y \cup Z) = (Mod(Y) \cap Mod(X_1)) \cup (Mod(Y) \cap Mod(X_2)) = Mod(Y \cup X_1) \cup Mod(Y \cup X_2)$. From this follows $Mod(Y \cup Z) \subseteq Mod(\phi)$ and this yields finally $\phi \in C_L(Y \cup Z)$.

(2) Let (L, C_L) be auto-distributiv and absolute. We show that $Mod(X_1) \cup Mod(X_2) = Mod(X_1 \cap X_2)$, we may assume that $X_1 \cap X_2$ is consistent, i.e. $C_L(X_1 \cap X_2) \neq L$. It is sufficient to prove that $Mod(X_1 \cap X_2) \subseteq Mod(X_1) \cup Mod(X_2)$. Assume, there is a model $m \in Mod(X_1 \cap X_2)$ such that $m \notin Mod(X_1) \cup Mod(X_2)$. By the definition of the LT-semantics m is a maximal consistent set such that $X_1 \cap X_2 \subseteq m$ but $X_1 \not\subseteq m$ and $X_2 \not\subseteq m$. This implies that $C_L(X_1 \cup m) = C_L(X_2 \cup m) = L$. By auto-distributivity it follows that $C_L(m \cup X_1) \cap C_l(m \cup X_2) \subseteq C_L(m \cup (X_1 \cap X_2))$. This implies $M = L$, which gives a contradiction. $\square$.

Definition 7 *Let $IF(\mathcal{F})$ be the inference frame associated to the semantical frame $\mathcal{F}$.*

1. *An inference frame $\mathcal{I} = (L, C_L, C)$ is said to be complete with respect to a semantical frame $(L, M, \models, \Phi)$ iff (L, C_L) is complete with respect to $(L, M, \models)$ and $C = C_\Phi$.*

2. *A class $\mathbf{K}$ of inference frames is representable in a class $\mathbf{F}$ of semantical frames iff $\mathbf{K} \subseteq \{IF(\mathcal{F}) : \mathcal{F} \in \mathbf{F}\}$; if in addition $\{IF(\mathcal{F}) : \mathcal{F} \in \mathbf{F}\} \subseteq \mathbf{K}$ then $\mathbf{F}$ is said to be adequate for $\mathbf{K}$.*

We introduce following abbreviations for classes of inference frames (i.f.).

- F_{la} - class of all left-absorbing i.f.;
- F_{fa} - class of all deductive i.f.;
- F_{cum} - class of all cumulative deductive i.f.;
- F_{loop} - class of all cumulative deductive i.f. satisfying loop;
- F_{dist} - class of all distributive deductive i.f.;
- F^t_{dist} - class of all distributive topological deductive i.f.;
- F_{cond} - class of all deductive cumulative frames satisfying conditionalization;
- F_{rat} - class of all rational deductive i.f.

Representation theorems for such classes of deductive frames can be formulated in terms of adequateness.

Proposition 4 *1. Let be $\mathcal{F} = (L, C_L, C)$ an inference frame satisfying left absorption. Then there exists a semantical frame $\mathcal{SF} = (L, M, \models, \Phi)$ such that $\mathcal{F}$ is complete with respect to $\mathcal{SF}$.*

2. For any semantical frame $(L, M, \models, \Phi)$, the inference frame $(L, C^{\models}, C_\Phi)$ is left absorbing.

Proof: 1. Let $(L, M, \models)$ be the Lindenbaum semantics for (L, C_L) and $\Phi(X) = \{m : m \in M \text{ and } C(X) \subseteq m\}$. It is easily to show that $C = C_\Phi$. Now let $(L, M, \models)$ be an arbitrary semantics representing (L, C_L). Define $\Phi(X) = Mod^{\models}(C(X))$; then $C_\Phi(X) = Th^{\models}(Mod^{\models}(C(X)) = C(X)$ since $C^{\models}(C(X)) = C(X)$.

2. $C^{\models}(Th^{\models}(\Phi(X))) = Th^{\models}(\Phi(X)) = C_\Phi(X)$. □

Left absorption does not imply congruence. From this follows that the class of semantical frames is not adequate for the class of deductive frames. We get an adequateness result for deductive frames by using selection frames or invariant semantical frames.

Proposition 5 *1. Let $\mathcal{F} = (L, C_L, C)$ be a deductive frame. Then there exists a semantical frame $\mathcal{S} = (L, M, \models, \Phi)$ such that Φ is a model selector and $\mathcal{S}$ represents $\mathcal{F}$.*

2. If Φ is a model selector for the logical system $(L, M, \models)$ then $(L, C^{\models}, C_\Phi)$ is a deductive frame.

Proof: (1) Let $(L, M, \models)$ be the Lindenbaum semantics for (L, C_L), and $\Sigma(K) = Mod^{\models}(C(Th^{\models}(K)))$ for every $K \in Im(M)$. Obviously,
$C_\Sigma(X) = Th^{\models}(\Sigma(Mod^{\models}(X)) =$

$Th^{\models}(Mod^{\models}(C(Th(Mod^{\models}(X))))) = C(Th^{\models}(Mod^{\models}(X)) =$
$C(C^{\models}(X)) = C(X)$.
(2) Let $\Phi(X) = \Sigma(Mod^{\models}(X))$ for a certain selection function $\Sigma : Im(M) \to M$. Then $C_{\Phi}(C_L(X)) = Th^{\models}(\Phi(C_L(X)) = Th^{\models}(\Sigma(Mod^{\models}(C_L(X)))) =$
$Th^{\models}(\Sigma(Mod^{\models}(X))) = C_{\Phi}(X)$. $\Box$

In the sequel we classify deductive frames. Since every deductive frame can be represented by a selection frame it is sufficient to describe subclasses of selection frames which are adequate for certain classes of deductive frames. We may further distinguish deductive frames with respect to the topological properties of the underlying deductive system. A deductive frame is said to be topological iff the underlying deductive system presents a topological space. Analogous, a semantical frame $(L, C^{\models}, C_{\Phi})$ is said to be topological if the elementary classes present a topological space. Let $(M, cl(M))$ be a topological space presented by the closed subsets $cl(M) \subseteq Pow(M)$. A function $f : cl(M) \to cl(M)$ is said to be contractive if $f(U) \subseteq U$ for all $U \in cl(U)$. We may assume that the selection function Σ is contractive.

Definition 8 *Let $(L, M, \models, \Sigma)$ a closed selection frame. The binary relation $\leq_{\Sigma} \subseteq cl(M) \times cl(M)$ is defined as follows.*

1. *$K \leq_{\Sigma} K_1$ iff $K = K_1$ or there is a $K_2 \supseteq K_1$, $K_2 \in cl(M)$ such that $\Sigma(K_2) = K$.*
2. *$\Omega(X) = \{K : K \in cl(M)$ and $K \subseteq Mod(X)\}$.*

It can easily be shown that $\Sigma(\mathrm{Mod}(X))$ is the $\leq_{\Sigma}$-minimum in $\Omega(X)$.

Proposition 6 *Let $(L, M, \models, \Sigma)$ be a selection frame such that the relation $\leq_{\Sigma}$ is anti-symmetric. Then $(L, C^{\models}, C_{\Sigma})$ is a cumulative deductive frame.*

Proof: It is $C_{\Sigma}(X) = Th(\Sigma(\mathrm{Mod}(X)))$. Since $(L, C^{\models}, C_{\Sigma})$ is a deductive frame , it remains to show $X \subseteq Y \subseteq C_{\Sigma}(X) \Rightarrow C_{\Sigma}(Y) = C_{\Sigma}(X)$. Assume $X \subseteq Y$; then $\mathrm{Mod}(X) \supseteq \Sigma(\mathrm{Mod}(Y))$. By definition of $\leq_{\Sigma}$ this implies $\Sigma(\mathrm{Mod}(X)) \leq_{\Sigma} \Sigma(\mathrm{Mod}(Y))$. Furthermore, from $X \subseteq Y \subseteq C_{\Sigma}(X)$ follows $X \subseteq Y \subseteq Th(\Sigma(Mod(X)))$, then $\mathrm{Mod}(Th(\Sigma(\mathrm{Mod}(X)$ $))\subseteq\mathrm{Mod}(Y)$, and finally $\Sigma(\mathrm{Mod}(X)) \subseteq \mathrm{Mod}(Y)$. By definition of $\leq_{\Sigma}$ it follows $\Sigma(Mod(Y))\leq_{\Sigma}$ $\Sigma(\mathrm{Mod}(X))$, and by anti-symmetry of $\leq_{\Sigma}$ we get $\Sigma(\mathrm{Mod}(X)) = \Sigma(\mathrm{Mod}(Y))$, hence $C_{\Sigma}(X) = C_{\Sigma}(Y)$.$\Box$

Proposition 7 *Let (L, C_L, C) be a cumulative deductive frame. Then there exists a closed selection frame $(L, M, \models, \Sigma)$ whose relation $\leq_{\Sigma}$ is anti-symmetric such that $C_L = C^{\models}$ and $C = C_{\Sigma}$.*

Proof: Let $(L, M, \models)$ the LT-representation for (L, C_L). For $K \in cl(M)$ we define: $\Sigma(K) = Mod^{\models}(C(Th(K)))$, and $C_{\Sigma}(X) = Th(\Sigma(Mod^{\models}(X)))$. Obviously, $C_{\Sigma}(X) = Th(Mod^{\models}(C(Th(Mod^{\models}(X)))))= C(Th(Mod^{\models}(X)) = C(C^{\models}(X)) = C(X)$. It remains to show that the relation $\leq_{\Sigma}$ is asymmetric. Let $K_1 \leq_{\Sigma} K_2$ and $K_2 \leq_{\Sigma} K_1$. By definition $\Sigma(K_i) = \mathrm{Mod}(C(Th(K_i)))$, i = 1,2. By definition of $\leq_{\Sigma}$ there is a

$K^1 \supseteq K_2$ such that $\overline{\Sigma}(K^1) = K_1$, i.e. $\mathrm{Mod}(C(Th(K^1))) = K_1$ and a $K^2 \supseteq K_1$ such that $\Sigma(K^2) = K_2$. Since C_Σ is cumulative it holds: $X \subseteq Y \subseteq C_\Sigma(X) \Rightarrow C_\Sigma(X) = C_\Sigma(Y)$. From $K_1 \subseteq K^2$ it follows $Th(K^2) \subseteq Th(K_1)$, and because of $K_1 = \mathrm{Mod}(C(Th(K^1)))$ we get $Th(K^2) \subseteq Th(\mathrm{Mod}(C(Th(K^1)))) = C_\Sigma(Th(K^1))$. Then $Th(K^1) \subseteq (Th(K^1) \cup Th(K^2) \subseteq C_\Sigma(Th(K^1)$, and by cumulativity of C_Σ it follows:

$$(1)\ C_\Sigma(Th(K^1)) = C_\Sigma(Th(K^1) \cup Th(K^2)),$$

Similar one shows:

$$(2)\ C_\Sigma(Th(K^2)) = C_\Sigma(Th(K^1) \cup Th(K^2)).$$

Hence, $C_\Sigma(Th(K^1)) = C_\Sigma(Th(K^2))$ and it follows
$K_1 = \mathrm{Mod}(C_\Sigma(Th(K^1))) = Mod(C_\Sigma(Th(K^2))) = K_2$. □

We introduce the following abbreviations for classes of semantical frames:

- Λ - semantical frames in general;
- Λ_S - selection frames;
- Λ_{cum} - anti-symmetric selection frames;
- Λ_{ord} - semantical frames obtained from a partial ordering
- Λ^t_{dist} - topological selection frames whose selection function satisfies the condition $\Sigma(K_1 \cup K_2) \subseteq \Sigma(K_1) \cup \Sigma(K_2)$.

We summarize the main results.

Proposition 8 *1. The class Λ of all semantical frames is adequate for F_{la}.*

2. Λ_S is adequate for F_{fa}.

3. Λ_{cum} is adequate for F_{cum}.

4. Λ^t_{dist} is adequate for F^t_{dist}.

In the papers [Ma89], [KL90], and [FL93] classes of cumulative deductive frames have been extensively investigated. Adequateness results were proved for classes of deductive cumulative frames by using selection frames where the selection function Σ (and thereby the operator Φ) is defined by a preferential relation between states.
Further adequateness results concerning the classes F_{cond}, F_{rat} are proved in [FL93]. All these results can be reconstructed in our framework. We conclude this section with some structural results on deductive frames. The previous representation results assume an inference frame (L, C_L, C) containing a deductive system (L, C_L) and an inference system (L, C). Conversely, we may start with an inference system (L, C) (a deductive system (L, C_L)) and try to expand it to an inference frame (L, C_L, C) by adding suitable deductive operations C_L (inference operation C). Of course, there is always the trivial expansion (L, C_{id}, C) $((L, C_L, C_L))$, where $C_{id}(X) = X$. Are there natural non-trivial expansions? Let (L, C_L, C) be an inference frame. Obviously, cut of C implies left absorption and cumulativity of C implies full absorption.

Lemma 9 *Let $\mathcal{F} = (L, C_L, C)$ be an inference frame satisfying cut. Then $\mathcal{F}$ satisfies left absorption. If $\mathcal{F}$ satisfies cumulativity then $\mathcal{F}$ satisfies full absorption.*

Proof: Let $X \subseteq L$; by supra-deductivity $X \subseteq c_L(C(X)) \subseteq C(C(X))$. By cut $C(C(X)) = C(X)$, hence $C_L(C(X)) \subseteq C(X)$; obviuosly $C(X) \subseteq C_L(C(X))$; this yields $C_L(C(X)) = C(X)$. Now assume that C is cumulative. By supra-deductivity $X \subseteq C_L(X) \subseteq C(X)$, then by cautious monotony $C(X) \subseteq C(C_L(X))$, and by cut $C(C_L(X)) \subseteq C(X)$, hence $C(C_L(X)) = C(X)$. $\Box$

Proposition 10 *[Di94] (1)Let (L, C) be an inference system satisfying inclusion. Then the set $\{C_0 : C_0 \leq C$ and (L, C_0, C) is an inference frame satisfying left absorption $\}$ has a greatest element.*
(2)There are inference systems (L, C) satisfying inclusion such that the set $\{C_0 : C_0 \leq C$ and (L, C_0, C) is a deductive frame $\}$ has no greatest element.

Proof: (1) Let $\Gamma_{la}(C) = \{C_0 \mid C_0 \leq C$ and (L, C_0, C) is an inference frame satisfying left absorption $\}$. Obviously, $\Gamma_{la}(C) \neq \emptyset$, because $C_{id} \in \Gamma_{la}(C)$, where $C_{id}(X) = X$ for all $X \subseteq L$. Now we show that the set $\Gamma_{la}(C)$ contains maximal element with respect to the partial ordering $\leq$, where $C_1 \leq C_2$ if and only if for all $X \subseteq L$: $C_1(X) \subseteq C_2(X)$. Let $C_1 \leq C_2 \leq \ldots \leq C_n \leq \ldots$ be an increasing chain of elements in Γ_{la}, and let $C_L(X) = \bigcup_{i<\omega} C_i(X)$. Obviously, C_L is an upper bound for $\{C_i \mid i < \omega\}$. We prove that $C_L \in \Gamma_{la}$. It is easy to see that C_L satisfies monotony and inclusion. It holds also $C_L(C_L(X)) = C_L(X)$. Let $\phi \in C_L^2(X)$, then, by definition of C_L there is an index $i0$ such that $\phi \in C_{i0}(C_L(X))$. By the compactness theorem for C_{i0} there is a finite subset $X_f \subseteq C_L(X)$ such that $\phi \in C_{i0}(X_f)$. Obviously, there is an index $m0$ such that $X_f \subseteq C_{m0}(X)$. This implies $\phi \in C_{i0}(C_{m0}(X)) \subseteq C^2_{max(i0,m0)}(X) = C_{max(i0,m0)}(X) \subseteq C_L(X)$, hence C_L is idempotent. C_L is also compact: $\phi \in C_L(X)$ implies $\phi \in C_{i0}(X)$ for an index $i0 < \omega$, and hence $\phi \in C_{i0}(X_f)$ for a certain finite subset $X_f \subseteq X$. (L, C_L, C) satisfies left absorption since the condition $\phi \in C_L(C(X))$ implies the existence of an index $i0$ such that $\phi \in C_{i0}(C(X)) = C(X)$. By Zorn's lemma there exists a maximal element in $\Gamma_{la}(C)$.[2] Assume now, that $\Gamma_{la}(C)$ contains two different maximal elements C_1, C_2. Define $C'(X) = \bigcup_{n<\omega}(C_1C_2)^n(X)$. It is easily to show that $C' \in \Gamma_{la}(C)$. This gives a contradiction because $C_1 \leq C'$ and $C_2 \leq C'$.

(2) Let be $L = L_P \cup L_Q$, where $L_P = \{p_i | i < \omega\}$; $L_Q = \{q_i | i < \omega\}$. Let $mp(X) = sup\{i | p_i \in X\}$; $mq(X) = sup\{i | q_i \in X\}$. The inference operations C, C_i, C_ω, $i < \omega$ are defined as follows: $C(X) = X \cup \{q_{mp(X)}\}$, if $mp(X) < \omega$, else $C(X) = X \cup L_P$. $C_k(X) = X \cup \{p_j | j \leq mp(X) + k\}$, if $mp(X) < \omega$, else $C_k(X) = X \cup L_P$. $C_\omega(X) = X \cup L_P$. It is easy to see that the operations C_k and C_ω are compact inference operations satisfying $C_K \leq C$ and $C_\omega \leq C$. Both frames (L, C_k, C) and (L, C_ω, C) are left absorbing. Furthermore, (L, C_k, C) ist full absorbing for every $k < \omega$. But (L, C_ω, C) does not satisfy congruence. We show that the set $\{C_k : k < \omega\}$ has no upper bound in $\Gamma_{fa}(C)$. Assume Cn is an upper bound for C_k, and (L, Cn, C) is a deductive frame. Obviously, $C_\omega \leq Cn$. This implies $CC_\omega = C$, which gives a contradiction. $\Box$

[2] The presented proof can be immediately generalized to chains of cofinality greater than ω.

Proposition 11 *[Di94] If the inference system* (L, C) *is cumulative then the set* $\{C_0 : C_0 \leq C$ *and* (L, C_0, C) *is a deductive frame* $\}$ *has a greatest element.*

Proof: By prop.10 the set $\Gamma_{la}(C)$ has a greatest element C_L. Cumulativity of C implies full absorption, hence (L, C_L, C) is a deductive frame. □

The largest deductive base for a cumulative inference operation $C : 2^L \to 2^L$ seems to define the adequate underlying monotonic logic for C. This is stressed by the fact that important properties of deductive frames are preserved upwards. In particular, if (L, C_i, C), $i = 1, 2$ are deductive frames such that $C_1 \leq C_2$ and (L, C_1, C) is distributive then (L, C_2, C) is distributive too. For the subsequent considerations we use the following notation. Let (L, C) be an inference system, and $\Gamma(C) = \{C_L | (L, C_L, C)$ is a inference frame $\}$; $\Gamma_{la} = \{C_L | (L, C_L, C)$ is an inference frame satisfying left absorption $\}$; $\Gamma_{fa}(C) = \{C_L | (L, C_L, C)$ is an inference frame satisfying full absorption $\}$. Let P be a property of inference frames. P is preserved upwards if for every inference operation $C, C_1, C_2 \in \Gamma(C)$, and $C_1 \leq C_2$ holds: if (L, C_1, C) has property P then (L, C_2, C) has. Analogously we define preservation downwards. The following proposition is obvious.

Proposition 12 *The properties of rationality, conditionalization and consistency preserving are preserved upwards. Left absorption and congruence are preserved downwards.*

According to Makinson [Ma93] an inference operation C can be understood as logical if (L, Cn, C) satisfies full absorption, where Cn is classical logic. This concept is too narrow to understand the logical character of an inference operation C, because only classical logic (L, Cn) is admitted. We use the following notion of logicality. An inference system (L, C) is said to be *logical* iff $\Gamma_{fa}(C)$ has a greatest element. Then we get

Proposition 13 *Every cumulative inference operation is logical.*

The converse of this proposition is not true: there are logical inference operations not being cumulative. If the inference operation C is not logical its *logical content* is presented by $\Gamma_{fa}(C)$.

5 Compactness and Finitary Restrictions

Firstly, we given an overview about compactness properties which were studied in the literature. Let $\mathcal{F} = (L, M, \models \Phi)$ be a semantical frame. C_Φ is *semantically compact* if for every set $X \subseteq L$ the following holds: if $\Phi(X_f) \neq \emptyset$ for every finite subset $X_f \subseteq X$ then $\Phi(X) \neq \emptyset$. In classical logic compactness and semantical compactness coincide. For arbitrary semantical frames this is not longer true. The following facts clarify the relation between compactness and semantical compactness. Φ is *strongly semantical compact* iff for every set $X \subseteq L$ and formula $\phi \in L$ the following holds: if $\Phi(X_f) \cap \mathrm{Mod}(\phi) \neq \emptyset$ for every finite subset $X_f \subseteq X$ then $\Phi(X) \cap \mathrm{Mod}(\phi) \neq \emptyset$. The following proposition shows the interrelation between these properties.

Proposition 14 *Let $\mathcal{F} = (L, M, \models, \Phi)$ be a semantical frame.*

1. *Assume L has contradiction. If C_Φ is compact then it is semantically compact.*

2. *Assume L has negation. Then C_Φ is strongly compact if and only if it is compact.*

Proof: (1): Let C_Φ be compact and assume C_Φ is not semantical compact. Then there is a set $X \subseteq L$ such that for every finite subset $X_f \subseteq X : \Phi(X_f) \neq \emptyset$, but $\Phi(X) = \emptyset$. Then $\perp \in C_\Phi(X)$, since $C_\Phi(X) = L$. By compactness of C_Φ there is a finite subset $Y \subseteq X$ such that $\perp \in C_\Phi(Y)$. But then $\Phi(Y) = \emptyset$, a contradiction. ($2 \leftarrow 1$): Assume that C_Φ satisfies compactness and ϕ does not satisfy strong semantical compactness. Then there is a set X and a formula $\phi \in L$ such that $\Phi(X_0) \cap \mathrm{Mod}(\phi) \neq \emptyset$ for every finite subset $X_0 \subseteq X$ but $\Phi(X) \cap Mod(\phi) = \emptyset$. Then $\Phi(X) \subseteq Mod(\neg\phi)$, ie. $\neg\phi \in C_\Phi(X)$. By compactness of C_Φ there is a finite subset $Y \subseteq X$ such that $\neg\phi \in C_\Phi(Y)$, and from this follows $\Phi(Y) \subseteq \mathrm{Mod}(\neg\phi)$. This implies the condition $\Phi(Y) \cap Mod(\phi) = \emptyset$, a contradiction.
($2 \rightarrow$): Assume Φ is strongly semantical compact and $X \models_\Phi \phi$, and assume that for every finite subset $X_0 \subseteq X$ holds $\phi \notin C_\Phi(X_0)$. Then $\Phi(X_0) \cap Mod(\neg\phi) \neq \emptyset$. Since Φ is strongly compact it follows $\Phi(X) \cap \mathrm{Mod}(\neg\phi) \neq \emptyset$, and this is a contradiction to $\phi \in C_\Phi(X)$. $\square$

We distinguish two cases.

(1) C is monotonic. Then let $\Delta_0(C_f) = \bigcup_{Y \in Fin(X)} C_f(Y)$. Δ_0 can be considered as an operator extending finitary inference operation to infinitary ones, and if C is monotonic then $\Delta_0(C_f) \leq C$. If C is monotonic and compact then $\Delta_0(C_f) = C$, ie. C is uniquely defined by its finitary restriction via Δ_0. In case C is not compact $\Delta_0(C)$ gives an approximation of C from below. If C represents first order logic then $\Delta_0(C_f) = C$; if C is second order logic then $\Delta_0(C_f) \neq C$.

(2) C does not satisfy monotony. Then there is no operator Δ allowing to reconstruct every cumulative inference operation from its finitary restriction. However, one could seek for a family of such operators generating a class of inference operations from its finitary elements. Thus, it seems an interesting task to investigate such operators in a general setting. For this purpose we use the following notions from [He94]. [3]

Definition 9 *Let (L, C_L) be a deductive system. $\mathcal{D}(L, C_L) = \{C : (L, C_L, C)$ is a deductive frame $\}$; $\mathcal{D}_f(L, C_L) = \{C : C$ is finitary and (L, C_L, C) is a deductive frame $\}$; $\mathcal{I}(L, C_L) = \{C : (L, C_L, C)$ is an inference frame $\}$.*

1. *A functor $\Delta : \mathcal{D}_f(L, C_L) \to \mathcal{I}(L, C_L)$ is said to be an extension operator if for every $C \in \mathcal{D}_f(L, C_L)$ the conditions $dom(\Delta(C)) = 2^L$ and $\Delta(C) \downarrow Fin(L) = C$ are satisfied. Δ is called deductive if $im(\Delta) \subseteq \mathcal{D}(L, C_L)$.*

2. *An inference operation $C : 2^L \to 2^L$ is Δ-compact iff $C \subseteq \Delta(C_f)$; C is completely Δ-compact iff $C = \Delta(C_f)$, where $C_f = C \downarrow Fin(C)$.*

[3] *Freund/Lehmann* [FL93] ask whether there is a reasonable notion of *pseudo-compactness* for which any finitary operation has an unique pseudo-compact extension to an infinitary one. It seems that there is no such notion. However, one should be more liberal and admit a well-behaved family of extension operators solving this problem.

3. *Let P be a property on deductive frames and $\mathcal{D}(P)$ the class of all deductive frames satisfying P. An extension operator Δ preserves the property P if for every $C \in dom(\Delta) \cap \mathcal{D}(P)$ the condition $\Delta(C) \in \mathcal{D}(P)$ is fulfilled.*

Abstract compactness properties can be expressed by conditions $cp(C_L, C, Fin(L))$ depending on C, C_L and the finite subsets of the language L. Important compactness properties are summarized in the following definition.

Definition 10 *Let (L, C_L, C) be a deductive frame.*

1. *C is weakly compact iff for every $X \subseteq L$, $\phi \in C(X)$ there is a finite subset $A \subseteq C_L(X)$ such that $\phi \in C(A)$.*
2. *C is supracompact iff for every $X \subseteq L$, $\phi \in C(X)$ there is a finite subset $A \subseteq X$ such that $\phi \in C(A \cup B)$ for every subset $B \subseteq C(A)$.*
3. *C is finitary supracompact iff for every $X \subseteq L$, $\phi \in C(X)$ there is a finite set $A \subseteq X$ such that for every finite $B \subseteq C_L(X)$ the condition $\phi \in C(A \cup B)$ is satisfied.*
4. *C is weakly supracompact iff for every $X \subseteq L$, $\phi \in C(X)$ and every finite $A \subseteq C_L(X)$ there is a finite set B, $A \subseteq B \subseteq C_L(X)$ such that $\phi \in C(B)$.*

The notion of supracompactness was introduced by *Freund*[Fr91]; finitary supracompactness was studied in *Freund/Lehmann* [FL93]. We collect some examples of extension operators, denoted by $\Delta_i, i = 1, 2, 3, 4$. Let (L, C_L, C) be a deductive frame. Then the transform of C is defined as follows: $C^t(X) = \{\phi \in L :$ there is a finite set $A \subseteq X$ such that for all $Y : A \subseteq Y \subseteq C(X) : \phi \in C(Y)\}$. It is easily to see that C^t is compact and $C_f = (C^t)_f$.

Definition 11 *Let (L, C_L, C) be a deductive frame, and F a finitary and cumulative inference operation which is logical over (L, C_L).*

1. *$\Delta_1(F)(X) = F(A)$, if there is a finite subset $A \subseteq C_L(X)$ such that $C_L(X) \subseteq F(A)$, and $\Delta_1(F)(X) = C_L(X)$ o.w.*
2. *$\Delta_2(F)(X) = \{\phi \in L :$ there is a finite subset $A \subseteq X$ such that for every finite $B \subseteq C_L(X); \phi \in F(A \cup B)\}$.*
3. *Let $C_0 = \Delta_2(F)$, $C_{i+1} = (C_i)^t$, and $\Delta_3(F) = \bigcap_{i<\omega} C_i$.*
4. *$\Delta_4(F)(X) = \{\phi :$ for every finite $A \subseteq C_L(X)$ there is a finite B such that $A \subseteq B \subseteq C_L(X)$ and $\phi \in F(B)\}$.*

The extension operators $\Delta_1, \Delta_2, \Delta_3$ were studied in [FL 93]. The deductive extension operator Δ_1 was invented by *Makinson* and has the following interesting property. Let (L, C_L, F) be a cumulative and finitary deductive frame. Then $\Delta_1(F)$ is the smallest extension of F such that $(L, C_L, \Delta_1(F))$ is a cumulative deductive frame.

Unfortunately, all these three operators do not cover one of the most natural nonmonotonic systems: *minimal reasoning.* Minimal reasoning is not Δ_i-compact $i \in \{1,2,3\}$ and is neither supracompact nor finitary supracompact. The operator Δ_4 was introduced and presented in [He93]. There are following relations between the operators Δ_i, $i = 1,2,3,4$. $\Delta_1(F) \leq \Delta_3(F) \leq \Delta_2(F) \leq \Delta_4(F)$. From this follows: if C is Δ_i-compact, $i \in \{1,2,3\}$, then C is Δ_4-compact. Using the transform C^t further extension operators can be derived: $C_0 = \Delta_4(F)$, $C_{i+1} = (C_i)^t$, $\Delta_5(F) = \bigcap_{i<\omega} C_i$.

In the following we show that the extension operator Δ_4 is suitable for analysing minimal reasoning in propositional and predicate logic.

6 Properties of Nonmonotonic Systems

In this section we start a systematic study of wellknown nonmonotonic systems. In the present paper we restrict the investigation to minimal reasoning and propositional Poole systems.

6.1 Classical Propositional Calculus

Let $\mathcal{L}(Prop) = (L(Var), M, \models)$ the classical propositional logic with standard semantics, ie. $M = 2^{Var}$. The language $L(Var)$ is generated from the set Var of propositional variables by the functors $\{\wedge, \vee, \neg\}$. For a set $X \subseteq L(Var)$ let $var(X)$ be the set of all variables appearing in X. For $V \subseteq Var$ let $L(V)$ be the sublanguage of $L(Var)$ generated by V. Let $Mod_V(X)\{w \downarrow V : w \in \mathrm{Mod}(X)\}$. As usual for $v, w \in M; v \leq w \Leftrightarrow (\forall p \in Var)(v(p) \leq w(p))$. If v, w are partial functions on Var then $v \sqsubseteq w$ iff $dom(v) \subseteq dom(w)$ and $v(p) = w(p)$ for every $p \in dom(v)$. A model v of X is minimal iff there is no model $w \in \mathrm{Mod}(X)$ such that $w < v$: $Min(X)$ denotes the set of all minimal models of X. The inference operation C_m of minimal reasoning is defined by $C_m(X) = Th(Min(X))$, and let $X \models_{min} \phi$ iff $\phi \in C_m(X)$. The following three propositions are wellknown.

Proposition 15 *Let X be a set $\subseteq L(Var)$ and $V \subseteq Var$ a finite subset of $var(X)$. Then there is a finite subset $B \subseteq C_L(X)$, $var(B) = V$ such that $Mod_V(X) = Mod_V(B)$.*

Proposition 16 *If $\phi \in L(V)$ then $X \models \phi$ iff $Mod_V(X) \subseteq Mod(\{\phi\})$.*

Proposition 17 *Every model $v \in Mod(X)$ is an extension of a minimal model of X.*

Obviously, the model operator $\Phi_m(X) = Min(X)$ is semantical compact, since for every classically consistent set X the condition $\Phi_m(X) \neq \emptyset$ is satisfied. The following example shows that the inference operation of minimal reasoning C_m is not compact. Let $X = \{p_1 \wedge \ldots \wedge p_i \wedge (p_{i+1} \vee p_0) : 1 \leq i < \omega\}$. Then $X \models_m \neg(p_0 \leftrightarrow p_1)$. If $m \in Min(X)$ then $m \models \neg p_0, m \models p_1$, hence $m \models \neg(p_0 \leftrightarrow p_1)$. For every finite subset $X_f \subseteq X$ holds $X_f \not\models_{min} \neg(p_0 \leftrightarrow p_1)$. [4]

[4]Another, more complicated example is presented in [PW90]

Proposition 18 *The propositional semantical frame* $\mathcal{LF} = (\mathcal{L}(Prop), \Phi_m)$, *where* $\Phi_m(X) = Min(X)$, *is weakly supracompact.*

Proof Let $X \models_m \phi$, $A \subseteq C_L(X)$, A finite and $var(A) \cup var(\phi) = \{p_1, \ldots, p_s\} = V$. By prop. 15 there is a finite subset $B \subseteq C_L(X)$ such that $var(B) = \{p_1, \ldots, p_s\}$ and $Mod_V(X) = Mod_V(A \cup B)$. Let $w \in Min_V(A \cup B)$, then $w \in 2^V$. w can be extended to a model $d \in \mathrm{Mod}(X)$. By prop. 17 there is a minimal model $e \in Min(X)$ such that $e \leq d$. By assumption $e \models \phi$. It is $e \downarrow V = w$, since w is minimal for $A \cup B$. By prop. 16: $w \models \phi \Leftrightarrow e \models \phi$, hence $A \cup B \models_m \phi$. □

Proposition 19 *Let* $X \subseteq L(Var)$ *and* $\phi \in L(Var)$. *Following conditions are equivalent:*

1. $X \models_m \phi$,
2. *for every finite subset* $A \subseteq C_L(X)$, $var(\phi) \subseteq A$, *there exists a finite subset* $B \subseteq C_L(X)$ *such that* $var(B) \subseteq var(A)$ *and* $A \cup B \models_m \phi$.

Proof: The implication (1) $\rightarrow$ (2) follows as in the proof of prop. 18.

We show (2) $\rightarrow$ (1). Let $\{p_i | i < \omega\}$ be an enumeration of the variables in Var and let $V_i = var(\phi) \cup \{p_k \mid k \leq i\}$. By prop. 15 there is a finite set $A_i \subseteq L(V_i) \cap Cn(X)$ such that $Mod_{V_i}(X) = Mod(A_i)$. Obviously, for every $B \subseteq Cn(X)$ satisfying $var(B) \subseteq var(A_i)$ the condition $Mod(A_i) = Mod(A_i \cup B)$ is true. By condition (2) we may assume that $A_i \models_{min} \phi$ for every i=1,2, ... We show that $X \models_{min} \phi$. Assume, this is not the case. Then there is a minimal model $m \in Min(X)$ such that $m \not\models \phi$, hence $m \models \neg\phi$. Then for every $i < \omega$ we have $m \downarrow V_i \notin Min(A_i)$ (otherwise $m \models \phi$ by $A_i \models_{min} \phi$). Since $A_i \models_{min} \phi$ there is a $n \in Min(A_i)$, $n < m \downarrow V_i$, such that $n \models \phi$. Hence, the set $\mathcal{M}_i = \{n \in 2^{V_i} \mid n \models A_i \cup \{\phi\} \wedge n < m \downarrow V_i\}$ is non-empty for every $i < \omega$. Take $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$ and $\mathcal{T} = (\mathcal{M}, \sqsubseteq)$. Then $\mathcal{T}$ is a finite branching tree. Also, for every $i < \omega$ there is an element $f \in \mathcal{M}$ of height i. By Koenig's lemma $\mathcal{T}$ has an infinite branch $b^* : b_1 \sqsubseteq b_2 \sqsubseteq \ldots$. Take $s = \bigcup_{i<\omega} b_i$. Then $s \models X$, because $s \models A_i$ for every $i < \omega$. Also $s \models \phi$ by definition of $\mathcal{M}_i$, $i < \omega$. We show that $s \leq m$. Let $v \in Var$ and let i be the least number such that $v \in V_i$. By definition of s there is a number $j \geq i$ and an $g \in \mathcal{M}_j$ such that $s \downarrow V_i = g \downarrow V_i$. By definition of $\mathcal{M}_j$ we have $g < m \downarrow V_j$, and this implies $g(v) = s(v) \leq m(v)$. The condition $m \models \neg\phi$ implies $m \neq s$, hence $s < m$, and this yields a contradiction to the minimality of m. □.

Proposition 20 (Corollary) *Let C be the inference operator of minimal reasoning in classical propositional logic. Then C is completely Δ_4-compact, ie.* $C = \Delta_4(C_f)$.

Proof: Let $C_1 = \Delta_4(C_f)$: We prove that for every subset $X \subseteq L : C(X) = C_1(X)$.

(1) $C(X) \subseteq C_1(X)$. Let $\phi \in C(X)$, then for every finite subset $A \subseteq Cn(X)$ there is a finite set $B \subseteq Cn(X)$, $var(B) \subset var(A)$, such that $\phi \in C(A \cup B)$. This implies the condition $\phi \in C_1(X)$ by proposition 19 and the construction of Δ_4.

(2) $C_1(X) \subseteq C(X)$. Let $\phi \in C_1(X)$. By definition: for every finite $A \subseteq Cn(X)$ there is a finite subset $B \subseteq Cn(X)$ such that $A \subseteq B$ and $\phi \in C_f(B)$; we may assume

that $var(\phi) \cup var(B) \subseteq var(A)$. Since B is finite, then $C_f(B) = C(B)$. This implies by prop. 19 $\phi \in C(X)$. $\Box$

Using the results in $[Di94]$ one can prove the following result. Let (L, Cn, C) be the deductive frame corresponding to $(\mathcal{L}(Prop), \Phi_m)$ then (L, Cn) is the greatest deductive basis for (L, C).

6.2 Many-valued Propositional Logic

The results about the classical propositional logic can be generalized to certain many-valued propositional logics. We summarize the crucial points of this generalization.

Proposition 21 *Let $\mathcal{P} = (\sigma, \mathcal{N})$ be a propositional base with finite truth algebra $\mathcal{N}$ and let $(L, M, \models)$ be the logical system based on $\mathcal{P}$. Then every model $v \in Mod(X)$ is an extension of a minimal model of X.*

Proof: Let $v \in \mathrm{Mod}(X)$ and $W(v) = \{w : w \in \mathrm{Mod}(X) \text{ and } w \leq v\}$. Let $\{v_i : i < \omega\}$, $v_{i+1} \leq v_i$, $i < \omega$ a decreasing sequence of elements from $W(v)$. Since $(N \leq)$ is a finite set, the set $\{v_i(p) : i < \omega\}$ has a minimum. We define $u(p) = min\{v_i(p) : i < \omega\}$. Then $u \leq v_i, i < \omega$. It is easily to prove that for every variable $p \in V$ there is a $j < \omega$ such that $u(p) = v_k(p)$ for all $k \geq j$. Let $\phi(p_1, \ldots, p_n) \in PFm(\sigma)$, then there is a $j < \omega$ such that for every $k \geq j$: $u(p_i) = v_k(p_i)$, $i = 1, \ldots, n$. Since $v_k \models \phi$ for all $k \geq j$ it follows $u \models \phi$. $\Box$

The following proposition can similarly proved as in the case of classical logic.

Proposition 22 *Let $\mathcal{P} = (\sigma, \mathcal{N})$ be a finite propositional base and $(L, M, \models)$ the logical system based on $\mathcal{P}$. We assume that the truth algebra $\mathcal{N}$ is definitionally complete, ie. every n-ary function on N can be defined by a term $t \in Tm(\sigma)$. Then the following holds. Let X be a set $\subseteq PFm(\sigma)$ and $V \subseteq Var$ is a finite subset of $var(X)$. Then there is a finite subset $B \subseteq C^{\models}(X)$, $var(B) = V$ such that $Mod_V(X) = Mod_V(B)$.*

Remark: The condition of the definitional completeness of the truth algebra $\mathcal{N}$ in prop. 20 can be weakened. It is sufficient that there are σ-definable functions $f_\wedge, f_\vee, f_\leftrightarrow, f_a$ satisfying following conditions:
$f_\wedge(x, y) \in T$ iff $\{x, y\} \subseteq T$;
$f_\vee(x, y) \in T$ iff $\{x, y\} \cap T \neq \emptyset$;
$f_\leftrightarrow(x, y) \in T$ iff $x = y$;
$f_a(x) = a$ for every $a \in N$.
Using the same ideas we may prove the following generalization.

Proposition 23 *Let $\mathcal{P} = (\sigma, \mathcal{N})$ be a finite propositional base which is definitionally complete, $(L, M, \models)$ the logical system based on $\mathcal{P}$ and $\Phi_m(X) = \{v : v \models X \text{ and } v$ is minimal in $\leq\}$, $C(X) = Th^{\models}(\Phi_m(X))$. Then the deductive system $(L, C^{\models}, C)$ is completely Δ_4-compact, ie. $C = \Delta_4(C_f)$.*

Example 1 *(3-valued logic): Consider the truth algebra* $\mathcal{N}_3 := (N, T, \{f_i\})$, *where* $N := \{1, 0, \perp\}$, $T := \{1\}$, *and* $\{f_i\}$ *is any set of truth functions which ensures functional completeness.* $\mathcal{N}$ *can be ordered by the following relations:* $\leq_t = \{(0,1), (0,\perp), (\perp,1)\}$, $\leq_k = \{(\perp,0), (\perp,1)\}$, *called truth- resp. knowledge order. Then* $\leq_t$ *and* $\leq_k$ *induce relations between valuations, and* $Min_{\leq t}(Min_{\leq k}$ *denote the* $\leq_t$ $(\leq_k)$*-minimal models of* X. *Let* (L, C_3) *be the deductive system defined by the truth algebra* $\mathcal{N}_3$, $C_t(X) := Th(Min_{\leq t}(X))$ *and* $C_k(X) = Th(Min_{\leq k}(X))$. *Then both frames* (L, C_3, C_k) *and* (L, C_3, C_t) *are completely* Δ_4*-compact.*

Example 2 *(Bilattice FOUR): If we consider Ginsberg' s bilattice FOUR we get another example. The truth algebra* $\mathcal{N}_4 := (N, T, \{f_i\})$ *is the diamond structure defined by* $N := \{1, 0, \perp, \top\}$, $T = \{1\}$, *and* $\{f_i\}$ *is any set of functions ensuring functional completeness. The truth order is defined by* $\leq_t = \{(0,1),\ (0,\perp),\ (\perp,1),\ (0,\top)\}$, *the knowledge order by* $\leq_k = \{(\perp,0), (\perp,1), (0,\top), (\perp,\top)\}$. C_4 *is the deductive operation defined by the truth algebra* $\mathcal{N}_4$, *and* C_t, C_k *are defined as above. Then the frames* (L, C_4, C_k) *and* (L, C_4, C_t) *are completely* Δ_4*-compact.*

6.3 Propositional Poole Systems

We conclude the propositonal calculus with the considerations of Poole systems. Let (L, C_L) be a deductive base. A *Poole system* is a pair (D, K) of sets of formulas. D is the set of defaults, K the set of constraints. The system (D, K) is said to be finite if D is finite. A subset $\delta \subseteq D$ is said to be a basis for $X \subseteq L$ iff $C_L(X \cup \delta \cup K) \neq L$ and δ is maximal with this property. Let $B_{(D,K)}(X) = \{\delta : \delta \subseteq D$ and δ is a basis for $X\}$, and $C_{(D,K)}(X) = \bigcap\{C_L(X \cup \delta) : \delta \in B_{(D,K)}(X)\}$. Then $\mathcal{P} = (L, C_L, C_{(D,K)})$ is a deductive frame. In [FL93] there is proved the following proposition.

Proposition 24 *Let* (L, C_L) *be a deductive system having contradiction, denoted by* $\perp$, *and* (D, K) *a finite Poole system. Then* $(L, C_L, C_{(D,K)})$ *is completely* Δ_2*- and completely* Δ_3*-compact.*

We may extend this result to the Δ_4-operator.

Proposition 25 *If* (L, C_L) *has contradiction and* (D, K) *is a finite Poole system, then the deductive frame* $(L, C_L, C_{(D,K)})$ *is completely* Δ_4*-compact.*

Proof Let $C = C_{(D,K)}$, we have to prove that $C = \Delta_4(C_f)$.
(1) $C \leq \Delta_4(C_f)$.
Assume $\phi \in C(X)$; by Δ_2-compactness of (L, C_L, C) there is a finite set $A_0 \subseteq X$ such that for every finite subset $B \subseteq C_L(X)$ the condition $\phi \in C_f(A_0 \cup B)$ is satisfied. Then for every finite set $A \subseteq C_L(X)$ there is a finite set $B \subseteq C_L(X)$ such that $\phi \in C_f(B)$, hence $\phi \in \Delta_4(C_f)(X)$.
(2) $\Delta_4(C_f) \leq C$.
Assume $\phi \in \Delta_4(C_f)(X)$, but $\phi \notin C(C)$. Then there is a basis $\delta \in D$ for X, ie $\delta \in B_{(D,K)}(X)$ such that $\phi \in C_L(X \cup \delta)$. Let $\Omega(\delta) = \{\rho : \rho \subseteq D$ and $\delta \subseteq \rho$ and $\rho \neq \delta\}$.

Obviously, the set $\Omega(\delta)$ is finite. We distinguish two cases.
Case 1. $\Omega(\delta) = \emptyset$.
Then $D \in B_{(D,K)}(X)$ and $C(X) = C_L(X \cup D)$. By assumption $\phi \in C(X)$, and hence $\phi \in C_f(A) = C_L(A \cup D)$ for every finite set $A \subseteq C_L(X)$. This implies $\phi \notin \Delta_4(C_f)(X)$.
Case 2. $\Omega(\delta) \neq \emptyset$. Then the sets $X \cup \rho \cup K$, $\rho \in \Omega(\delta)$, are inconsistent and it follows $X \cup \rho \cup K \vdash_L \bot$. By compactness there are finite sets $A(\rho) \subseteq X$, $\rho \in \Omega(\delta)$, such that $A(\rho) \cup \rho \cup K \vdash_L \bot$. Let $A = \bigcup\{A(\rho) : \rho \in \Omega(\delta)\}$. Then $A \subseteq X$ is a finite set and $A \cup \rho \cup K$ is inconsistent for every $\rho \in \Omega(\delta)$. From this follows that δ is a Poole basis for A and furthermore, $\delta \in B_{(D,K)}(A)$ for every set $Y \subseteq C_L(X)$. From this follows that $\delta \in B_{(D,K)}(B)$ for every finite set $B, A \subseteq B \subseteq C_L(X)$, and hence $\phi \notin C_f(B)$ for every such B. This finally shows that $\phi \notin \Delta_4(C_f)(X)$. $\square$

Now we consider several classes of infinite Poole-systems in classical propositional logic. We assume that for (D, K) the set K of contraints is empty, and $D \subseteq Var \cup Var^c$, where $Var^c = \{\neg p : p \in Var\}$, ie. D is a set of literals. Let X be a set formulas and $\delta \in B_D(X)$. Then the following partial function c_δ can be introduced. $dom(c_\delta) = \{p : \{p, \neg p\} \cap \delta \neq \emptyset\}$ and $c_\delta(p) = 0$ if $\neg p \in \delta$ and $c_\delta(p) = 1$ iff $p \in \delta$. Furthermore, let $m(\delta) = \{w : c_\delta \sqsubseteq w$ and $w \in 2^{Var}$ and $w \models X\}$. Let $Mod_D(X) = \bigcup\{m(\delta) : \delta \in B_D(X)\}$. Then $\phi \in C_D(X)$ iff $Mod_D(X) \subseteq \mathrm{Mod}(X)$.

Proposition 26 *Let $D = \{\neg p : p \in Var\}$, $K = \emptyset$. Then $Mod_D(X) = Min(X)$, and hence $C(X)$ represents minimal reasoning in classical propositional logic.*

Proposition 27 *There are non-compact Poole-systems.*

Proof: this follows immediately from the non-compactness of minimal reasoning in classical propositional logic. $\square$

We now study the problem whether Poole systems in classical propositional logic are weakly compact or even Δ_4-compact. Using the proof of prop. 25 one can show the following.

Proposition 28 *For every set $D \subseteq Var \cup Var^c$ of literals the frame (L, Cn, C_D) based on the Poole-system $(D, \emptyset)$ is completely Δ_4-compact.*

Proposition 28 cannot be extended to more general cases as the following examples show.

Example 3 *There are Poole systems $(\emptyset, D)$ such that the deductive frame $(L, C_0, C_{(\emptyset,D)})$ is not weakly compact; (L, C_0) is classical propositional logic.*
Put $X = \{\bigwedge_{i=1}^{n} \wedge(p_{i+1} \vee p_0) : n \geq 1\}$, $A = \neg(p_0 \leftrightarrow p_1)$, $D = \{\neg p_0\} \cup \{d_i : i > 1\}$, where $d_i = \neg p_i \wedge (p_0 \vee p_i) \wedge (p_1 \vee p_i)$. There is only one maximal subset $D_1 \subseteq D$ such that $X \cup D_1$ is consistent, $D_1 = \{\neg p_0\}$. Hence $C_0(X \cup \{\neg p_0\}) = C_{(\emptyset,D)}(X)$. From this follows that $A \in C_{(\emptyset,D)}(X)$. Let $X_f \subseteq X$ a finite subset and p_N a variable satisfying $p_N \notin Var(X_f)$. It is easily to show that $X_f \cup \{d_N\}$ is consistent since it has a model. Let $D_1 \subseteq D$ a maximal subset such that $d_N \in D_1$ and $X_f \cup D_1$ is consistent. Obviously, $A \notin C_0(X_f \cup D_1)$. Hence $A \notin C_{(\emptyset,D)}(X_f)$.

Example 4 *There are Poole systems* (K, D) *with constraints and a set of defaults containing literals only such that* $(L, C_0, C_{(K,D)})$ *is not weakly compact. Put* $X = \{\bigwedge_{i=1}^n p_i \wedge (p_{i+1} \vee p_0) : i \geq 1\}$, $A = \neg(p_0 \leftrightarrow p_1)$, $D = \{\neg p_i : i \geq 0\}$, $K = \{p_0 \vee p_i : i > 1\} \cup \{p_1 \vee p_i : i \geq 1\}$. *Similarly as in example 3 one can show that* $A \in C_{(K,D)}(X)$ *but for every fsinite subset* $X_f \subseteq C_{(K,D)}(X)$ *we have* $A \notin C_{(K,D)}(X_f)$.

6.4 Predicate Logic

Now we investigate the problem whether these results can be generalized to first order predicate logic. Let $\mathcal{L}(Pred) = (L, \bigcup_{\tau \in \mathcal{T}} Str(\tau), \models_L)$ be the logical system corresponding to first order predicate logic over the signatures from $\mathcal{T}$. $\mathcal{LF}(Pred) = (\mathcal{L}(Pred), \Phi_m)$ is the predicative semantical frame determined by $\mathcal{L}(Pred)$ and the operator Φ_m which selects minimal models, ie. $\Phi_m(X) = \{\mathcal{B} : \mathcal{B}$ is a minimal model of $X\}$. The minimality of a model is defined as follows. $\mathcal{B} = (U, (R_i)_{i \in I}, (f_j)_{j \in J})$ is a minimal model of X if $\mathcal{B} \models X$ and for every $\mathcal{C} = (U, (R_i^*)_{i \in I}, (f_j)_{j \in J})$: if $R_i^* \subseteq R_i$ for all $i \in I$ and there is an $i \in I$ such that $R_i^* \neq R_i$ then $\mathcal{B} \not\models X$. In contradistinction to propositional logic minimal reasoning in predicate logic is not semantically compact. This is shown by the following example. Let be $\sigma = (P(x), s(x))$, P a unary predicate symbol, s a unary function symbol. Let $Z(n) :=$ "There is a s-circle of length n", and $\phi := \forall x(P(x) \rightarrow P(s(x)) \wedge \forall x \exists y(s(y) = x) \wedge \forall x \forall y(s(x) = s(y) \rightarrow x = y) \wedge \exists x P(x)$. $Ax(\infty)$ is the following set of axioms: $\{\phi \wedge \neg Z(n) : n < \omega\}$.
Let $X \subseteq Ax(\infty)$ be a finite subset. Then there is a number $k < \omega$ such that for every $j \geq k : \phi \wedge Z(j) \notin X$. Obviously, X has a minimal model: choose a circle A of length $k \geq j$ and define $x \in P \Leftrightarrow x \in A$. From this follows $\Phi_m(X) \neq \emptyset$. But $\Phi_m(Ax(\infty)) = \emptyset$: take an arbitrary model $\mathcal{A} \models Ax(\infty)$; then $P(\mathcal{A}) \neq \emptyset$. It is easily to show that $\mathcal{A}$ cannot be a minimal model of $Ax(\infty)$.

Proposition 29 *The predicative semantical frame* $\mathcal{LF}(Pred) = (\mathcal{L}(Pred), \Phi_m)$ *is not weakly compact.*

Proof: Let σ the following signature: $\sigma = (\cup, \cap, -, I(x), 0, 1)$, where $\cup, \cap$ are binary operation symbols, $-$ is a unary operation symbol, I is an unary predicate, and $0, 1$ are constants. Let $Ax(BA)$ be a finite set of sentences axiomatizing the theory of atomic Boolean algebras. We introduce following definitions and formulas:

- $at(x) \leftrightarrow x$ *is an atom*
- $atomic(x) \leftrightarrow x = sup(y : y \leq x \wedge at(y))$.
- $\phi_1 := \forall xy(I(x) \wedge I(y) \rightarrow I(x \cup y)) \wedge \forall xy(I(x) \rightarrow I(x \cap y))$
- $\phi_2 := \forall x(at(x) \rightarrow I(x))$
- $a(i) \leftrightarrow$ *there exist at least* i *many atoms*

Obviously, all these conditions can be formalized in first order predicate logic. Let T be the following theory: $T = \{\phi_1, \phi_2\} \cup Ax(BA) \cup \{a(i) : i \in \omega\}$. Let ψ be the following sentence: $\phi_1 \wedge \phi_2 \wedge \bigwedge\{Ax(BA)\} \wedge \forall x \exists y(I(x) \rightarrow (x < y \wedge I(y))$.

Then $T \models_m \psi$. Let $\mathcal{A} = (U, \cup, \cap, -, 0, 1, I)$ be a minimal model of T. Then $\mathcal{A}$ is an atomic Boolean algebra having an infinite number of atoms and I is an ideal. Since I is a minimal predicate I is a proper ideal and hence I has no maximal element. From this follows that $\mathcal{A} \models \forall x \exists y (I(x) \rightarrow (x < y \wedge I(y))$, and hence $\mathcal{A} \models \psi$. We show that for every finite subset $S_0 \subseteq T^{\models}$ holds $S_0 \not\models_m \psi$. Let $S_0 \subseteq T^{\models}$ be a finite subset. By compactness there is a finite subset $T_0 \subseteq T$ such that $T_0 \models S_0$. Since T is not finitely axiomatizable there exists a number n such that $S_0 \not\models a(j)$ for every $j \geq n$. Let j_0 maximal satisfying $S_0 \models a(j_0)$. We may assume that every model of S_0 is a Boolean algebra (otherwise $S_0 \not\models \psi$). Now there exists a model $\mathcal{A} \models S_0$ having exactly j_0 many atoms, hence $\mathcal{A}$ is finite. Then $\mathcal{A} \not\models \psi$, since $\mathcal{A} \not\models \forall x (I(x) \rightarrow \exists y (x < y \wedge I(y))$. $\Box$

If we restrict T and ϕ in a suitable way, then weak compactness is true. Let T be an universal theory of signature σ, ϕ a quantifier free sentence; $B(T)$ the set of all atomic sentences. Then ϕ is a Boolean combination of sentences from $B(T)$: let $At(\phi)$ the set of all atomic sentences appearing in ϕ. A model I of T can be represented by a subset $I \subseteq B(T)$ such that $I \models T$. For a set $P \subseteq B(\sigma)$ and $I \subseteq B(\sigma)$ let $I \downarrow P =_{df} I \cap P$. The following proposition is obvious.

Proposition 30 *Let $I \subseteq B(\sigma)$, ϕ a quantifier free P-sentence, ie. $at(\phi) \subseteq P$. Then $I \models \phi$ if and only if $I \cap P \models \phi$.*

For every finite subset $P \subseteq B(\sigma)$ and universal theory T let $Mod_P(T) = \{I \cap P : I \in \mathrm{Mod}(T)\}$. The following proposition is obvious.

Proposition 31 *Let T be an universal theory, $P \subseteq B(\sigma)$ a finite set. Then there is a quantifier free sentence ϕ such that $at(\phi) \subseteq P$, and $Mod_P(\phi) = Mod_P(T)$.*

Proposition 32 *Let T be a set of universal sentences, and ϕ a quantifier free sentence. Then the following holds: if $T \models_m \phi$ then there exists a finite subset $A \subseteq Th(T)$ such that $A \models_m \phi$.*

Proof: Let $at(\phi) = P$. Then there is a sentence ψ such that $at(\psi) = P$ and $Mod_P(\psi) = Mod_P(T)$. Obviously, $T \models \psi$. We show $\{\psi\} \models_m \phi$. Let I be a minimal model of ψ. Then by $Mod_P(\psi) = Mod_P(T)$ I can be extended to a model J of T. Since T is universal,J is an extension of a minimal model J_0. [5] Then $J_0 \downarrow P = I$ (this follows from the fact that J is an extension of I: if $J_0 \downarrow P < I$ then this contradicts the minimality of I because $J_0 \cap P \in Mod_P(\psi)$). Since J_0 is minimal, by assumption, $J_0 \models \phi$ and since $at(\phi) = P$ finally, $I \models \phi$. Altogether, $\{\psi\} \models_m \phi$. $\Box$.

7 Conclusions and other Approaches

We have presented a framework for the investigation of nonmonotonic inference operations. This framework uses the notion of a deductive frame, describing the syntactical level, and the notion of a semantical frame, formalizing the semantics of a nonmonotonic system. The deductive basis represents a monotonic logic which is considered as

[5] It is wellknown that every model of a universal theory is an extension of a minimal model

equally important to the nonmonotonic inference operation. Such an abstract handling of monotonic systems was introduced in [FL94]. The present approach is even more general and emphasizes the notion of a deductive frame as central.

Then we studied new versions of compactness allowing to analyse minimal reasoning in propositional and predicate calculus; the previously known concepts of compactness are not adequate to describe these systems. We compare our approach to similar approaches in the literature.

(1) In [FL94] a fixed basic logic (L, Cn) with model theory is assumed. Then nonmonotonic inference operations $C : 2^L \to 2^L$ are studied, which satisfy inclusion, left absorption and right absorption, ie the system (L, Cn, C) is a deductive frame. The inverse task to construct compact logics for given nonmonotonic inference systems (L, C) resulting in a deductive frame (L, Cn, C) is out of the scope of the paper [FL94]. The semantics in [FL94] is based on cumulative models which can be understood as a special case of a definable model operator in our sense. It is evident that our approach is more general.

(2) In the paper of *S. Lindstrom*, [Li91] the semantics of a nonmonotonic inference operation is formulated in terms of selection functions. Selection functions can be understood as special case of model operators. Similar as in [FL94] a deductive system $(L, \vdash)$ is fixed beforehand. The inverse problem is not considered. Again, the notion of a model operator is more general than a selection function.

(3) A very general framework for nonmonotonic logics was introduced by *J. Bell* [Be91]; these are called pragmatic logics. A generalized pragmatic logic can be understood as a semantical frame $(L, M, \models, \Phi)$ such that the requirement $\Phi(X) \subseteq \mathrm{Mod}(X)$ is dropped. Inverse problems are not considered.

8 Problems and Future Research

We conclude the paper with a collection of of open problems.

Representation Problems

1. Which inference systems (L, C) can be expanded to maximal deductive frames (L, C_L, C) satisfying distributivity, conditionalization, rationality?

2. Which properties are satisfied by natural nonmonotonic systems? Find abstract characterisation theorems for nonmonotonic standard systems, in particular for minimal reasoning, Poole-systems, autoepistemic logic and default logic.
   Example of a characterization theorem:
   The consequence operation $CN : 2^L \longrightarrow 2^L$ of classical propositional logic is the least operator satisfying the following conditions:

   - Cn is a compact closure operator;
   - $Cn(\emptyset) =$ set of all tautologies;
   - Cn satisfies the deduction theorem, i.e. $\phi \to \psi \in Cn(X)$ iff $\psi \in Cn(X \cup \{\phi\})$

3. What can be understood by a deductive standard basis for a cumulative system (L, C)? One possibility is to take the greatest deductive basis C_{max} below C. Which additional properties are satisfied by (L, C_{max})? Is the deductive system (L, C_{max}) absolute, ie is every r-maximal set maximally consistent?

4. Investigation of the space $\Gamma_{fa}(C)$= set of all deductive bases for C. Characterize those C for wich $\Gamma_{fa}(C)$ has a graetest element. Describe the maximal elemenst of $\Gamma_{fa}(C)$.

Compactness Properties.

1. Investigation of classical nonmonotonic systems, in particular default logic, nonmonotonic modal logics and Poole systems. Are propositional Poole systems completely Δ_4-compact? Is the paraconsistent system lp described in [Pr91] weakly compact? Also, cf. [We97].

2. Let (L, C) be a cumulative inference system. Does there exist a deductive basis (L, C_L) such that (L, C_L, C) is a deductive frame and C is Δ-compact for a certain extension operator? Is there a finite set $\Delta_1, \ldots, \Delta_n$ of extension operators such that for every cumulative deductive frame (L, C_L, C) the inference operation C is Δ_i-compact for a certain $i \leq n$.

3. Find natural constructions Δ which extends finitary i.o. to infinitary ones. Of importance is a notion of *rule compactness*. One may try to associate to every finitary deductive frame (L, C_L, C) a system $\mathcal{R}$ of finitary nonmonotonic derivation rules. Then define an an infinitary inference operation $\Delta_{\mathcal{R}}(F)$ based on these rules. C is said to be *rule-compact* iff $C \leq \Delta_{\mathcal{R}}(C_f)$, and *rule-complete* if $C = \Delta_{\mathcal{R}}(C_f)$. Which inference operations are rule-compact, rule-complete?

Definability.
How inference operations $C : 2^L \to 2^L$ and model operators $\Phi : 2^L \to 2^M$ can be described, specified? A paradigm for the solution of this problem is the following. Let (L, Cn, Φ) the deductive frame of minimal reasoning for classical propositional logic and X a set of propositional formulas. Let $L^{prop}_{\infty,\infty}(\forall, \exists)$ be infitary propositonal calculus with quantifications of propositional variables, and arbitrary infinite conjunctions and disjunctions, and arbitrary well-ordered sequences of quantifications. Then there is an infinitary formula $\theta(X) \in L^{prop}_{\infty\infty}(\forall, \exists)$ such that $\Phi_m(X) = \{m : m \models \theta(X)\}$. The formula is constructed as follows. Let $X(\overline{V})$ be a set of formulas depending on the sequence $\overline{V}$ of propositional variables; let $\overline{W}$ be a well-ordered sequence of propositional variables of the same type λ as $\overline{V}$ such that $\overline{W} \cap \overline{V} = \emptyset$. Let $\overline{W} \leq \overline{V}$ the following formula $\bigwedge_{\alpha<\lambda} w_\alpha \leq v_\alpha$ and $\theta(X) = \bigwedge X(V) \wedge \forall \overline{W}(\overline{W} \leq \overline{V} \wedge \bigwedge X(\overline{W}) \to \overline{W} = \overline{V}))$. Obviously, the formula $\theta(X)$ satisfies the desired condition and represents a description of Φ_m. A similar construction can be carried out for the predicate calculus. In this case we get a principle of transfinite second order circumscription which is a generalization of circumscription, [Mc80].

The general idea behind these examples is the following. Let $(L, M, \models, \Phi)$ be a semantical frame and $(L_1, M, \models_1)$ an extension of $(L, M, \models)$, ie. $\models_1\downarrow (M \times L)$ equals

$\models$ and $L \subseteq L_1$. The model operator Φ is definable in $(L_1, \models_1, M)$ if there is a formula $\theta(X)$ depending on $X \subseteq L$ (such a dependency can be described by assuming that $\bigwedge X$ is a suitable subformula of $\theta(X)$) such that $\Phi(X) = \{m : m \models_1 \theta(X)\}$.

Application to Logic Programs.
By a refinement of the abstract theory of nonmontonic inference operations one can investigate nonmonotonic operators for logic programs. Then there is the open problem of finding an adequate underlying logic, say for the well-founded semantics or stable semantics of normal logic programs. Let us discuss stable semantics. Let P be a normal logic program, considered as a set of propositional formulas of the form $a_1 \wedge \ldots a_k \wedge \neg b_1 \wedge \ldots \wedge b_l \longrightarrow c$, and $St(P)$ the set of all stable models of P. Define $C_{st} = \{\phi | St(P) \models \phi\}$, then $\mathcal{F} = (L, Cn, C_{st})$ is an inference frame; note that C_{st} is not defined for every subset $X \subseteq L$. $\mathcal{F}$ does not satisfy congruence, hence Cn is not a deductive basis for C_{st}. It is interesting task to analyse the space $\Gamma_{fa}(C_{st})$ in detail. Are there maximal elements? D. Pearce has proved that Nelson's constructive logic is a deductive basis for C_{st}, cf. [Pe95].

Develop a model-theoretical definability theory for several semantics of general logic programs.

Abstract Model Theory.
Investigation of nonmonotonic generalizations of finite model theory. Let T be a first order theory and $\mathcal{U}(T)$ the set of all complete consistent extensions of T; $\mathcal{U}(T)$ is considered as a topological space with a basis of closed-open sets. For a closed subset X of $\mathcal{U}(T)$ let X' be the set of all limit points of X. The Bendixson derivatives are defined as usual: $\mathcal{U}(T)^0 = \mathcal{U}(T)$, $\mathcal{U}^{\alpha+1}(T) = (\mathcal{U}^{\alpha}(T))'$, $U^{\lambda}(T) = \bigcap_{\alpha<\lambda} \mathcal{U}^{\alpha}(T)$, if λ is a limit ordinal. Let $Is^{\alpha}(T) = \mathcal{U}^{\alpha}(T) - \mathcal{U}^{\alpha+1}(T)$, and $J^{\alpha}(T) = \mathcal{U}(T) - \mathcal{U}^{\alpha}(T)$. Then we may introduce the following inference operations: $C^{\alpha}(T) = \bigcap Is^{\alpha}(T)$, and $D^{\alpha}(T) = \bigcap J^{\alpha}(T)$. These inference operations can be understood as generalizations of the operator $C_{fin}(X) = Th(\{\mathcal{A} : \mathcal{A} \models X \text{ and } \mathcal{A} \text{ is finite }\})$. These operators are nonmonotonic, and represent a generalization of finite model theory.

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