On the production of labour value and use value in capitalist and pre-capitalist worlds

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Introduction

Within the past 100 years the Marxian Labour Theory of Value (LTV) was the target of fierce debates and controversies. As Morishima[2] correctly stated already in 1972, it was a "great misfortune that economists have for a long time been divided between 'orthodox' and Marxian camps as a result of cliquishness; each school has lost touch with the others and has become inbred." Today, after the implosion of the Socialist Countries, we can start a fresh try taking stocks and asking which parts of the Labour Theory of Value can be carried on into the future and which ones we have to abandon and throw them on the heap of garbage of history. In the author's opinion one should not stop here and only refer to the reproduction of past positions of the LTV, but instead modify and, if necessary, (re)create and develop further parts of the Theory according to the changed basic conditions of the contemporary political-economic, ecological and social environment and taking into account the innovations in science and technology.

This paper intends to illustrate the state of the art of the LTV by input-output data from Austria and to present new theoretical insights on values, prices and the transformation problem from a geometric perspective in a Leontief economy.

Basic features

First some basics: The LTV is based on the assumption that human beings - to convert natural substances into human materials - have to labour, in order to perform the metabolism of society with nature and with their fellow men. This conversion creates "values in use"[3]. The meaning of the term "use value" is essential for a correct analysis. "Value in use" is a very extensive and broad concept. It includes any good or service useful to somebody. Values in use can meet consumers' needs directly as consumer goods or services, but also indirectly as means of production, intermediary goods and business services. The production of values in use is done by a combination of means of production, auxiliary materials and manpower, but they might also be produced by nature. In principle, we can distinguish between two different kinds of values in use: Material values in use which are reified in goods and therefore remain fixed over a certain period of time, and "immaterial" values in use which disappear in the moment of their production. In common terms we can call them "goods" on the one hand, and "services" on the other. This difference is crucial not only for the understanding of the following paragraphs, but also for

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[2] Michio Morishima: Marx's Economics – A Dual Theory of Value and Growth. Cambridge University Press, Cambridge, 1973: 1

[3] "The word VALUE, it is to be observed, has two different meanings, and sometimes expresses the utility of some particular object, and sometimes the power of purchasing other goods which the possession of that object conveys. The one may be called 'value in use ;' the other, 'value in exchange.' The things which have the greatest value in use have frequently little or no value in exchange; and on the contrary, those which have the greatest value in exchange have frequently little or no value in use. Nothing is more useful than water: but it will purchase scarce any thing; scarce any thing can be had in exchange for it. A diamond, on the contrary, has scarce any value in use; but a very great quantity of other goods may frequently be had in exchange for it." Adam Smith: An Inquiry into the Nature and Causes of the Wealth of Nations. Methuen and Co., Ltd., ed. Edwin Cannan, London 1904. Fifth edition. Volume I, Chapter IV. http://www.econlib.org/LIBRARY/Smith/smWN.html

certain empirical properties of a real economy like the maximum possible growth rate or the maximum rate of profit. While material products or goods (we use the terms synonymously) can be part of the surplus product (which in my opinion is always material), services cannot. Both kinds of use values meet human needs, but services are restricted to consumption, and excluded from capital investment. This does not mean, however, that services would not be important for economic development; on the contrary, many services are more essential for qualitative changes of the economy (like research activities, education and training, fine arts, literature) than material goods.

Already with the emergence of humankind women and men were producing goods and services for each other. Their work could be directly seen and understood as a kind of care taking of their fellow (wo)men. In fact, one can say that division of labour separated people from each other – but at the same time they are bound to each other – creating communities and - society. In the course of history people have always created "values in use" - although different in quantities and qualities – more or less irrespective of the social structures they lived and worked in. Today capitalistic relations of production dominate, i.e. capitalists acquire the command over human labour by paying the workers their costs of reproduction, and combine the work of the labourers with machines, buildings, auxiliary materials, and services, in order to let them produce machines, buildings, auxiliary materials and services again, however in different composition. If technology and wage rates permit it, the value of the output capitalists can sell via the market is higher than the costs of its production: The difference forms the basis for their profits.

In his analysis of capitalistic society Marx identified the commodity as the basic element of wealth. According to Aristotle, Adam Smith, Karl Marx and many others, a commodity has "value in use" and - if exchanged - "value in exchange". Exchange value is not an individual property of a commodity. In the market the commodities of a certain kind produced by individual firms are compared with each other, and in the ideal case a uniform price is determined. In analogy to the theological discourse of the Last Judgement of Roman Catholics, the bad ones are punished for being lazy (or inefficient), the good ones can proceed to the (neoliberal) paradise, i.e. those who produce high quality at low costs receive an extra profit, while the others make losses and are expelled from the market. Thus the market (under conditions of competition) works as a societal machinery of increasing efficiency which forces the entrepreneur to innovate permanently and to do everything to reduce production cost.

In the first volume of "Kapital", Marx looked for the qualitative and quantitative conditions at which individual goods can be exchanged in the market. In agreement with Adam Smith and David Ricardo he identified labour as being the essential ingredient commodities have in common, and labour time as their quantitative measure. Starting from there, Marx elaborated his theory and identified the labour value of a commodity as socially necessary labour time. According to Marx, the labour value (or "Wert" in German), w, consists of two additive components, of the value c of constant capital, needed for the production of the commodity, and the newly added labour time or life labour, n. n is central for the understanding of Marx' theory of exploitation: Under capitalistic conditions it is split into two parts, variable capital v, which is given as wages to the workers, and surplus value m, which remains with the entrepreneur. v is called variable, because it is the only element in the production process creating more value than is needed for its production. Thus we end up with the well known formula of trinity: The labour value w of a commodity is made up of the sum of constant capital c, variable capital v and surplus value m:

$$w = c + v + m$$

With these basic definitions the essential variables of Marxian theory can be stated: *rate of surplus value* m/v, *organic composition of capital* $v/(c + v)$ and *rate of profit* $r = m/(c + v)$. The product of rate of surplus value times organic composition of capital equals the rate of profit. Many speculations of political economists have been linked to this term. Marx himself formulated a tendency of the rate of profit to fall. Some of his successors used this formula to predict the break-down of capitalism with the following argument: If the rate of surplus value remains constant (an assumption frequently used by Marx himself) and capital accumulation and technological change proceeds, organic composition would have to fall and the rate of profit would have to go to zero: end of capitalism. But as one can easily show, this is wishful thinking. If we take for granted that all components of value are revaluated at replacement value, all components can be reproduced at less and less labour time (because of more and more automated production). In such a case all components of the rate of profit are linear functions of life labour n. Nominator and denominator depend equally on n. If n goes to zero, r is an indefinite value $r = 0/0$. This does not mean that r becomes zero. As one can see the quotient of the first derivatives of nominator and denominator

$$\lim(n\text{->}0)\ r = \lim(n\text{->}0)\ m/(c + v) = \lim(n\text{->}0)\ [(a1*n + b1\ )\ /\ (a2*n + b2 + a3*n + b3)]$$

becomes $a1/(a2 + a3)$ or – if the bi's are not zero - $b1/(b2 + b3)$.

To modernize Marx' economic theory in terms of mathematics it has been an essential step forward to apply linear algebra and the matrix calculus to it (like many have done before: e.g. Leontief[4], Brody[5] and Morishima[6]). From there it becomes evident that Marx' theory can be seen in a dual way, from the point of view of (i) values in use and of (ii) unit values in exchange (or unit prices). A third level (iii) is possible in terms of "turnover values" – meaning unit values – or unit prices – times the number of values in use – or number of units.

Let us present the corresponding formulae in the following order: Firstly, from the point of view of values in use at level (i). The matrix of technical coefficients A represents the technology of the economy. Gross output x (a column vector) contains the amounts of values in use in the economy. By kind of use x can be split into intermediate goods Ax and final demand y.

$$Ax + y = x \tag{1}$$

Secondly, from a dual point of view on level (ii) one can break down the unit price/value p (a row vector) into the costs of intermediate goods pA used in production plus the unit value added v.

$$pA + v = p \tag{2}$$

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[4] Wassily W Leontief: Input-output economics. In: Scientific American, October 1951a: 15–21; Wassily W Leontief: Input-Output Economics. 2nd ed., Oxford University Press, New York, 1986. First ed. 1966.
[5] Andras Brody: Proportions, Prices and Planning: A mathematical restatement of labor theory of value. North-Holland Pub. Co., Budapest, Amsterdam,1970.
[6] Michio Morishima: Marx's Economics – A Dual Theory of Value and Growth. Cambridge University Press, Cambridge, 1973.

Finally, on level (iii), we can write a synthesis (on the layer of turnover):

$$\text{diag}(p) \, A \, \text{diag}(x) \, 1 + \text{diag}(p) \, y = \text{diag}(p)x \qquad (3a)$$
$$\text{and}$$

$$1'\text{diag}(p) \, A \, \text{diag}(x) + v \, \text{diag}(x) = p \, \text{diag}(x) \qquad (3b)$$

diag(x) is a square matrix with the elements of the vector x as main diagonal. 1 is a column vector of ones, 1' is the transpose of this vector.

Under the condition that unit labour values w (a row vector) are nothing else than a special price system (we will go into the details later), equation (2) can be written in the following form:

$$wA + n = w \qquad (2a)$$

In (2a) we replace the vector of unit value added v by the row vector n (life labour per unit of output). wA is the cost of production expressed in labour time. A simple matrix transformation with the so called Leontief-Inverse $(E - A)^{-1}$ allows to solve (2a) for w (E means the identity matrix with ones only in the main diagonal)

$$w = n \, (E - A)^{-1}. \qquad (4)$$

w is more or less identical with Marx' labour value (per unit) of the first volume of Kapital. His idea is that in an economy where commodities are exchanged according to their labour value, the value content before and after an exchange remains the same. Values in use produced by a certain amount of socially necessary labour time can be exchanged against other use values produced within the same amount of labour time. The exchange allows replacing A's goods by B's goods which were produced in the same labour time. While the qualities of use values change, the quantities of labour values remain the same before and after the exchange. This is the condition of the so called principle of „equivalent exchange". One could imagine that this principle had ruled some types of pre-capitalist societies, e.g. societies of small commodity producers. Even if one does not believe in its existence as historical truth, one could us it as a useful hypothesis.

Material production vs. services
If some of the industries do not produce material products, but services, the principle of "equivalent exchange" would be violated even in a pre-capitalist economy (where capital advanced does not influence the formation of prices), if one allows the service sectors to acquire surplus value. Let us show this by means of a thought experiment: Let us assume that the first 10 sectors of an economy with 20 branches produce material products, and the last 10 services. If we take stock of the surplus product, we can see that only the sectors of material production contribute to it in physical form by applying a certain amount of labour time. If now – as is also the case in a capitalist economy - also service sectors are able to make profits and to do investment, they would buy parts of the surplus product. But then it is impossible for the producers of the surplus product (the first ten branches) to buy goods at the full value they have produced, because they have to share the surplus product (capital investments) with the service sectors. There is only one possibility how to save the principle of "equivalent exchange": One can set up a price system which does not allow the service sectors to make profits. Their output is evaluated at reproduction costs. The necessary goods for the production of services are

completely produced outside the service sectors. But this also means that in service sectors there is no contribution to the labour values of the economy. Workers in the service sectors produce neither labour value nor surplus value. On the contrary, in a first approximation they consume values from the sectors of material production.

In the discussion of economists this difference between value producing and value creating sectors was reflected in various, but somewhat hidden ways: The first line of discussion was centred about "productive" and "non-productive labour"[7]. One can trace this line back to Adam Smith: "The labour of some of the most respectable orders in the society is, like that of menial servants, unproductive of any value, and does not fix or realize itself in any permanent subject; or vendible commodity, which endures after that labour is past, and for which an equal quantity of labour could afterwards be procured. … Their service, how honourable, how useful, or how necessary so ever, produces nothing for which an equal quantity of service can afterwards be procured. … In the same class must be ranked, some both of the gravest and most important, and some of the most frivolous professions: churchmen, lawyers, physicians, men of letters of all kinds; players, buffoons, musicians, opera-singers, opera-dancers, &c. …Like the declamation of the actor, the harangue of the orator, or the tune of the musician, the work of all of them perishes in the very instant of its production."[8] This assessment is closely linked to the way "productivity" is defined. In my opinion, it seems to be useful to distinguish three kinds of productivity:

- Productivity(i) could be measured in the number of use values per life labour time independent of relations of production.
- Productivity(ii) could be measured by total labour value created over life labour spent. This is the meaning of productivity by Adam Smith.
- Productivity(iii) could be measured by the amount of profit acquired by the workers (measured in hours or in wages) for their master. The latter is the measure of profitability applied under capitalistic rule and it was called "productive" by Marx.

There are of course different variants of measurement. Important is, that productivity(ii) is zero with respect to services which do not add to the surplus product or capital investment. Nevertheless, in real capitalism, service production can gain profits via productivity(iii).

The second line of discussion can be located around the different systems of national economic accounting, the System of National Accounts (SNA), used in the capitalistic world, and the Material Product System (MPS), used in the imploded Socialist Countries. While, "in the SNA, commodities are, broadly speaking, marketed products whether these be goods and services; whereas, in the MPS, the concept is restricted to material products and excludes many of the services included among commodities in the SNA."[9] In the terminology we use throughout this paper one could state that SNA is counting "values in use" measured at market prices, irrespective of goods or services. In contrast to SNA, MPS focused mainly on goods production, and included only a few service sectors which were linked to material production like transport or

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[7] Zentralinstitut für Wirtschaftswissenschaften der Akademie der Wissenschaften der DDR: Produktive und unproduktive Arbeit im Sozialismus – Literaturbericht über die Diskussion in sozialistischen Ländern, Berlin 1986. This publication was supported by the research project „economic growth and structural change (Nr. 2702)" which was financed by the Anniversary Fund of the Austrian National Bank.

[8] Adam Smith, An Inquiry into the Nature and Causes of the Wealth of Nations f Book II, Chapter III, Of the Accumulation of Capital, or of Productive and Unproductive Labour
http://www.econlib.org/LIBRARY/Smith/smWN.html

[9] Richard Stone: Mathematical Models of the economy and other essays. Chapman and Hall, London 1970. Chapter XIII: A Comparison of the SNA with the MPS: 201-233; 201.

with the "reproduction of the labour force" like health care. Today, SNA is the surviving system more or less exclusively used and promoted by the United Nations, and in a specialised form by the Member States of the European Union.

If we take stock of all the economic activities of one country under the perspective of material products only, one could represent them by their (re)production cost. But then it becomes evident, that the origin of the material goods used for the production of services (intermediary products and consumption) is limited to the sectors of material production. And from this perspective, services would represent a second (or more) counting of material products, because the material products used for the production of services were already included in the accounts of the sectors of material production. The only kind of goods which is never counted double or multiple is the surplus product – in a very rigid definition of the term. Precise double accounting would be the case if services would not use services also (meaning the diagonal sub-matrices of services are zero).
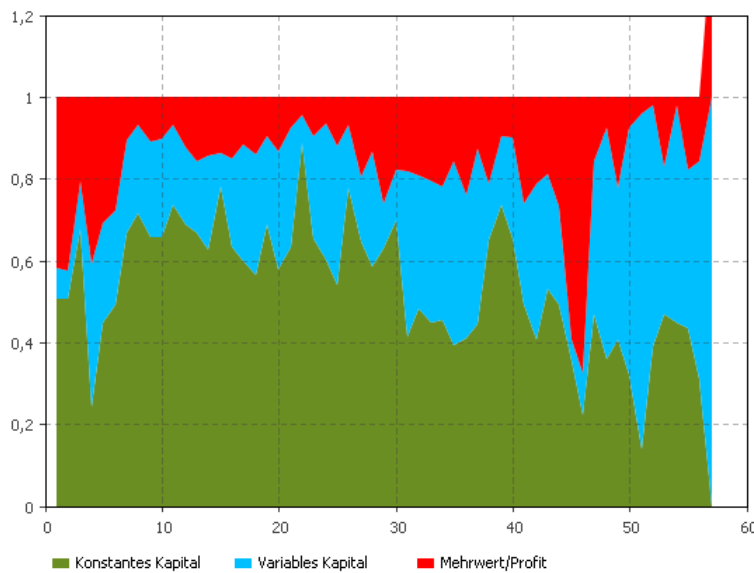

Figure 1: Actual prices, Austria 2003, 57 branches of production

Empirical illustration

Before we illustrate the computation of labour values on the basis of empirical data from Austria, I would like to present Austrian gross output data[10] at actual prices 2003 and their decomposition for 57 branches of production based on the Austrian input-output statistics. Figure 1 shows the structure of gross output at actual prices divided by constant capital[11] (green), variable capital (blue) and surplus value (red). The branch of production with the highest relative surplus value is „leasing" (no. 46), followed by „real estate" (no. 45)[12]. It is also worth mentioning that the last

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[10] See http://www.statistik.at/web_de/statistiken/volkswirtschaftliche_gesamtrechnungen/input-output-statistik/index.html

[11] Because of lack of data constant fixed capital was neglected. Mathematically speaking would it be easy to include also fixed constant capital and not only circulating capital.

[12] The names of 59 branches of production one can find at http://www.statistik.at/web_de/static/aufkommens-_und_verwendungstabelle_2003_019869.xls, where three branches are aggregated to one sector: "Crude oil, gas and ore mining" to end up at 57

branch of production, services of private households, shows negative surplus values (therefore variable capital surpasses the 100% line, surplus value have to be deducted from the highest point in the graph in downward direction).

Figure 2 presents the rate of profit, the rate of surplus value and the organic composition of capital fort he 57 branches of production in terms of observed prices. One can see high rates of surplus values also in the branches no. 1 and 2, agriculture and forestry, because of a low fraction of salaried labour.
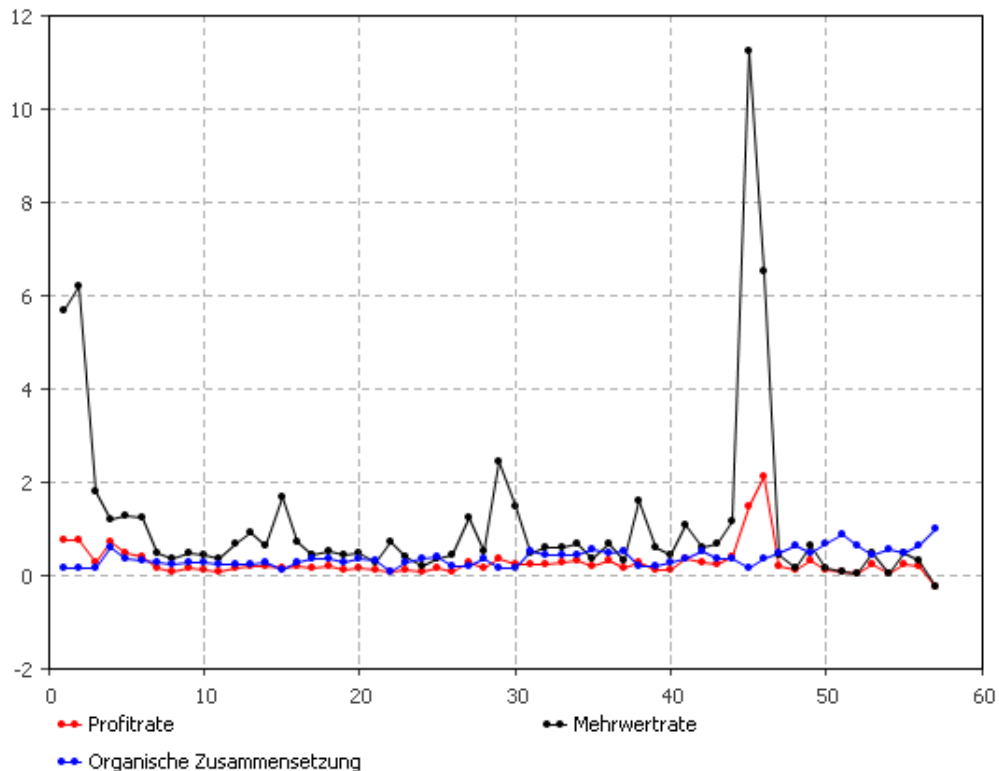
Figure 2: Actual prices, Austria 2003, 57 branches of production, Marxian indicators

Labour values in Austria 2003 (only material production creates value)
Without presenting details of the mathematics behind, in figures 3 we show what would be the structure of the labour values in an economy where only sectors which are productive(ii) are seen as value creating. In the following figure one can easily see that there is no surplus value in the sectors 33 to 57 (all those represent services). Positive surplus values only come up in the sectors 1 to 32. Maybe it is worth to mention that for the computation of figure 3 the purchases of services were no longer seen as purchases out of the surplus value, but are accounted as intermediary goods which increase the value of constant capital, and by the change in accounting surplus values of the material sectors are reduced. In my opinion the method described above could be a way determining labour values without violating the principle of „equivalent exchange". This consistency comes at a certain price: Services are excluded from acquiring profits which is not the case in real capitalist economies. Up to this moment we are dealing with the economy on a rather abstract, pre capitalist level. If we allow service sectors making profits and to invest them we have to transform labour values into prices of production or to apply any

similar transformation. But, in the result, each of these transformations violates the principle of „equivalent exchange".
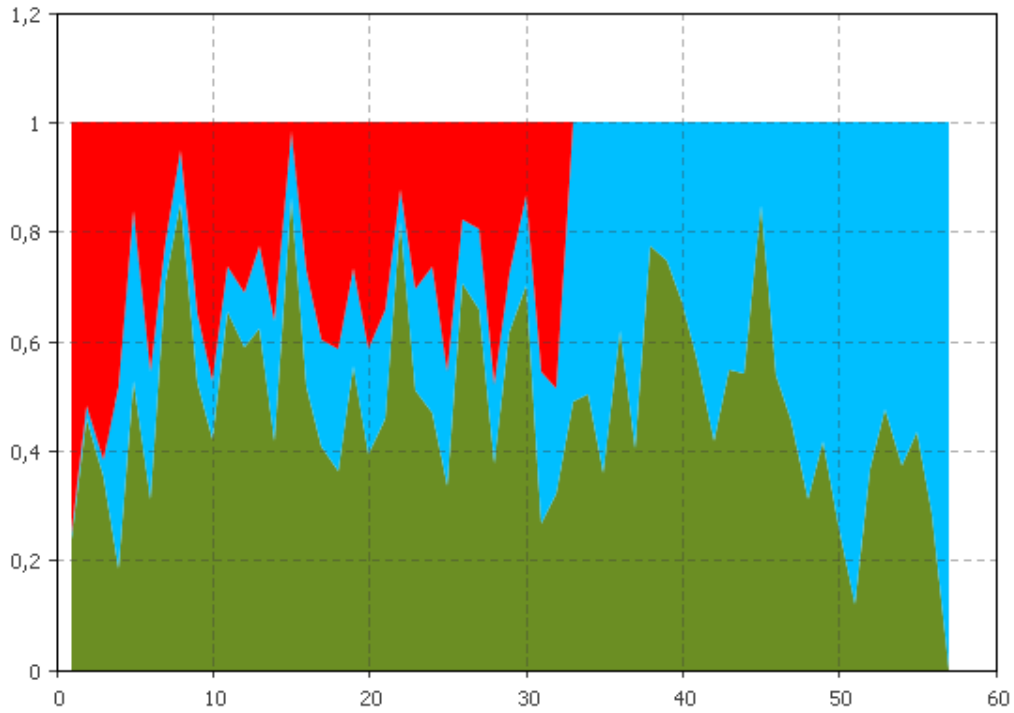

Figure 3: Labour values (only material production creates value), Austria 2003, 57 branches of production
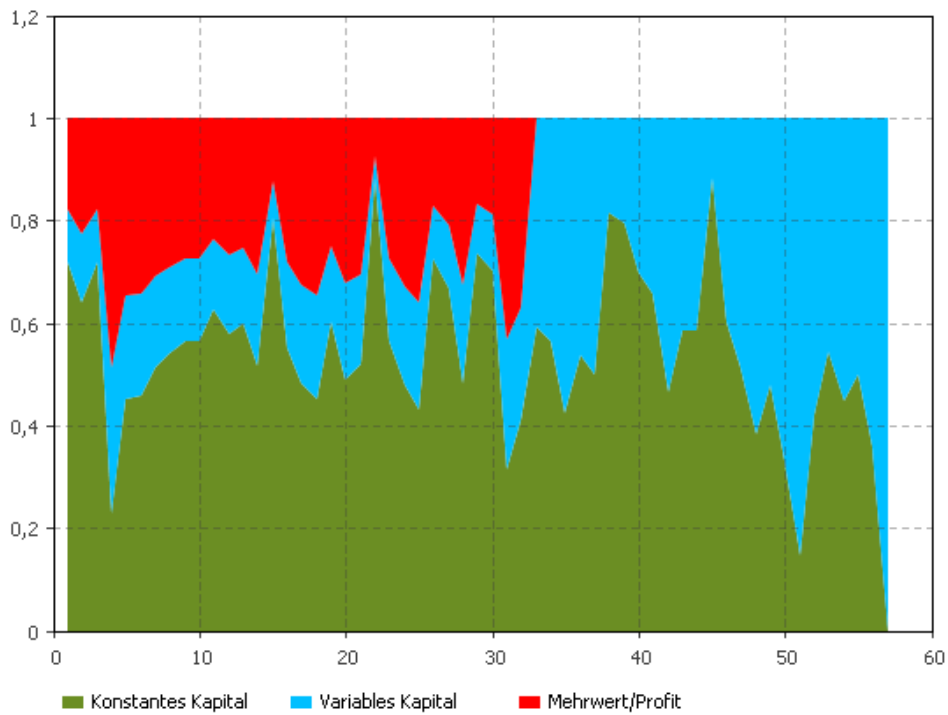

Figure 4: Labour values (only material production creates value), equal rates of surplus value, Austria 2003

Figures 4 and 5 present the Austrian economy on the highest level of abstraction possible. They do not only abstract from the actual price system and replace it by a price system generated by labour time, but it abstracts also from differences in the rates of surplus value (Marx frequently assumed a uniform rate of surplus value).
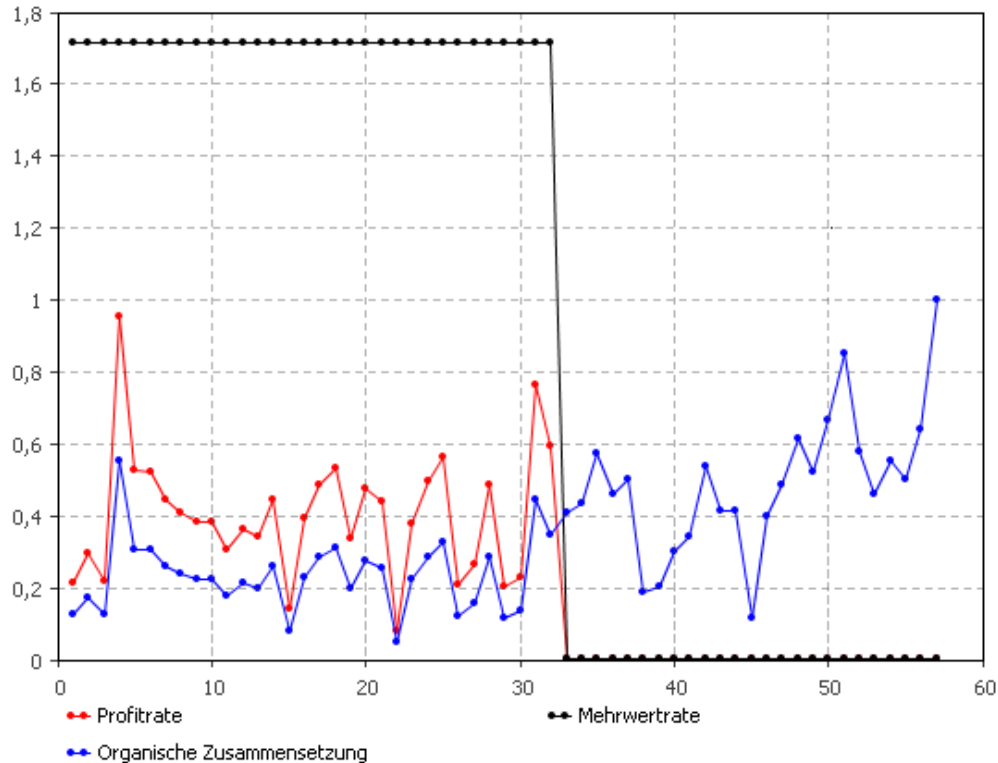

Figure 5: Labour values (only material production creates value), equal rates of surplus value, Marxian indicators

Prices of production

Figure 6 shows the value structure in Austria after the transformation of labour values into prices of production[13] which is nothing else than a solution to the so called "transformation problem". The idea behind goes back to Marx: Because of competition between capitalists seeking for higher rates of profit, capital moves between the branches of production. In an idealized way these movements would lead to equalized profit rates. Although it might not be true that in reality profit rates are equal, the construction of a price system with equalized rates of profits is theoretically very useful. It is interesting to note that Marx's transformation did not end up at a "correct" system of prices of production (as von Bortkiewicz[14] has shown long ago, Marx ended up with output prices different from input prices), but that he in fact did an essential step into the right direction. If one repeats Marx' step by adding profits at equal rates expressed in terms of the input price system, one can show that after some iterations one ends up at the „correct" prices of production. At the same time it can be shown that it does not matter at what price or value system one starts: every iteration process will end up at the same prices of production. The result: all branches of production, sectors of material production and services gain profits. In the Austrian

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[13] Restrictive assumptions: only circulating capital, turnover time equals 1 year, surplus value includes taxes.

[14] Ladislaus von Bortkiewicz: Wertrechung und Preisrechung im Marxschen System. Published in three parts in: Archiv für Sozialwissenschaft und Sozialpolitik, Part I: Vol. XXIII, Heft 1, 1906: 1-50. The remaining parts were published in: Vol. XXV, 1907: 10-51 und 445-488.

example, the rate of profit is high (25%), maybe for the reason that for sake of simplification we did not explicitly exclude taxes and governmental activities from profits or surplus values.
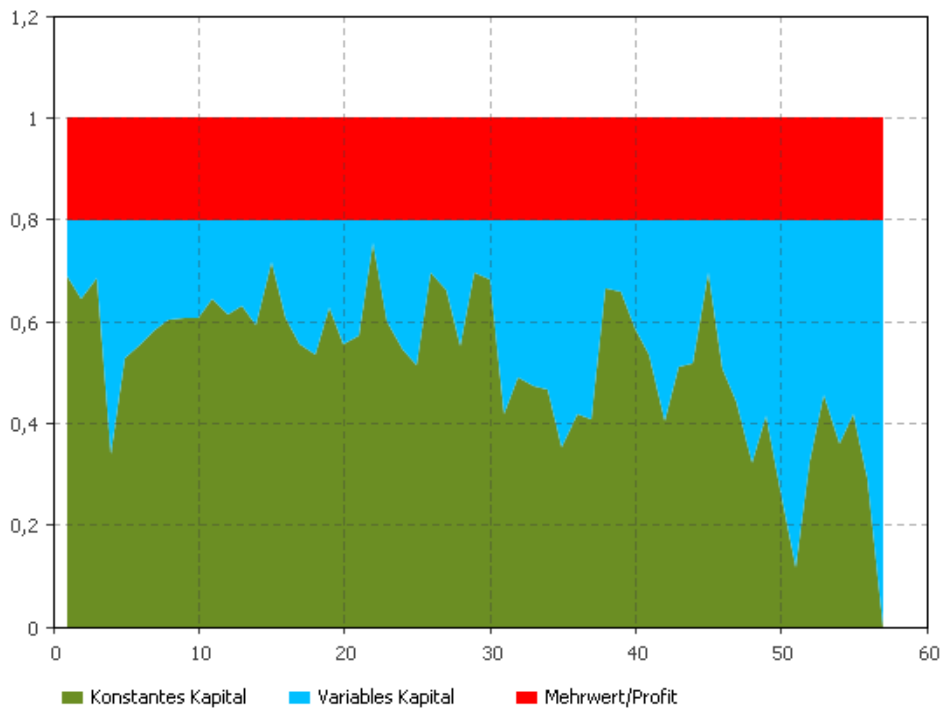

Figure 6: Prices of Production, Austria 2003, 57 branches of production

If we follow Marx' example and determine output prices by adding profits with equal profit rates to capital advanced, we get figure 7 as the result of the first iteration.
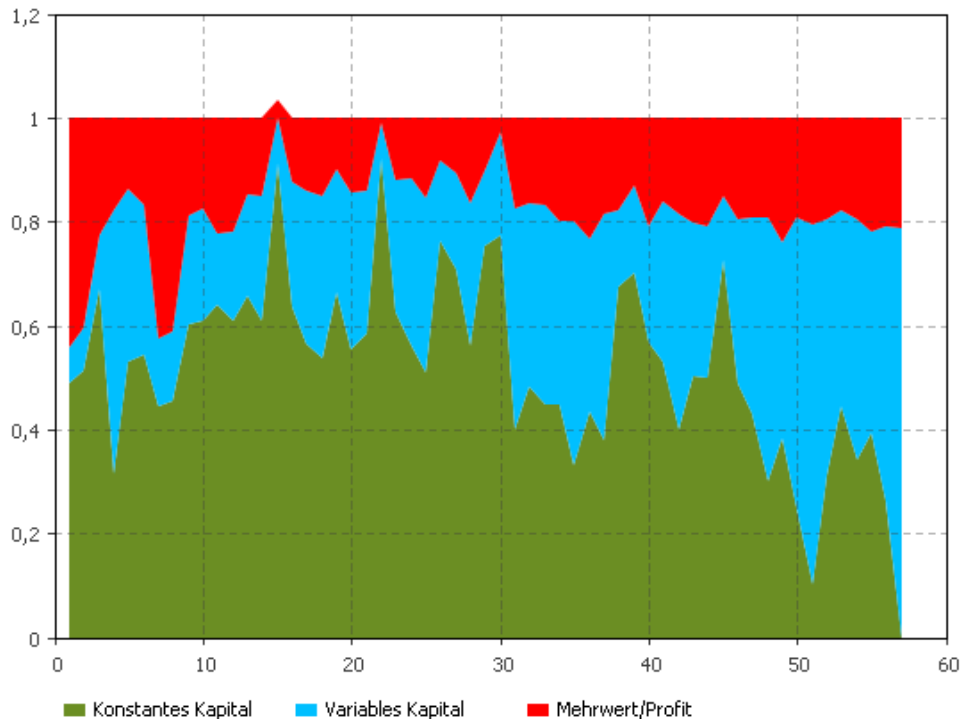

Figure 7: Marx' solution of the transformation problem (first iteration only),
Austria 2003, 57 branches of production

If we continue the iteration process, one can see that the correlation coefficient between actual prices and the iterative prices up to iteration no. 5 goes up to its maximum of 0.9537. Further iterations show a slight decline of the correlation coefficient towards 0.9524 (Figure 8). In our findings for Austria, prices of production (in particular at iteration 5) are the best predictor of observed prices.[15]

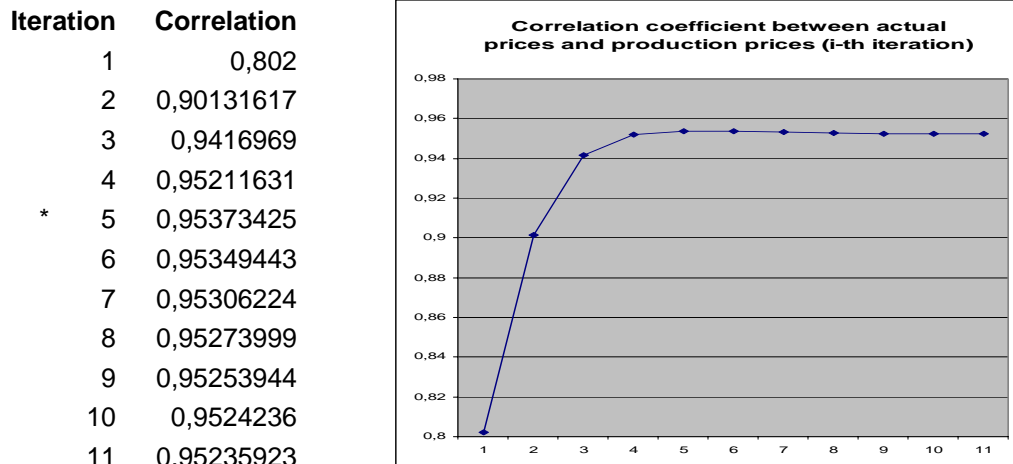|	Iteration	Correlation
	1	0,802
	2	0,90131617
	3	0,9416969
	4	0,95211631
*	5	0,95373425
	6	0,95349443
	7	0,95306224
	8	0,95273999
	9	0,95253944
	10	0,9524236
	11	0,95235923

Figure 8: Correlation coefficient between actual prices and production prices by iteration step
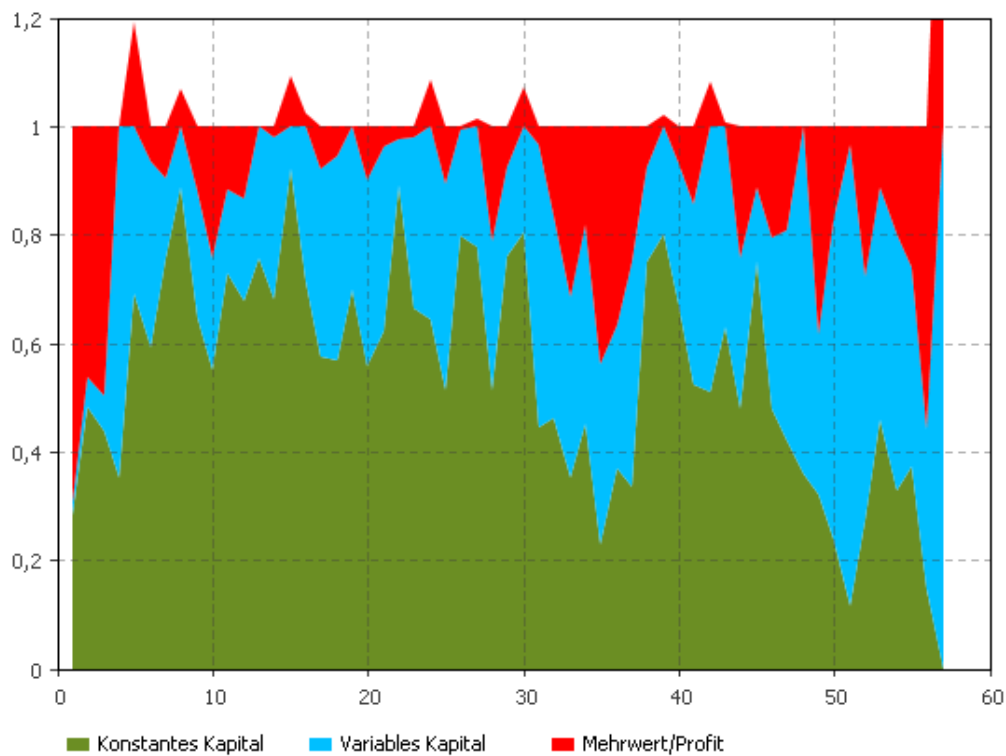
Figure 9: Labour values (all sectors create value), Austria 2003, 57 branches of production

[15] Marx transformed values into prices of production keeping x constant. One can find a solution of the transformation problem with variable x in my paper "The Marxian transformation problem revisited" (see http://www.peter.fleissner.org/uploads/texte/Transformations_Problem_Beijing2.pdf)

Labour values computed under the assumption that all sectors create value have a correlation coefficient with actual prices of only 0.883 (Figure 9). It is interesting to see that this method for Austria results in eight additional branches of production with negative profits which does not speak for a good approximation of actual prices by labour values. Labour values based only on branches of material production (which represent the "essence" of value creation, but are farer away from the "outer appearance" of the empirical surface) have a lower correlation coefficient (0.802) with actual prices. This result indicates that in the Austrian economy there are some forces of equilibration towards uniform rates of profit, but the best correlation does not come up at a situation of total uniformity. The highest correlation can be seen somewhere in between labour values and prices of production a la von Bortkiewicz.

Change of Perspective: Geometric Interpretation of a Leontief Economy
In the following paragraphs the point of view is moved away from a side by side representation of indicators of the branches of production of an economy towards a perspective where the indicators of all branches of production are represented in one single mathematical entity. It is possible to interpret e.g. a price system or gross output not only as a collection of numbers in a row, but at the same time as coordinates of a point in an n-dimensional space. n is the number of branches of production. This means that gross output can be represented as one point of an n-dimensional vector space[16] or more precisely as an n-dimensional vector. The space should have a *norm* to describe the length of each vector. We know that relative prices are defined only up to a constant factor. For this reason we can choose freely the length of the price vector, either as 1 or as the total turnover of all the economy (which would mean that x becomes the unit vector 1).
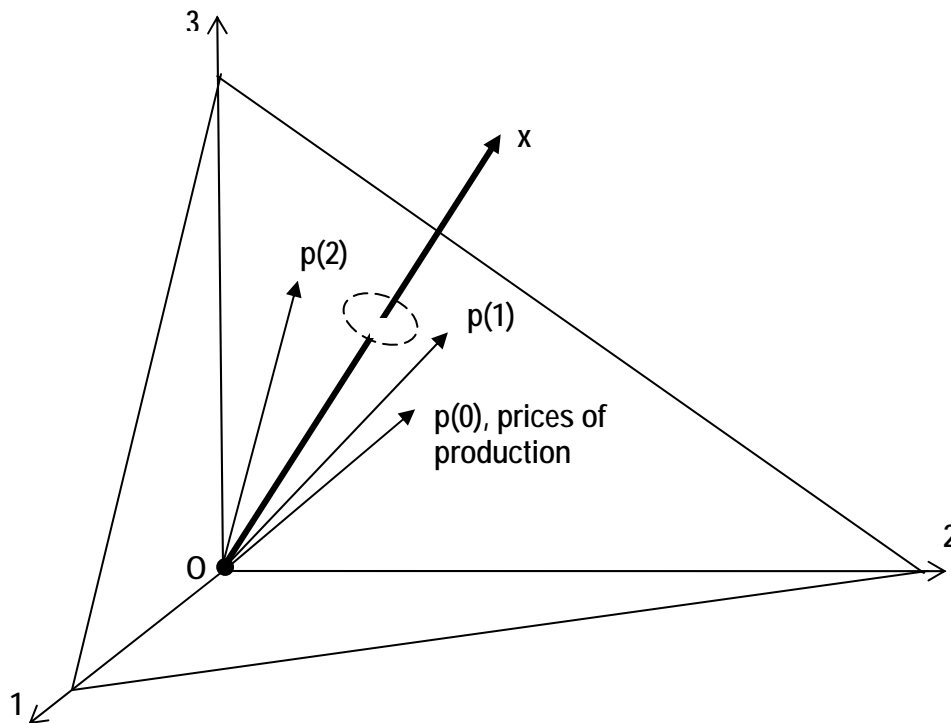

Figure 10: The location of all systems of relative prices p(i) on a hyperplane
(three-dimensional case)

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[16] Luenberger, D. G: Optimization by Vector Space Methods, Wiley, New York 1969, pp.11-45

The condition implemented by Marx to transform values into prices[17] says that the sum of all the values before and after any transformation of price systems remains invariant (px = const). For expressing this property we have to extend the concept of normed vector space by allowing for an inner product (this is nothing else than a generalization of the dot product of two vectors in analytical geometry. Two vectors are defined orthogonal if their inner product is zero). A normed vector space together with an inner product is called a Hilbert space. The geometric interpretation is easy: Marx' invariance condition means that the price vectors related to a given x are located on a hyperplane (with a degree of freedom of n-1, where n is the number of branches of production). The hyperplane is orthogonal to the vector x (figure 10). We will come back to this hyperplane later.
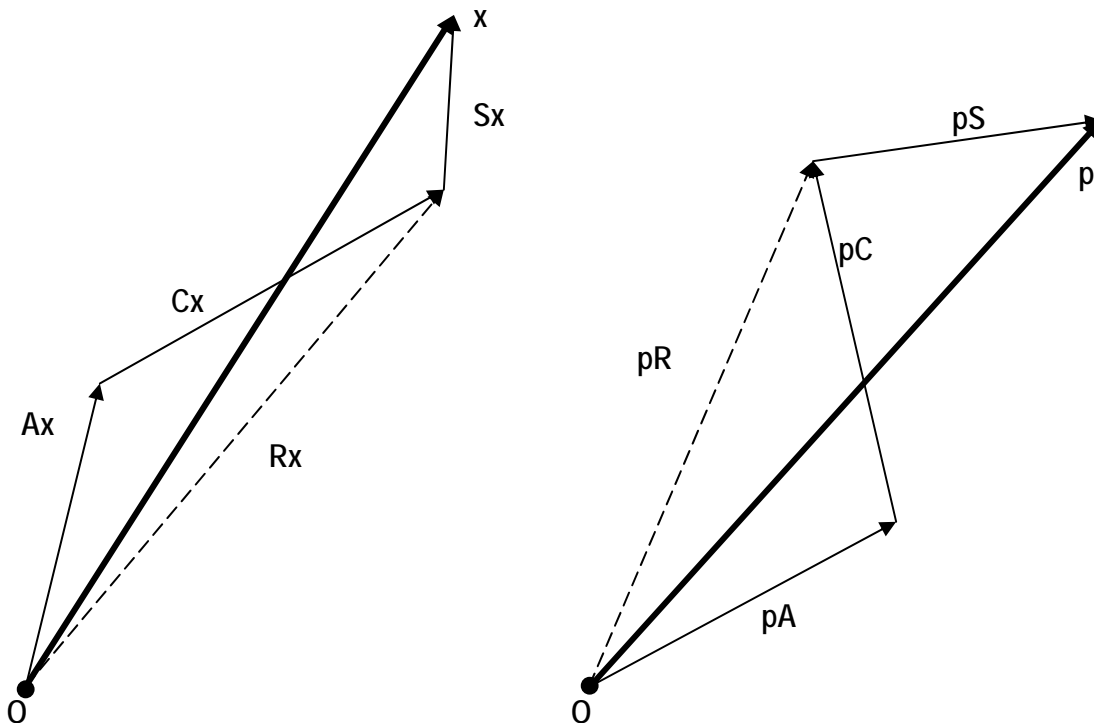

Figure 11: Decomposition of output x (left) and unit prices (right)

A second example of geometric interpretation is the decomposition of unit prices p and of the output x. If we construct a consumption matrix C (in a close analogy to the matrix A of technical coefficients containing the amounts of consumer goods of type i per unit of output j), a "reproduction matrix" R = A + C, and a surplus matrix S (containing the amounts of investment goods of type i per unit of output j financed out of profits of sector j), the equation

$$Ax + Cx + Sx = Rx + Sx = x$$

allows for a decomposition of x as a vector sum of intermediary goods Ax, consumption Cx and surplus product s = Sx. The dual equation

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[17] as assumed in the numerical examples of Chapter 9 of Marx' Capital's Volume III.

$$p = pA + pC + pS = pR + pS$$

allows for a decomposition of p as a vector sum of the various kinds of costs, constant capital pA, variable capital pC and surplus value pS (both dual geometric representations are shown in figure 11).

For prices of production which are the left-eigenvectors of the reproduction matrix R = A + C holds that p, pR (the total unit costs of reproduction) and pS (the vector of unit surplus values or unit profits) are parallel in the n-dimensional vector space:

$$p = pR (1 + r) = pR + r\, pR = pR + pS$$

This means that the quotient of individual profit and capital advanced is equal for each branch of production. $(1 + r)$ is nothing else than the inverse of the largest eigenvalue of the of reproduction matrix R.

According to the principle of duality the analogue to the prices of production can be presented also for the use-values: If x is the right eigenvector of the matrix of reproduction R, x describes relative amounts of use-values necessary for equilibrium growth of the rate g. The largest eigenvalue of matrix R is unique, thus the rate of growth is equal to the rate of profit (r = g)
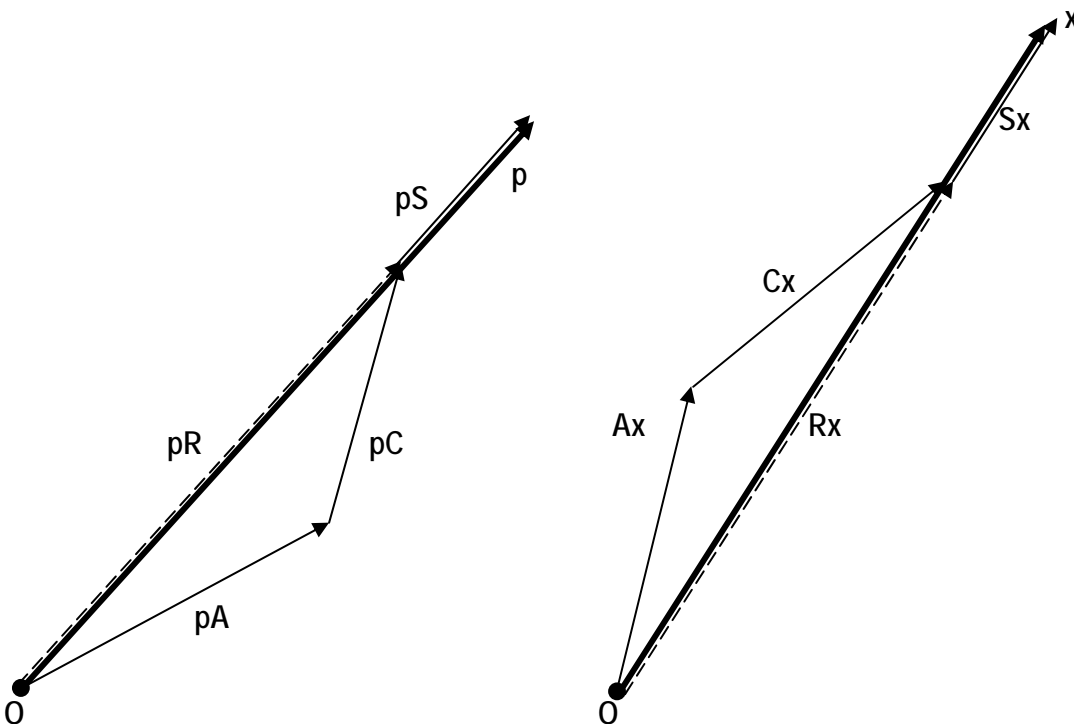
$$x = Rx (1 + g) = Rx + r\, Rx = Rx + Sx$$

Figure 12: Decomposition of output x (left) for equilibrium growth
and unit prices (right) for unit prices of production

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These considerations can be used as a basis for the analysis of Marx' schemes of simple and extended reproduction (figure 12). After a short investigation of these schemes it becomes evident that Marx' assumptions were far from eigenvectors, and therefore the rates of growth he determined in his examples are far below the growth rates possible (according to his choice of the reproduction matrix R). This is the reason why he had great difficulties to establish a clear cut analysis of equilibrium growth. But nevertheless he asked the appropriate questions.

If we choose the n-dimensional space as a space of turnover (in the dimension unit price times number of use-values), both aspects of duality can be presented in one single figure (figure 13). We draw Marx' variables „c", „v" and „m" as vectors in an n-dimensional space and add them up as vectors to the total value „w".
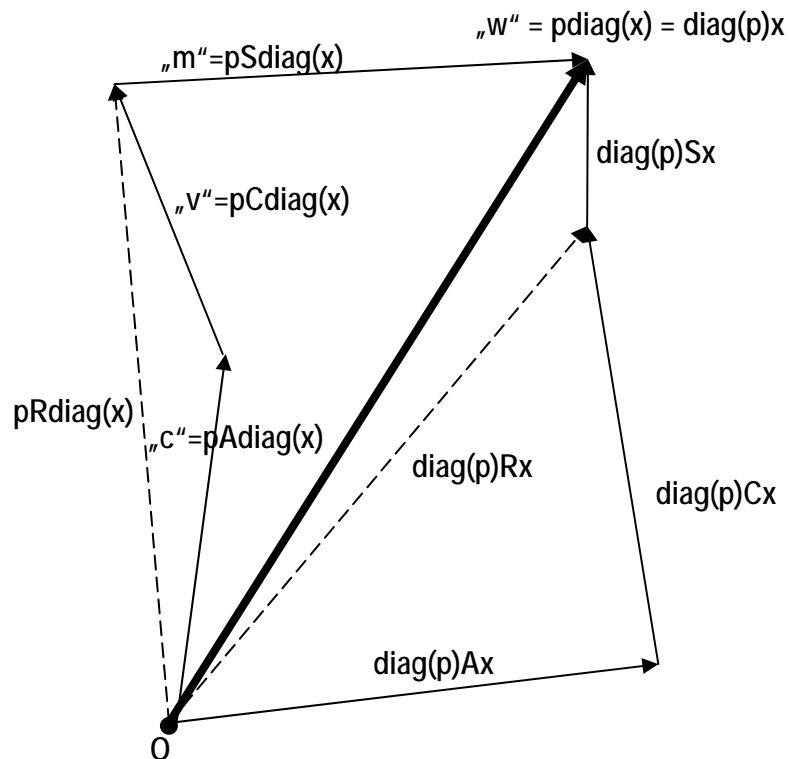

Figure 13: Decomposition of turnover (output x times unit price p)

The sums of the components of the dual vectors are always equal to each other, irrespective if they were derived from a perspective of use-values or unit prices:

$$1'\text{diag}(p)Ax = pA\text{diag}(x)1 = pAx$$

$$1'\text{diag}(p)Cx = pC\text{diag}(x)1 = pCx$$

$$1'\text{diag}(p)Sx = pS\text{diag}(x)1 = pSx$$

$$1'\text{diag}(p)x = p\text{diag}(x)1 = px$$

Because according to our assumptions matrices A, C (and R = A + C) and the vector x are given, only S, and – related to the choice of S, p can be chosen. But even this choice cannot be done without restrictions: For each matrix S and unit price vector p (characterized by an index i, i = 1,2,….) it holds that the total sum of profits p(i)S(i)diag(x)1 has to be equal to the value p(i)s = 1'diag(p(i))S(i)x of the surplus product, where the surplus product s

$$s = (E – R)x$$

is a given invariant.[18]

For each S(i)
$$S(i)x = s,$$

i.e. the surplus product is invariant. And for each S(i) and unit price system p(i)

$$p(i)S(i)x = p(i)s = p(i)(E – R)x,$$

i.e. the total value of the sum of profits has always to be equal to the total value of the surplus product. If profits or surplus values are known, we can construct a specific matrix of the surplus product S(i). This is the case, whenever we start from known unit prices p(i). We use the simplifying idea that the surplus product is proportionally allocated to the branches of production according to the specific surplus values in each of the branches of production. The surplus product itself remains constant.

$$S(i) = s \, m(i) / m(i)1 = s \, p(i)(E – R) / p(i)(E – R)x$$
$$= (E – R)1 \, p(i)(E – R) / p(i)(E – R)x$$

Adversely, the unit prices p(i) are the left eigenvector of the matrix equation with eigenvalue 1

$$p(i) [R + S(i)] = p(i).$$

The interplay of price systems in geometric perspective
A final remark is appropriate which is not really surprising within the context of matrix calculus, but in the context of the transformation problem it seems to be interesting: As already Samuelson has correctly observed, the rate of profit and the prices of production are determined simultaneously. If we go for an iterative method for solving the above equation, it is irrelevant from which price vector we start. The only pre-condition is that the start vector is a feasible price vector, meaning that it is located on the hyperplane shown in figures 10 and 14 and is also a solution[19] of the equation

$$p(i) [A + C + S(i)] = p(i).$$

We could start at the actual price system, or at labour values determined on the basis of all sectors or only on the sectors of material production, or at any other feasible price system: All iterations

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[18] This invariance holds only for a given price system, it does not hold during the transformation of labour values into prices of production.
[19] This will create a subset of feasible price systems in the hyperplane. Its precise shape is not investigated here.

that apply Marx' method of allocating equal profit rates to each of the sectors (by linking total profits per unit capital advanced as a mark-up to the amount of capital advanced in each of the sectors) after a few iterations end up at the same vector of prices of production (see figure 14 for a two-dimensional illustration of the hyperplane. Dotted lines represent possible pathways of convergence toward prices of production within the hyperplane. For defining the precise shapes of these pathways further investigation is needed).

Therefore, in 1971 Paul Samuelson could write with some truth in it:

„Contemplate the two mutually-exclusive alternatives of 'values' and 'prices'. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. *Voila!* You have completed your transformation algorithm." [20]

In history and in other evolutionary processes it happens rather frequently that a certain traditional, older regime is replaced by a new one. In such a case the new system does not carry around any longer the mechanisms of its predecessor. It needs scientific methods to show the traces of the perished system within the surviving one. In the case of the transformation problem we can reconstruct the former value structure by mathematical transformations of the price system. Matrix algebra provides tools similar to a time machine to look back to the logical or maybe temporal history of the capitalistic structure of the surface.
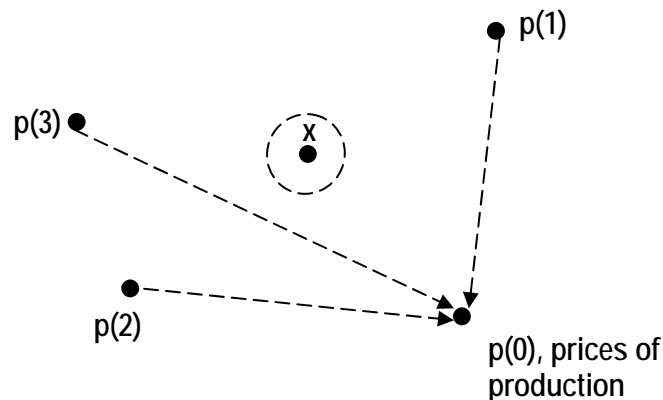

Figure 14: Two-dimensional illustration of the hyperplane and the location of all feasible price systems on it.

Although from a mathematical perspective it might be sufficient to observe actual prices only or to determine directly the numerical values of prices of production based on the reproduction matrix R. But this restricted view of the economy would separate any social scientist from a more comprehensive analysis, and it would leave the economist in an ontological vacuum. The quality (in a philosophic perspective) of prices cannot be linked any longer to essential human activities like work, and further considerations about the nature of the economy would be suppressed. Neither would it be possible to link economics to the non quantitative realms of anthropology, history or philosophy.

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[20] Paul Samuelson: Understanding the Marxian Notion of Exploitation: A Summary of the So-Called Transformation Problem Between Marxian Values and Competitive Prices. Journal of Economic Literature 1971 9 2: 399–431

As Hegel and others have stated frequently, to understand any phenomenon in a deeper way you should know its history and origin. If we – like Samuelson – destroy all the links to the origin of prices, we destroy all the possibilities of deeper investigation.

If we keep up the road of logical or historical emergence of prices, we have to choose which price system we select as relevant for further investigation. Within the labour theory of value two variants of price systems can be offered: (1) values created only by means of material branches of production or (2) values created by all sectors of production of the economy. We can show that in the second case the principle of equivalent exchange is violated, while in case (1) it is not. Thus, my preference would be to go back the pathway of abstraction from the phenomena at the surface in a consistent way and look for a theoretical description of value creation free of contradictions. This can only be done on the basis of the principle of equivalent exchange, meaning that the producers of labour value can acquire the same amount of value in terms of other use values in exchange.