Weighted HOM-Problem for Nonnegative Integers

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HOM-Problem:

regular tree language $\xrightarrow{\text{tree homomorphism}}$ image regular?

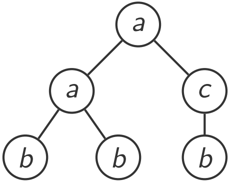
weighted HOM-Problem:

weighted regular tree language $\xrightarrow{\text{tree homomorphism}}$ image regular?

weighted HOM-Problem:

weighted regular tree language $\xrightarrow{\text{tree homomorphism}}$ image regular?

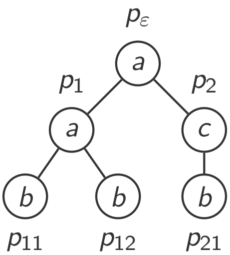
weighted tree automaton over $\mathbb{N}_0$



weighted HOM-Problem:

weighted regular tree language $\xrightarrow{\text{tree homomorphism}}$ image regular?

weighted tree automaton over $\mathbb{N}_0$



weighted HOM-Problem:

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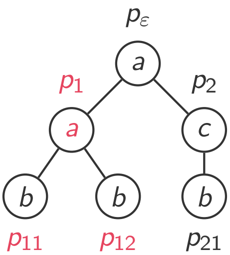

(p_{11}, p_{12}, a, p_1)

weighted HOM-Problem:

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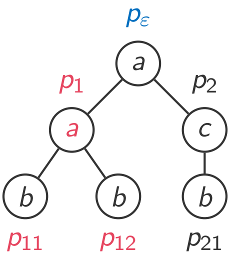

final weights

transition weights

in $\mathbb{N}_0$

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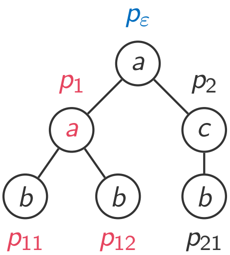

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weight of run $=$

$\prod$ transition weights $\cdot$ final weight

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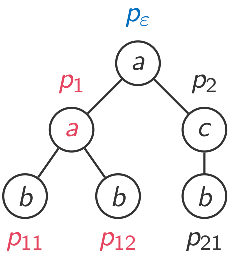

final weights
transition weights
in $\mathbb{N}_0$

(p_{11}, p_{12}, a, p_1)

weight of run =
$\prod$ transition weights · final weight

weight of tree =
sum over runs on tree

alphabets: $\Sigma = \{\sigma^{(1)} \quad \gamma^{(1)} \quad \beta^{(0)}\}$ $\Delta = \{a^{(2)} \quad c^{(1)} \quad b^{(0)}\}$

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homomorphism h from Σ to Δ:

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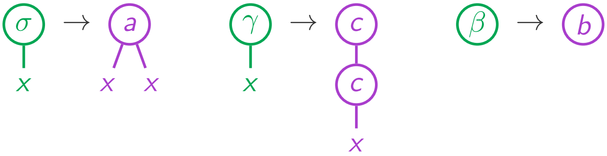

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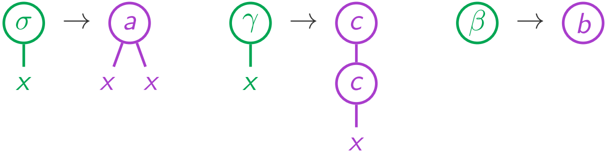

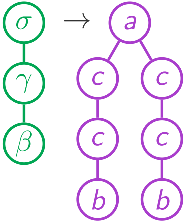

apply rules from bottom to top

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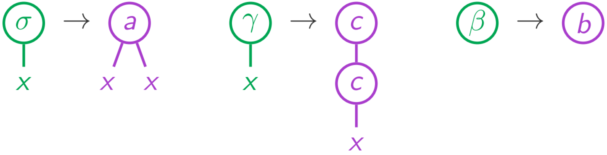

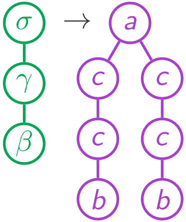

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if $\mathcal{A}$ assigns weights to trees t over Σ

$\Rightarrow s \mapsto \sum_{t \in h^{-1}(s)} \mathcal{A}(t)$ assigns weights to trees s over Δ

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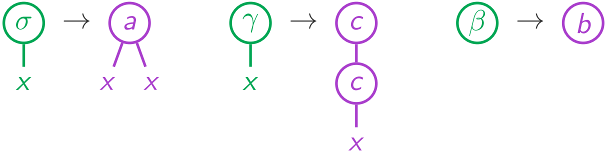

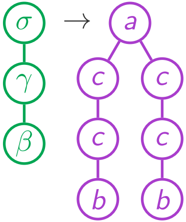

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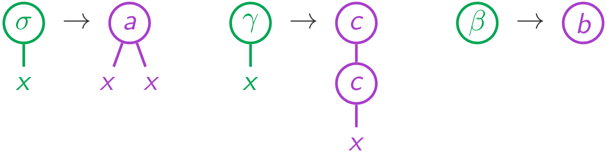

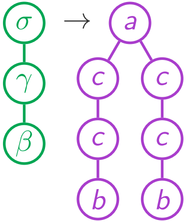

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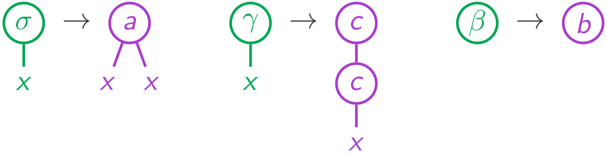

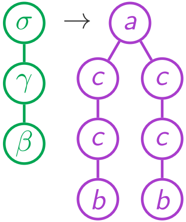

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decidable in P

Proving non-regularity

for $L = \{$  $\mid t$ tree $\}$ $\qquad 1_L$ non-regular $\qquad$ (non-regular support)

Proving non-regularity

for $L = \left\{ \begin{array}{c} a \\ / \ \backslash \\ t \quad t \end{array} \;\middle|\; t \text{ tree} \right\}$ $\quad 1 + 1_L$ non-regular? $\quad \mathcal{A} + 1_L$ non-regular?

Proving non-regularity

for $L = \left\{ a(t, t) \;\middle|\; t \text{ tree} \right\}$ $\qquad 1 + 1_L$ non-regular? $\qquad \mathcal{A} + 1_L$ non-regular?

$$\mathcal{A}\big(a(s, t)\big) = \sum_{\text{run on } a(s,t)} \prod \text{transitions in run} \cdot \text{final weight}$$

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$r(p, q)$

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$$= \sum_{p,q \in Q} \sum_{\substack{r \in Q \\ \text{run on } s \text{ to } p \\ \text{run on } t \text{ to } q}} \prod \text{transitions on } s \cdot \prod \text{transitions on } t \cdot \mathrm{wt}(p,q,a,r) \cdot \mathrm{final}(r)$$

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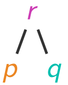

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r
p q

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$$= v \cdot M \cdot w$$

vector · Matrix · vector

Proving non-regularity

for $L = \{$  $\mid t$ tree $\}$ $\qquad$ $\mathcal{A} + 1_L$ non-regular? $\qquad$ $t_1, t_2, t_3, \ldots$ pairwise distinct

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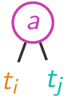
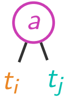
for $L = \Big\{$  $\Big|\ t$ tree $\Big\}$ $\quad \mathcal{A} + 1_L$ non-regular? $\quad t_1, t_2, t_3, \ldots$ pairwise distinct

assume $\mathcal{A} + 1_L$ is regular

$(\mathcal{A} + 1_L)\Big($  $\Big) = \mathcal{A}\Big($  $\Big) + \delta_{ij}$

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$$v_i' M' v_j' = \qquad (\mathcal{A} + 1_L)\left(a(t_i, t_j) \right) = \mathcal{A}\left(a(t_i, t_j) \right) + \delta_{ij} \qquad = v_i M v_j + \delta_{ij}$$

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$$\begin{pmatrix} v_1 \\ v_1' \end{pmatrix}, \begin{pmatrix} v_2 \\ v_2' \end{pmatrix}, \ldots \text{ vectors in finite-dim vector space } \mathbb{Q}^n \quad \Rightarrow \quad \begin{pmatrix} v_{K+1} \\ v_{K+1}' \end{pmatrix} = \sum_{i=1}^{K} \alpha_i \begin{pmatrix} v_i \\ v_i' \end{pmatrix}$$

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