Review of the book

> M. Bronstein: **Symbolic Integration I – Transcendental Functions**. Algorithms and Computation in Mathematics Vol. 1, 2nd edition, Berlin: Springer Verlag 2005, xv + 325 pp., ISBN 3-540-21493-3.

for the journal „Zeitschrift für Analysis und ihre Anwendungen“.

The new edition of the monograph of a well known, strong researcher – who suddenly and much to early passed away this spring – is a slightly enhanced and revised version of the first volume of Springer's “Algorithms and Computation in Mathematics”, a high level standard textbook about constructive aspects of symbolic integration of transcendental functions written by one of the leading experts in that area.

As central topic the author discusses constructive solutions to the following question: For each given elementary function $f(x)$ find an elementary first integral $g(x) = \int f(x)\,\mathrm{d}x$ or a proof that no such function exists. Elementary functions are functions defined by rational expressions containing transcendental functions with rational functions as derivations as, e.g., ln or tan, or towers of such constructions.

The prospect to find an exhaustive constructive solution to that question was judged very pessimistic yet at the beginning of the 20th century. A first complete decision procedure that solves the integration problem was described by R. H. Risch [3] in 1969. It required some more years of concentrated efforts to turn that approach into a computer implementation and to compute integrals within reasonable large subclasses of elementary functions by Computer Algebra Systems (CAS). Besides J. H. Davenport [1] and A. C. Norman [2] it was M. Bronstein who had a big impact on these developments.

The fundamental difficulty of a constructive approach to compute integrals of elementary functions occurs already for first integrals $F(x) = \int \frac{f(x)}{g(x)}\,\mathrm{d}x$ of rational functions. The classical approach of Leibniz-Bernoulli (described in ch. 2.1) starts with a decomposition into partial fractions with linear and quadratic factors in the denominators and can be found in most calculus textbooks. But such a partial fraction decomposition requires a complete factorization of $g(x)$ – a difficult, if not hopeless task from a constructive point of view: it is equivalent to find symbolic expressions for all zeros of $g(x)$, i.e., to compute the splitting field as a tower of field extensions. A first constructive simplification provides the approach of Horowitz-Ostrogradski (ch. 2.3), that starts from a decomposition of $g(x)$ into irreducible over the ground field factors and reduces the problem to the integration of the essential logarithmic part, i.e., with irreducible denominator $g(x)$ and numerator $f(x)$ with $\deg(f) < \deg(g)$. The classical theory implies that the logarithmic part may be written as

$$\sum_{c\,:\,g(c)=0} h(c) \cdot \log(x - c),$$

and it turns out that $h(x)$ is polynomial itself: $h(x) \equiv f(x) \cdot g'(x)^{-1} \pmod{g(x)}$. But even such a solution may introduce extra algebraic numbers: For $f(x) = g'(x)$ the answer is $F(x) = \log(g(x))$ and does not require additional algebraic elements. This problem is addressed and finally solved by the Rioboo-Trager-Rothstein algorithm (ch. 2.5).

Along this main road one can decide existence and compute elementary first integrals also for elementary functions in general. The monograph under review contains a very detailed and precisely elaborated explanation of all important aspects of a constructive approach to this question. It starts with a survey of basic facts and algorithms from constructive algebra (gcd, resultants and subresultants, polynomial remainder sequences, squarefree factorization) required in the following, continues in ch. 2 with first integrals of rational functions, develops in ch. 3 and 4 basics of the Risch calculus (differential fields, order functions and residues), and finally handles the core part of the theory of integration of elementary functions in great detail (ch. 5 and 6). In the remaining chapters the author discusses parametric problems (ch. 7), integration over the reals (ch. 8) and structure theorems (ch. 9). The new edition contains in an additional ch. 10 apects of parallel integration, i.e., how to avoid a recursive approach that is very natural for Risch integration but turns out to be a computational bottleneck in practical applications.

The explanations are very detailed and carried on up to descriptions of the algorithms in pseudo code notation that can – beyond questions of efficiency – almost immediately be implemented in a CAS. The book contains many examples and exercises for the reader to get a feeling not only of the theoretical but also the practical aspects of the algorithms. It is designed as a textbook and well suited for courses about these constructive aspects of calculus. A “must have” for the bookshelf of every library.

[1] J.H. Davenport: On the integration of algebraic fucntions. LNCS vol. 102, Springer, Heidelberg 1981.

[2] A.C. Norman, P.M.A. Moore: *Implementing the New Risch Algorithm*, In: Proc. 4th International Symposium on Advanced Comp. Methods in Theor. Phys., CNRS, Marseilles, 1977.

[3] R. Risch: *The problem of integration in finite terms.* Trans. Amer. Math. Soc. **139** (1969), 167–189.

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