A Feferman-Vaught Decomposition Theorem for Weighted MSO Logic*

Manfred Droste
Institute of Computer Science, Leipzig University, 04109 Leipzig, Germany
droste at informatik.uni-leipzig.de

Erik Paul
Institute of Computer Science, Leipzig University, 04109 Leipzig, Germany
epaul at informatik.uni-leipzig.de

Abstract

We prove a weighted Feferman-Vaught decomposition theorem for disjoint unions and products of finite structures. The classical Feferman-Vaught Theorem describes how the evaluation of a first order sentence in a generalized product of relational structures can be reduced to the evaluation of sentences in the contributing structures and the index structure. The logic we employ for our weighted extension is based on the weighted MSO logic introduced by Droste and Gastin to obtain a Büchi-type result for weighted automata. We show that for disjoint unions and products of structures, the evaluation of formulas from two respective fragments of the logic can be reduced to the evaluation of formulas in the contributing structures. We also prove that the respective restrictions are necessary. Surprisingly, for the case of disjoint unions, the fragment is the same as the one used in the Büchi-type result of weighted automata. In fact, even the formulas used to show that the respective restrictions are necessary are the same in both cases. However, here proving that they do not allow for a Feferman-Vaught-like decomposition is more complex and employs Ramsey's Theorem. We also show how translation schemes can be applied to go beyond disjoint unions and products.

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1 Introduction

The Feferman-Vaught Theorem [6] is one of the fundamental theorems in model theory. The theorem describes how the computation of the truth value of a first order sentence in a generalized product of relational structures can be reduced to the computation of truth values of first order sentences in the contributing structures and the evaluation of a monadic second order sentence in the index structure. The theorem itself has a long-standing history. It builds upon work of Mostowski [17], and was later shown to hold true for monadic second order logic (MSO logic) as well [5, 8, 9, 12, 21]. For a survey and more background information, see [13].

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In this paper, we show that under appropriate assumptions, the Feferman-Vaught Theorem also holds true for a weighted MSO logic with arbitrary commutative semirings as weight structure. The logic we employ is based on the weighted logic by Droste and Gastin [3]. In this logic, formulas can take values which convey a quantitative meaning. The logic's connectives and quantifiers hence also adopt quantitative roles. The disjunction becomes a sum, the conjunction a product. The existential quantifier, instead of only checking whether some element with a certain property exists, now takes the truth value of this property for every element in the universe and sums over these values. Under appropriate assumptions, the result of this summation can for instance be the exact number of elements that satisfy the given property. One example of a property which can be expressed using this logic is the number of cliques of a given size in an undirected graph. In [3], the authors prove a Büchi-like result for a specific fragment of the MSO logic, showing that for finite and infinite words, this fragment is expressively equivalent to semiring-weighted automata [20]. The study of a weighted Feferman-Vaught Theorem for disjoint unions, employing the same logic as we do, was initiated by Ravve et al. in [19], where the authors also point out several algorithmic uses and possible applications of a weighted Feferman-Vaught Theorem.

The classical Feferman-Vaught Theorem considers finite and infinite structures without any need for distinction between them. This results from the fact that, in the Boolean setting, infinite joins and meets are well-defined. In particular, existential and universal quantification, which are essentially joins and meets ranging over the whole universe of a structure, are well-defined for finite and infinite structures alike. However, for arbitrary semirings, infinite sums and products are usually not defined. For lack of space, here we consider only finite structures and finite disjoint unions and products of these structures. We note that an extension to infinite structures is possible by employing *bicomplete* semirings. Bicomplete semirings are equipped with infinite sum and product operations that naturally extend their respective finite operations. Our main results are the following.

- We provide a Feferman-Vaught Theorem for disjoint unions of structures with our weighted MSO logic, where the first order product quantifier is restricted to quantify only over formulas which do not contain any sum or product quantifier themselves. Surprisingly, this restriction and the resulting fragment are the same as the one working for the Büchi-like result of [3].
- We show that no similar theorem can hold for disjoint unions if the first order product quantifier is not restricted. The formulas we employ for this in fact also occurred in [3] and [4] as examples of weighted formulas whose semantics could not be described by weighted automata. While in these papers, it was elementary to show that the formulas given define weighted languages not recognizable by weighted automata, here proving that they do not allow for a Feferman-Vaught-like decomposition is more complex and employs a weak version of Ramsey's Theorem [18].
- We show that a Feferman-Vaught Theorem also holds for products of structures for the product-quantifier-free first order fragment of our logic.
- We show that no similar theorem can hold for products if we include the first order product quantifier.
- We show that our theorems are also true for more general disjoint unions and products defined by *translation schemes* [13, 22, 2].

With respect to our proofs, here we just note that in comparison to the universal quantifier of the Boolean setting, the product quantifier requires a separate and new consideration. While universal quantification can simply be expressed using negation and existential quantification, it is in general not possible to express multiplication by addition.

Translation schemes are a model theoretic tool to "translate" structures over one logical signature into structures over another signature in a well behaved fashion, namely in an MSO-defined fashion. They can be applied, for example, to translate between *texts* and *trees* [11], and between *nested words*, *alternating texts*, and *hedges* [16]. These particular translations were employed in [15, 14, 16] to prove that weighted automata over texts, hedges, and nested words are expressively equivalent to weighted logics over these structures. Translation schemes are a rather natural concept and therefore they have been frequently rediscovered and named differently [13, 22, 2]. Our notion of a translation scheme is mostly due to [13].

Related work. A concept related to weighted logics is that of many-valued logics. In both models the evaluation of a formula on a structure produces a quantitative piece of information. In many approaches to many-valued logics, values are taken in the interval $[0, 1]$, cf. [10, 7]. In contrast to this, weights in weighted logics are taken from a semiring and may occur as atomic formulas which enables the modeling of quantitative properties.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A *signature* σ is a pair $(\mathrm{Rel}_\sigma, \mathrm{ar}_\sigma)$ where Rel_σ is a set of relation symbols and $\mathrm{ar}_\sigma \colon \mathrm{Rel}_\sigma \to \mathbb{N}$ the arity function. A *σ-structure* $\mathfrak{A}$ is a pair $(\mathcal{U}_\mathfrak{A}, \mathcal{I}_\mathfrak{A})$ where $\mathcal{U}_\mathfrak{A}$ is a set, called the *universe of* $\mathfrak{A}$, and $\mathcal{I}_\mathfrak{A}$ is an *interpretation*, which maps every $R \in \mathrm{Rel}_\sigma$ to a set $R^\mathfrak{A} \subseteq \mathcal{U}_\mathfrak{A}^{\mathrm{ar}_\sigma(R)}$. A structure is called *finite* if its universe is a finite set. By $\mathrm{Str}(\sigma)$ we denote the class of all σ-structures.

For two σ-structures $\mathfrak{A} = (A, \mathcal{I}_\mathfrak{A})$ and $\mathfrak{B} = (B, \mathcal{I}_\mathfrak{B})$, we define the *product* $\mathfrak{A} \times \mathfrak{B} \in \mathrm{Str}(\sigma)$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}$ and the *disjoint union* $\mathfrak{A} \sqcup \mathfrak{B} \in \mathrm{Str}(\sigma)$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}$ as follows. For the product we let $\mathfrak{A} \times \mathfrak{B} = (A \times B, \mathcal{I}_{\mathfrak{A}\times\mathfrak{B}})$ with $R^{\mathfrak{A}\times\mathfrak{B}} = \{((a_1, b_1), \dots, (a_k, b_k)) \mid (a_1, \dots, a_k) \in R^\mathfrak{A}$ and $(b_1, \dots, b_k) \in R^\mathfrak{B}\}$. For the disjoint union, let $A \sqcup B$ be the disjoint union (i.e., the set theoretic coproduct) of A and B with inclusions ι_A and ι_B. Then $\mathfrak{A} \sqcup \mathfrak{B} = (A \sqcup B, \mathcal{I}_{\mathfrak{A}\sqcup\mathfrak{B}})$ with $R^{\mathfrak{A}\sqcup\mathfrak{B}} = \{(\iota_A(a_1), \dots, \iota_A(a_k)) \mid (a_1, \dots, a_k) \in R^\mathfrak{A}\} \cup \{(\iota_B(b_1), \dots, \iota_A(b_k)) \mid (b_1, \dots, b_k) \in R^\mathfrak{B}\}$. Throughout the paper, we identify $a \in A$ with $\iota_A(a) \in A \sqcup B$ and $b \in B$ with $\iota_B(b) \in A \sqcup B$.

A *commutative semiring* is a tuple $(S, +, \cdot, \mathbb{0}, \mathbb{1})$, abbreviated by S, with operations sum $+$ and product $\cdot$ and constants $\mathbb{0}$ and $\mathbb{1}$ such that $(S, +, \mathbb{0})$ and $(S, \cdot, \mathbb{1})$ are commutative monoids, multiplication distributes over addition, and $s \cdot \mathbb{0} = \mathbb{0}$ for every $s \in S$.

The following definitions are due to [3] in the form of [1]. We provide a countable set $\mathcal{V}$ of first and second order variables, where lower case letters like x and y denote first order variables and capital letters like X and Y denote second order variables. We define monadic second order formulas β over σ and weighted monadic second order formulas φ over σ and S through

$$\beta ::= \mathtt{false} \mid R(x_1, \dots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$
$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\varphi \mid \bigoplus X.\varphi \mid \bigotimes X.\varphi,$$

with $R \in \mathrm{Rel}_\sigma$, $n = \mathrm{ar}_\sigma(R)$, $x, x_1, \dots, x_n \in \mathcal{V}$ first order variables, $X \in \mathcal{V}$ a second order variable and $s \in S$. We also allow the usual abbreviations $\wedge$, $\forall$, $\rightarrow$, $\longleftrightarrow$ and $\mathtt{true}$. By $\mathrm{MSO}(\sigma)$ and $\mathrm{wMSO}(\sigma, S)$ we denote the sets of all monadic second order formulas over σ and all weighted monadic second order formulas over σ and S, respectively. The sets of first order formulas $\mathrm{FO}(\sigma)$ and weighted first order formulas $\mathrm{wFO}(\sigma, S)$ are defined as the sets of all formulas from $\mathrm{MSO}(\sigma)$ and $\mathrm{wMSO}(\sigma, S)$, respectively, which do not contain any subformulas of the form $x \in X$, $\exists X.\beta$, $\bigoplus X.\varphi$ and $\bigotimes X.\varphi$.

The notion of *free variables* is defined as usual, i.e., the operators $\exists, \forall, \bigoplus$ and $\bigotimes$ bind variables. We let $\mathrm{Free}(\varphi)$ be the set of all free variables of φ. A formula φ with $\mathrm{Free}(\varphi) = \emptyset$ is called a *sentence*. For a vector $\bar{\varphi} = (\varphi_1, \ldots, \varphi_n) \in \mathrm{wMSO}(\sigma, S)^n$, we define the set of free variables of $\bar{\varphi}$ as $\mathrm{Free}(\bar{\varphi}) = \bigcup_{i=1}^{n} \mathrm{Free}(\varphi_i)$.

We now define the semantics of MSO and wMSO. Let σ be a signature, $\mathfrak{A} = (A, \mathcal{I}_{\mathfrak{A}})$ a σ-structure and $\mathcal{V}$ a set of first and second order variables. A $(\mathcal{V}, \mathfrak{A})$-assignment ρ is a partial function $\rho \colon \mathcal{V} \nrightarrow A \cup \mathcal{P}(A)$ such that, whenever $x \in \mathcal{V}$ is a first order variable and $\rho(x)$ is defined, we have $\rho(x) \in A$, and whenever $X \in \mathcal{V}$ is a second order variable and $\rho(X)$ is defined, we have $\rho(X) \subseteq A$. The reason we consider partial functions is that in our Feferman-Vaught theorems for the disjoint union of structures we want to be able to restrict the range of a variable assignment to a subset of the universe. For a first order variable, this restriction may cause the variable to become undefined. Let $\mathrm{dom}(\rho)$ be the domain of ρ. For a first order variable $x \in \mathcal{V}$ and an element $a \in A$, the *update* $\rho[x \to a]$ is defined through $\mathrm{dom}(\rho[x \to a]) = \mathrm{dom}(\rho) \cup \{x\}$, $\rho[x \to a](\mathcal{X}) = \rho(\mathcal{X})$ for all $\mathcal{X} \in \mathcal{V} \setminus \{x\}$ and $\rho[x \to a](x) = a$. For a second order variable $X \in \mathcal{V}$ and a set $I \subseteq A$, the update $\rho[X \to I]$ is defined in a similar fashion. By $\mathfrak{A}_{\mathcal{V}}$ we denote the set of all $(\mathcal{V}, \mathfrak{A})$-assignments.

For $\rho \in \mathfrak{A}_{\mathcal{V}}$ and a formula $\beta \in \mathrm{MSO}(\sigma)$ the relation "$(\mathfrak{A}, \rho)$ satisfies β", denoted by $(\mathfrak{A}, \rho) \models \beta$, is defined as usual, with the minor addition that $(\mathfrak{A}, \rho)$ can satisfy $x \in X$ and $R(x_1, \ldots, x_n)$ only if all of the occurring variables are in $\mathrm{dom}(\rho)$. In the following, for all sums and products to be well-defined, we assume that A is finite. For a formula $\varphi \in \mathrm{wMSO}(\sigma, S)$ and a structure $\mathfrak{A} \in \mathrm{Str}(\sigma)$, the *(weighted) semantics* of φ is a mapping $[\![\varphi]\!](\mathfrak{A}, \cdot) \colon \mathfrak{A}_{\mathcal{V}} \to S$ inductively defined as

$$
\begin{aligned}
[\![\beta]\!](\mathfrak{A}, \rho) &= \begin{cases} \mathbb{1} & \text{if } (\mathfrak{A}, \rho) \models \beta \\ \mathbb{0} & \text{otherwise} \end{cases} \\
[\![s]\!](\mathfrak{A}, \rho) &= s \\
[\![\varphi_1 \oplus \varphi_2]\!](\mathfrak{A}, \rho) &= [\![\varphi_1]\!](\mathfrak{A}, \rho) + [\![\varphi_2]\!](\mathfrak{A}, \rho) \\
[\![\varphi_1 \otimes \varphi_2]\!](\mathfrak{A}, \rho) &= [\![\varphi_1]\!](\mathfrak{A}, \rho) \cdot [\![\varphi_2]\!](\mathfrak{A}, \rho)
\end{aligned}
\qquad
\begin{aligned}
[\![\textstyle\bigoplus x.\varphi]\!](\mathfrak{A}, \rho) &= \textstyle\sum_{a \in A} [\![\varphi]\!](\mathfrak{A}, \rho[x \to a]) \\
[\![\textstyle\bigotimes x.\varphi]\!](\mathfrak{A}, \rho) &= \textstyle\prod_{a \in A} [\![\varphi]\!](\mathfrak{A}, \rho[x \to a]) \\
[\![\textstyle\bigoplus X.\varphi]\!](\mathfrak{A}, \rho) &= \textstyle\sum_{I \subseteq A} [\![\varphi]\!](\mathfrak{A}, \rho[X \to I]) \\
[\![\textstyle\bigotimes X.\varphi]\!](\mathfrak{A}, \rho) &= \textstyle\prod_{I \subseteq A} [\![\varphi]\!](\mathfrak{A}, \rho[X \to I]).
\end{aligned}
$$

We will usually identify a pair $(\mathfrak{A}, \emptyset)$ with $\mathfrak{A}$. For a vector of formulas $\bar{\varphi} \in \mathrm{wMSO}(\sigma, S)^n$, we define $[\![\bar{\varphi}]\!](\mathfrak{A}, \rho) = ([\![\varphi_1]\!](\mathfrak{A}, \rho), \ldots, [\![\varphi_n]\!](\mathfrak{A}, \rho)) \in S^n$.

We give some examples of how weighted formulas can be interpreted. For more examples, see also [19].

▶ **Example 1.** If $S = \mathbb{B}$ is the Boolean semiring, we obtain the classical Boolean logic.

▶ **Example 2.** Assume that $S = (\mathbb{Q}, +, \cdot, 0, 1)$ is the field of rational numbers and that σ is the signature of an (undirected) graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}\}$ with edge binary. Then for every fixed $n \in \mathbb{N}$, we can count the number of n-cliques of a graph with no loops $\mathfrak{G} \in \mathrm{Str}(\sigma)$ using the formula $\varphi = \frac{1}{n!} \otimes \bigoplus x_1 \ldots \bigoplus x_n. \bigwedge_{i \neq j} (\mathrm{edge}(x_i, x_j) \vee \mathrm{edge}(x_j, x_i))$.

▶ **Example 3.** We consider the *minimum cut* of directed acyclic graphs. For this, we interpret these graphs as *flow networks* in the following way. Every vertex which does not have a predecessor is considered a *source*, every vertex without successors is considered a *drain*, and every edge is assumed to have a capacity of 1. Let $G = (V, E)$ be a directed acyclic graph where V is the set of vertices and $E \subseteq V \times V$ the set of edges. A cut (S, D) of G is a partition of V, i.e., $S \cup D = V$ and $S \cap D = \emptyset$, such that all sources of G are in S, and all drains of G are in D. The *minimum cut* of G is the smallest number $|E \cap (S \times D)|$ such that (S, D) is a cut of G.

We can express the minimum cut of directed acyclic graphs by a weighted formula as follows. We let σ be the signature from the previous example and this time interpret it as the signature of a directed graph. For our semiring, we choose the tropical semiring $\mathrm{Trop} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$. Then using the abbreviation

$$\mathrm{cut}(X,Y) = \forall x.\Big((x \in X \leftrightarrow \neg(x \in Y)) \wedge (\exists y.\mathrm{edge}(y,x) \vee x \in X) \wedge (\exists y.\mathrm{edge}(x,y) \vee x \in Y)\Big)$$

we can express the minimum cut of a directed acyclic graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$ using the formula

$$\varphi = \bigoplus X.\bigoplus Y.\Big(\mathrm{cut}(X,Y) \otimes \bigotimes x.\bigotimes y.(1 \oplus \neg(x \in X \wedge y \in Y \wedge \mathrm{edge}(x,y)))\Big).$$

For $\varphi \in \mathrm{wMSO}(\sigma, S)$ and a first order variable x which does not appear in φ as a bound variable, we define φ^{-x} as the formula obtained from φ by replacing all atomic subformulas containing x, i.e., all subformulas of the form $x \in X$ and $R(\dots, x, \dots)$ for $R \in \mathrm{Rel}_\sigma$, by `false`. It is easy to show by induction that for all σ-structures $\mathfrak{A} = (A, \mathcal{I}_\mathfrak{A})$ and $(\mathcal{V}, \mathfrak{A})$-assignments ρ with $x \notin \mathrm{dom}(\rho)$ we have $[\![\varphi]\!](\mathfrak{A}, \rho) = [\![\varphi^{-x}]\!](\mathfrak{A}, \rho)$. As in the sequel we will deal with disjoint unions and products of structures, we need to define the restrictions of a variable assignment to the contributing structures of the disjoint union or product. Fix two structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ with universes A and B. For a $(\mathcal{V}, \mathfrak{A} \sqcup \mathfrak{B})$-assignment ρ, we define the restriction $\rho|_\mathfrak{A} \colon \mathcal{V} \nrightarrow A$ as

$$\rho|_\mathfrak{A}(\mathcal{X}) = \begin{cases} \rho(\mathcal{X}) \cap A & \text{if } \mathcal{X} \text{ is a second order variable} \\ \rho(\mathcal{X}) & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \in A \\ \text{undefined} & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \notin A. \end{cases}$$

The restriction $\rho|_\mathfrak{B}$ is defined similarly.

For a $(\mathcal{V}, \mathfrak{A} \times \mathfrak{B})$-assignment ρ, we define the restrictions $\rho|_\mathfrak{A}$ and $\rho|_\mathfrak{B}$ by projection on the corresponding entries. That is, we let π_A be the projection on the first and π_B be the projection on the second entry of $A \times B$ and let $\rho|_\mathfrak{A} = \pi_A \circ \rho$ and $\rho|_\mathfrak{B} = \pi_B \circ \rho$.

The *union* of two assignments ρ and ς with $\mathrm{dom}(\rho) \cap \mathrm{dom}(\varsigma) = \emptyset$, denoted by $\rho \cup \varsigma$, is defined by $\mathrm{dom}(\rho \cup \varsigma) = \mathrm{dom}(\rho) \cup \mathrm{dom}(\varsigma)$, $(\rho \cup \varsigma)(\mathcal{X}) = \rho(\mathcal{X})$ for $\mathcal{X} \in \mathrm{dom}(\rho)$ and $(\rho \cup \varsigma)(\mathcal{X}) = \varsigma(\mathcal{X})$ for $\mathcal{X} \in \mathrm{dom}(\varsigma)$.

Fix two disjoint sets of variables $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$. For $n \in \mathbb{N}$ we define the set of *expressions* $\mathrm{Exp}_n(S)$ over a semiring S by the grammar

$$E ::= x_i \mid y_i \mid E \oplus E \mid E \otimes E,$$

where $i \in \{1, \dots, n\}$. The *(weighted) semantics* of an expression $E \in \mathrm{Exp}_n(S)$ is a mapping $\langle\!\langle E \rangle\!\rangle \colon S^n \times S^n \to S$ defined for $\bar{s}, \bar{t} \in S^n$ inductively by

$$\langle\!\langle x_i \rangle\!\rangle(\bar{s}, \bar{t}) = s_i \qquad \langle\!\langle E_1 \oplus E_2 \rangle\!\rangle(\bar{s}, \bar{t}) = \langle\!\langle E_1 \rangle\!\rangle(\bar{s}, \bar{t}) + \langle\!\langle E_2 \rangle\!\rangle(\bar{s}, \bar{t})$$
$$\langle\!\langle y_i \rangle\!\rangle(\bar{s}, \bar{t}) = t_i \qquad \langle\!\langle E_1 \otimes E_2 \rangle\!\rangle(\bar{s}, \bar{t}) = \langle\!\langle E_1 \rangle\!\rangle(\bar{s}, \bar{t}) \cdot \langle\!\langle E_2 \rangle\!\rangle(\bar{s}, \bar{t}).$$

For expressions over the Boolean semiring $\mathbb{B} = (\{\mathtt{false}, \mathtt{true}\}, \vee, \wedge, \mathtt{false}, \mathtt{true})$ we will usually write $\vee$ instead of $\oplus$ and $\wedge$ instead of $\otimes$.

▶ **Construction 4.** We call an expression $E \in \mathrm{Exp}_n(S)$ a *pure product* if

$$E = \mathrm{x}_1 \otimes \ldots \otimes \mathrm{x}_l \otimes \mathrm{y}_1 \otimes \ldots \otimes \mathrm{y}_m$$

with $\mathrm{x}_i \in \{x_1, \dots, x_n\}$ for $i \in \{1, \dots, l\}$ and $\mathrm{y}_j \in \{y_1, \dots, y_n\}$ for $j \in \{1, \dots, m\}$. We define a substitution procedure as follows. Let $\bar{\varphi}^1, \bar{\varphi}^2 \in \mathrm{wMSO}(\sigma, S)^n$ be given. Let $i \in \{1, \dots, l\}$

and assume $\mathrm{x}_i = x_k$ for some k, then we define $\xi_i = \varphi_k^{\mathbb{1}}$. Likewise, for $j \in \{1, \ldots, m\}$ and $\mathrm{y}_j = y_k$, we define $\theta_j = \varphi_k^2$. We let $\xi = \xi_1 \otimes \ldots \otimes \xi_l$ and $\theta = \theta_1 \otimes \ldots \otimes \theta_m$. Then for $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, every $(\mathcal{V}, \mathfrak{A})$-assignment ρ and every $(\mathcal{V}, \mathfrak{B})$-assignment ς we have

$$\langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho), [\![\bar{\varphi}^2]\!](\mathfrak{B}, \varsigma)) = [\![\xi]\!](\mathfrak{A}, \rho) \cdot [\![\theta]\!](\mathfrak{B}, \varsigma).$$

We define $\mathrm{PRD}^{\mathbb{1}}(E, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2) = \xi$ and $\mathrm{PRD}^2(E, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2) = \theta$.

Pure products $B \in \mathrm{Exp}_n(\mathbb{B})$ are also called *pure conjunctions.* For a pure conjunction $B \in \mathrm{Exp}_n(\mathbb{B})$, formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in \mathrm{MSO}(\sigma)$ and ξ_i, θ_j as above, we define the $\mathrm{MSO}(\sigma)$-formulas $\mathrm{CON}^{\mathbb{1}}(B, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2) = \xi = \xi_1 \wedge \ldots \wedge \xi_l$ and $\mathrm{CON}^2(B, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2) = \theta = \theta_1 \wedge \ldots \wedge \theta_m$. We then have

$$\langle\!\langle B \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho), [\![\bar{\varphi}^2]\!](\mathfrak{B}, \varsigma)) = \mathtt{true} \text{ iff } (\mathfrak{A}, \rho) \models \xi \text{ and } (\mathfrak{B}, \varsigma) \models \theta. \qquad \Diamond$$

We say that an expression $E \in \mathrm{Exp}_n(S)$ is in *normal form* if $E = E_1 \oplus \ldots \oplus E_m$ for some $m \geq 1$ and pure products E_i. By applying the laws of distributivity of the semiring S, every expression $E \in \mathrm{Exp}_n(S)$ can be transformed into normal form. More precisely, we have the following lemma.

▶ **Lemma 5.** *For every $E \in \mathrm{Exp}_n(S)$ there exists an expression $E' \in \mathrm{Exp}_n(S)$ in normal form with the same semantics as E.*

3 The classical Feferman-Vaught Theorem

For convenience, we recall the Feferman-Vaught Theorem for disjoint unions and products of two structures. Let σ be a signature.

▶ **Theorem 6** ([6]). *Let $\mathcal{V}$ be a set of first and second order variables and $\beta \in \mathrm{MSO}(\sigma)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, vectors of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^2 \in \mathrm{MSO}(\sigma)^n$ and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^2) \subseteq \mathrm{Free}(\beta)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ and all $(\mathcal{V}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ρ:*

$$(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho|_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho|_{\mathfrak{B}})) = \mathtt{true}.$$

▶ **Theorem 7** ([6]). *Let $\mathcal{V}$ be a set of first and second order variables and $\beta \in \mathrm{FO}(\sigma)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, vectors of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^2 \in \mathrm{FO}(\sigma)^n$ and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^2) \subseteq \mathrm{Free}(\beta)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ and all $(\mathcal{V}, \mathfrak{A} \times \mathfrak{B})$-assignments ρ:*

$$(\mathfrak{A} \times \mathfrak{B}, \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho|_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho|_{\mathfrak{B}})) = \mathtt{true}.$$

4 Translation schemes

Theorems 6 and 7 consider disjoint unions and products only. So far, there is no interaction between the two constituting structures. *Translation schemes* allow us to create such interactions in an MSO-defined manner. More precisely, translation schemes “translate” structures over one signature into structures over another signature. Applying this to disjoint unions and products, we can extend Theorems 6 and 7 to more complex constructs. The usefulness of such extensions by translation schemes was discussed in [13], which we follow here.

Let σ and τ be two signatures, $\mathcal{Z} = \{z, z_1, z_2, \ldots\}$ be a set of distinguished first order variables and $\mathcal{W}$ be a set of first and second order variables with $\mathcal{W} \cap \mathcal{Z} = \emptyset$. A *$\sigma$-$\tau$-translation scheme Φ over $\mathcal{W}$ and $\mathcal{Z}$* is a pair $(\phi_{\mathcal{U}}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ where $\phi_{\mathcal{U}}, \phi_T \in \mathrm{MSO}(\sigma)$, $\phi_{\mathcal{U}}$

has variables from $\mathcal{W} \cup \{z\}$ and ϕ_T has variables from $\mathcal{W} \cup \{z_1, \ldots, z_{\mathrm{ar}_\tau(T)}\}$. The variables from $\mathcal{Z}$ may not be used for quantification, i.e., all variables from $\mathcal{Z}$ must be free. We set $\mathrm{Free}(\Phi) = \mathrm{Free}(\phi_\mathcal{U}) \cup \bigcup_{T \in \mathrm{Rel}_\tau} \mathrm{Free}(\phi_T)$. The formulas $\phi_\mathcal{U}$ and $(\phi_T)_{T \in \mathrm{Rel}_\tau}$ depend on $\mathcal{Z}$ in the following way. For a first order variable x not occurring in $\phi_\mathcal{U}$, the formula $\phi_\mathcal{U}(x)$ is obtained from $\phi_\mathcal{U}$ by replacing all occurrences of z by x. Similarly, for $T \in \mathrm{Rel}_\tau$ and first order variables $x_1, \ldots, x_{\mathrm{ar}_\tau(T)}$ not occurring in ϕ_T, the formula $\phi_T(x_1, \ldots, x_{\mathrm{ar}_\tau(T)})$ is obtained from ϕ_T by replacing all occurrences of z_i by x_i for $i \in \{1, \ldots, \mathrm{ar}_\tau(T)\}$.

For a σ-structure $\mathfrak{A} = (A, \mathcal{I}_\mathfrak{A})$ and a $(\mathcal{W}, \mathfrak{A})$-assignment ς, we define the Φ-*induced* τ-*structure of* $\mathfrak{A}$ *and* ς, denoted by $\Phi^\star(\mathfrak{A}, \varsigma)$, as a τ-structure with universe $\mathcal{U}_\mathfrak{C}$ and interpretation $\mathcal{I}_\mathfrak{C}$ as follows.

$$\mathcal{U}_\mathfrak{C} = \{a \in A \mid (\mathfrak{A}, \varsigma[z \to a]) \models \phi_\mathcal{U}\} \qquad \mathcal{I}_\mathfrak{C}(T) = \{\bar{c} \in \mathcal{U}_\mathfrak{C}^{\mathrm{ar}_\tau(T)} \mid (\mathfrak{A}, \varsigma[\bar{z} \to \bar{c}]) \models \phi_T\}$$

▶ **Example 8.** A translation scheme can be used to cut a subtree from a given tree at a specified node in the tree. For this let $\sigma = \tau = (\{\mathrm{edge}\}, \mathrm{edge} \mapsto 2)$ be the signature of a directed graph. For a σ-structure $\mathfrak{G} = (V, \mathrm{edge} \mapsto E)$ let E' be the transitive closure of the relation $E \subseteq V \times V$. We say that $\mathfrak{G}$ is a directed rooted tree with root $r \in V$ if (1) E' is irreflexive, (2) $(r, v) \in E'$ for all $v \in V \setminus \{r\}$ and (3) for all $v \in V \setminus \{r\}$ there is exactly one $v' \in V$ with $(v', v) \in E$. We define the following abbreviation which describes the reflexive transitive closure of E.

$$(x \leq y) = \forall X\, (x \in X \wedge (\forall z.(\exists z'.z' \in X \wedge \mathrm{edge}(z', z)) \to z \in X)) \to y \in X$$

We define a σ-σ-translation scheme $\Phi = (\phi_\mathcal{U}, \phi_{\mathrm{edge}})$ through $\phi_\mathcal{U} = (x \leq z)$ and $\phi_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2)$. Then with $\mathfrak{G}$ as above and $v \in V$, the structure $\mathfrak{C} = \Phi^*(\mathfrak{G}, x \mapsto v)$ is the subtree of $\mathfrak{G}$ at the node v, i.e.,

$$\mathcal{U}_\mathfrak{C} = \{v\} \cup \{v' \in V \mid (v, v') \in E'\} \qquad\qquad \mathcal{I}_\mathfrak{C} = E \cap (\mathcal{U}_\mathfrak{C} \times \mathcal{U}_\mathfrak{C}).$$

We have the following fundamental property of translation schemes [13].

▶ **Lemma 9** ([13]). *Let* $\Phi = (\phi_\mathcal{U}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ *be a* σ-τ-*translation scheme over* $\mathcal{W}$ *and* $\mathcal{Z}$, $\mathcal{V}$ *be a set of first and second order variables such that* $\mathcal{V}$, $\mathcal{W}$, *and* $\mathcal{Z}$ *are pairwise disjoint, and* $\beta \in \mathrm{MSO}(\tau)$ *with variables from* $\mathcal{V}$. *Then there exists a formula* $\alpha \in \mathrm{MSO}(\sigma)$ *such that* $\mathrm{Free}(\alpha) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ *and for all structures* $\mathfrak{A} \in \mathrm{Str}(\sigma)$, *all* $(\mathcal{W}, \mathfrak{A})$-*assignments* ς *and all* $(\mathcal{V}, \Phi^\star(\mathfrak{A}, \varsigma))$-*assignments* ρ:

$$(\Phi^\star(\mathfrak{A}, \varsigma), \rho) \models \beta \textit{ iff } (\mathfrak{A}, \varsigma \cup \rho) \models \alpha.$$

Together with Theorems 6 and 7, this gives us the following Feferman-Vaught decomposition theorems for disjoint unions and products with translations schemes.

▶ **Theorem 10** ([13]). *Let* $\Phi = (\phi_\mathcal{U}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ *be a* σ-τ-*translation scheme over* $\mathcal{W}$ *and* $\mathcal{Z}$, $\mathcal{V}$ *be a set of first and second order variables such that* $\mathcal{V}$, $\mathcal{W}$, *and* $\mathcal{Z}$ *are pairwise disjoint, and* $\beta \in \mathrm{MSO}(\tau)$ *with variables from* $\mathcal{V}$. *Then there exist* $n \geq 1$, *vectors of formulas* $\bar{\beta}^\mathbb{1}, \bar{\beta}^2 \in \mathrm{MSO}(\sigma)^n$ *and an expression* $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ *such that* $\mathrm{Free}(\bar{\beta}^\mathbb{1}) \cup \mathrm{Free}(\bar{\beta}^2) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ *and for all structures* $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, *all* $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-*assignments* ς *and all* $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-*assignments* ρ:

$$(\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^\mathbb{1}]\!](\mathfrak{A}, (\varsigma \cup \rho)|_\mathfrak{A}), [\![\bar{\beta}^2]\!](\mathfrak{B}, (\varsigma \cup \rho)|_\mathfrak{B})) = \mathtt{true}.$$

▶ **Theorem 11** ([13]). *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\beta \in \mathrm{FO}(\tau)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, vectors of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{FO}(\sigma)^n$ and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ:*

$$(\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)|_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)|_{\mathfrak{B}})) = \mathtt{true}.$$

We give a short example to illustrate Theorem 10.

▶ **Example 12.** We consider the signature σ of a labeled graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}, \mathrm{lab}_a, \mathrm{lab}_b\}$ where edge has arity 2 and $\mathrm{lab}_a, \mathrm{lab}_b$ both have arity 1. Given two directed rooted labeled trees $\mathfrak{G}_1, \mathfrak{G}_2$ in this signature (see Example 8), we can use a translation scheme to add edges between all leaves of $\mathfrak{G}_1$ and the root of $\mathfrak{G}_2$ in $\mathfrak{G}_1 \sqcup \mathfrak{G}_2$. For this scenario we have to distinguish between the vertices from the first and the second graph, so the use of an intermediate signature is necessary. We define the signature σ' to be σ extended by the relation symbols G_1 and G_2 of arity 1. Then for $i \in \{1, 2\}$ we define a σ-σ'-translation scheme $\Phi_i = (\phi_{\mathcal{U}}, \phi'_{\mathrm{edge}}, \phi_{\mathrm{lab}_a}, \phi_{\mathrm{lab}_b}, \phi^i_{G_1}, \phi^i_{G_2})$ by

$$\phi_{\mathcal{U}} = \mathtt{true} \qquad \phi_{\mathrm{lab}_a} = \mathrm{lab}_a(z_1) \qquad \phi^i_{G_j} = \begin{cases} \mathtt{true} & \text{if } i = j \\ \mathtt{false} & \text{otherwise.} \end{cases}$$
$$\phi'_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2) \qquad \phi_{\mathrm{lab}_b} = \mathrm{lab}_b(z_1)$$

With the abbreviations $\mathrm{root}(x) = \neg\exists y.\mathrm{edge}(y, x)$ and $\mathrm{leaf}(x) = \neg\exists y.\mathrm{edge}(x, y)$ we then define the σ'-σ-translation scheme $\Phi = (\phi_{\mathcal{U}}, \phi_{\mathrm{edge}}, \phi_{\mathrm{lab}_a}, \phi_{\mathrm{lab}_b})$ through

$$\phi_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2) \vee (G_1(z_1) \wedge G_2(z_2) \wedge \mathrm{leaf}(z_1) \wedge \mathrm{root}(z_2)).$$

Then $\mathfrak{G} = \Phi^*(\Phi_1^*(\mathfrak{G}_1) \sqcup \Phi_2^*(\mathfrak{G}_2))$ is exactly $\mathfrak{G}_1 \sqcup \mathfrak{G}_2$ with the leaves of $\mathfrak{G}_1$ connected to the root of $\mathfrak{G}_2$. We now consider the formula

$$\beta = \exists x.\exists y.(\mathrm{edge}(x, y) \wedge \mathrm{lab}_a(x) \wedge \mathrm{lab}_b(y))$$

which asks whether there is some edge between an a-labeled and a b-labeled vertex. We can apply Lemma 9 and Theorem 10 to obtain the following decomposition of β. Let

$$\bar{\beta}^{\mathbb{1}} = (\beta, \exists x.\mathrm{lab}_a(x) \wedge \mathrm{leaf}(x)) \qquad B_\beta = x_1 \vee y_1 \vee (x_2 \wedge y_2).$$
$$\bar{\beta}^{2} = (\beta, \exists y.\mathrm{lab}_b(y) \wedge \mathrm{root}(y))$$

Then we have $\mathfrak{G} \models \beta$ iff $\langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{G}_1), [\![\bar{\beta}^{2}]\!](\mathfrak{G}_2)) = \mathtt{true}$.

5 Weighted Feferman-Vaught Decomposition Theorems

Our goal is to prove weighted versions of Theorems 10 and 11. That is, we would like to replace FO by wFO and MSO by wMSO in those theorems. This, however, is not possible as we will see in Sections 5.2 and 5.3. For disjoint unions, we have to restrict the use of the first order product quantifier and entirely remove the second order product quantifier in wMSO. For products, it is not possible to include the first order product quantifier at all.

5.1 Formulation of the theorems

Let σ be a signature and S a commutative semiring. We define two fragments of our logic and formulate our weighted versions of Theorems 10 and 11 for these fragments.

▶ **Definition 13** (Product-free weighted first order logic)**.** We define the *product-free* first order fragment $\mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, S)$ of our logic as the set of all formulas from $\mathrm{wFO}(\sigma, S)$ which do not contain any first order product quantifier. Using this fragment, we will formulate a weighted Feferman-Vaught decomposition theorem for products of structures.

▶ **Definition 14** (Product-restricted weighted monadic second order logic)**.** In order to define the *product-restricted* fragment of our weighted monadic second order logic, we first define the fragment of so-called *almost-Boolean* formulas through the grammar

$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \otimes \psi.$$

This fragment, which we denote by $\mathrm{wMSO}^{\text{a-bool}}(\sigma, S)$, already appeared in [3] in the form of *recognizable step functions*. To obtain the main theorem of [3], the product quantifier was restricted to quantify only over recognizable step functions. We employ the same restriction and define the product-restricted fragment of our logic through the grammar

$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\psi \mid \bigoplus X.\varphi,$$

where $\beta \in \mathrm{MSO}(\sigma)$ is a monadic second order formula, $s \in S$, x is a first order variable, X is a second order variable and $\psi \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, S)$ is an almost-Boolean formula. By $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)$ we denote the set of all such formulas. The set $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)$ contains all formulas from $\mathrm{wMSO}(\sigma, S)$ which do not contain any second order quantifier and where for every subformula of the form $\bigotimes x.\psi$ we have that ψ is an almost-Boolean formula. Our weighted Feferman-Vaught decomposition theorem for disjoint unions of structures will be formulated for this fragment. In [3] it was shown that for finite and infinite words, this fragment is expressively equivalent to weighted finite automata.

We note that the restrictions we impose on the product quantifier are necessary as we will show in Sections 5.2 and 5.3. We formulate the weighted versions of Theorems 10 and 11 as follows.[1] Let τ, $\mathcal{W}$ and $\mathcal{Z}$ be as in Section 4.

▶ **Theorem 15.** *Let S be a commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\tau, S)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, vectors of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^2) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$ and an expression $E_\varphi \in \mathrm{Exp}_n(S)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)|_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, (\varsigma \cup \rho)|_{\mathfrak{B}})).$$

▶ **Theorem 16.** *Let S be a commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that*

[1] In [19] a weighted version of Theorem 10 similar to ours is stated (without proof) to hold without any restriction on the first order product quantifier. However, in Subsection 5.2 we show that a restriction on the product quantifier is necessary.

$\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wFO}^{\bigotimes\text{-free}}(\tau, S)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, vectors of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, S)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$ and an expression $E_\varphi \in \mathrm{Exp}_n(S)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \times \mathfrak{B})$-assignments ς and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ we have

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)|_{\mathfrak{A}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)|_{\mathfrak{B}})).$$

The proofs of both theorems are deferred to Section 5.4. For formulas without free variables and a trivial translation scheme, i.e., $\phi_{\mathcal{U}} = \mathtt{true}$ and $\phi_T = T(z_1, \dots, z_{\mathrm{ar}_\tau(T)})$ for all $T \in \mathrm{Rel}_\tau$, the theorems reduce to the following, simplified versions.

▶ **Theorem 17.** *Let S be a commutative semiring and $\varphi \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)$ be a sentence. Then there exist $n \geq 1$, vectors of sentences $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^n$ and an expression $E_\varphi \in \mathrm{Exp}_n(S)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ we have*

$$[\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B})).$$

▶ **Theorem 18.** *Let S be a commutative semiring and $\varphi \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, S)$ be a sentence. Then there exist $n \geq 1$, vectors of sentences $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, S)^n$ and an expression $E_\varphi \in \mathrm{Exp}_n(S)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ we have*

$$[\![\varphi]\!](\mathfrak{A} \times \mathfrak{B}) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B})).$$

▶ **Example 19.** To illustrate Theorem 17 we consider the semiring of natural numbers $\mathbb{N}_0 = (\mathbb{N}_0, +, \cdot, 0, 1)$ and the signature σ of a labeled graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}, \mathrm{lab}_a, \mathrm{lab}_b\}$ with edge binary and $\mathrm{lab}_a, \mathrm{lab}_b$ both unary. Consider the following formula which multiplies the number of vertices labeled a with the number of edges between two vertices labeled b.

$$\Big(\underbrace{\bigoplus x.\mathrm{lab}_a(x)}_{=\varphi_a}\Big) \otimes \Big(\underbrace{\bigoplus x.\bigoplus y.\mathrm{edge}(x,y) \wedge \mathrm{lab}_b(x) \wedge \mathrm{lab}_b(y)}_{=\varphi_b}\Big)$$

The formula can be decomposed as follows. Let $\bar{\varphi}^{\mathbb{1}} = \bar{\varphi}^{2} = (\varphi_a, \varphi_b)$ and $E_\varphi = (x_1 \oplus y_1) \otimes (x_2 \oplus y_2)$. Then for all σ-structures $\mathfrak{G}_1, \mathfrak{G}_2$ we have $[\![\varphi]\!](\mathfrak{G}_1 \sqcup \mathfrak{G}_2) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{G}_1), [\![\bar{\varphi}^{2}]\!](\mathfrak{G}_2))$.

▶ **Example 20.** In [19], it is discussed how translation schemes can be applied for Feferman-Vaught-like decompositions of weighted properties. Theorems 15 and 16 show that this is possible for all properties which can be expressed by formulas in our weighted logic fragments.

5.2 Necessity of restricting the logic for disjoint unions

In this section, we show that the restrictions we impose on the product quantifiers are indeed necessary. For disjoint unions, we will prove that already Theorem 17 does not hold over the tropical semiring $\mathrm{Trop} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0)$ and over the arctic semiring $\mathrm{Arct} = (\mathbb{R}_{\geq 0} \cup \{-\infty\}, \max, +, -\infty, 0)$ for the formulas $\bigotimes x. \bigotimes y.1$ and $\bigotimes X.1$. Here, $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. To prove this, we employ Ramsey's Theorem. Then we show that for the formula $\bigotimes x. \bigoplus y.1$, Theorem 17 does not hold over the semiring $\mathbb{N}_0 = (\mathbb{N}_0, +, \cdot, 0, 1)$. We note that these types of formulas also occurred in [3] and [4] as examples of weighted formulas whose semantics could not be described by weighted automata.

We will employ the following version of Ramsey's Theorem. For a set X, we denote by $\left[\frac{X}{2}\right]$ the set of all subsets of X of size 2.

▶ **Theorem 21** ([18]). *Let* $f\colon \left[\frac{\mathbb{N}}{2}\right] \to \{1,\dots,k\}$ *be a function. Then there exists an infinite subset* $E \subseteq \mathbb{N}$ *such that* $f|_{\left[\frac{E}{2}\right]} \equiv i$ *for some* $i \in \{1,\dots,k\}$.

▶ **Theorem 22.** *Let* $S \in \{\mathrm{Trop}, \mathrm{Arct}\}$, $\sigma = (\emptyset, \emptyset)$ *be the empty signature and for* $l \in \mathbb{N}$ *consider the* σ*-structures* $\mathfrak{S}_l = (\{1,\dots,l\}, \emptyset)$*. Then for* $\varphi = \bigotimes x. \bigotimes y.1$ *there do not exist* $n \in \mathbb{N}$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in (\mathrm{wMSO}(\sigma, S))^n$ *and* $E_\varphi \in \mathrm{Exp}_n(S)$ *such that for all* $l, m \in \mathbb{N}$ *we have*

$$[\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)). \tag{5.1}$$

Proof (Sketch). First, consider $S = \mathrm{Trop}$. For contradiction, suppose that $n, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2$ and E_φ as above satisfying (5.1) exist. We may assume that $E_\varphi = E_1 \oplus \ldots \oplus E_k$ is in normal form with all E_i pure products. For $l \geq 1$ and $i \in \{1,\dots,k\}$ we let $a_{li} = [\![\mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$ and $b_{li} = [\![\mathrm{PRD}^2(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$. Then by assumption we have

$$(l+m)^2 = [\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \min_{i=1}^{k}(a_{li} + b_{mi}). \tag{5.2}$$

Given $l \geq 1$ and $m \geq 1$, for at least one index $j \in \{1,\dots,k\}$ we have $(l+m)^2 = a_{lj} + b_{mj}$. We define j_{lm} as the smallest such index. Then we define a function $f\colon \left[\frac{\mathbb{N}}{2}\right] \to \{1,\dots,k\}$ by $f(\{l,m\}) = j_{lm}$ for $l < m$. Now take $E \subseteq \mathbb{N}$ according to Ramsey's Theorem. As E is infinite, there are $l, \lambda, m, \mu \in E$ with $l < \lambda < m < \mu$. With $j = j_{lm}$, we thus have $(l+m)^2 = a_{lj} + b_{mj}$, $(\lambda+m)^2 = a_{\lambda j} + b_{mj}$, $(l+\mu)^2 = a_{lj} + b_{\mu j}$, and $(\lambda+\mu)^2 = a_{\lambda j} + b_{\mu j}$. Using the first three of these equalities, an elementary calculation shows that we have $a_{\lambda j} + b_{\mu j} < (\lambda+\mu)^2$. This is clearly a contradiction to the fourth equality. Therefore, $n, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2$ and E_φ as chosen cannot exist. To prove the theorem for the arctic semiring, it suffices to replace min by max in equation (5.2). ◀

With similar methods, we can show the following.

▶ **Theorem 23.** *Let* $S \in \{\mathrm{Trop}, \mathrm{Arct}\}$, $\sigma = (\emptyset, \emptyset)$ *be the empty signature and for* $l \in \mathbb{N}$ *consider the* σ*-structures* $\mathfrak{S}_l = (\{1,\dots,l\}, \emptyset)$*. Then for* $\varphi = \bigotimes X.1$ *there do not exist* $n \in \mathbb{N}$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in (\mathrm{wMSO}(\sigma, S))^n$ *and* $E_\varphi \in \mathrm{Exp}_n(S)$ *such that for all* $l, m \in \mathbb{N}$ *we have*

$$[\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)).$$

The nesting of a first order sum quantifier into the first order product quantifier also leads to formulas which do not allow for a Feferman-Vaught-like decomposition as the following theorem shows.

▶ **Theorem 24.** *Let* $S = (\mathbb{N}_0, +, \cdot, 0, 1)$, $\sigma = (\emptyset, \emptyset)$ *be the empty signature and for* $l \in \mathbb{N}$ *consider the* σ*-structures* $\mathfrak{S}_l = (\{1,\dots,l\}, \emptyset)$*. Then for* $\varphi = \bigotimes x. \bigoplus y.1$ *there do not exist* $n \in \mathbb{N}$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in (\mathrm{wMSO}(\sigma, \mathbb{N}_0))^n$ *and* $E_\varphi \in \mathrm{Exp}_n(\mathbb{N}_0)$ *such that for all* $l, m \in \mathbb{N}$ *we have*

$$[\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)). \tag{5.3}$$

Proof (Sketch). We proceed by contradiction and assume $n, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2$ and E_φ as above satisfying (5.3) exist. We may assume that $E_\varphi = E_1 \oplus \ldots \oplus E_k$ is in normal form with all E_i pure products. For $l \geq 1$ and $i \in \{1,\dots,k\}$ we let $a_{li} = [\![\mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$ and $b_{li} = [\![\mathrm{PRD}^2(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$. Then by assumption we have

$$(l+m)^{(l+m)} = [\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \textstyle\sum_{i=1}^{k}(a_{li} \cdot b_{mi}). \tag{5.4}$$

For every $j \in \{1, \dots, k\}$ we choose $L_j \geq 1$ such that $a_{L_j j} \neq 0$, or let $L_j = 0$ if for all $l \geq 1$ we have $a_{lj} = 0$. Assume $m \geq 1$ and $j \in \{1, \dots, k\}$ with $L_j \neq 0$, then $a_{L_j j} \geq 1$, hence

$$(L_j + m)^{(L_j+m)} = \sum_{i=1}^{k}(a_{L_j i} \cdot b_{mi}) \geq (a_{L_j j} \cdot b_{mj}) \geq b_{mj}.$$

In particular, with $L = \max\{L_i \mid i \in \{1, \dots, k\}\}$, we have that for every $j \in \{1, \dots, k\}$ either (i) $b_{mj} \leq (L+m)^{(L+m)}$ for all $m \geq 1$ or (ii) $a_{lj} = 0$ for all $l \geq 1$. Note that from equation (5.4) it follows that $L = 0$ is impossible. In the same fashion, we can find $M \geq 1$ such that for every $l \geq 1$ and every $j \in \{1, \dots, k\}$ either (i) $a_{lj} \leq (l+M)^{(l+M)}$ for all $l \geq 1$ or (ii) $b_{mj} = 0$ for all $m \geq 1$.

Now consider (5.4) for $l = m$. If $j \in \{1, \dots, k\}$ such that either $a_{lj} = 0$ for all $l \geq 1$ or $b_{mj} = 0$ for all $m \geq 1$, then clearly also $(a_{lj} \cdot b_{lj}) = 0$ for all l. If j is not like this, we have

$$(a_{lj} \cdot b_{lj}) \leq (l+M)^{(l+M)} \cdot (L+l)^{(L+l)} \leq (l+C)^{2(l+C)}$$

for $C = \max\{L, M\}$. It follows that $(2l)^{2l} \leq k(l+C)^{2(l+C)}$ for every $l \geq 1$. Elementary calculus can be used to show that this is not true. ◀

5.3 Necessity of restricting the logic for products

The proof of Theorem 22 can also be used to show that no Feferman-Vaught-like theorem holds for products if the first order product quantifier is included in the weighted logic. More precisely, already Theorem 18 does not hold over the tropical and arctic semirings for the formula $\varphi = \bigotimes x.1$ even if $\bar{\varphi}^{\mathbb{1}}$ and $\bar{\varphi}^{2}$ are allowed to be from $\mathrm{wMSO}(\sigma, S)$.

▶ **Theorem 25.** *Let $S \in \{\mathrm{Trop}, \mathrm{Arct}\}$, $\sigma = (\emptyset, \emptyset)$ be the empty signature and for $l \in \mathbb{N}$ consider the σ-structures $\mathfrak{S}_l = (\{1, \dots, l\}, \emptyset)$. Then for $\varphi = \bigotimes x.1$ there do not exist $n \in \mathbb{N}$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in (\mathrm{wMSO}(\sigma, S))^n$ and $E_\varphi \in \mathrm{Exp}_n(S)$ such that for all $l, m \in \mathbb{N}$ we have*

$$[\![\varphi]\!](\mathfrak{S}_l \times \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^{2}]\!](\mathfrak{S}_m)).$$

5.4 Proofs of Theorems 15 and 16

We now come to the proof of Theorems 15 and 16. By the following result, we can reduce the proofs to the case where the translation scheme is the identity.

▶ **Lemma 26.** *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}(\tau, S)$ with variables from $\mathcal{V}$. Then there exists a formula $\psi \in \mathrm{wMSO}(\sigma, S)$ with $\mathrm{Free}(\psi) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$ such that the following holds. For all finite structures $\mathfrak{A} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A})$-assignments ς and all $(\mathcal{V}, \Phi^\star(\mathfrak{A}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A}, \varsigma), \rho) = [\![\psi]\!](\mathfrak{A}, \varsigma \cup \rho).$$

If φ is from $\mathrm{wMSO}^{\bigotimes\text{-res}}(\tau, S)$ or $\mathrm{wFO}^{\bigotimes\text{-free}}(\tau, S)$, then ψ can also be chosen as a formula from $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)$ or $\mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, S)$, respectively.

Lemma 26 can be proved by induction on the structure of formulas.

Proof of Theorem 15 (Sketch). We proceed by induction. By Lemma 26 it suffices to prove the case $\tau = \sigma$ and $\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma) = \mathfrak{A} \sqcup \mathfrak{B}$.

■ Assume $\varphi = \beta$ for some $\beta \in \mathrm{MSO}(\sigma)$. We apply Theorem 6 to the formula β and obtain $l \geq 1$, vectors of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{MSO}(\sigma)^l$ and an expression $B_\beta \in \mathrm{Exp}_l(\mathbb{B})$ such that

$$(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \text{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho|_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho|_{\mathfrak{B}})) = \mathtt{true}.$$

We may assume that $B_\beta = B_1 \vee \ldots \vee B_m$ is in normal form with all B_i pure conjunctions. We let $\gamma_i = \mathrm{CON}^{\mathbb{1}}(B_i, \bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2})$ and $\delta_i = \mathrm{CON}^{2}(B_i, \bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2})$ for $i \in \{1, \ldots, m\}$ (see Construction 4). We set $n = 2m$ and define

$$\bar{\varphi}^{\mathbb{1}} = (\gamma_1, \ldots, \gamma_m, \neg\gamma_1, \ldots, \neg\gamma_m) \qquad\qquad \bar{\varphi}^{2} = (\delta_1, \ldots, \delta_m, \neg\delta_1, \ldots, \neg\delta_m).$$

Intuitively, we would now define the expression E_φ as $x_1 \otimes y_1 \oplus \ldots \oplus x_m \otimes y_m$, but this expression is not necessarily evaluated to $\mathbb{1}$ in S if $\gamma_i \wedge \delta_i$ is true for more than one index i. Instead, we define expressions $E_k \in \mathrm{Exp}_n(S)$ for $k \in \{1, \ldots, m\}$ inductively by $E_1 = x_1 \otimes y_1$ and

$$E_k = (E_{k-1} \otimes ((x_{k+m} \otimes y_k) \oplus y_{k+m})) \oplus (x_k \otimes y_k)$$

and set $E_\varphi = E_m$. In a sense, E_k is evaluated to $\mathbb{1}$ if $\gamma_k \wedge \delta_k$ is true, and otherwise, if either γ_k or δ_k does not hold, it is evaluated to E_{k-1}.

■ Assume $\varphi = s$ for some $s \in S$. We let $n = 1$, $\varphi_1^{\mathbb{1}} = \varphi_1^{2} = s$ and $E_\varphi = x_1$.

■ For $\varphi = \zeta \oplus \eta$, we assume the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^l$ and $E_\zeta \in \mathrm{Exp}_l(S)$, and for η with $\bar{\eta}^{\mathbb{1}}, \bar{\eta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^m$ and $E_\eta \in \mathrm{Exp}_m(S)$. We set $\bar{\varphi}^{\mathbb{1}} = (\zeta_1^{\mathbb{1}}, \ldots, \zeta_l^{\mathbb{1}}, \eta_1^{\mathbb{1}}, \ldots, \eta_m^{\mathbb{1}})$, $\bar{\varphi}^{2} = (\zeta_1^{2}, \ldots, \zeta_l^{2}, \eta_1^{2}, \ldots, \eta_m^{2})$ and $E_\varphi = E_\zeta \oplus E'_\eta$, where E'_η is obtained from E_η by replacing every variable x_i by x_{i+l} and every variable y_i by y_{i+l}.

■ For $\varphi = \zeta \otimes \eta$, the proof is the same as for the previous case, only that here we define $E_\varphi = E_\zeta \otimes E'_\eta$.

■ For $\varphi = \bigoplus x.\zeta$, we assume the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^l$ and $E_\zeta \in \mathrm{Exp}_l(S)$. We may assume that $E_\zeta = E_1 \oplus \ldots \oplus E_m$ is in normal form with all E_i pure products and that x does no occur as a bound variable in any of the $\zeta_i^{\mathbb{1}}$ or ζ_i^{2}. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$ and $\theta_i = \mathrm{PRD}^{2}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$. We set $n = 2m$ and define

$$\begin{aligned} \varphi^{\mathbb{1}} &= (\textstyle\bigoplus x.\xi_1, \ldots, \bigoplus x.\xi_m, \xi_1^{-x}, \ldots, \xi_m^{-x}) \\ \varphi^{2} &= (\textstyle\bigoplus x.\theta_1, \ldots, \bigoplus x.\theta_m, \theta_1^{-x}, \ldots, \theta_m^{-x}) \end{aligned} \qquad E_\varphi = \textstyle\bigoplus_{i=1}^{m}((x_i \otimes y_{m+i}) \oplus (x_{m+i} \otimes y_i)).$$

■ For $\varphi = \bigoplus X.\zeta$, we assume that the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, S)^l$ and $E_\zeta \in \mathrm{Exp}_l(S) = E_\zeta = E_1 \oplus \ldots \oplus E_m$ in normal form with all E_i pure products. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$ and $\theta_i = \mathrm{PRD}^{2}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$. We set $n = m$ and define

$$\begin{aligned} \varphi_j^{\mathbb{1}} &= (\textstyle\bigoplus X.\xi_1, \ldots, \bigoplus X.\xi_m) \\ \varphi_j^{2} &= (\textstyle\bigoplus X.\theta_1, \ldots, \bigoplus X.\theta_m) \end{aligned} \qquad E_\varphi = \textstyle\bigoplus_{i=1}^{m}(x_i \otimes y_i).$$

■ Assume $\varphi = \bigotimes x.\zeta$ with $\zeta \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, S)$ almost boolean. Using the laws of distributivity in S and the fact that for two boolean formulas $\alpha, \beta \in \mathrm{MSO}(\sigma)$ we have $[\![\alpha]\!] \cdot [\![\beta]\!] \equiv [\![\alpha \otimes \beta]\!]$, we may assume that $\zeta = (s_1 \otimes \beta_1) \oplus \ldots \oplus (s_l \otimes \beta_l)$ for some $l \geq 1$, $s_i \in S$ and $\beta_i \in \mathrm{MSO}(\sigma)$. Applying a simple construction, we may even assume that $\beta_1, \ldots, \beta_l$ form a partition, i.e., that for all $(\mathcal{V}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ρ' there is exactly one $i \in \{1, \ldots, l\}$ with

$(\mathfrak{A} \sqcup \mathfrak{B}, \rho') \models \beta_i$. Let $X_1, \dots, X_l \in \mathcal{V}$ be second order variables not occurring in ζ. We define the abbreviation

$$(x \in X_i) \rhd s_i = ((x \in X_i) \otimes s_i) \oplus \neg(x \in X_i).$$

One can check elementarily that

$$[\![\varphi]\!] \equiv [\![\bigoplus X_1 \dots \bigoplus X_l. \left(\bigwedge_{i=1}^{l} \forall x.(x \in X_i \leftrightarrow \beta_i)\right) \otimes \bigotimes_{i=1}^{l} \bigotimes x.((x \in X_i) \rhd s_i)]\!].$$

Therefore, it suffices to show this case for formulas of the form

$$\varphi = \bigotimes x.((x \in X) \rhd s).$$

For this, we let $n = 1$ and define $\varphi^1 = \varphi^2 = (\bigotimes x.((x \in X) \rhd s))$ and $E_\varphi = x_1 \otimes y_1$. ◀

Proof of Theorem 16 (Sketch). Again we proceed by induction and assume that $\tau = \sigma$ and $\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma) = \mathfrak{A} \times \mathfrak{B}$. The proofs for the cases $\varphi = \beta$, $\varphi = s$, $\varphi = \zeta \oplus \eta$ and $\varphi = \zeta \otimes \eta$ are identical to the ones used in the proof of Theorem 15 for the corresponding cases. For the case $\varphi = \bigoplus x.\zeta$ we proceed as for the case $\varphi = \bigoplus X.\zeta$ in the proof of Theorem 15. ◀

6 Conclusion

We have derived a weighted version of the Feferman-Vaught Theorem for disjoint unions and products of finite structures. We just mention here three possible extensions that were left out due to lack of space. First, Theorems 15 and 16 also hold for infinite structures if the commutative semiring S is bicomplete, i.e., if it is equipped with infinite sum and product operations that naturally extend its finite sum and product operations. Second, in the particular case that the semiring S is a De Morgan algebra, both theorems hold without any need for restrictions on the product quantifiers; that is, Theorem 15 holds for the full weighted MSO logic, and Theorem 16 holds for the full weighted FO logic. Third, both theorems may be extended to employ transductions as defined by Courcelle [2] in place of the present translation schemes.

References

1 Benedikt Bollig, Paul Gastin, and Benjamin Monmege. Weighted specifications over nested words. In Frank Pfenning, editor, *Proc. FoSSaCS*, volume 7794 of *LNCS*, pages 385–400. Springer, 2013.

2 Bruno Courcelle. Monadic second-order definable graph transductions: a survey. *Theor. Comput. Sci.*, 126(1):53–75, 1994.

3 Manfred Droste and Paul Gastin. Weighted automata and weighted logics. *Theor. Comput. Sci.*, 380(1-2):69–86, 2007.

4 Manfred Droste and George Rahonis. Weighted automata and weighted logics with discounting. *Theor. Comput. Sci.*, 410(37):3481–3494, 2009.

5 Andrzej Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math.*, 49(2):129–141, 1961.

6 Solomon Feferman and Robert L. Vaught. The first order properties of products of algebraic systems. *Fund. Math.*, 47:57–103, 1959.

7 Siegfried Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, 2001.

8 Yuri Gurevich. Modest theory of short chains. I. *J. Symbolic Logic*, 44(4):481–490, 12 1979.

9 Yuri Gurevich. Chapter XIII: Monadic second-order theories. In Jon Barwise and Solomon Feferman, editors, *Model-Theoretic Logics*, volume 8 of *Perspect. Math. Logic*, pages 479–506. Springer, 1985.

10 Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, 1998.

11 Hendrik Jan Hoogeboom and Paulien ten Pas. Monadic second-order definable text languages. *Theory Comput. Syst.*, 30(4):335–354, 1997.

12 Hans Läuchli and John Leonard. On the elementary theory of linear order. *Fund. Math.*, 59(1):109–116, 1966.

13 Johann A. Makowsky. Algorithmic uses of the Feferman–Vaught theorem. *Ann. Pure Appl. Logic*, 126(1):159–213, 2004.

14 Christian Mathissen. Definable transductions and weighted logics for texts. In Tero Harju, Juhani Karhumäki, and Arto Lepistö, editors, *Proc. DLT*, pages 324–336. Springer, 2007.

15 Christian Mathissen. Weighted logics for nested words and algebraic formal power series. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfsdóttir, and Igor Walukiewicz, editors, *Proc. ICALP*, volume 5126 of *LNCS*, pages 221–232. Springer, 2008.

16 Christian Mathissen. *Weighted Automata and Weighted Logics over Tree-like Structures*. PhD thesis, Leipzig University, Germany, 2009. URL: `http://www.dr.hut-verlag.de/978-3-86853-180-0.html`.

17 Andrzej Mostowski. On direct products of theories. *J. Symbolic Logic*, 17(1):1–31, 1952.

18 Frank P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30(1):264–286, 1930.

19 Elena V. Ravve, Zeev Volkovich, and Gerhard-Wilhelm Weber. Effective optimization with weighted automata on decomposable trees. *Optimization*, 63(1):109–127, 2014.

20 Marcel-Paul Schützenberger. On the definition of a family of automata. *Inform. Control*, 4(2–3):245 – 270, 1961.

21 Saharon Shelah. The monadic theory of order. *Ann. Math.*, 102(3):379–419, 1975.

22 Alan S. Stern, Jan Mycielski, and Pavel Pudlák. *A Lattice of Chapters of Mathematics: Interpretations between Theorems*, volume 426 of *Mem. Amer. Math. Soc.* American Math. Soc., 1990.