Nivat's Theorem for Turing Machines Based on Unsharp Quantum Logic

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Post measurement collapse of states
repeated measurement of incompatible observables
$\rightsquigarrow$ change of observables

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$\Rightarrow p \wedge (q \vee r) \not\equiv (p \wedge q) \vee (p \wedge r)$ $\quad$ distributivity fails

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Borel sets $\to$ projectors

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probability that measurement of A in state ψ is in $X \subseteq \mathbb{R}$

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spectral measure $P_A\colon \mathcal{B}(\mathbb{R}) \to \mathbb{P}(H)$ for observable A

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negation: $P' = I - P$ $\Rightarrow$ Quantum Logic

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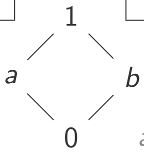

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$a \wedge b = \min\{a, b\}$

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Quantum Computing

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Grover '96 $\mathcal{O}(\sqrt{n})$ algorithm for search in unsorted database

QMV Turing machine

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$I, F\colon Q \to \mathscr{E}$	initial / finial state function

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$\delta\colon Q \times \Gamma \times Q \times \Gamma \times \{L, S, R\} \to \mathscr{E}$	transition function
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A Nivat Theorem for QMV-Turing machines

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$\Leftarrow$ show closures, recognizability of homomorphic languages