A Feferman-Vaught Decomposition Theorem for Weighted MSO Logic

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Abstract

We prove a weighted Feferman-Vaught decomposition theorem for disjoint unions and products of finite structures. The classical Feferman-Vaught Theorem describes how the evaluation of a first order sentence in a generalized product of relational structures can be reduced to the evaluation of sentences in the contributing structures and the index structure. The logic we employ for our weighted extension is based on the weighted MSO logic introduced by Droste and Gastin to obtain a Büchi-type result for weighted automata. We show that for disjoint unions and products of structures, the evaluation of formulas from two respective fragments of the logic can be reduced to the evaluation of formulas in the contributing structures. We also prove that the respective restrictions are necessary. Surprisingly, for the case of disjoint unions, the fragment is the same as the one used in the Büchi-type result of weighted automata. In fact, even the formulas used to show that the respective restrictions are necessary are the same in both cases. However, here proving that they do not allow for a Feferman-Vaught-like decomposition is more complex and employs Ramsey's Theorem. We also show how translation schemes can be applied to go beyond disjoint unions and products.

1 Introduction

The Feferman-Vaught Theorem [10] is one of the fundamental theorems in model theory. The theorem describes how the computation of the truth value of a first order sentence in a generalized product of relational structures can be reduced to the computation of truth values of first order sentences in the contributing structures and the evaluation of a monadic second order sentence in the index structure. The theorem itself has a long-standing history. It builds upon work of Mostowski [22], and was later shown to hold true for monadic second order logic (MSO logic) as well [7, 12, 13, 17, 27]. For a survey and more background information, see [18].

In this paper, we show that under appropriate assumptions, the Feferman-Vaught Theorem also holds true for a weighted MSO logic with arbitrary commutative semirings as weight structure. The logic we employ is based on the weighted logic by Droste and Gastin [4]. In this logic, formulas can take values which convey a quantitative meaning. The logic's connectives and quantifiers hence also adopt quantitative roles. The disjunction becomes a sum, the conjunction a product. The existential quantifier, instead of only verifying whether some element with a certain property exists, now takes the truth value of this property for every element in the universe and sums over these values. Under appropriate assumptions, the result of this summation can for instance be the exact number of elements that satisfy the given property. One example of a property which can be expressed using this logic is the number of cliques of a given size in an undirected graph. In [4], the authors prove a Büchi-like result for a specific fragment of the MSO logic, showing that for finite and infinite words, this fragment is expressively equivalent to semiring-weighted automata [26]. The study of a weighted Feferman-Vaught Theorem for disjoint unions, employing the same logic as we do, was initiated by Ravve et al. in [25], where the authors also point out several algorithmic uses and possible applications of a weighted Feferman-Vaught Theorem.

The classical Feferman-Vaught Theorem considers finite and infinite structures without any need for distinction between them. This results from the fact that, in the Boolean setting, infinite joins and meets are well-defined. In particular, existential and universal quantification, which are essentially joins and meets ranging over the whole universe of a structure, are well-defined for finite and infinite structures alike. However, for arbitrary semirings, infinite sums and products are usually not defined. To allow for infinite structures, we therefore also consider *bicomplete* semirings, which are equipped with infinite sum and product operations. Our main results are the following.

- We provide a Feferman-Vaught Theorem for disjoint unions of structures with our weighted MSO logic, where the first order product quantifier is restricted to quantify only over formulas which do not contain any sum or product quantifier themselves. Surprisingly, this restriction and the resulting fragment are the same as the one working for the Büchi-like result of [4].
- We show that no similar theorem can hold for disjoint unions if the first order product quantifier is not restricted. The formulas we employ for this in fact also occurred in [4] and [5] as examples of weighted formulas whose semantics cannot be described by weighted automata. While in these papers, it was elementary to show that the formulas given define weighted languages not recognizable by weighted automata, here proving that they do not allow for a Feferman-Vaught-like decomposition is more complex and employs a weak version of Ramsey's Theorem [24].
- We show that a Feferman-Vaught Theorem also holds for products of structures for the product-quantifier-free first order fragment of our logic.
- We show that no similar theorem can hold for products if we include the first order product quantifier.
- We show that our theorems are also true for more general disjoint unions and products defined by *translation schemes* [18, 23, 3].

With respect to our proofs, here we just note that in comparison to the universal quantifier of the Boolean setting, the product quantifier requires a separate and new consideration. While universal quantification can simply be expressed using negation and existential quantification, it is in general not possible to express multiplication by addition.

Translation schemes are a model theoretic tool to "translate" structures over one logical signature into structures over another signature in a well behaved fashion, namely in an MSO-defined fashion. They can be applied, for example, to translate between *texts* and *trees* [15], and between *nested words*, *alternating texts*, and *hedges* [21, 20, 19]. These particular translations were employed in [19] to prove that weighted automata over texts, hedges, and nested words are expressively equivalent to weighted logics over these structures. Translation schemes are a rather natural concept and therefore they have been frequently rediscovered and named differently [18, 23, 3]. Our notion of a translation scheme is mostly due to [18].

Related work. A concept related to weighted logics is that of many-valued logics. In both models the evaluation of a formula on a structure produces a quantitative piece of information. In many approaches to many-valued logics, values are taken in the interval $[0, 1]$, cf. [14, 11]. In contrast to this, weights in weighted logics are taken from a semiring and may occur as atomic formulas which enables the modeling of quantitative properties.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. For a set X, we denote the power set of X by $\mathcal{P}(X)$ and the cardinality of X by $|X|$. For two sets X and Y and a mapping $f\colon X \to Y$, we call X the *domain* of f, denoted by $\mathrm{dom}(f)$, and Y the *range* of f, denoted by $\mathrm{range}(f)$. For a subset $X' \subseteq X$, we call the set $f(X') := \{y \in Y \mid \exists x \in X'\colon f(x) = y\}$ the *image* or *range of X' under f*. The *restriction of f to X'*, denoted by $f{\restriction_{X'}}$, is the mapping $f{\restriction_{X'}}\colon X' \to Y$ defined by $f{\restriction_{X'}}(x) := f(x)$ for every $x \in X'$. We also call a mapping $g\colon X' \to Y$ a *partial* mapping from X to Y, denoted by $g\colon X \nrightarrow Y$. For a subset $Y' \subseteq Y$, we call the set $f^{-1}(Y') := \{x \in X \mid f(x) \in Y'\}$ the *preimage of Y' under f*. For an element $y \in Y$, we define the preimage of y under f by $f^{-1}(y) := f^{-1}(\{y\})$. For a second mapping $h\colon X \to Y$, we write $f = h$ if for all $x \in X$ we have $f(x) = h(x)$.

A *signature* σ is a pair $(\mathrm{Rel}_\sigma, \mathrm{ar}_\sigma)$ where Rel_σ is a set of relation symbols and $\mathrm{ar}_\sigma\colon \mathrm{Rel}_\sigma \to \mathbb{N}_+$ the arity function. A *σ-structure* $\mathfrak{A}$ is a pair $(\mathcal{U}_\mathfrak{A}, \mathcal{I}_\mathfrak{A})$ where $\mathcal{U}_\mathfrak{A}$ is a set, called the *universe of* $\mathfrak{A}$, and $\mathcal{I}_\mathfrak{A}$ is an *interpretation*, which maps every symbol $R \in \mathrm{Rel}_\sigma$ to a set $R^\mathfrak{A} \subseteq \mathcal{U}_\mathfrak{A}^{\mathrm{ar}_\sigma(R)}$. A structure is called *finite* if its universe is a finite set. By $\mathrm{Str}(\sigma)$ we denote the class of all σ-structures.

For two σ-structures $\mathfrak{A} = (A, \mathcal{I}_\mathfrak{A})$ and $\mathfrak{B} = (B, \mathcal{I}_\mathfrak{B})$, we define the *product* $\mathfrak{A} \times \mathfrak{B} \in \mathrm{Str}(\sigma)$ *of $\mathfrak{A}$ and $\mathfrak{B}$* and the *disjoint union* $\mathfrak{A} \sqcup \mathfrak{B} \in \mathrm{Str}(\sigma)$ *of $\mathfrak{A}$ and $\mathfrak{B}$* as follows. For the product, we let $\mathfrak{A} \times \mathfrak{B} = (A \times B, \mathcal{I}_{\mathfrak{A}\times\mathfrak{B}})$ with $R^{\mathfrak{A}\times\mathfrak{B}} = \{((a_1, b_1), \ldots, (a_k, b_k)) \mid (a_1, \ldots, a_k) \in R^\mathfrak{A}$ and $(b_1, \ldots, b_k) \in R^\mathfrak{B}\}$. For the disjoint union, we let $A \sqcup B$ be the disjoint union (i.e., the set theoretic coproduct) of A and B with inclusions ι_A and ι_B. Then we define $\mathfrak{A} \sqcup \mathfrak{B} := (A \sqcup B, \mathcal{I}_{\mathfrak{A}\sqcup\mathfrak{B}})$ by $R^{\mathfrak{A}\sqcup\mathfrak{B}} := \{(\iota_A(a_1), \ldots, \iota_A(a_k)) \mid$

$(a_1,\dots,a_k)\in R^{\mathfrak{A}}\}\cup\{(\iota_B(b_1),\dots,\iota_A(b_k))\mid(b_1,\dots,b_k)\in R^{\mathfrak{B}}\}$. Throughout the paper, we identify $a\in A$ with $\iota_A(a)\in A\sqcup B$ and $b\in B$ with $\iota_B(b)\in A\sqcup B$.

A *commutative semiring* is a tuple $(K,+,\cdot,\mathbb{0},\mathbb{1})$, abbreviated by K, with operations sum $+$ and product $\cdot$ and constants $\mathbb{0}$ and $\mathbb{1}$ such that $(K,+,\mathbb{0})$ and $(K,\cdot,\mathbb{1})$ are commutative monoids, multiplication distributes over addition, and $\kappa\cdot\mathbb{0}=\mathbb{0}$ for every $\kappa\in K$.

Next, assume that the semiring K is equipped, for every index set I, with an infinitary sum operation $\sum_I\colon K^I\to K$ such that for every family $(\kappa_i)_{i\in I}$ of elements of K and $\kappa\in K$ we have

$$\sum_{i\in\emptyset}\kappa_i=0,\quad \sum_{i\in\{j\}}\kappa_i=\kappa_j,\quad \sum_{i\in\{j,l\}}\kappa_i=\kappa_j+\kappa_l \text{ for } j\neq l, \tag{2.1}$$

$$\sum_{j\in J}\Big(\sum_{i\in I_j}\kappa_i\Big)=\sum_{i\in I}\kappa_i,\ \text{ if } \textstyle\bigcup_{j\in J}I_j=I \text{ and } I_j\cap I_{j'}=\emptyset \text{ for } j\neq j', \tag{2.2}$$

$$\sum_{i\in I}(\kappa\cdot\kappa_i)=\kappa\cdot\Big(\sum_{i\in I}\kappa_i\Big),\quad \sum_{i\in I}(\kappa_i\cdot\kappa)=\Big(\sum_{i\in I}\kappa_i\Big)\cdot\kappa. \tag{2.3}$$

Then the semiring K together with the operations $\sum_I$ is called *complete* [8, 16].

If in addition, K is endowed, for every index set I, with an infinitary product operation $\prod_I\colon K^I\to K$ such that for every family $(\kappa_i)_{i\in I}$ of elements of K we have

$$\prod_{i\in\emptyset}\kappa_i=\mathbb{1},\quad \prod_{i\in\{j\}}\kappa_i=\kappa_j,\quad \prod_{i\in\{j,l\}}\kappa_i=\kappa_j\cdot\kappa_l \text{ for } j\neq l,\quad \prod_{i\in I}\mathbb{1}=\mathbb{1}, \tag{2.4}$$

$$\prod_{j\in J}\Big(\prod_{i\in I_j}\kappa_i\Big)=\prod_{i\in I}\kappa_i,\ \text{ if } \textstyle\bigcup_{j\in J}I_j=I \text{ and } I_j\cap I_{j'}=\emptyset \text{ for } j\neq j', \tag{2.5}$$

then we call K *bicomplete*. We just want to mention here that there exists a different notion of semirings with infinite sums and products, namely the notion of *totally complete* semirings [9]. The main difference between these two notions lies in the definition of infinite products. For totally complete semirings, only products over countable index sets need to be defined, but the infinite products are required to be completely distributive over the infinite sums. We do not require this infinitary distributivity here.

Example 1. Examples of bicomplete semirings include

- the *Boolean semiring* $\mathbb{B}=(\{0,1\},\vee,\wedge,0,1)$,
- the *max-plus* or *arctic semiring* $\mathrm{Arct}=(\mathbb{R}\cup\{-\infty\},\max,+,-\infty,0)$ and the *min-plus* or *tropical semiring* $\mathrm{Trop}=(\mathbb{R}\cup\{\infty\},\min,+,\infty,0)$, where $\mathbb{R}$ denotes the set of real numbers,
- the semiring of extended natural numbers $(\mathbb{N}\cup\{\infty\},+,\cdot,0,1)$ where $0\cdot\infty=0$,
- any complete distributive lattice $(L,\vee,\wedge,0,1)$ which satisfies the distributivity law (2.3). For instance, every complete Boolean algebra B satisfies (2.3) (see [1, page 167]), so then B is bicomplete but may not be completely distributive and therefore not totally complete.

The following definitions are due to [4] in the form of [2]. We provide a countable set $\mathcal{V}$ of first and second order variables, where lower case letters like x and y denote first order variables and capital letters like X and Y denote second order variables. We define first order formulas β over a signature σ and weighted first order formulas φ over σ and a semiring K, respectively, by the grammars

$$\beta ::= \mathtt{false}\mid R(x_1,\dots,x_n)\mid\neg\beta\mid\beta\vee\beta\mid\exists x.\beta$$
$$\varphi ::= \beta\mid\kappa\mid\varphi\oplus\varphi\mid\varphi\otimes\varphi\mid\textstyle\bigoplus x.\varphi\mid\bigotimes x.\varphi,$$

where $R\in\mathrm{Rel}_\sigma$, $n=\mathrm{ar}_\sigma(R)$, $x,x_1,\dots,x_n\in\mathcal{V}$ are first order variables, and $\kappa\in K$. Likewise, we define monadic second order formulas β over σ and weighted monadic second order formulas φ over σ and K through

$$\beta ::= \mathtt{false}\mid R(x_1,\dots,x_n)\mid x\in X\mid\neg\beta\mid\beta\vee\beta\mid\exists x.\beta\mid\exists X.\beta$$
$$\varphi ::= \beta\mid\kappa\mid\varphi\oplus\varphi\mid\varphi\otimes\varphi\mid\textstyle\bigoplus x.\varphi\mid\bigotimes x.\varphi\mid\bigoplus X.\varphi\mid\bigotimes X.\varphi,$$

with $R\in\mathrm{Rel}_\sigma$, $n=\mathrm{ar}_\sigma(R)$, $x,x_1,\dots,x_n\in\mathcal{V}$ first order variables, $X\in\mathcal{V}$ a second order variable, and $\kappa\in K$. We also allow the usual abbreviations $\wedge$, $\forall$, $\rightarrow$, $\longleftrightarrow$, and $\mathtt{true}$. By $\mathrm{FO}(\sigma)$ and $\mathrm{wFO}(\sigma,K)$ we denote the sets of all first order formulas over σ and all weighted first order formulas over σ and K, respectively, and by $\mathrm{MSO}(\sigma)$ and $\mathrm{wMSO}(\sigma, K)$ we denote the sets of all monadic second order formulas over σ and all weighted monadic second order formulas over σ and K, respectively.

The notion of *free variables* is defined as usual, i.e., the operators $\exists, \forall, \bigoplus$, and $\bigotimes$ bind variables. We let $\mathrm{Free}(\varphi)$ be the set of all free variables of φ. A formula φ with $\mathrm{Free}(\varphi) = \emptyset$ is called a *sentence*. For a tuple $\bar{\varphi} = (\varphi_1, \dots, \varphi_n) \in \mathrm{wMSO}(\sigma, K)^n$, we define the set of free variables of $\bar{\varphi}$ as $\mathrm{Free}(\bar{\varphi}) = \bigcup_{i=1}^n \mathrm{Free}(\varphi_i)$.

We define the semantics of MSO and wMSO as follows. Let σ be a signature, $\mathfrak{A} = (A, \mathcal{I}_{\mathfrak{A}})$ a σ-structure, and $\mathcal{V}$ a set of first and second order variables. A $(\mathcal{V}, \mathfrak{A})$-assignment ρ is a partial function $\rho\colon \mathcal{V} \nrightarrow A \cup \mathcal{P}(A)$ such that, whenever $x \in \mathcal{V}$ is a first order variable and $\rho(x)$ is defined, we have $\rho(x) \in A$, and whenever $X \in \mathcal{V}$ is a second order variable and $\rho(X)$ is defined, we have $\rho(X) \subseteq A$. The reason we consider partial functions is that in our Feferman-Vaught theorems for the disjoint union of structures we want to be able to restrict the range of a variable assignment to a subset of the universe. For a first order variable, this restriction may cause the variable to become undefined. Let $\mathrm{dom}(\rho)$ be the domain of ρ. For a first order variable $x \in \mathcal{V}$ and an element $a \in A$, the *update* $\rho[x \to a]$ is defined through $\mathrm{dom}(\rho[x \to a]) = \mathrm{dom}(\rho) \cup \{x\}$, $\rho[x \to a](\mathcal{X}) = \rho(\mathcal{X})$ for all $\mathcal{X} \in \mathcal{V} \setminus \{x\}$, and $\rho[x \to a](x) = a$. For a second order variable $X \in \mathcal{V}$ and a set $I \subseteq A$, the update $\rho[X \to I]$ is defined in a similar fashion. By $\mathfrak{A}_{\mathcal{V}}$ we denote the set of all $(\mathcal{V}, \mathfrak{A})$-assignments.

For $\rho \in \mathfrak{A}_{\mathcal{V}}$ and a formula $\beta \in \mathrm{MSO}(\sigma)$ the relation "$(\mathfrak{A}, \rho)$ satisfies β", denoted by $(\mathfrak{A}, \rho) \models \beta$, is defined as

$$
\begin{array}{lcl}
(\mathfrak{A}, \rho) \models \texttt{false} & & \text{never holds} \\
(\mathfrak{A}, \rho) \models R(x_1, \dots, x_n) & \iff & x_1, \dots, x_n \in \mathrm{dom}(\rho) \text{ and } (\rho(x_1), \dots, \rho(x_n)) \in R^{\mathfrak{A}} \\
(\mathfrak{A}, \rho) \models x \in X & \iff & x, X \in \mathrm{dom}(\rho) \text{ and } \rho(x) \in \rho(X) \\
(\mathfrak{A}, \rho) \models \neg\beta & \iff & (\mathfrak{A}, \rho) \models \beta \text{ does not hold} \\
(\mathfrak{A}, \rho) \models \beta_1 \vee \beta_2 & \iff & (\mathfrak{A}, \rho) \models \beta_1 \text{ or } (\mathfrak{A}, \rho) \models \beta_2 \\
(\mathfrak{A}, \rho) \models \exists x.\beta & \iff & (\mathfrak{A}, \rho[x \to a]) \models \beta \text{ for some } a \in A \\
(\mathfrak{A}, \rho) \models \exists X.\beta & \iff & (\mathfrak{A}, \rho[X \to I]) \models \beta \text{ for some } I \subseteq A.
\end{array}
$$

Example 2. Let σ is the signature of a graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}\}$ with edge binary. We call a graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$ *undirected* if its interpretation of edge is a symmetric relation on the universe of $\mathfrak{G}$. For every undirected graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$ and a subset I of its universe, we can check whether the nodes from I form a clique in $\mathfrak{G}$ using the MSO formula

$$\mathrm{clique}(X) := \forall x \forall y \Big(\big(x \in X \wedge y \in X \wedge x \neq y\big) \to \mathrm{edge}(x, y) \Big).$$

Here, the formula $x \neq y$ is an abbreviation for $\exists Y (y \in Y \wedge \neg(x \in Y))$. We have that $(\mathfrak{G}, [X \to I])$ satisfies $\mathrm{clique}(X)$ if and only if I is a clique in $\mathfrak{G}$.

In the following, for all sums and products to be well-defined, we assume that either the universe A is finite, or that K is bicomplete. For a formula $\varphi \in \mathrm{wMSO}(\sigma, K)$ and a structure $\mathfrak{A} \in \mathrm{Str}(\sigma)$, the *(weighted) semantics* of φ is a mapping $[\![\varphi]\!](\mathfrak{A}, \cdot)\colon \mathfrak{A}_{\mathcal{V}} \to K$ inductively defined as

$$
\begin{array}{lcl}
[\![\beta]\!](\mathfrak{A}, \rho) & = & \begin{cases} \mathbb{1} & \text{if } (\mathfrak{A}, \rho) \models \beta \\ \mathbb{0} & \text{otherwise} \end{cases} \\
[\![\kappa]\!](\mathfrak{A}, \rho) & = & \kappa \\
[\![\varphi_1 \oplus \varphi_2]\!](\mathfrak{A}, \rho) & = & [\![\varphi_1]\!](\mathfrak{A}, \rho) + [\![\varphi_2]\!](\mathfrak{A}, \rho) \\
[\![\varphi_1 \otimes \varphi_2]\!](\mathfrak{A}, \rho) & = & [\![\varphi_1]\!](\mathfrak{A}, \rho) \cdot [\![\varphi_2]\!](\mathfrak{A}, \rho) \\
[\![\bigoplus x.\varphi]\!](\mathfrak{A}, \rho) & = & \displaystyle\sum_{a \in A} [\![\varphi]\!](\mathfrak{A}, \rho[x \to a]) \\
[\![\bigotimes x.\varphi]\!](\mathfrak{A}, \rho) & = & \displaystyle\prod_{a \in A} [\![\varphi]\!](\mathfrak{A}, \rho[x \to a]) \\
[\![\bigoplus X.\varphi]\!](\mathfrak{A}, \rho) & = & \displaystyle\sum_{I \subseteq A} [\![\varphi]\!](\mathfrak{A}, \rho[X \to I]) \\
[\![\bigotimes X.\varphi]\!](\mathfrak{A}, \rho) & = & \displaystyle\prod_{I \subseteq A} [\![\varphi]\!](\mathfrak{A}, \rho[X \to I]).
\end{array}
$$

We will usually identify a pair $(\mathfrak{A}, \emptyset)$ with $\mathfrak{A}$. For a tuple of formulas $\bar{\varphi} \in \mathrm{wMSO}(\sigma, K)^n$, we define $[\![\bar{\varphi}]\!](\mathfrak{A}, \rho) = ([\![\varphi_1]\!](\mathfrak{A}, \rho), \ldots, [\![\varphi_n]\!](\mathfrak{A}, \rho)) \in K^n$.

We give some examples of how weighted formulas can be interpreted. For more examples, see also [25].

Example 3. If $K = \mathbb{B}$ is the Boolean semiring, we obtain the classical Boolean logic.

Example 4. Using the arctic semiring $\mathrm{Arct} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, we can describe the size of the largest clique in a graph as follows. We reuse the signature σ of a graph and the MSO formula $\mathrm{clique}(X)$ from Example 2 and define a wMSO formula as follows.

$$\varphi := \bigoplus X. \Big(\mathrm{clique}(X) \otimes \bigotimes x. \big(0 \oplus (1 \otimes x \in X)\big)\Big)$$

Then for every undirected graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$, we have that $[\![\varphi]\!](\mathfrak{G})$ is the size of the largest clique in $\mathfrak{G}$.

Example 5. Assume that $K = (\mathbb{Q}, +, \cdot, 0, 1)$ is the field of rational numbers and that σ is the signature from the previous example. Then for every fixed $n \in \mathbb{N}_+$, we can count the number of n-cliques of an undirected graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$ using the wMSO formula

$$\varphi_n := \frac{1}{n!} \otimes \bigoplus x_1 \ldots \bigoplus x_n. \bigwedge_{i \neq j} \Big((x_i \neq x_j) \wedge \mathrm{edge}(x_i, x_j)\Big).$$

Here, $x_i \neq x_j$ again is an abbreviation for $\exists Y (x_j \in Y \wedge \neg(x_i \in Y))$.

Example 6. We consider the *minimum cut* of directed acyclic graphs. For this, we interpret these graphs as *flow networks* in the following way. Every vertex which does not have a predecessor is considered a *source*, every vertex without successors is considered a *drain*, and every edge is assumed to have a capacity of 1. Let $G = (V, E)$ be a directed acyclic graph where V is the set of vertices and $E \subseteq V \times V$ the set of edges. A cut (S, D) of G is a partition of V, i.e., $S \cup D = V$ and $S \cap D = \emptyset$, such that all sources of G are in S, and all drains of G are in D. The *minimum cut* of G is the smallest number $|E \cap (S \times D)|$ such that (S, D) is a cut of G.

We can express the minimum cut of directed acyclic graphs by a weighted formula as follows. We let σ be the signature from the previous two examples and as our semiring, we choose the tropical semiring $\mathrm{Trop} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$. Then using the abbreviation

$$\mathrm{cut}(X, Y) := \forall x. \Big((x \in X \leftrightarrow \neg(x \in Y)) \wedge (\exists y. \mathrm{edge}(y, x) \vee x \in X) \wedge (\exists y. \mathrm{edge}(x, y) \vee x \in Y)\Big)$$

we can express the minimum cut of a directed acyclic graph $\mathfrak{G} \in \mathrm{Str}(\sigma)$ using the formula

$$\varphi := \bigoplus X. \bigoplus Y. \Big(\mathrm{cut}(X, Y) \otimes \bigotimes x. \bigotimes y. (1 \oplus \neg(x \in X \wedge y \in Y \wedge \mathrm{edge}(x, y)))\Big).$$

Example 7 ([4])**.** Let $K = (\mathbb{N}, +, \cdot, 0, 1)$ be the semiring of natural numbers and let $\varphi \in \mathrm{wMSO}(\sigma, K)$ be a formula which does not contain any constants $\kappa \in K$. Then we may understand $[\![\varphi]\!](\mathfrak{A}, \rho)$ as the number of proofs we have that $(\mathfrak{A}, \rho)$ satisfies φ assuming that we interpret the weighted operators in the following way. For Boolean formulas, we simply consider satisfaction to give us one proof, and otherwise we have no proof. The sum $[\![\varphi_1 \oplus \varphi_2]\!]$ is the number of proofs we have that $\varphi_1 \vee \varphi_2$ is true. This says that, if we have n proofs for φ_1 and m proofs for φ_2, then we interpret this as having $n + m$ proofs for the fact that $\varphi_1 \vee \varphi_2$ is true. Likewise, we interpret the product $[\![\varphi_1 \otimes \varphi_2]\!]$ as the number of proofs we have that $\varphi_1 \wedge \varphi_2$ is true. Similar interpretations apply for the weighted quantifiers.

For $\varphi \in \mathrm{wMSO}(\sigma, K)$ and a first order variable x which does not appear in φ as a bound variable, we define φ^{-x} as the formula obtained from φ by replacing all atomic subformulas containing x, i.e., all subformulas of the form $x \in X$ and $R(\ldots, x, \ldots)$ for $R \in \mathrm{Rel}_\sigma$, by `false`. It is easy to show by induction that for all σ-structures $\mathfrak{A} = (A, \mathcal{I}_\mathfrak{A})$ and $(\mathcal{V}, \mathfrak{A})$-assignments ρ with $x \notin \mathrm{dom}(\rho)$ we have

$$[\![\varphi]\!](\mathfrak{A}, \rho) = [\![\varphi^{-x}]\!](\mathfrak{A}, \rho).$$

As in the sequel we will deal with disjoint unions and products of structures, we need to define the restrictions of a variable assignment to the contributing structures of the disjoint union or product. Fix two structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ with universes A and B. For a $(\mathcal{V}, \mathfrak{A} \sqcup \mathfrak{B})$-assignment ρ, we define the restriction $\rho\restriction_{\mathfrak{A}} \colon \mathcal{V} \nrightarrow A$ as

$$\rho\restriction_{\mathfrak{A}}(\mathcal{X}) = \begin{cases} \rho(\mathcal{X}) \cap A & \text{if } \mathcal{X} \text{ is a second order variable} \\ \rho(\mathcal{X}) & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \in A \\ \text{undefined} & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \notin A. \end{cases}$$

The restriction $\rho\restriction_{\mathfrak{B}}$ is defined similarly.

For a $(\mathcal{V}, \mathfrak{A} \times \mathfrak{B})$-assignment ρ, we define the restrictions $\rho\restriction_{\mathfrak{A}}$ and $\rho\restriction_{\mathfrak{B}}$ by projection on the corresponding entries. That is, we let π_A be the projection on the first and π_B be the projection on the second entry of $A \times B$ and let $\rho\restriction_{\mathfrak{A}} = \pi_A \circ \rho$ and $\rho\restriction_{\mathfrak{B}} = \pi_B \circ \rho$.

The *union* of two assignments ρ and ς with $\mathrm{dom}(\rho) \cap \mathrm{dom}(\varsigma) = \emptyset$, denoted by $\rho \cup \varsigma$, is defined by $\mathrm{dom}(\rho \cup \varsigma) = \mathrm{dom}(\rho) \cup \mathrm{dom}(\varsigma)$, $(\rho \cup \varsigma)(\mathcal{X}) = \rho(\mathcal{X})$ for $\mathcal{X} \in \mathrm{dom}(\rho)$ and $(\rho \cup \varsigma)(\mathcal{X}) = \varsigma(\mathcal{X})$ for $\mathcal{X} \in \mathrm{dom}(\varsigma)$.

Fix two disjoint sets of variables $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$. For $n \in \mathbb{N}_+$, we define the set of *expressions* $\mathrm{Exp}_n(K)$ over a semiring K by the grammar

$$E ::= x_i \mid y_i \mid E \oplus E \mid E \otimes E,$$

where $i \in \{1, \dots, n\}$. The *(weighted) semantics* of an expression $E \in \mathrm{Exp}_n(K)$ is a mapping $\langle\!\langle E \rangle\!\rangle \colon K^n \times K^n \to K$ defined for $\bar{\kappa}, \bar{\lambda} \in K^n$ inductively by

$$\begin{array}{lcl} \langle\!\langle x_i \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) & = & \kappa_i \\ \langle\!\langle y_i \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) & = & \lambda_i \\ \langle\!\langle E_1 \oplus E_2 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) & = & \langle\!\langle E_1 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) + \langle\!\langle E_2 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) \\ \langle\!\langle E_1 \otimes E_2 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) & = & \langle\!\langle E_1 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) \cdot \langle\!\langle E_2 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}). \end{array}$$

For expressions over the Boolean semiring $\mathbb{B} = (\{\mathtt{false}, \mathtt{true}\}, \vee, \wedge, \mathtt{false}, \mathtt{true})$ we will usually write $\vee$ instead of $\oplus$ and $\wedge$ instead of $\otimes$.

Construction 8. We call an expression $E \in \mathrm{Exp}_n(K)$ a *pure product* if

$$E = \mathrm{x}_1 \otimes \ldots \otimes \mathrm{x}_l \otimes \mathrm{y}_1 \otimes \ldots \otimes \mathrm{y}_m$$

with $\mathrm{x}_i \in \{x_1, \dots, x_n\}$ for $i \in \{1, \dots, l\}$ and $\mathrm{y}_j \in \{y_1, \dots, y_n\}$ for $j \in \{1, \dots, m\}$. We define a substitution procedure as follows. Let $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}(\sigma, K)^n$ be given. Let $i \in \{1, \dots, l\}$ and assume $\mathrm{x}_i = x_k$ for some k, then we define $\xi_i = \varphi^{\mathbb{1}}_k$. Likewise, for $j \in \{1, \dots, m\}$ and $\mathrm{y}_j = y_k$, we define $\theta_j = \varphi^{2}_k$. We let $\xi = \xi_1 \otimes \ldots \otimes \xi_l$ and $\theta = \theta_1 \otimes \ldots \otimes \theta_m$. Then for $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, every $(\mathcal{V}, \mathfrak{A})$-assignment ρ, and every $(\mathcal{V}, \mathfrak{B})$-assignment ς we have

$$\begin{aligned} & \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, \varsigma)) \\ &= [\![\xi_1]\!](\mathfrak{A}, \rho) \cdot \ldots \cdot [\![\xi_l]\!](\mathfrak{A}, \rho) \cdot [\![\theta_1]\!](\mathfrak{B}, \varsigma) \cdot \ldots \cdot [\![\theta_m]\!](\mathfrak{B}, \varsigma) \\ &= [\![\xi]\!](\mathfrak{A}, \rho) \cdot [\![\theta]\!](\mathfrak{B}, \varsigma). \end{aligned}$$

We define $\mathrm{PRD}^{\mathbb{1}}(E, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2}) = \xi$ and $\mathrm{PRD}^{2}(E, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2}) = \theta$.

Pure products $B \in \mathrm{Exp}_n(\mathbb{B})$ are also called *pure conjunctions*. For a pure conjunction $B \in \mathrm{Exp}_n(\mathbb{B})$, formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{MSO}(\sigma)$ and ξ_i, θ_j as above, we define the $\mathrm{MSO}(\sigma)$-formulas $\mathrm{CON}^{\mathbb{1}}(B, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2}) = \xi = \xi_1 \wedge \ldots \wedge \xi_l$ and $\mathrm{CON}^{2}(B, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2}) = \theta = \theta_1 \wedge \ldots \wedge \theta_m$. We then have

$$\langle\!\langle B \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, \varsigma)) = \mathtt{true} \text{ iff } (\mathfrak{A}, \rho) \models \xi \text{ and } (\mathfrak{B}, \varsigma) \models \theta. \qquad \Diamond$$

We say that an expression $E \in \mathrm{Exp}_n(K)$ is in *normal form* if

$$E = E_1 \oplus \ldots \oplus E_m$$

for some $m \geq 1$ and pure products E_i. By applying the laws of distributivity of the semiring K, every expression $E \in \mathrm{Exp}_n(K)$ can be transformed into normal form. More precisely, we have the following lemma.

Lemma 9. *For every $E \in \mathrm{Exp}_n(K)$ there exists an expression $E' \in \mathrm{Exp}_n(K)$ in normal form with the same semantics as E.*

Proof. We proceed by induction. Let $E \in \mathrm{Exp}_n(K)$. If $E = x_i$ or $E = y_i$ for some $i \in \{1, \dots, n\}$, then E is in normal form. If E is of the form $E_1 \oplus E_2$ or $E_1 \otimes E_2$ for two expressions $E_1, E_2 \in \mathrm{Exp}_n(K)$, we can find by induction two expressions $E_1', E_2' \in \mathrm{Exp}_n(K)$ in normal form with $\langle\!\langle E_1 \rangle\!\rangle = \langle\!\langle E_1' \rangle\!\rangle$ and $\langle\!\langle E_2 \rangle\!\rangle = \langle\!\langle E_2' \rangle\!\rangle$. In the first case, we see that $E' := E_1' \oplus E_2'$ is also in normal form and we have $\langle\!\langle E \rangle\!\rangle = \langle\!\langle E' \rangle\!\rangle$.

For the case that $E = E_1 \otimes E_2$, we write $E_1' = E_1^{(1)} \oplus \dots \oplus E_l^{(1)}$ and $E_2' = E_1^{(2)} \oplus \dots \oplus E_m^{(2)}$ with $E_1^{(1)}, \cdots, E_l^{(1)}$ and $E_1^{(2)}, \cdots, E_m^{(2)}$ pure products. Then we see that $E' := \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{m} E_i^{(1)} \otimes E_j^{(2)}$ is in normal form and due to the distributivity of K, we have $\langle\!\langle E \rangle\!\rangle = \langle\!\langle E' \rangle\!\rangle$. $\square$

3 The classical Feferman-Vaught Theorem

For convenience, we recall the Feferman-Vaught Theorem for disjoint unions and products of two structures and prove both cases. For this section, let σ be a signature.

3.1 A Feferman-Vaught Decomposition Theorem for disjoint unions

First, we state and prove the classical Feferman-Vaught Theorem for disjoint unions in the framework we will also employ for our weighted extension.

Theorem 10 ([10]). *Let $\mathcal{V}$ be a set of first and second order variables and $\beta \in \mathrm{MSO}(\sigma)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{MSO}(\sigma)^n$, and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ and all $(\mathcal{V}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ρ:*

$$(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{true}.$$

Proof. We proceed by induction.

■ $\beta = R(x_1, \dots, x_k)$ for a relation symbol $R \in \mathrm{Rel}_\sigma$ of arity k
Set $n = 1$, $\beta_1^{\mathbb{1}} = \beta_1^{2} = R(x_1, \dots, x_k)$, and $B_\beta = x_1 \vee y_1$.

■ $\beta = (x \in X)$
Set $n = 1$, $\beta_1^{\mathbb{1}} = \beta_1^{2} = (x \in X)$, and $B_\beta = x_1 \vee y_1$.

■ $\beta = \neg\alpha$
Assume the theorem is true for α with $\bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^{2} \in \mathrm{MSO}(\sigma)^l$ and $B_\alpha \in \mathrm{Exp}_l(\mathbb{B})$. We may assume that $B_\alpha = B_1 \vee \ldots \vee B_m$ is in normal form with all B_i pure conjunctions. We let $\gamma_i = \mathrm{CON}^{\mathbb{1}}(B_i, \bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^{2})$ and $\delta_i = \mathrm{CON}^{2}(B_i, \bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^{2})$ (see Construction 8) and set

$$\begin{aligned}
\bar{\beta}^{\mathbb{1}} &= (\neg\gamma_1, \dots, \neg\gamma_m) \\
\bar{\beta}^{2} &= (\neg\delta_1, \dots, \neg\delta_m) \\
B_\beta &= \bigwedge_{i=1}^{m} (x_i \vee y_i).
\end{aligned}$$

Then we have

$$\begin{aligned}
(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta &\Leftrightarrow (\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \alpha \text{ does not hold} \\
&\Leftrightarrow \langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{false} \\
&\Leftrightarrow \langle\!\langle B_i \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{false} \text{ for all } i \in \{1, \dots, m\} \\
&\Leftrightarrow (\mathfrak{A}, \rho\restriction_{\mathfrak{A}}) \models \gamma_i \text{ does not hold or } (\mathfrak{B}, \rho\restriction_{\mathfrak{B}}) \models \delta_i \text{ does not hold for all } i \in \{1, \dots, m\} \\
&\Leftrightarrow \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{true}.
\end{aligned}$$

Furthermore,

$$\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\bar{\alpha}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\alpha}^{2}) \subseteq \mathrm{Free}(\alpha) = \mathrm{Free}(\beta).$$

■ $\beta = \alpha \vee \gamma$
Assume the theorem is true for α with $\bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^{2} \in \mathrm{MSO}(\sigma)^l$ and $B_\alpha \in \mathrm{Exp}_l(\mathbb{B})$, and for γ with $\bar{\gamma}^{\mathbb{1}}, \bar{\gamma}^{2} \in$

$\mathrm{MSO}(\sigma)^m$ and $B_\gamma \in \mathrm{Exp}_m(\mathbb{B})$. Then we set $\bar{\beta}^{\mathbb{1}} = (\alpha_1^{\mathbb{1}}, \ldots, \alpha_l^{\mathbb{1}}, \gamma_1^{\mathbb{1}}, \ldots, \gamma_m^{\mathbb{1}})$, $\bar{\beta}^2 = (\alpha_1^2, \ldots, \alpha_l^2, \gamma_1^2, \ldots, \gamma_m^2)$, and $B_\beta = B_\alpha \vee B'_\gamma$, where B'_γ is obtained from B_γ by replacing every variable x_i by x_{i+l} and every variable y_i by y_{i+l}. Then we have

$$\begin{aligned}
&(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \\
\Leftrightarrow\ &(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \alpha \text{ or } (\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \gamma \\
\Leftrightarrow\ &\langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\alpha}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true} \text{ or } \langle\!\langle B_\gamma \rangle\!\rangle([\![\bar{\gamma}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\gamma}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true} \\
\Leftrightarrow\ &\langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true} \text{ or } \langle\!\langle B'_\gamma \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true} \\
\Leftrightarrow\ &\langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true}.
\end{aligned}$$

Also,

$$\mathrm{Free}(\bar{\alpha}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\alpha}^2) \cup \mathrm{Free}(\bar{\gamma}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\gamma}^2) \subseteq \mathrm{Free}(\alpha) \cup \mathrm{Free}(\gamma) = \mathrm{Free}(\beta).$$

■ $\beta = \exists x.\alpha$

Assume the theorem is true for α with $\bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^2 \in \mathrm{MSO}(\sigma)^l$ and $B_\alpha \in \mathrm{Exp}_l(\mathbb{B})$. We may assume that $B_\alpha = B_1 \vee \ldots \vee B_m$ is in normal form with all B_i pure conjunctions and that x does no occur as a bound variable in any of the $\alpha_i^{\mathbb{1}}$ or α_i^2. We let $\gamma_i = \mathrm{CON}^{\mathbb{1}}(B_i, \bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^2)$ and $\delta_i = \mathrm{CON}^2(B_i, \bar{\alpha}^{\mathbb{1}}, \bar{\alpha}^2)$ and set

$$\begin{aligned}
\bar{\beta}^{\mathbb{1}} &= (\exists x.\gamma_1, \ldots, \exists x.\gamma_m, \gamma_1^{-x}, \ldots, \gamma_m^{-x}) \\
\bar{\beta}^2 &= (\exists x.\delta_1, \ldots, \exists x.\delta_m, \delta_1^{-x}, \ldots, \delta_m^{-x}) \\
B_\beta &= \bigvee_{i=1}^{m} ((x_i \wedge y_{m+i}) \vee (x_{m+i} \wedge y_i)).
\end{aligned}$$

Then we have

$$\begin{aligned}
&(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \\
\Leftrightarrow\ &(\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c]) \models \alpha \text{ for some } c \in A \sqcup B && \text{let } \rho_c = \rho[x \to c] \\
\Leftrightarrow\ &\langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_c\restriction_{\mathfrak{A}}), [\![\bar{\alpha}^2]\!](\mathfrak{B}, \rho_c\restriction_{\mathfrak{B}})) = \mathtt{true} \text{ for some } c \in A \sqcup B \\
\Leftrightarrow\ &\langle\!\langle B_i \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_c\restriction_{\mathfrak{A}}), [\![\bar{\alpha}^2]\!](\mathfrak{B}, \rho_c\restriction_{\mathfrak{B}})) = \mathtt{true} \text{ for some } c \in A \sqcup B \text{ and } i \in \{1, \ldots, m\} \\
\Leftrightarrow\ &(\mathfrak{A}, \rho_c\restriction_{\mathfrak{A}}) \models \gamma_i \text{ and } (\mathfrak{B}, \rho_c\restriction_{\mathfrak{B}}) \models \delta_i \text{ for some } c \in A \sqcup B \text{ and } i \in \{1, \ldots, m\} \\
\Leftrightarrow\ &(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}[x \to a]) \models \gamma_i \text{ and } (\mathfrak{B}, \rho\restriction_{\mathfrak{B}}) \models \delta_i^{-x} \text{ for some } a \in A \text{ and } i \in \{1, \ldots, m\} \text{ or} \\
&(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}) \models \gamma_i^{-x} \text{ and } (\mathfrak{B}, \rho\restriction_{\mathfrak{B}}[x \to b]) \models \delta_i \text{ for some } b \in B \text{ and } i \in \{1, \ldots, m\} \\
\Leftrightarrow\ &(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}) \models \exists x.\gamma_i \text{ and } (\mathfrak{B}, \rho\restriction_{\mathfrak{B}}) \models \delta_i^{-x} \text{ for some } i \in \{1, \ldots, m\} \text{ or} \\
&(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}) \models \gamma_i^{-x} \text{ and } (\mathfrak{B}, \rho\restriction_{\mathfrak{B}}) \models \exists x.\delta_i \text{ for some } i \in \{1, \ldots, m\} \\
\Leftrightarrow\ &\langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^2) &\subseteq (\mathrm{Free}(\bar{\alpha}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\alpha}^2)) \setminus \{x\} \\
&\subseteq \mathrm{Free}(\alpha) \setminus \{x\} \\
&= \mathrm{Free}(\beta).
\end{aligned}$$

■ $\beta = \exists X.\alpha$

We reuse the notation from first order existential quantification and set

$$\begin{aligned}
\bar{\beta}^{\mathbb{1}} &= (\exists X.\gamma_1, \ldots, \exists X.\gamma_m) \\
\bar{\beta}^2 &= (\exists X.\delta_1, \ldots, \exists X.\delta_m) \\
B_\beta &= \bigvee_{i=1}^{m} (x_i \wedge y_i).
\end{aligned}$$

Then we have

$$(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta$$

$$\begin{aligned}
&\Leftrightarrow (\mathfrak{A} \sqcup \mathfrak{B}, \rho[X \to I]) \models \alpha \text{ for some } I \subseteq A \sqcup B && \text{let } \rho_I = \rho[X \to I]\\
&\Leftrightarrow \langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_I{\restriction_{\mathfrak{A}}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho_I{\restriction_{\mathfrak{B}}})) = \texttt{true} \text{ for some } I \subseteq A \sqcup B\\
&\Leftrightarrow \langle\!\langle B_i \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_I{\restriction_{\mathfrak{A}}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho_I{\restriction_{\mathfrak{B}}})) = \texttt{true} \text{ for some } I \subseteq A \sqcup B \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho_I{\restriction_{\mathfrak{A}}}) \models \gamma_i \text{ and } (\mathfrak{B}, \rho_I{\restriction_{\mathfrak{B}}}) \models \delta_i \text{ for some } I \subseteq A \sqcup B \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}[X \to I]) \models \gamma_i \text{ and } (\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}[X \to J]) \models \delta_i \text{ for some } I \subseteq A,\ J \subseteq B, \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}) \models \exists X.\gamma_i \text{ and } (\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}) \models \exists X.\delta_i \text{ for some } i \in \{1, \ldots, m\}\\
&\Leftrightarrow \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}})) = \texttt{true}.
\end{aligned}$$

In the same way as for first order existential quantification, we obtain $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta)$. ☐

3.2 A Feferman-Vaught Decomposition Theorem for products

Here, we state and prove the classical Feferman-Vaught Theorem for products in the framework we will also employ for our weighted extension.

Theorem 11 ([10]). *Let $\mathcal{V}$ be a set of first and second order variables and $\beta \in \mathrm{FO}(\sigma)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{FO}(\sigma)^n$, and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ and all $(\mathcal{V}, \mathfrak{A} \times \mathfrak{B})$-assignments ρ:*

$$(\mathfrak{A} \times \mathfrak{B}, \rho) \models \beta \textit{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}})) = \texttt{true}.$$

Proof. We proceed by induction.

■ $\beta = R(x_1, \ldots, x_k)$ for a relation symbol $R \in \mathrm{Rel}_\sigma$ of arity k
Set $n = 1$, $\beta_1^{\mathbb{1}} = \beta_1^{2} = R(x_1, \ldots, x_k)$, and $B_\beta = x_1 \wedge y_1$.

■ The proofs for $\beta = \neg\alpha$ and $\beta = \alpha \vee \gamma$ are the same as in Theorem 10.

■ $\beta = \exists x.\alpha$
We reuse the notation from first order existential quantification of Theorem 10 and set

$$\begin{aligned}
\bar{\beta}^{\mathbb{1}} &= (\exists x.\gamma_1, \ldots, \exists x.\gamma_m)\\
\bar{\beta}^{2} &= (\exists x.\delta_1, \ldots, \exists x.\delta_m)\\
B_\beta &= \bigvee_{i=1}^{m} (x_i \wedge y_i).
\end{aligned}$$

Then we have

$$\begin{aligned}
&(\mathfrak{A} \times \mathfrak{B}, \rho) \models \beta\\
&\Leftrightarrow (\mathfrak{A} \times \mathfrak{B}, \rho[x \to c]) \models \alpha \text{ for some } c \in A \times B && \text{let } \rho_c = \rho[x \to c]\\
&\Leftrightarrow \langle\!\langle B_\alpha \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_c{\restriction_{\mathfrak{A}}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho_c{\restriction_{\mathfrak{B}}})) = \texttt{true} \text{ for some } c \in A \times B\\
&\Leftrightarrow \langle\!\langle B_i \rangle\!\rangle([\![\bar{\alpha}^{\mathbb{1}}]\!](\mathfrak{A}, \rho_c{\restriction_{\mathfrak{A}}}), [\![\bar{\alpha}^{2}]\!](\mathfrak{B}, \rho_c{\restriction_{\mathfrak{B}}})) = \texttt{true} \text{ for some } c \in A \times B \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho_c{\restriction_{\mathfrak{A}}}) \models \gamma_i \text{ and } (\mathfrak{B}, \rho_c{\restriction_{\mathfrak{B}}}) \models \delta_i \text{ for some } c \in A \times B \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}[x \to a]) \models \gamma_i \text{ and } (\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}[x \to b]) \models \delta_i \text{ for some } a \in A,\ b \in B, \text{ and } i \in \{1, \ldots, m\}\\
&\Leftrightarrow (\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}) \models \exists x.\gamma_i \text{ and } (\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}) \models \exists x.\delta_i \text{ for some } i \in \{1, \ldots, m\}\\
&\Leftrightarrow \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}})) = \texttt{true}.
\end{aligned}$$

Again, we easily obtain $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta)$. ☐

4 Translation schemes

Theorems 10 and 11 consider disjoint unions and products only. So far, there is no interaction between the two constituting structures. *Translation schemes* allow us to create such interactions in an MSO-defined manner. More precisely, translation schemes "translate" structures over one signature into structures over another signature. Applying this to disjoint unions and products, we can extend Theorems 10 and 11 to more complex constructs. The usefulness of such extensions by translation schemes was discussed in [18], which we follow here.

Let σ and τ be two signatures, $\mathcal{Z} = \{z, z_1, z_2, \ldots\}$ be a set of distinguished first order variables, and $\mathcal{W}$ be a set of first and second order variables with $\mathcal{W} \cap \mathcal{Z} = \emptyset$. A *$\sigma$-$\tau$-translation scheme Φ over $\mathcal{W}$ and $\mathcal{Z}$* is a pair $(\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ where $\phi_{\mathcal{U}}, \phi_T \in \mathrm{MSO}(\sigma)$, $\phi_{\mathcal{U}}$ has variables from $\mathcal{W} \cup \{z\}$, and ϕ_T has variables from $\mathcal{W} \cup \{z_1, \ldots, z_{\mathrm{ar}_\tau(T)}\}$. The variables from $\mathcal{Z}$ may not be used for quantification, i.e., all variables from $\mathcal{Z}$ must be free.

Intuitively, the formula $\phi_{\mathcal{U}}$ is a filter for the new universe, i.e., the universe of our new τ-structure will contain all elements a of our σ-structure which satisfy $\phi_{\mathcal{U}}$ when z is mapped to a. Likewise, every formula ϕ_T defines the relation T of our new τ-structure, i.e., the interpretation of T will contain all tuples $(a_1, \ldots, a_{\mathrm{ar}_\tau(T)})$ which satisfy ϕ_T when each z_i is mapped to a_i.

We set $\mathrm{Free}(\Phi) = \mathrm{Free}(\phi_{\mathcal{U}}) \cup \bigcup_{T\in\mathrm{Rel}_\tau} \mathrm{Free}(\phi_T)$. The formulas $\phi_{\mathcal{U}}$ and $(\phi_T)_{T\in\mathrm{Rel}_\tau}$ depend on $\mathcal{Z}$ in the following way. For a first order variable x not occurring in $\phi_{\mathcal{U}}$, the formula $\phi_{\mathcal{U}}(x)$ is obtained from $\phi_{\mathcal{U}}$ by replacing all occurrences of z by x. Similarly, for $T \in \mathrm{Rel}_\tau$ and first order variables $x_1, \ldots, x_{\mathrm{ar}_\tau(T)}$ not occurring in ϕ_T, the formula $\phi_T(x_1, \ldots, x_{\mathrm{ar}_\tau(T)})$ is obtained from ϕ_T by replacing all occurrences of z_i by x_i for $i \in \{1, \ldots, \mathrm{ar}_\tau(T)\}$.

For a σ-structure $\mathfrak{A} = (A, \mathcal{I}_{\mathfrak{A}})$ and a $(\mathcal{W}, \mathfrak{A})$-assignment ς, we define the *Φ-induced τ-structure of $\mathfrak{A}$ and ς*, denoted by $\Phi^\star(\mathfrak{A}, \varsigma)$, as a τ-structure with universe $\mathcal{U}_{\mathfrak{C}}$ and interpretation $\mathcal{I}_{\mathfrak{C}}$ as follows.

$$\mathcal{U}_{\mathfrak{C}} = \{a \in A \mid (\mathfrak{A}, \varsigma[z \to a]) \models \phi_{\mathcal{U}}\}$$
$$\mathcal{I}_{\mathfrak{C}}(T) = \{\bar{c} \in \mathcal{U}_{\mathfrak{C}}^{\mathrm{ar}_\tau(T)} \mid (\mathfrak{A}, \varsigma[\bar{z} \to \bar{c}]) \models \phi_T\}$$

Example 12. We can use a translation scheme to connect a specified vertex in a graph to a set of vertices of the graph. For this let $\sigma = \tau = (\{\mathrm{edge}\}, \mathrm{edge} \mapsto 2)$ be the signature of a directed graph like in Example 2. We define a σ-σ-translation scheme $\Phi = (\phi_{\mathcal{U}}, \phi_{\mathrm{edge}})$ through

$$\phi_{\mathcal{U}} = \mathtt{true}$$
$$\phi_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2) \vee (z_1 = x \wedge z_2 \in X),$$

where $z_1 = x$ is an abbreviation for $\forall Y (z_1 \in Y \to x \in Y)$. Let $\mathfrak{G} = (V, \mathrm{edge} \mapsto E) \in \mathrm{Str}(\sigma)$ be a graph, $v \in V$ a vertex, and $I \subseteq V$ a set of vertices. Then the graph $\Phi^*(\mathfrak{G}, \{x \mapsto v, X \mapsto I\})$ is exactly the graph $\mathfrak{G}$ with an edge added between v and every vertex $v' \in I$.

Example 13. A translation scheme can also be used to cut a subtree from a given tree at a specified vertex in the tree. As in the previous example, let σ be the signature of a directed graph. For a graph $\mathfrak{G} = (V, \mathrm{edge} \mapsto E) \in \mathrm{Str}(\sigma)$, let E' be the transitive closure of the relation $E \subseteq V \times V$. We say that $\mathfrak{G}$ is a directed rooted tree with root $r \in V$ if (1) E' is irreflexive, (2) $(r, v) \in E'$ for all $v \in V \setminus \{r\}$ and (3) for all $v \in V \setminus \{r\}$ there is exactly one $v' \in V$ with $(v', v) \in E$. We define the following abbreviation which describes the reflexive transitive closure of E.

$$(x \leq y) = \forall X\, (x \in X \wedge (\forall z.(\exists z'.z' \in X \wedge \mathrm{edge}(z', z)) \to z \in X)) \to y \in X$$

We define a σ-σ-translation scheme $\Phi = (\phi_{\mathcal{U}}, \phi_{\mathrm{edge}})$ through

$$\phi_{\mathcal{U}} = (x \leq z)$$
$$\phi_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2).$$

Then with $\mathfrak{G}$ as above and $v \in V$, the graph $\mathfrak{C} = \Phi^*(\mathfrak{G}, x \mapsto v)$ is the subtree of $\mathfrak{G}$ at the vertex v, i.e.,

$$\mathcal{U}_{\mathfrak{C}} = \{v\} \cup \{v' \in V \mid (v, v') \in E'\}$$
$$\mathcal{I}_{\mathfrak{C}} = E \cap (\mathcal{U}_{\mathfrak{C}} \times \mathcal{U}_{\mathfrak{C}}).$$

We have the following fundamental property of translation schemes [18].

Lemma 14 ([18])**.** *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\beta \in \mathrm{MSO}(\tau)$ with variables from $\mathcal{V}$. Then there exists a formula $\alpha \in \mathrm{MSO}(\sigma)$ such that* $\mathrm{Free}(\alpha) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ *and for all structures $\mathfrak{A} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A}, \varsigma))$-assignments ρ:*

$$(\Phi^\star(\mathfrak{A}, \varsigma), \rho) \models \beta \textit{ iff } (\mathfrak{A}, \varsigma \cup \rho) \models \alpha.$$

Proof. We indicate the proof for the convenience of the reader. We proceed by induction. In the following, we will assume that for formulas β', β_1, and β_2, the theorem holds by induction with the formulas α', α_1 and α_2, respectively.

For $\beta = (x \in X)$, we let $\alpha = (x \in X)$. For $\beta = T(x_1, \ldots, x_k)$ for some $T \in \mathrm{Rel}_\tau$, we let $\alpha = \phi_T(x_1, \ldots, x_k)$. For $\beta = \neg\beta'$, we let $\alpha = \neg\alpha'$. For $\beta = \beta_1 \vee \beta_2$, we let $\alpha = \alpha_1 \vee \alpha_2$. For $\beta = \exists x.\beta'$, we let $\alpha = \exists x.(\alpha' \wedge \phi_{\mathcal{U}}(x))$ and for $\beta = \exists X.\beta'$, we let $\alpha = \exists X.(\alpha' \wedge \forall x.(x \in X \to \phi_{\mathcal{U}}(x)))$. □

Together with Theorems 10 and 11, this gives us the following Feferman-Vaught decomposition theorems for disjoint unions and products with translations schemes.

Theorem 15 ([18]). *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\beta \in \mathrm{MSO}(\tau)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{MSO}(\sigma)^n$, and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ:*

$$(\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) \models \beta \text{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})) = \mathtt{true}.$$

Proof. By Lemma 14 we know that there is a formula $\alpha \in \mathrm{MSO}(\sigma)$ such that

$$(\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) \models \beta \text{ iff } (\mathfrak{A} \sqcup \mathfrak{B}, \varsigma \cup \rho) \models \alpha.$$

We then use Theorem 10 for the formula α to obtain $n \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{MSO}(\sigma)^n$, and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ as required. □

Theorem 16 ([18]). *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\beta \in \mathrm{FO}(\tau)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{FO}(\sigma)^n$, and an expression $B_\beta \in \mathrm{Exp}_n(\mathbb{B})$ such that $\mathrm{Free}(\bar{\beta}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\beta}^{2}) \subseteq \mathrm{Free}(\beta) \cup \mathrm{Free}(\Phi)$ and for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ:*

$$(\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) \models \beta \text{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\beta}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})) = \mathtt{true}.$$

Proof. We proceed as in the proof of Theorem 15 and combine Lemma 14 and Theorem 11. □

Example 17. We consider the signature σ of a labeled graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}, \mathrm{label}_a, \mathrm{label}_b\}$ where edge has arity 2 and $\mathrm{label}_a, \mathrm{label}_b$ both have arity 1. Given two directed rooted labeled trees $\mathfrak{G}_1, \mathfrak{G}_2$ in this signature (see Example 13), we can use a translation scheme to add edges between all leaves of $\mathfrak{G}_1$ and the root of $\mathfrak{G}_2$ in $\mathfrak{G}_1 \sqcup \mathfrak{G}_2$. For this scenario, we have to distinguish between the vertices from the first and the second graph, so the use of an intermediate signature is necessary. We define the signature σ' to be σ extended by the relation symbols G_1 and G_2 of arity 1. Then for $i \in \{1, 2\}$, we define a σ-σ'-translation scheme $\Phi_i = (\phi_{\mathcal{U}}, \phi'_{\mathrm{edge}}, \phi_{\mathrm{label}_a}, \phi_{\mathrm{label}_b}, \phi^i_{G_1}, \phi^i_{G_2})$ as

$$\begin{aligned}
\phi_{\mathcal{U}} &= \mathtt{true} \\
\phi'_{\mathrm{edge}} &= \mathrm{edge}(z_1, z_2) \\
\phi_{\mathrm{label}_a} &= \mathrm{label}_a(z_1) \\
\phi_{\mathrm{label}_b} &= \mathrm{label}_b(z_1) \\
\phi^i_{G_j} &= \begin{cases} \mathtt{true} & \text{if } i = j \\ \mathtt{false} & \text{otherwise.} \end{cases}
\end{aligned}$$

With the abbreviations

$$\begin{aligned}
\mathrm{root}(x) &= \neg\exists y.\mathrm{edge}(y, x) \\
\mathrm{leaf}(x) &= \neg\exists y.\mathrm{edge}(x, y)
\end{aligned}$$

we then define the σ'-σ-translation scheme $\Phi = (\phi_{\mathcal{U}}, \phi_{\mathrm{edge}}, \phi_{\mathrm{label}_a}, \phi_{\mathrm{label}_b})$ through

$$\phi_{\mathrm{edge}} = \mathrm{edge}(z_1, z_2) \vee (G_1(z_1) \wedge G_2(z_2) \wedge \mathrm{leaf}(z_1) \wedge \mathrm{root}(z_2)).$$

Then $\mathfrak{G} = \Phi^*(\Phi_1^*(\mathfrak{G}_1) \sqcup \Phi_2^*(\mathfrak{G}_2))$ is exactly $\mathfrak{G}_1 \sqcup \mathfrak{G}_2$ with the leaves of $\mathfrak{G}_1$ connected to the root of $\mathfrak{G}_2$. We now consider the formula

$$\beta = \exists x.\exists y.(\mathrm{edge}(x, y) \wedge \mathrm{label}_a(x) \wedge \mathrm{label}_b(y))$$

which asks whether there is some edge between an a-labeled and a b-labeled vertex. We can apply Lemma 14 and Theorem 15 to obtain the following decomposition of β. Let

$$\begin{aligned} \bar{\beta}^1 &= (\beta, \exists x.\mathrm{label}_a(x) \wedge \mathrm{leaf}(x)) \\ \bar{\beta}^2 &= (\beta, \exists y.\mathrm{label}_b(y) \wedge \mathrm{root}(y)) \\ B_\beta &= x_1 \vee y_1 \vee (x_2 \wedge y_2). \end{aligned}$$

Then we have

$$\mathfrak{G} \models \beta \text{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^1]\!](\mathfrak{G}_1), [\![\bar{\beta}^2]\!](\mathfrak{G}_2)) = \mathtt{true}.$$

5 Weighted Feferman-Vaught Decomposition Theorems

Our goal is to prove weighted versions of Theorems 15 and 16. That is, we would like to replace FO by wFO and MSO by wMSO in those theorems. This, however, is not possible as we will see in Sections 5.2 and 5.3. For disjoint unions, we have to restrict the use of the first order product quantifier and entirely remove the second order product quantifier in wMSO. For products, it is not possible to include the first order product quantifier at all.

5.1 Formulation of the theorems

Let σ be a signature and K a commutative semiring. We define two fragments of our logic and formulate our weighted versions of Theorems 15 and 16 for these fragments.

Definition 18 (Product-free weighted first order logic)**.** We define the *product-free* first order fragment of our logic through the grammar

$$\varphi ::= \beta \mid \kappa \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi,$$

where $\beta \in \mathrm{FO}(\sigma)$ is a first order formula, $\kappa \in K$, and x is a first order variable. By $\mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)$, we denote the set of all such formulas. In fact, $\mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)$ is the set of all formulas from $\mathrm{wFO}(\sigma, K)$ which do not contain any first order product quantifier. Using this fragment, we will formulate a weighted Feferman-Vaught decomposition theorem for products of structures.

Definition 19 (Product-restricted weighted monadic second order logic)**.** In order to define the *product-restricted* fragment of our weighted monadic second order logic, we first define the fragment of so-called *almost-Boolean* formulas through the grammar

$$\psi ::= \beta \mid \kappa \mid \psi \oplus \psi \mid \psi \otimes \psi,$$

where $\beta \in \mathrm{MSO}(\sigma)$ is a monadic second order formula and $\kappa \in K$. This fragment, which we denote by $\mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$, already appeared in [4] in the form of *recognizable step functions.* To obtain the main theorem of [4], the product quantifier was restricted to quantify only over recognizable step functions. We employ the same restriction and define the product-restricted fragment of our logic through the grammar

$$\varphi ::= \beta \mid \kappa \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\psi \mid \bigoplus X.\varphi,$$

where $\beta \in \mathrm{MSO}(\sigma)$ is a monadic second order formula, $\kappa \in K$, x is a first order variable, X is a second order variable, and $\psi \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$ is an almost-Boolean formula. By $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)$ we denote the set of all such formulas. The set $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)$ contains all formulas from $\mathrm{wMSO}(\sigma, K)$ which do not contain any second order quantifier and where for every subformula of the form $\bigotimes x.\psi$ we have that ψ is an almost-Boolean formula. Our weighted Feferman-Vaught decomposition theorem for disjoint unions of structures will be formulated for this fragment. In [4] it was shown that for finite and infinite words, this fragment is expressively equivalent to weighted finite automata.

We note that the restrictions we impose on the product quantifier are necessary as we will show in Sections 5.2 and 5.3. We formulate the weighted versions of Theorems 15 and 16 as follows.[1] Let τ, $\mathcal{W}$, and $\mathcal{Z}$ be as in Section 4.

Theorem 20. *Let K be a commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho){\restriction_{\mathfrak{A}}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho){\restriction_{\mathfrak{B}}})).$$

Theorem 21. *Let K be a commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wFO}^{\bigotimes\text{-free}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A} \times \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho){\restriction_{\mathfrak{A}}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho){\restriction_{\mathfrak{B}}})).$$

The proofs of both theorems are deferred to Section 5.4. For formulas without free variables and a trivial translation scheme, i.e., $\sigma = \tau$, $\phi_{\mathcal{U}} = \mathtt{true}$, and $\phi_T = T(z_1, \ldots, z_{\mathrm{ar}_\tau(T)})$ for all $T \in \mathrm{Rel}_\tau$, the theorems reduce to the following, simplified versions.

Theorem 22. *Let K be a commutative semiring and $\varphi \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)$ be a sentence. Then there exist $n \geq 1$, tuples of sentences $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^n$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if K is bicomplete, we have*

$$[\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B})).$$

Theorem 23. *Let K be a commutative semiring and $\varphi \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)$ be a sentence. Then there exist $n \geq 1$, tuples of sentences $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)^n$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if K is bicomplete, we have*

$$[\![\varphi]\!](\mathfrak{A} \times \mathfrak{B}) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B})).$$

Example 24. To illustrate Theorem 22, we consider the semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ and the signature σ of a labeled graph, i.e., $\mathrm{Rel}_\sigma = \{\mathrm{edge}, \mathrm{label}_a, \mathrm{label}_b\}$ with edge binary and $\mathrm{label}_a, \mathrm{label}_b$ both unary. Consider the following formula which multiplies the number of vertices labeled a with the number of edges between two vertices labeled b.

$$\left(\underbrace{\bigoplus x.\mathrm{label}_a(x)}_{=\varphi_a}\right) \otimes \left(\underbrace{\bigoplus x.\bigoplus y.\mathrm{edge}(x, y) \wedge \mathrm{label}_b(x) \wedge \mathrm{label}_b(y)}_{=\varphi_b}\right)$$

The formula can be decomposed as follows. Let

$$\begin{aligned} \bar{\varphi}^{\mathbb{1}} = \bar{\varphi}^{2} &= (\varphi_a, \varphi_b) \\ E_\varphi &= (x_1 \oplus y_1) \otimes (x_2 \oplus y_2). \end{aligned}$$

Then for every two σ-structures $\mathfrak{G}_1, \mathfrak{G}_2$ we have

$$[\![\varphi]\!](\mathfrak{G}_1 \sqcup \mathfrak{G}_2) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{G}_1), [\![\bar{\varphi}^{2}]\!](\mathfrak{G}_2)).$$

[1] In [25] a weighted version of Theorem 15 similar to ours is stated (without proof) to hold without any restriction on the first order product quantifier. However, in Subsection 5.2 we show that a restriction on the product quantifier is necessary.

Example 25. In Example 7, we interpreted $[\![\varphi]\!](\mathfrak{A},\rho)$ as the number of proofs we have that $(\mathfrak{A},\rho)$ satisfies φ, assuming that φ does not contain constants. Applying Theorem 20 in this scenario means that the number of proofs that $(\mathfrak{A}\sqcup\mathfrak{B},\rho)$ satisfies a formula φ can be computed from the number of proofs we have that $(\mathfrak{A},\rho{\restriction}_{\mathfrak{A}})$ satisfies some formulas $\varphi_1^{\mathbb{1}},\dots,\varphi_n^{\mathbb{1}}$ and the number of proofs we have that $(\mathfrak{B},\rho{\restriction}_{\mathfrak{B}})$ satisfies some formulas $\varphi_1^2,\dots,\varphi_n^2$ by combining these numbers only through an expression.

Example 26. In [25], it is discussed how translation schemes can be applied for Feferman-Vaught-like decompositions of weighted properties. Theorems 20 and 21 show that this is possible for all properties which can be expressed by formulas in our weighted logic fragments.

5.2 Necessity of restricting the logic for disjoint unions

In this section, we show that the restrictions we impose on the product quantifiers are indeed necessary. For disjoint unions, we will prove that already Theorem 22 does not hold over the tropical semiring $\mathrm{Trop}=(\mathbb{R}\cup\{\infty\},\min,+,\infty,0)$ and over the arctic semiring $\mathrm{Arct}=(\mathbb{R}\cup\{-\infty\},\max,+,-\infty,0)$ for the formulas $\bigotimes x.\bigotimes y.1$ and $\bigotimes X.1$. To prove this, we employ Ramsey's Theorem. Then we show that for the formula $\bigotimes x.\bigoplus y.1$, Theorem 22 does not hold over the semiring $(\mathbb{N},+,\cdot,0,1)$. We note that these types of formulas also occurred in [4] and [5] as examples of weighted formulas whose semantics could not be described by weighted automata.

We will employ the following version of Ramsey's Theorem. For a set X, we denote by $\left[\frac{X}{2}\right]$ the set of all subsets of X of size 2.

Theorem 27 ([24])**.** *Let X be an infinite set, $k\geq 1$ a positive integer, and $f\colon\left[\frac{\mathbb{N}}{2}\right]\to\{1,\dots,k\}$ a mapping. Then there exists an infinite subset $E\subseteq\mathbb{N}$ such that $f{\restriction}_{\left[\frac{E}{2}\right]}\equiv i$ for some $i\in\{1,\dots,k\}$.*

Theorem 28. *Let $K\in\{\mathrm{Trop},\mathrm{Arct}\}$, $\sigma=(\emptyset,\emptyset)$ be the empty signature, and for $l\in\mathbb{N}_+$ consider the σ-structures $\mathfrak{S}_l=(\{1,\dots,l\},\emptyset)$. Then for $\varphi=\bigotimes x.\bigotimes y.1$ there do not exist $n\geq 1$, $\bar{\varphi}^{\mathbb{1}},\bar{\varphi}^2\in(\mathrm{wMSO}(\sigma,K))^n$, and $E_\varphi\in\mathrm{Exp}_n(K)$ such that for all $l,m\in\mathbb{N}_+$ we have*

$$[\![\varphi]\!](\mathfrak{S}_l\sqcup\mathfrak{S}_m)=\langle\!\langle E_\varphi\rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l),[\![\bar{\varphi}^2]\!](\mathfrak{S}_m)). \tag{5.1}$$

Proof. First, consider $K=\mathrm{Trop}$. For contradiction, suppose that $n,\bar{\varphi}^{\mathbb{1}},\bar{\varphi}^2$, and E_φ as above satisfying (5.1) exist. We may assume that $E_\varphi=E_1\oplus\ldots\oplus E_k$ is in normal form with all E_i pure products. For $l\geq 1$ and $i\in\{1,\dots,k\}$ we let $a_{li}=[\![\mathrm{PRD}^{\mathbb{1}}(E_i,\bar{\varphi}^{\mathbb{1}},\bar{\varphi}^2)]\!](\mathfrak{S}_l)$ and $b_{li}=[\![\mathrm{PRD}^2(E_i,\bar{\varphi}^{\mathbb{1}},\bar{\varphi}^2)]\!](\mathfrak{S}_l)$. Then by assumption we have

$$(l+m)^2=[\![\varphi]\!](\mathfrak{S}_l\sqcup\mathfrak{S}_m)=\min_{i=1}^{k}(a_{li}+b_{mi}). \tag{5.2}$$

Given $l\geq 1$ and $m\geq 1$, for at least one index $j\in\{1,\dots,k\}$ we have $(l+m)^2=a_{lj}+b_{mj}$. We define j_{lm} as the smallest such index. Then we define a function $f\colon\left[\frac{\mathbb{N}_+}{2}\right]\to\{1,\dots,k\}$ by $f(\{l,m\})=j_{lm}$ for $l<m$. Then we take $E\subseteq\mathbb{N}_+$ according to Ramsey's Theorem. As E is infinite, there are $l,\lambda,m,\mu\in E$ with $l<\lambda<m<\mu$. With $j=j_{lm}$, we thus have

$$\begin{aligned}(l+m)^2&=a_{lj}+b_{mj}\\(\lambda+m)^2&=a_{\lambda j}+b_{mj}\\(l+\mu)^2&=a_{lj}+b_{\mu j}\\(\lambda+\mu)^2&=a_{\lambda j}+b_{\mu j}.\end{aligned}$$

This implies that

$$\begin{aligned}(\lambda+\mu)^2&=(\lambda+m)^2+(l+\mu)^2-(l+m)^2\\&=\lambda^2+\mu^2+2\lambda m+2l\mu-2lm\\&=(\lambda+\mu)^2-2(\lambda-l)(\mu-m)\\&<(\lambda+\mu)^2,\end{aligned}$$

a contradiction. Therefore, $n,\bar{\varphi}^{\mathbb{1}},\bar{\varphi}^2$, and E_φ as chosen cannot exist.

The proof for the arctic semiring is in fact identical, the only difference is that equations (5.2) become

$$(l+m)^2=[\![\varphi]\!](\mathfrak{S}_l\sqcup\mathfrak{S}_m)=\max_{i=1}^{k}(a_{li}+b_{mi}).$$

$\square$

Theorem 29. *Let $K \in \{\mathrm{Trop}, \mathrm{Arct}\}$, $\sigma = (\emptyset, \emptyset)$ be the empty signature, and for $l \in \mathbb{N}_+$ consider the σ-structures $\mathfrak{S}_l = (\{1, \ldots, l\}, \emptyset)$. Then for $\varphi = \bigotimes X.1$ there do not exist $n \geq 1$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in (\mathrm{wMSO}(\sigma, K))^n$, and $E_\varphi \in \mathrm{Exp}_n(K)$ such that for all $l, m \in \mathbb{N}_+$ we have*

$$[\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)).$$

Proof. We proceed as in the proof of Theorem 28 and by contradiction obtain a system of equations

$$2^{l+m} = [\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \min_{i=1}^{k}(a_{li} + b_{mi}).$$

Also employing Ramsey's Theorem in the same way, we obtain $l < \lambda < m < \mu$ and $j \in \{1, \ldots, k\}$ such that

$$\begin{aligned} 2^{l+m} &= a_{lj} + b_{mj} \\ 2^{\lambda+m} &= a_{\lambda j} + b_{mj} \\ 2^{l+\mu} &= a_{lj} + b_{\mu j} \\ 2^{\lambda+\mu} &= a_{\lambda j} + b_{\mu j}, \end{aligned}$$

which gives us the equality

$$2^{\lambda+\mu} = 2^{\lambda+m} + 2^{l+\mu} - 2^{l+m}.$$

By dividing by 2^{l+m} we obtain

$$2^{(\lambda-l)+(\mu-m)} = 2^{\lambda-l} + 2^{\mu-m} - 1. \tag{5.3}$$

However, we have

$$\begin{aligned} 2^{(\lambda-l)+(\mu-m)} &\geq 2^{\lambda-l} + 2^{\mu-m} \\ &> 2^{\lambda-l} + 2^{\mu-m} - 1, \end{aligned}$$

which contradicts equation (5.3). □

Theorem 30. *Let $K = (\mathbb{N}, +, \cdot, 0, 1)$, $\sigma = (\emptyset, \emptyset)$ be the empty signature, and for $l \in \mathbb{N}_+$ consider the σ-structures $\mathfrak{S}_l = (\{1, \ldots, l\}, \emptyset)$. Then for $\varphi = \bigotimes x.\bigoplus y.1$ there do not exist $n \geq 1$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in (\mathrm{wMSO}(\sigma, \mathbb{N}))^n$, and $E_\varphi \in \mathrm{Exp}_n(\mathbb{N})$ such that for all $l, m \in \mathbb{N}_+$ we have*

$$[\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^2]\!](\mathfrak{S}_m)). \tag{5.4}$$

Proof. We proceed by contradiction and assume $n, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2$, and E_φ as above satisfying (5.4) exist. We may assume that $E_\varphi = E_1 \oplus \ldots \oplus E_k$ is in normal form with all E_i pure products. For $l \geq 1$ and $i \in \{1, \ldots, k\}$ we let $a_{li} = [\![\mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$ and $b_{li} = [\![\mathrm{PRD}^2(E_i, \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2)]\!](\mathfrak{S}_l)$. Then by assumption we have

$$(l+m)^{(l+m)} = [\![\varphi]\!](\mathfrak{S}_l \sqcup \mathfrak{S}_m) = \sum_{i=1}^{k}(a_{li} \cdot b_{mi}). \tag{5.5}$$

For every $j \in \{1, \ldots, k\}$ we choose $L_j \geq 1$ such that $a_{L_j j} \neq 0$, or let $L_j := 0$ if for all $l \geq 1$ we have $a_{lj} = 0$. Assume $m \geq 1$ and $j \in \{1, \ldots, k\}$ with $L_j \neq 0$, then $a_{L_j j} \geq 1$, hence

$$(L_j + m)^{(L_j+m)} = \sum_{i=1}^{k}(a_{L_j i} \cdot b_{mi}) \geq (a_{L_j j} \cdot b_{mj}) \geq b_{mj}.$$

In particular, with $L := \max\{L_i \mid i \in \{1, \ldots, k\}\}$, we have that for every $j \in \{1, \ldots, k\}$ either (i) $b_{mj} \leq (L+m)^{(L+m)}$ for all $m \geq 1$ or (ii) $a_{lj} = 0$ for all $l \geq 1$. Note that from equation (5.5) it follows that $L = 0$ is impossible. In the same fashion, we can find $M \geq 1$ such that for every $l \geq 1$ and every $j \in \{1, \ldots, k\}$ either (i) $a_{lj} \leq (l+M)^{(l+M)}$ for all $l \geq 1$ or (ii) $b_{mj} = 0$ for all $m \geq 1$.

Now, for arbitrary $l \geq 1$, consider the special case

$$(l+l)^{(l+l)} = \sum_{i=1}^{k}(a_{li} \cdot b_{li}).$$

If $j \in \{1, \dots, k\}$ such that either $a_{lj} = 0$ for all $l \geq 1$ or $b_{mj} = 0$ for all $m \geq 1$, then clearly also $(a_{lj} \cdot b_{lj}) = 0$. If j is not like this, we have

$$(a_{lj} \cdot b_{lj}) \leq (l+M)^{(l+M)} \cdot (L+l)^{(L+l)} \leq (l+C)^{2(l+C)}$$

for $C := \max\{L, M\}$. In summary, we have

$$(2l)^{2l} \leq k(l+C)^{2(l+C)}$$

for every $l \geq 1$. Now if l is of the form NC for some $N \in \mathbb{N}$, we have

$$\begin{aligned} & & (2l)^{2l} &\leq k(l+C)^{2(l+C)} \\ \Leftrightarrow & & (2NC)^{NC} &\leq \sqrt{k}((N+1)C)^{(N+1)C} \\ \Leftrightarrow & & (2N)^{N} &\leq \sqrt[2C]{k}C(N+1)^{(N+1)} \\ \Leftrightarrow & & \frac{2^N}{N+1}\left(\frac{N}{N+1}\right)^N &\leq \sqrt[2C]{k}C. \end{aligned}$$

However, this inequality cannot hold for all $N \in \mathbb{N}$, as

$$\frac{2^N}{N+1} \xrightarrow{N\to\infty} +\infty \quad \text{and} \quad \left(\frac{N}{N+1}\right)^N \xrightarrow{N\to\infty} e^{-1}.$$

□

5.3 Necessity of restricting the logic for products

The proof of Theorem 28 can also be used to show that no Feferman-Vaught-like theorem holds for products if the first order product quantifier is included in the weighted logic. More precisely, already Theorem 23 does not hold over the tropical and arctic semirings for the formula $\varphi = \bigotimes x.1$ even if $\bar{\varphi}^{\mathbb{1}}$ and $\bar{\varphi}^{2}$ are allowed to be from $\mathrm{wMSO}(\sigma, K)$.

Theorem 31. *Let $K \in \{\mathrm{Trop}, \mathrm{Arct}\}$, $\sigma = (\emptyset, \emptyset)$ be the empty signature, and for $l \in \mathbb{N}_+$ consider the σ-structures $\mathfrak{S}_l = (\{1, \dots, l\}, \emptyset)$. Then for $\varphi = \bigotimes x.1$ there do not exist $n \geq 1$, $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in (\mathrm{wMSO}(\sigma, K))^n$, and $E_\varphi \in \mathrm{Exp}_n(K)$ such that for all $l, m \in \mathbb{N}_+$ we have*

$$[\![\varphi]\!](\mathfrak{S}_l \times \mathfrak{S}_m) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{S}_l), [\![\bar{\varphi}^{2}]\!](\mathfrak{S}_m)).$$

Proof. Like in the proof of Theorem 28, for $K = \mathrm{Trop}$ we reduce the problem to a system of equations

$$lm = [\![\varphi]\!](\mathfrak{S}_l \times \mathfrak{S}_m) = \min_{i=1}^{k}(a_{li} + b_{mi}).$$

Employing Ramsey's Theorem, we again obtain $l < \lambda < m < \mu$ and $j \in \{1, \dots, k\}$ such that

$$\begin{aligned} lm &= a_{lj} + b_{mj} \\ \lambda m &= a_{\lambda j} + b_{mj} \\ l\mu &= a_{lj} + b_{\mu j} \\ \lambda\mu &= a_{\lambda j} + b_{\mu j}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \lambda\mu &= \lambda m + l\mu - lm \\ &= \lambda\mu - (\lambda - l)(\mu - m) \\ &< \lambda\mu, \end{aligned}$$

which is a contradiction. For $K = \mathrm{Arct}$, the proof is again analogous. □

5.4 Proofs of Theorems 20 and 21

We now turn to the proof of Theorems 20 and 21. We can reduce the proofs to the case where the translation scheme is the identity.

Lemma 32. *Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}(\tau, K)$ with variables from $\mathcal{V}$. Then there exists a formula $\psi \in \mathrm{wMSO}(\sigma, K)$ with $\mathrm{Free}(\psi) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$ such that the following holds. For all finite structures $\mathfrak{A} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A} \in \mathrm{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A}, \varsigma), \rho) = [\![\psi]\!](\mathfrak{A}, \varsigma \cup \rho).$$

If φ is from $\mathrm{wMSO}^{\bigotimes\text{-res}}(\tau, K)$ or $\mathrm{wFO}^{\bigotimes\text{-free}}(\tau, K)$, then ψ can also be chosen as a formula from $\mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)$ or $\mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)$, respectively.

Proof. We proceed by induction. In the sequel we will assume that for formulas φ', φ_1, and φ_2, the lemma holds by induction with the formulas ψ', ψ_1, and ψ_2, respectively.

For $\varphi = \beta \in \mathrm{MSO}(\tau)$, we obtain ψ by applying Lemma 14 to β. For $\varphi = \kappa \in K$, we let $\psi = \kappa$. For $\varphi = \varphi_1 \oplus \varphi_2$ or $\varphi = \varphi_1 \otimes \varphi_2$, we define $\psi = \psi_1 \oplus \psi_2$ or $\psi = \psi_1 \otimes \psi_2$, respectively.
For $\varphi = \bigoplus x.\varphi'$, we let $\psi = \bigoplus x.(\psi' \otimes \phi_{\mathcal{U}}(x))$.
For $\varphi = \bigoplus X.\varphi'$, we let $\psi = \bigoplus X.(\psi' \otimes \forall x.(x \in X \to \phi_{\mathcal{U}}(x)))$.
For $\varphi = \bigotimes x.\varphi'$, we let $\psi = \bigotimes x.((\psi' \otimes \phi_{\mathcal{U}}(x)) \oplus \neg\phi_{\mathcal{U}}(x))$.
For $\varphi = \bigotimes X.\varphi'$, we define $\beta = \forall x.(x \in X \to \phi_{\mathcal{U}}(x))$ and let $\psi = \bigotimes X.((\psi' \otimes \beta) \oplus \neg\beta)$.

Note that for the cases of infinite sums and products, we need that $\prod_I \mathbb{1} = \mathbb{1}$ and $\sum_I \mathbb{0} = \mathbb{0}$ for every index set I. The first is an axiom of our infinite products, the latter follows from the distributivity of the infinite sum. □

Proof of Theorem 20. We proceed by induction. By Lemma 32, it suffices to prove the case $\tau = \sigma$ and $\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma) = \mathfrak{A} \sqcup \mathfrak{B}$.

■ $\varphi = \beta$ for some $\beta \in \mathrm{MSO}(\sigma)$
We apply Theorem 10 to the formula β and obtain $l \geq 1$, tuples of formulas $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{2} \in \mathrm{MSO}(\sigma)^l$, and an expression $B_\beta \in \mathrm{Exp}_l(\mathbb{B})$ such that

$$(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta \text{ iff } \langle\!\langle B_\beta \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true}.$$

We may assume that $B_\beta = B_1 \vee \ldots \vee B_m$ is in normal form with all B_i pure conjunctions. We let $\gamma_i = \mathrm{CON}^{\mathbb{1}}(B_i, \bar{\beta}^{\mathbb{1}}, \bar{\beta}^2)$ and $\delta_i = \mathrm{CON}^2(B_i, \bar{\beta}^{\mathbb{1}}, \bar{\beta}^2)$ for $i \in \{1, \ldots, m\}$ (see Construction 8). We set $n = 2m$ and define

$$\bar{\varphi}^{\mathbb{1}} = (\gamma_1, \ldots, \gamma_m, \neg\gamma_1, \ldots, \neg\gamma_m)$$
$$\bar{\varphi}^{2} = (\delta_1, \ldots, \delta_m, \neg\delta_1, \ldots, \neg\delta_m).$$

Intuitively, we would now define the expression E_φ as $(x_1 \otimes y_1) \oplus \ldots \oplus (x_m \otimes y_m)$, but this expression is not necessarily evaluated to $\mathbb{1}$ in K if $\gamma_i \wedge \delta_i$ is true for more than one index i. Instead, we define expressions $E_k \in \mathrm{Exp}_n(K)$ for $k \in \{1, \ldots, m\}$ inductively by $E_1 = x_1 \otimes y_1$ and

$$E_k = (E_{k-1} \otimes ((x_{k+m} \otimes y_k) \oplus y_{k+m})) \oplus (x_k \otimes y_k)$$

for $k \geq 2$ and set $E_\varphi = E_m$. The expression E_k is evaluated to $\mathbb{1}$ if $\gamma_k \wedge \delta_k$ holds, and otherwise, if either γ_k or δ_k does not hold, it is evaluated to E_{k-1}. We show by induction that for all $k \in \{1, \ldots, m\}$ we have

$$\langle\!\langle E_k \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) =$$
$$\begin{cases} \mathbb{1} & \text{if } \langle\!\langle B_i \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), [\![\bar{\beta}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathtt{true} \text{ for some } i \in \{1, \ldots, k\} \\ \mathbb{0} & \text{otherwise.} \end{cases}$$

Let $\bar{\kappa} = [\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho\restriction_{\mathfrak{A}})$ and $\bar{\lambda} = [\![\bar{\varphi}^2]\!](\mathfrak{B}, \rho\restriction_{\mathfrak{B}})$. For $k = 1$ we have, due to the fact that $\bar{\kappa}, \bar{\lambda} \in \{\mathbb{0}, \mathbb{1}\}^{2m}$,

$$\langle\!\langle x_1 \otimes y_1 \rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{1}$$

$$\begin{aligned}&\Leftrightarrow \kappa_1 = \mathbb{1} \text{ and } \lambda_1 = \mathbb{1}\\&\Leftrightarrow (\mathfrak{A}, \rho\restriction_{\mathfrak{A}}) \models \gamma_1 \text{ and } (\mathfrak{B}, \rho\restriction_{\mathfrak{A}}) \models \delta_1\\&\Leftrightarrow \langle\!\langle B_1\rangle\!\rangle(\llbracket\bar{\beta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\beta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{true}\end{aligned}$$

and $\langle\!\langle E_1\rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{0}$ otherwise. For $k > 1$ and $\langle\!\langle B_i\rangle\!\rangle(\llbracket\bar{\beta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\beta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{false}$ for all $i \in \{1, \ldots, k\}$, we have $\langle\!\langle E_{k-1}\rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{0}$ by induction and at least one of κ_k, λ_k is $\mathbb{0}$. It is easy to see that in this case $\langle\!\langle E_k\rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{0}$. Otherwise either $\langle\!\langle E_{k-1}\rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{1}$ by induction or $\kappa_k = \lambda_k = \mathbb{1}$. Taking into account that the values for x_k, x_{k+m} and y_k, y_{k+m} are always "dual", a simple case distinction shows that in this case $\langle\!\langle E_k\rangle\!\rangle(\bar{\kappa}, \bar{\lambda}) = \mathbb{1}$. In conclusion, we have $\llbracket\varphi\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \in \{\mathbb{0}, \mathbb{1}\}$ and

$$\begin{aligned}&\llbracket\varphi\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho) = \mathbb{1}\\\Leftrightarrow\ &(\mathfrak{A} \sqcup \mathfrak{B}, \rho) \models \beta\\\Leftrightarrow\ &\langle\!\langle B_\beta\rangle\!\rangle(\llbracket\bar{\beta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\beta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{true}\\\Leftrightarrow\ &\langle\!\langle B_i\rangle\!\rangle(\llbracket\bar{\beta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\beta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \texttt{true} \text{ for some } i \in \{1, \ldots, m\}\\\Leftrightarrow\ &\langle\!\langle E_\varphi\rangle\!\rangle(\llbracket\bar{\varphi}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\varphi}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) = \mathbb{1},\end{aligned}$$

hence $\llbracket\varphi\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho) = \langle\!\langle E_\varphi\rangle\!\rangle(\llbracket\bar{\varphi}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\varphi}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}}))$.

■ $\varphi = \kappa$ for some $\kappa \in K$
We let $n = 1$, $\varphi_1^{\mathbb{1}} = \varphi_1^{2} = \kappa$ and $E_\varphi = x_1$.

■ $\varphi = \zeta \oplus \eta$
We assume the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^l$ and $E_\zeta \in \mathrm{Exp}_l(K)$, and for η with $\bar{\eta}^{\mathbb{1}}, \bar{\eta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^m$ and $E_\eta \in \mathrm{Exp}_m(K)$. We let $\bar{\varphi}^{\mathbb{1}} = (\zeta_1^{\mathbb{1}}, \ldots, \zeta_l^{\mathbb{1}}, \eta_1^{\mathbb{1}}, \ldots, \eta_m^{\mathbb{1}})$, $\bar{\varphi}^{2} = (\zeta_1^{2}, \ldots, \zeta_l^{2}, \eta_1^{2}, \ldots, \eta_m^{2})$, and $E_\varphi = E_\zeta \oplus E'_\eta$, where E'_η is obtained from E_η by replacing every variable x_i by x_{i+l} and every variable y_i by y_{i+l}. We have

$$\begin{aligned}&\llbracket\varphi\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho)\\=\ &\llbracket\zeta\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho) + \llbracket\eta\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho)\\=\ &\langle\!\langle E_\zeta\rangle\!\rangle(\llbracket\bar{\zeta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\zeta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) + \langle\!\langle E_\eta\rangle\!\rangle(\llbracket\bar{\eta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\eta}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}}))\\=\ &\langle\!\langle E_\zeta\rangle\!\rangle(\llbracket\bar{\varphi}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\varphi}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})) + \langle\!\langle E'_\eta\rangle\!\rangle(\llbracket\bar{\varphi}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\varphi}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}}))\\=\ &\langle\!\langle E_\varphi\rangle\!\rangle(\llbracket\bar{\varphi}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho\restriction_{\mathfrak{A}}), \llbracket\bar{\varphi}^{2}\rrbracket(\mathfrak{B}, \rho\restriction_{\mathfrak{B}})).\end{aligned}$$

■ $\varphi = \zeta \otimes \eta$
The proof is the same as for the previous case, only that here we define $E_\varphi = E_\zeta \otimes E'_\eta$.

■ $\varphi = \bigoplus x.\zeta$
We assume the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^l$ and $E_\zeta \in \mathrm{Exp}_l(K)$. We may assume that $E_\zeta = E_1 \oplus \ldots \oplus E_m$ is in normal form with all E_i pure products and that x does no occur as a bound variable in any of the $\zeta_i^{\mathbb{1}}$ or ζ_i^{2}. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$ and $\theta_i = \mathrm{PRD}^{2}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$. We set $n = 2m$ and define

$$\begin{aligned}\bar{\varphi}^{\mathbb{1}} &= (\textstyle\bigoplus x.\xi_1, \ldots, \bigoplus x.\xi_m, \xi_1^{-x}, \ldots, \xi_m^{-x})\\\bar{\varphi}^{2} &= (\textstyle\bigoplus x.\theta_1, \ldots, \bigoplus x.\theta_m, \theta_1^{-x}, \ldots, \theta_m^{-x})\\E_\varphi &= \textstyle\bigoplus_{i=1}^{m}((x_i \otimes y_{m+i}) \oplus (x_{m+i} \otimes y_i)).\end{aligned}$$

Then we have

$$\begin{aligned}&\llbracket\varphi\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho)\\=\ &\sum_{c \in A \sqcup B} \llbracket\zeta\rrbracket(\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c])\\=\ &\sum_{c \in A \sqcup B} \langle\!\langle E_\zeta\rangle\!\rangle(\llbracket\bar{\zeta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho[x \to c]\restriction_{\mathfrak{A}}), \llbracket\bar{\zeta}^{2}\rrbracket(\mathfrak{B}, \rho[x \to c]\restriction_{\mathfrak{B}}))\\=\ &\sum_{c \in A \sqcup B} \sum_{i=1}^{m} \langle\!\langle E_i\rangle\!\rangle(\llbracket\bar{\zeta}^{\mathbb{1}}\rrbracket(\mathfrak{A}, \rho[x \to c]\restriction_{\mathfrak{A}}), \llbracket\bar{\zeta}^{2}\rrbracket(\mathfrak{B}, \rho[x \to c]\restriction_{\mathfrak{B}}))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{c\in A\sqcup B}\sum_{i=1}^{m}[\![\xi_i]\!](\mathfrak{A},\rho[x\to c]{\restriction_{\mathfrak{A}}})\cdot[\![\theta_i]\!](\mathfrak{B},\rho[x\to c]{\restriction_{\mathfrak{B}}})\\
&= \sum_{a\in A}\sum_{i=1}^{m}[\![\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}[x\to a])\cdot[\![\theta_i^{-x}]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}) + \sum_{b\in B}\sum_{i=1}^{m}[\![\xi_i^{-x}]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}})\cdot[\![\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}[x\to b])\\
&= \sum_{i=1}^{m}\left(\sum_{a\in A}[\![\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}[x\to a])\right)\cdot[\![\theta_i^{-x}]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}) + [\![\xi_i^{-x}]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}})\cdot\left(\sum_{b\in B}[\![\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}[x\to b])\right)\\
&= \sum_{i=1}^{m}[\![\textstyle\bigoplus x.\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}})\cdot[\![\theta_i^{-x}]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}) + [\![\xi_i^{-x}]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}})\cdot[\![\textstyle\bigoplus x.\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}})\\
&= \langle\!\langle E_\varphi\rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}),[\![\bar{\varphi}^{2}]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}})).
\end{aligned}
$$

■ $\varphi = \bigoplus X.\zeta$

As for the first order sum quantifier, we assume that the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^l$ and $E_\zeta \in \mathrm{Exp}_l(K)$ such that $E_\zeta = E_1 \oplus \ldots \oplus E_m$ is in normal form with all E_i pure products. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$ and $\theta_i = \mathrm{PRD}^{2}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{2})$. We set $n = m$ and define

$$
\begin{aligned}
\bar{\varphi}^{\mathbb{1}} &= (\textstyle\bigoplus X.\xi_1, \ldots, \bigoplus X.\xi_m)\\
\bar{\varphi}^{2} &= (\textstyle\bigoplus X.\theta_1, \ldots, \bigoplus X.\theta_m)\\
E_\varphi &= \textstyle\bigoplus_{i=1}^{m}(x_i \otimes y_i).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&[\![\varphi]\!](\mathfrak{A}\sqcup\mathfrak{B},\rho)\\
&= \sum_{I\subseteq A\sqcup B}[\![\zeta]\!](\mathfrak{A}\sqcup\mathfrak{B},\rho[X\to I])\\
&= \sum_{I\subseteq A\sqcup B}\langle\!\langle E_\zeta\rangle\!\rangle([\![\bar{\zeta}^{\mathbb{1}}]\!](\mathfrak{A},\rho[X\to I]{\restriction_{\mathfrak{A}}}),[\![\bar{\zeta}^{2}]\!](\mathfrak{B},\rho[X\to I]{\restriction_{\mathfrak{B}}}))\\
&= \sum_{I\subseteq A\sqcup B}\sum_{i=1}^{m}[\![\xi_i]\!](\mathfrak{A},\rho[X\to I]{\restriction_{\mathfrak{A}}})\cdot[\![\theta_i]\!](\mathfrak{B},\rho[X\to I]{\restriction_{\mathfrak{B}}})\\
&= \sum_{i=1}^{m}\sum_{I\subseteq A}\sum_{J\subseteq B}[\![\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}[X\to I])\cdot[\![\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}[X\to J])\\
&= \sum_{i=1}^{m}\left(\sum_{I\subseteq A}[\![\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}[X\to I])\right)\cdot\left(\sum_{J\subseteq B}[\![\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}}[X\to J])\right)\\
&= \sum_{i=1}^{m}[\![\textstyle\bigoplus X.\xi_i]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}})\cdot[\![\textstyle\bigoplus X.\theta_i]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}})\\
&= \langle\!\langle E_\varphi\rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A},\rho{\restriction_{\mathfrak{A}}}),[\![\bar{\varphi}^{2}]\!](\mathfrak{B},\rho{\restriction_{\mathfrak{B}}})).
\end{aligned}
$$

■ $\varphi = \bigotimes x.\zeta$ with $\zeta \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$ almost Boolean

Using the laws of distributivity in K and the fact that for two MSO formulas $\alpha, \beta \in \mathrm{MSO}(\sigma)$ we have $[\![\alpha \otimes \beta]\!] = [\![\alpha \wedge \beta]\!]$, we may assume that $\zeta = (\kappa_1 \otimes \beta_1) \oplus \ldots \oplus (\kappa_l \otimes \beta_l)$ for some $l \geq 1$, $\kappa_i \in K$, and $\beta_i \in \mathrm{MSO}(\sigma)$. First, we will show that we may even assume that $\beta_1, \ldots, \beta_l$ form a partition, i.e., that for every $(\mathcal{V}, \mathfrak{A}\sqcup\mathfrak{B})$-assignment ρ' there is exactly one $i \in \{1, \ldots, l\}$ with $(\mathfrak{A}\sqcup\mathfrak{B}, \rho') \models \beta_i$.

For this, let $\Omega = \{\beta_1, \neg\beta_1\} \times \ldots \times \{\beta_l, \neg\beta_l\}$. For every $\bar{\omega} = (\omega_1, \ldots, \omega_l) \in \Omega$ we define a formula $\alpha_{\bar{\omega}}$ and $\kappa_{\bar{\omega}} \in K$ as follows.

$$
\alpha_{\bar{\omega}} = \bigwedge_{i=1}^{l}\omega_i \qquad\qquad \kappa_{\bar{\omega}} = \sum_{\substack{1\leq i\leq l\\ \omega_i=\beta_i}}\kappa_i
$$

The empty sum is $\mathbb{0}$ by convention. It is clear that for every $(\mathcal{V}, \mathfrak{A}\sqcup\mathfrak{B})$-assignment ρ' there exists a unique $\bar{\omega} \in \Omega$ with $(\mathfrak{A}\sqcup\mathfrak{B}, \rho') \models \alpha_{\bar{\omega}}$. Moreover, for $i \in \{1, \ldots, l\}$ we have $(\mathfrak{A}\sqcup\mathfrak{B}, \rho') \models \beta_i$ if and only if

$(\mathfrak{A} \sqcup \mathfrak{B}, \rho') \models \alpha_{\bar{\omega}}$ for some $\bar{\omega} \in \Omega$ with $\omega_i = \beta_i$, and in this case $\bar{\omega}$ is unique. We therefore have

$$\begin{aligned}
[\![\zeta]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho') &= \sum_{i=1}^{l} \kappa_i \cdot [\![\beta_i]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho') \\
&= \sum_{i=1}^{l} \kappa_i \cdot \sum_{\substack{\bar{\omega}\in\Omega \\ \omega_i=\beta_i}} [\![\alpha_{\bar{\omega}}]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho') \\
&= \sum_{i=1}^{l} \sum_{\substack{\bar{\omega}\in\Omega \\ \omega_i=\beta_i}} \kappa_i \cdot [\![\alpha_{\bar{\omega}}]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho') \\
&= \sum_{\bar{\omega}\in\Omega} \left(\sum_{\substack{1\le i\le l \\ \omega_i=\beta_i}} \kappa_i \right) \cdot [\![\alpha_{\bar{\omega}}]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho') \\
&= \sum_{\bar{\omega}\in\Omega} \kappa_{\bar{\omega}} \cdot [\![\alpha_{\bar{\omega}}]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho').
\end{aligned}$$

Thus, $[\![\zeta]\!] = [\![\bigoplus_{\bar{\omega}\in\Omega} \kappa_{\bar{\omega}} \otimes \alpha_{\bar{\omega}}]\!]$ and the family $(\alpha_{\bar{\omega}})_{\bar{\omega}\in\Omega}$ forms a partition in the above sense. In the following, we simply assume that $\zeta = (\kappa_1 \otimes \beta_1) \oplus \ldots \oplus (\kappa_l \otimes \beta_l)$ and that $\beta_1, \ldots, \beta_l$ form a partition.

For every $i \in \{1, \ldots, l\}$, let $X_i \in \mathcal{V}$ be a second order variable not occurring in ζ. We define the abbreviation

$$((x \in X_i) \rhd \kappa_i) := ((x \in X_i) \otimes \kappa_i) \oplus \neg(x \in X_i).$$

We write all of the X_i into a tuple $\bar{X}$ and for sets $I_i \subseteq A \sqcup B$ ($i \in \{1, \ldots, l\}$), we let $\bar{I}$ be the corresponding tuple of sets. Then for $c \in A \sqcup B$ and sets $I_i \subseteq A \sqcup B$ we have

$$[\![(x \in X_i) \rhd \kappa_i]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho[\bar{X} \to \bar{I}, x \to c]) = \begin{cases} \kappa_i & \text{if } c \in I_i \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Now consider the formula

$$\left(\bigwedge_{i=1}^{l} \forall x.(x \in X_i \leftrightarrow \beta_i) \right) \otimes \bigotimes_{i=1}^{l} \bigotimes x.((x \in X_i) \rhd \kappa_i).$$

For sets $I_i \subseteq A \sqcup B$ ($i \in \{1, \ldots, l\}$) we have

$$\begin{aligned}
&[\![\bigwedge_{i=1}^{l} \forall x.(x \in X_i \leftrightarrow \beta_i)]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho[\bar{X} \to \bar{I}]) \\
&= \begin{cases} \mathbb{1} & \text{if for all } c \in A \sqcup B \text{ and all } i \in \{1, \ldots, l\}\text{: } c \in I_i \text{ iff } (\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c]) \models \beta_i \\ \mathbb{0} & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence, the above is evaluated to $\mathbb{1}$ if and only if $I_i = \{c \in A \sqcup B \mid (\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c]) \models \beta_i\}$ for all $i \in \{1, \ldots, l\}$. In this case, the family $(I_i)_{1\le i\le l}$ is a partition of $A \sqcup B$, since the family $(\beta_i)_{1\le i\le l}$ forms a partition. Therefore, in this case we have

$$\begin{aligned}
&[\![\bigotimes_{i=1}^{l} \bigotimes x.((x \in X_i) \rhd \kappa_i)]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho[\bar{X} \to \bar{I}]) \\
&= \prod_{i=1}^{l} \prod_{c\in I_i} \kappa_i \\
&= \prod_{c\in A\sqcup B} \sum_{i=1}^{l} \kappa_i \cdot [\![\beta_i]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c]) \\
&= \prod_{c\in A\sqcup B} [\![\zeta]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho[x \to c])
\end{aligned}$$

$$= [\![\varphi]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho).$$

In conclusion, we have

$$[\![\varphi]\!] = [\![\bigoplus X_1.\bigoplus X_2 \ldots \bigoplus X_l. \left(\bigwedge_{i=1}^{l} \forall x.(x \in X_i \leftrightarrow \beta_i)\right) \otimes \bigotimes_{i=1}^{l} \bigotimes x.((x \in X_i) \rhd \kappa_i)]\!].$$

Therefore, it suffices to show this case of the induction for formulas of the form

$$\varphi = \bigotimes x.((x \in X) \rhd \kappa).$$

We let $n = 1$ and define $\bar{\varphi}^{\mathbb{1}} = \bar{\varphi}^{\mathbb{2}} = (\bigotimes x.((x \in X) \rhd \kappa))$ and $E_\varphi = x_1 \otimes y_1$. Then we have

$$\begin{aligned}
&[\![\bigotimes x.((x \in X) \rhd \kappa)]\!](\mathfrak{A} \sqcup \mathfrak{B}, \rho) \\
&= [\![\bigotimes x.((x \in X) \rhd \kappa)]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}) \cdot [\![\bigotimes x.((x \in X) \rhd \kappa)]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}) \\
&= \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}), [\![\bar{\varphi}^{\mathbb{2}}]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}})).
\end{aligned}$$

□

Proof of Theorem 21. Again we proceed by induction and assume that $\tau = \sigma$ and $\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma) = \mathfrak{A} \times \mathfrak{B}$. The proofs for the cases $\varphi = \beta$, $\varphi = \kappa$, $\varphi = \zeta \oplus \eta$, and $\varphi = \zeta \otimes \eta$ are identical to the ones used in the proof of Theorem 20 for the corresponding cases.

For the case $\varphi = \bigoplus x.\zeta$ we proceed as for the case $\varphi = \bigoplus X.\zeta$ in the proof of Theorem 20 as follows. We assume that the theorem is true for ζ with $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{\mathbb{2}} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^l$ and $E_\zeta = E_1 \oplus \ldots \oplus E_m$ is in normal form with all E_i pure products. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{\mathbb{2}})$ and $\theta_i = \mathrm{PRD}^{\mathbb{2}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^{\mathbb{2}})$. We set $n = m$ and define

$$\begin{aligned}
\bar{\varphi}^{\mathbb{1}} &= (\bigoplus x.\xi_1, \ldots, \bigoplus x.\xi_m) \\
\bar{\varphi}^{\mathbb{2}} &= (\bigoplus x.\theta_1, \ldots, \bigoplus x.\theta_m) \\
E_\varphi &= \textstyle\bigoplus_{i=1}^{m}(x_i \otimes y_i).
\end{aligned}$$

Then we have

$$\begin{aligned}
&[\![\varphi]\!](\mathfrak{A} \times \mathfrak{B}, \rho) \\
&= \sum_{c \in A \times B} [\![\zeta]\!](\mathfrak{A} \times \mathfrak{B}, \rho[x \to c]) \\
&= \sum_{c \in A \times B} \langle\!\langle E_\zeta \rangle\!\rangle([\![\bar{\zeta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho[x \to c]{\restriction_{\mathfrak{A}}}), [\![\bar{\zeta}^{\mathbb{2}}]\!](\mathfrak{B}, \rho[x \to c]{\restriction_{\mathfrak{B}}})) \\
&= \sum_{c \in A \times B} \sum_{i=1}^{m} [\![\xi_i]\!](\mathfrak{A}, \rho[x \to c]{\restriction_{\mathfrak{A}}}) \cdot [\![\theta_i]\!](\mathfrak{B}, \rho[x \to c]{\restriction_{\mathfrak{B}}}) \\
&= \sum_{i=1}^{m} \sum_{a \in A} \sum_{b \in B} [\![\xi_i]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}[x \to a]) \cdot [\![\theta_i]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}[x \to b]) \\
&= \sum_{i=1}^{m} \left(\sum_{a \in A} [\![\xi_i]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}[x \to a])\right) \cdot \left(\sum_{b \in B} [\![\theta_i]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}[x \to b])\right) \\
&= \sum_{i=1}^{m} [\![\bigoplus x.\xi_i]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}) \cdot [\![\bigoplus x.\theta_i]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}}) \\
&= \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction_{\mathfrak{A}}}), [\![\bar{\varphi}^{\mathbb{2}}]\!](\mathfrak{B}, \rho{\restriction_{\mathfrak{B}}})).
\end{aligned}$$

□

This concludes the proofs of Theorems 20 and 21.

5.5 De Morgan algebras

In this section, we want to consider the special case where our semiring can be extended by a unary operation $\neg$ to form a *De Morgan algebra* $(L, \vee, \wedge, \neg, 0, 1)$. A tuple $(L, \vee, \wedge, \neg, 0, 1)$ is called a De Morgan algebra if $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\neg\colon L \to L$ is an involution satisfying De Morgan's laws, i.e., we have $\neg(x\vee y) = \neg x\wedge\neg y$, $\neg(x\wedge y) = \neg x\vee\neg y$, and $\neg\neg x = x$ for all $x, y \in L$. If $\leq$ is the induced order of the lattice $(L, \vee, \wedge, 0, 1)$, it follows that $\neg\colon (L, \leq) \to (L, \leq)$ is an order-antiisomorphism. In particular, $\neg 0 = 1$ and $\neg 1 = 0$. A De Morgan algebra is called *complete* if $(L, \vee, \wedge, 0, 1)$ is a complete lattice. Since $\neg$ is an order-antiisomorphism, it follows that the equalities $\neg \bigwedge_{x\in X} x = \bigvee_{x\in X} \neg x$ and $\neg \bigvee_{x\in X} x = \bigwedge_{x\in X} \neg x$ hold for every subset $X \subseteq L$ of a complete lattice L.

Example 33. Examples of De Morgan algebras include

- all Boolean algebras, in particular, the two element Boolean algebra $\mathbb{B}$,
- *Kleene* or *Priest logic*, i.e., the three element Kleene algebra $(\{F, I, T\}, \vee, \wedge, \neg, F, T)$ where $F \leq I \leq T$ describes the lattice and $\neg I = I$, $\neg F = T$ the negation,
- *Belnap* or *Dunn logic* $(\{F, B, N, T\}, \vee, \wedge, \neg, F, T)$ where $F \leq B \leq T$, $F \leq N \leq T$, and B and N are incomparable, and the negation is given by $\neg B = B$, $\neg N = N$, $\neg F = T$, and
- the *Łukasiewicz logics*, for example $L_\infty = ([0, 1], \max, \min, \neg, 0, 1)$ where $\neg x = 1 - x$.

Whenever we are dealing with a De Morgan algebra, we can include the operator $\neg$ into our weighted logic.

Definition 34 (De Morgan-extension)**.** Let σ be a signature. We define the *De Morgan-extensions* of our weighted first order and monadic second order logics through the grammars

$$\varphi ::= \beta \mid \kappa \mid \neg\varphi \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\varphi,$$

where $\beta \in \mathrm{FO}(\sigma)$ is a first order formula, $\kappa \in L$, and x is a first order variable, and

$$\varphi ::= \beta \mid \kappa \mid \neg\varphi \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\varphi \mid \bigoplus X.\varphi \mid \bigotimes X.\varphi,$$

where $\beta \in \mathrm{MSO}(\sigma)$ is a monadic second order formula, $\kappa \in L$, x is a first order variable, and X is a second order variable, respectively. The semantics of $\neg\varphi$ is defined by $[\![\neg\varphi]\!](\mathfrak{A}, \rho) = \neg[\![\varphi]\!](\mathfrak{A}, \rho)$ for a σ-structure $\mathfrak{A}$ and a variable assignment ρ. By $\mathrm{wFO}^\neg(\sigma, L)$ and $\mathrm{wMSO}^\neg(\sigma, L)$, we denote the sets of all such formulas, respectively. Weighted logics for *words* over bounded lattices were also considered in [6], where the authors showed that Kleene-type and Büchi-like results hold for these logics.

Since L is a De Morgan algebra, it is easy to see that for every formula $\varphi \in \mathrm{wMSO}^\neg(\sigma, L)$ the formulas $\bigotimes x.\varphi$ and $\neg\bigoplus x.\neg\varphi$ are semantically equivalent. The same holds true for the formulas $\bigotimes X.\varphi$ and $\neg\bigoplus X.\neg\varphi$. Therefore, in this scenario we do not need any restriction to formulate weighted Feferman-Vaught decomposition theorems. Let τ, $\mathcal{W}$, and $\mathcal{Z}$ be as in Section 4.

Theorem 35. *Let L be a De Morgan algebra, $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}^\neg(\tau, L)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}^\neg(\sigma, L)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(L)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if L is complete, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})).$$

Theorem 36. *Let L be a De Morgan algebra, $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wFO}^\neg(\tau, L)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^\neg(\sigma, L)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(L)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$ if L is complete, all $(\mathcal{W}, \mathfrak{A} \times \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})).$$

Proof. We proceed as in the proofs of Theorems 20 and 21. To see that we can assume the translation scheme to be trivial, note that the inductive proof of Lemma 32 can easily be extended to $\mathrm{wMSO}^{\neg}(\sigma, L)$: if $\varphi = \neg\varphi'$ and the lemma is true for φ' with the formula ψ', then we can choose $\psi = \neg\psi'$.

Using the inductive steps of the proofs for Theorems 20 and 21 and the above rewriting of product quantifiers into sum quantifiers through a double weighted negation, we see that it only remains to show the inductive step for the weighted negation $\varphi = \neg\zeta$ as follows.

We proceed as in the proof for the Boolean case. Also, the proofs for the disjoint union and the product are the same, so in the following let $\mathfrak{C} = \mathfrak{A} \sqcup \mathfrak{B}$ or $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$. We assume the theorem is true for ζ with $E_\zeta \in \mathrm{Exp}_l(L)$ and $\bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^2$ from $\mathrm{wFO}^{\neg}(\sigma, L)^l$ or from $\mathrm{wMSO}^{\neg}(\sigma, L)^l$. We may assume that $E_\zeta = E_1 \oplus \ldots \oplus E_m$ is in normal form with all E_i pure products. We let $\xi_i = \mathrm{PRD}^{\mathbb{1}}(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^2)$ and $\theta_i = \mathrm{PRD}^2(E_i, \bar{\zeta}^{\mathbb{1}}, \bar{\zeta}^2)$ and define

$$\begin{aligned}
\bar{\varphi}^{\mathbb{1}} &= (\neg\xi_1, \ldots, \neg\xi_m)\\
\bar{\varphi}^2 &= (\neg\zeta_1, \ldots, \neg\zeta_m)\\
E_\varphi &= \bigwedge_{i=1}^{m} (x_i \vee y_i).
\end{aligned}$$

Then we have

$$\begin{aligned}
[\![\varphi]\!](\mathfrak{C}, \rho) &= \neg\langle\!\langle E_\zeta \rangle\!\rangle([\![\bar{\zeta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction}_{\mathfrak{A}}), [\![\bar{\zeta}^2]\!](\mathfrak{B}, \rho{\restriction}_{\mathfrak{B}}))\\
&= \neg \bigvee_{i=1}^{m} \langle\!\langle E_i \rangle\!\rangle([\![\bar{\zeta}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction}_{\mathfrak{A}}), [\![\bar{\zeta}^2]\!](\mathfrak{B}, \rho{\restriction}_{\mathfrak{B}}))\\
&= \neg \bigvee_{i=1}^{m} [\![\xi_i]\!](\mathfrak{A}, \rho{\restriction}_{\mathfrak{A}}) \wedge [\![\theta_i]\!](\mathfrak{B}, \rho{\restriction}_{\mathfrak{B}})\\
&= \bigwedge_{i=1}^{m} [\![\neg\xi_i]\!](\mathfrak{A}, \rho{\restriction}_{\mathfrak{A}}) \vee [\![\neg\theta_i]\!](\mathfrak{B}, \rho{\restriction}_{\mathfrak{B}})\\
&= \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, \rho{\restriction}_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, \rho{\restriction}_{\mathfrak{B}})).
\end{aligned}$$

□

5.6 Weakly biaperiodic semirings

In this section, we show that Theorems 20 and 21 hold true without the need for any restriction whenever our weights are taken from a *weakly biaperiodic* commutative semiring. A monoid is called *weakly aperiodic* if for every element x there exists a positive integer n such that $x^n = x^{n+1}$. We call a semiring $(K, +, \cdot, \mathbb{0}, \mathbb{1})$ weakly biaperiodic if both its additive monoid $(K, +, \mathbb{0})$ and its multiplicative monoid $(K, \cdot, \mathbb{1})$ are weakly aperiodic. Weighted logics for *words* over weakly biaperiodic semirings were also considered in [4, 6].

Example 37. Examples of weakly biaperiodic semirings include

- Every De Morgan algebra, in particular, all semirings from Example 33,
- the Łukasiewicz semiring $([0, 1], \max, \otimes, 0, 1)$ where $x \otimes y = \max\{0, x + y - 1\}$,
- the truncated min-plus semiring $([0, d], \min, +_d, d, 0)$ for a real number $d > 0$, where $x +_d y = \min\{d, x + y\}$.

For weakly biaperiodic semirings, we can show that every quantifier, when quantifying over an almost Boolean formula, again models an almost Boolean formula. The proof for this employs explicit case distinctions to compute the outcomes of the quantifiers. By induction, it follows that for weakly biaperiodic semirings, every wMSO formula is semantically equivalent to an almost Boolean formula, i.e., a formula containing no weighted quantifiers. We thus have the following lemma.

Lemma 38. *Let K be a weakly biaperiodic commutative semiring and σ a signature. Then for every formula $\varphi \in \mathrm{wMSO}(\sigma, K)$, there exists a formula $\psi \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$ with $[\![\varphi]\!] = [\![\psi]\!]$.*

Proof. We proceed by induction. For the cases $\varphi = \beta \in \mathrm{MSO}(\sigma, K)$, $\varphi = \kappa \in K$, $\varphi = \psi_1 \oplus \psi_2$, and $\varphi = \psi_1 \otimes \psi_2$ with $\psi_1, \psi_2 \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$, the statement is clear.

For the cases $\varphi = \bigoplus x.\varphi'$, $\varphi = \bigoplus X.\varphi'$, $\varphi = \bigotimes x.\varphi'$, and $\varphi = \bigotimes X.\varphi'$ with $\varphi' \in \mathrm{wMSO}(\sigma, K)$, we proceed as follows. By induction, we assume that there exists an almost Boolean formula $\psi' \in \mathrm{wMSO}^{\text{a-bool}}(\sigma, K)$ such that $[\![\varphi']\!] = [\![\psi']\!]$. We assume that ψ' is of the form $\psi' = (\kappa_1 \otimes \beta_1) \oplus \ldots \oplus (\kappa_l \otimes \beta_l)$, where $\beta_1, \ldots, \beta_l$ form a partition like in the proof of Theorem 20. By the assumption that K is weakly biaperiodic, there exists for every $i \in \{1, \ldots, l\}$ a number $n_i \in \mathbb{N}_+$ such that $\sum_{j=1}^{n_i} \kappa_i = \sum_{j=1}^{n_i+1} \kappa_i$. We let $N_1 := \max_{i=1}^{l} n_i$. Likewise, there exist $n_i \in \mathbb{N}_+$ such that $\prod_{j=1}^{n_i} \kappa_i = \prod_{j=1}^{n_i+1} \kappa_i$ for every $i \in \{1, \ldots, l\}$. We let $N_2 := \max_{i=1}^{l} n_i$. Then with $N = \max\{N_1, N_2\}$ we have $\sum_{j=1}^{N} \kappa_i = \sum_{j=1}^{N+1} \kappa_i$ and $\prod_{j=1}^{N} \kappa_i = \prod_{j=1}^{N+1} \kappa_i$ for all $i \in \{1, \ldots, l\}$. Furthermore, we define abbreviations as follows. For first order variables y_1 and y_2 and second order variables Y_1 and Y_2, we let

$$(y_1 = y_2) := \forall Z.(y_1 \in Z \leftrightarrow y_2 \in Z)$$
$$(Y_1 = Y_2) := \forall z.(z \in Y_1 \leftrightarrow z \in Y_2).$$

Now let $\beta \in \mathrm{MSO}(\sigma, K)$ be a monadic second order formula. For a first order variable y, we denote by $\beta(y)$ the formula which results from β by renaming every free occurrence of the first order variable x to y. For a second order variable Y, we denote by $\beta(Y)$ the formula which results from β by renaming every free occurrence of the second order variable X to Y. Then for $m \in \mathbb{N}_+$ and $\mathcal{X} \in \{x, X\}$, we define the abbreviations

$$\begin{aligned}
\exists^{\geq m} \mathcal{X}.\beta &:= \exists \mathcal{X}_1 \ldots \exists \mathcal{X}_m. \Big(\bigwedge_{i=1}^{m} \beta(\mathcal{X}_i) \wedge \bigwedge_{i \neq j} \neg(\mathcal{X}_i = \mathcal{X}_j) \Big) \\
\exists^{m} \mathcal{X}.\beta &:= \exists^{\geq m} \mathcal{X}.\beta \wedge \neg(\exists^{\geq m+1} \mathcal{X}.\beta) \\
\bar{\exists}^{m} \mathcal{X}.\beta &:= \begin{cases} \exists^{m} \mathcal{X}.\beta & \text{if } m < N \\ \exists^{\geq N} \mathcal{X}.\beta & \text{if } m \geq N, \end{cases}
\end{aligned}$$

where $\mathcal{X}_1, \ldots, \mathcal{X}_m \notin \mathrm{Free}(\psi')$ are first order variables if $\mathcal{X} = x$, and they are second order variables if $\mathcal{X} = X$. For every $\bar{v} \in \{0, \ldots, N\}^l$, we define the constants

$$\kappa_{\bar{v}} := \sum_{i=1}^{l} \sum_{j=1}^{v_i} \kappa_i \qquad\qquad \lambda_{\bar{v}} := \prod_{i=1}^{l} \prod_{j=1}^{v_i} \kappa_i.$$

Again, the empty sum is defined as $\mathbb{0}$ and the empty product as $\mathbb{1}$. Then for the case $\varphi = \bigoplus \mathcal{X}.\varphi'$ with $\mathcal{X} \in \{x, X\}$, we define the formula

$$\psi := \bigoplus_{\bar{v} \in \{0, \ldots, N\}^l} \kappa_{\bar{v}} \otimes \bigwedge_{i=1}^{l} \bar{\exists}^{v_i} \mathcal{X}.\beta_i.$$

By the definition of $\kappa_{\bar{v}}$ and the choice of N, we have $[\![\varphi]\!] = [\![\psi]\!]$ and ψ is almost Boolean. For the case $\varphi = \bigotimes \mathcal{X}.\varphi'$ with $\mathcal{X} \in \{x, X\}$, we define

$$\psi := \bigoplus_{\bar{v} \in \{0, \ldots, N\}^l} \lambda_{\bar{v}} \otimes \bigwedge_{i=1}^{l} \bar{\exists}^{v_i} \mathcal{X}.\beta_i.$$

Again, we have $[\![\varphi]\!] = [\![\psi]\!]$ and ψ is almost Boolean. $\square$

We let τ, $\mathcal{W}$, and $\mathcal{Z}$ be as in Section 4. Then we have the following theorems.

Theorem 39. *Let K be a weakly biaperiodic commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T \in \mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wMSO}(\sigma, K)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^{2}) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^{2}]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})).$$

Theorem 40. *Let K be a weakly biaperiodic commutative semiring. Let $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wFO}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in \mathrm{wFO}(\sigma, K)^n$ with $\mathrm{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \mathrm{Free}(\bar{\varphi}^2) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_n(K)$ such that the following holds. For all structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\sigma)$, all $(\mathcal{W}, \mathfrak{A} \times \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, (\varsigma \cup \rho)\restriction_{\mathfrak{B}})).$$

Proof. This can be shown using the exact same methods as in the proofs of Lemma 32 and Theorems 20 and 21. Note that, since every formula $\varphi \in \mathrm{wMSO}(\sigma, K)$ is equivalent to some almost Boolean formula, we do not need any assumptions on the finiteness of our structures. ☐

5.7 Beyond disjoint unions and products of two structures

So far, we considered induced disjoint unions and products of only two structures. Here, we shortly point out an extension of Theorems 20 and 21 to disjoint unions and products of more than two structures.

For $m, n \geq 1$ we define the set of expressions $\mathrm{Exp}_{m\times n}(K)$ similar to $\mathrm{Exp}_n(K)$ but with variables from $\{x_i^j \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ and the semantics extended in the obvious fashion. Then we have the following theorem for disjoint unions.

Theorem 41. *Let $m \geq 2$, $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^1, \ldots, \bar{\varphi}^m \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^n$ with $\bigcup_{i=1}^m \mathrm{Free}(\bar{\varphi}^i) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$, and an expression $E_\varphi \in \mathrm{Exp}_{m\times n}(K)$ such that the following holds. For all finite structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_m \in \mathrm{Str}(\sigma)$, or, for all structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_m \in \mathrm{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m)$-assignments ς, and all $(\mathcal{V}, \Phi^\star(\mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}_1, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_1}), \ldots, [\![\bar{\varphi}^m]\!](\mathfrak{A}_m, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_m})).$$

Proof. We proceed by induction. For $m = 2$ the theorem follows from Theorem 20. Let $m \geq 2$. By Theorem 20, we we can find $l \geq 1$, $\bar{\zeta}, \bar{\eta} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^l$, and $E_0 \in \mathrm{Exp}_l(K)$ such that

$$\begin{gathered}[\![\varphi]\!](\Phi^\star(\mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m \sqcup \mathfrak{A}_{m+1}, \varsigma), \rho) = \\ \langle\!\langle E_0 \rangle\!\rangle([\![\bar{\zeta}]\!](\mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_1\sqcup\ldots\sqcup\mathfrak{A}_m}), [\![\bar{\eta}]\!](\mathfrak{A}_{m+1}, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_{m+1}})).\end{gathered}$$

By induction, for every $i \in \{1, \ldots, l\}$, we can find $n_i \geq 1$, tuples of formulas $\bar{\zeta}_i^1, \ldots, \bar{\zeta}_i^m \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^{n_i}$, and an expression $E_{\zeta_i} \in \mathrm{Exp}_{m\times n_i}(K)$ such that

$$[\![\zeta_i]\!](\Phi^\star(\mathfrak{A}_1 \sqcup \ldots \sqcup \mathfrak{A}_m, \varsigma), \rho) = \langle\!\langle E_{\zeta_i} \rangle\!\rangle([\![\bar{\zeta}_i^1]\!](\mathfrak{A}_1, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_1}), \ldots, [\![\bar{\zeta}_i^m]\!](\mathfrak{A}_m, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_m})).$$

For $j \in \{1, \ldots, m\}$ we define

$$\bar{\varphi}^j = (\bar{\zeta}_1^j, \ldots, \bar{\zeta}_l^j)$$

and with $n = \sum_{i=1}^l n_i$ we define $\bar{\varphi}^{m+1} \in \mathrm{wMSO}^{\bigotimes\text{-res}}(\sigma, K)^n$ as

$$\bar{\varphi}^{m+1} = (\bar{\eta}, \mathtt{true}, \ldots, \mathtt{true}).$$

For every $i \in \{1, \ldots, l\}$, we define the expression E'_{ζ_i} as the expression obtained from E_{ζ_i} by replacing every variable x_ι^j by $x^j_{\iota+\sum_{k=1}^{i-1} n_k}$. We define $E_\varphi \in \mathrm{Exp}_{m+1\times n}(K)$ as the expression obtained from E_0 by replacing every variable x_i by the expression E'_{ζ_i} and every variable y_i by the variable x_i^{m+1}. With these definitions, the theorem holds. ☐

With an essentially identical proof, we have the following theorem for induced products of more than two structures.

Theorem 42. *Let $m \geq 2$, $\Phi = (\phi_{\mathcal{U}}, (\phi_T)_{T\in\mathrm{Rel}_\tau})$ be a σ-τ-translation scheme over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \mathrm{wFO}^{\bigotimes\text{-free}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^1, \ldots, \bar{\varphi}^m \in \mathrm{wFO}^{\bigotimes\text{-free}}(\sigma, K)^n$*

with $\bigcup_{i=1}^{m} \mathrm{Free}(\bar{\varphi}^i) \subseteq \mathrm{Free}(\varphi) \cup \mathrm{Free}(\Phi)$*, and an expression* $E_\varphi \in \mathrm{Exp}_{m\times n}(K)$ *such that the following holds. For all finite structures* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m \in \mathrm{Str}(\sigma)$*, or, for all structures* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m \in \mathrm{Str}(\sigma)$ *if* K *is bicomplete, all* $(\mathcal{W}, \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_m)$*-assignments* ς*, and all* $(\mathcal{V}, \Phi^\star(\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_m, \varsigma))$*-assignments* ρ *we have*

$$[\![\varphi]\!](\Phi^\star(\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_m, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathfrak{A}_1, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_1}), \ldots, [\![\bar{\varphi}^m]\!](\mathfrak{A}_m, (\varsigma \cup \rho)\restriction_{\mathfrak{A}_m})).$$

5.8 Translation schemes and Courcelle's transductions

Like translation schemes, *transductions* provide a tool to translate structures over one signature into structures over another signature. Transductions extend our notion of translation scheme by allowing multiple copies of the given universe. More precisely, a σ-τ-translation scheme is a 1-σ-τ transduction in the sense defined below. In the following, we show that, with some adjustments, our weighted Feferman-Vaught Theorems can also be applied to transductions. For a survey on transductions, see [3].

Definition 43 ([3])**.** Let $k > 0$ be a natural number, $[k] = \{1, \ldots, k\}$, and

$$\tau * k = \{(T, \bar{\imath}) \mid T \in \mathrm{Rel}_\tau \text{ and } \bar{\imath} \in [k]^{\mathrm{ar}_\tau(T)}\}.$$

A *k-σ-τ-transduction Ψ over $\mathcal{W}$ and $\mathcal{Z}$* is a tuple $(\psi_{\mathcal{U}}^1, \ldots, \psi_{\mathcal{U}}^k, (\psi_w)_{w \in \tau * k})$ where $\psi_{\mathcal{U}}^i, \psi_w \in \mathrm{MSO}(\sigma)$ are formulas with variables from $\mathcal{W} \cup \mathcal{Z}$. The variables from $\mathcal{Z}$ may not be used for quantification, i.e., all variables from $\mathcal{Z}$ must be free.

Intuitively, the formulas $\psi_{\mathcal{U}}^1, \ldots, \psi_{\mathcal{U}}^k$ are filters for the copies of the universe, i.e., for an element a from the universe of the σ-structure, there will be one copy of a in the universe of the new τ-structure for each $\psi_{\mathcal{U}}^i$ which is satisfied when the free variable z is mapped to a. Likewise, the formulas $\psi_{(T,\bar{\imath})}$ are used to define the interpretation of T for the new τ-structure, where $\bar{\imath}$ determines from which copy of the universe each entry of the new tuple has to be.

For a σ-structure $\mathfrak{A} = (A, \mathcal{I}_{\mathfrak{A}})$ and a $(\mathcal{W}, \mathfrak{A})$-assignment ς, the *Ψ-induced τ-structure of $\mathfrak{A}$ and ς*, denoted by $\Psi^\star(\mathfrak{A}, \varsigma)$, is defined as a τ-structure with universe $\mathcal{U}_{\mathfrak{C}}$ and interpretation $\mathcal{I}_{\mathfrak{C}}$ as follows. For $i \in \{1, \ldots, k\}$ we define

$$A_i = \{a \in A \mid (\mathfrak{A}, \varsigma[z \to a]) \models \psi_{\mathcal{U}}^i\}$$

and let $\iota_i \colon A_i \to A_1 \sqcup \ldots \sqcup A_k$ be the inclusions. Then we let

$$\mathcal{U}_{\mathfrak{C}} = A_1 \sqcup \ldots \sqcup A_k$$

$$\mathcal{I}_{\mathfrak{C}}(T) = \bigcup_{\bar{\imath} \in [k]^{\mathrm{ar}_\tau(T)}} \{(\iota_{i_1}(a_1), \ldots, \iota_{i_{\mathrm{ar}_\tau(T)}}(a_{\mathrm{ar}_\tau(T)})) \mid (a_1, \ldots, a_{\mathrm{ar}_\tau(T)}) \in A_{i_1} \times \ldots \times A_{i_{\mathrm{ar}_\tau(T)}} \text{ and } (\mathfrak{A}, \varsigma[\bar{z} \to \bar{a}]) \models \psi_{(T,\bar{\imath})}\}$$

where $\bar{\imath} = (i_1, \ldots, i_{\mathrm{ar}_\tau(T)})$ and $\bar{a} = (a_1, \ldots, a_{\mathrm{ar}_\tau(T)})$.

We refrain from restricting the domain of the transduction, as it does not make any difference for our purpose.

We can prove an analogue of Lemma 32 for transductions. Therefore, Theorems 20 and 21 are true for transductions as well. However, we have to make two small concessions. First, for the Boolean fragment of our first order logic, we need a new atomic formula `def`(x), where x is a first order variable. This formula is evaluated to `true` if the variable x is defined, and to `false` otherwise. For our second order logic, we use `def`(x) as an abbreviation for the formula $\exists X.(x \in X)$. We denote by `def`-wFO(σ, K) the first order logic where `def`(x) is allowed as an atomic formula. Second, the variables of the formula we want to "translate" do usually not suffice for the translated formula. In particular, the translated formula potentially has more free variables than the formula to translate.

For a set of first and second order variables $\mathcal{V}$ and $k > 0$, we let $\mathcal{V}^{\sqcup k} = \{\mathcal{X}^i \mid \mathcal{X} \in \mathcal{V}, i \in \{1, \ldots, k\}\}$ be the set of variables containing k copies of every variable from $\mathcal{V}$. Then, with the above notation, we define for a $(\mathcal{V}, \Psi^\star(\mathfrak{A}, \varsigma))$-assignment ρ the $(\mathcal{V}^{\sqcup k}, \mathfrak{A})$-assignment $\rho^\#$ by

$$\rho^\#(\mathcal{X}^i) = \begin{cases} \iota_i^{-1}(\rho(\mathcal{X}) \cap \iota_i(A)) & \text{if } \mathcal{X} \text{ is a second order variable} \\ \iota_i^{-1}(\rho(\mathcal{X})) & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \in \iota_i(A) \\ \text{undefined} & \text{if } \mathcal{X} \text{ is a first order variable and } \rho(\mathcal{X}) \notin \iota_i(A). \end{cases}$$

Then we have the following lemma.

Lemma 44. *Let K be a commutative semiring. Let $\Psi = (\psi^1_{\mathcal{U}}, \dots, \psi^k_{\mathcal{U}}, (\psi_w)_{w \in \tau * k})$ be a k-σ-τ-transduction over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \texttt{def}\text{-wFO}(\tau, K)$ or $\varphi \in \text{wMSO}(\tau, K)$ with variables from $\mathcal{V}$. Then there exists a formula $\psi \in \texttt{def}\text{-wFO}(\sigma, K)$ or $\psi \in \text{wMSO}(\sigma, K)$, respectively, with $\text{Free}(\psi) \subseteq \text{Free}(\varphi)^{\sqcup k} \cup \text{Free}(\Psi)$ such that the following holds. For all structures $\mathfrak{A} \in \text{Str}(\sigma)$, or, for all structures $\mathfrak{A} \in \text{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A})$-assignments ς, and all $(\mathcal{V}, \Psi^\star(\mathfrak{A}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Psi^\star(\mathfrak{A}, \varsigma), \rho) = [\![\psi]\!](\mathfrak{A}, \varsigma \cup \rho^\#).$$

If φ is from $\text{wMSO}^{\bigotimes\text{-res}}(\tau, K)$ or $\texttt{def}\text{-wFO}^{\bigotimes\text{-free}}(\tau, K)$, then ψ can also be chosen as a formula from $\text{wMSO}^{\bigotimes\text{-res}}(\sigma, K)$ or $\texttt{def}\text{-wFO}^{\bigotimes\text{-free}}(\sigma, K)$, respectively. Furthermore, if φ does not contain free variables, ψ can be chosen to not contain any subformula of the form $\texttt{def}(x)$.

Proof. We proceed by induction and first cover the Boolean case.

If $\varphi = T(x_1, \dots, x_n)$ for some $T \in \text{Rel}_\tau$, we let $\psi = \bigvee_{\bar{\imath} \in [k]^n} \Big(\psi_{(T,\bar{\imath})}(x_1^{i_1}, \dots, x_n^{i_n}) \wedge \bigwedge_{j=1}^n \texttt{def}(x_j^{i_j})\Big)$. If $\varphi = (x \in X)$, we let $\psi = \bigvee_{i=1}^k x^i \in X^i$. If $\varphi = \texttt{def}(x)$ we let $\psi = \bigvee_{i=1}^k \texttt{def}(x^i)$.

Now we assume that by induction, the theorem holds for the formulas φ_1, φ_2, and φ' with the formulas ψ_1, ψ_2, and ψ'. If $\varphi = \varphi_1 \vee \varphi_2$, we let $\psi = \psi_1 \vee \psi_2$, and if $\varphi = \neg\varphi'$, we let $\psi = \neg\psi'$.

If $\varphi = \exists x.\varphi'$, we define for $i \in \{1, \dots, k\}$ the formula ψ'^{+i} as the formula obtained by replacing all atomic subformulas in ψ' that contain a variable x^j with $j \neq i$ by $\texttt{false}$. Then we let $\psi = \bigvee_{i=1}^k \exists x^i. \big(\psi^i_{\mathcal{U}}(x^i) \wedge \psi'^{+i}\big)$.

If $\varphi = \exists X.\varphi'$, we let $\psi = \exists X^1 \dots \exists X^k. \Big(\psi' \wedge \bigwedge_{i=1}^k \forall x.(x \in X^i \to \psi^i_{\mathcal{U}}(x))\Big)$, where x is a new first order variable.

We now turn to the weighted case.

If $\varphi = \kappa \in K$, we let $\psi = \kappa$.

If $\varphi = \varphi_1 \oplus \varphi_2$ or $\varphi = \varphi_1 \otimes \varphi_2$, we let $\psi = \psi_1 \oplus \psi_2$ or $\psi = \psi_1 \otimes \psi_2$, respectively.

If $\varphi = \bigoplus x.\varphi'$, we let $\psi = \bigoplus_{i=1}^k \bigoplus x^i. \big(\psi^i_{\mathcal{U}}(x^i) \otimes \psi'^{+i}\big)$.

If $\varphi = \bigoplus X.\varphi'$, we let $\psi = \bigoplus X^1 \dots \bigoplus X^k. \Big(\psi' \otimes \bigwedge_{i=1}^k \forall x.(x \in X^i \to \psi^i_{\mathcal{U}}(x))\Big)$, where x is a new first order variable.

If $\varphi = \bigotimes x.\varphi'$, we let $\psi = \bigotimes_{i=1}^k \bigotimes x^i.((\psi'^{+i} \otimes \psi^i_{\mathcal{U}}(x^i)) \oplus \neg\psi^i_{\mathcal{U}}(x^i))$.

If $\varphi = \bigotimes X.\varphi'$ we define $\beta = \bigwedge_{i=1}^k \forall x.(x \in X^i \to \psi^i_{\mathcal{U}}(x))$, where x is a new first order variable, and let $\psi = \bigotimes X^1 \dots \bigotimes X^k.((\psi' \otimes \beta) \oplus \neg\beta)$.

To see that all atomic subformulas $\texttt{def}(x)$ in ψ can be removed if φ does not contain free variables, note that every subformula $\texttt{def}(x)$ can be replaced by $\texttt{true}$ without changing the semantics of ψ if x is a bound variable. $\square$

With this, we have the following versions of Theorems 20 and 21 for transductions.

Theorem 45. *Let K be a commutative semiring. Let $\Psi = (\psi^1_{\mathcal{U}}, \dots, \psi^k_{\mathcal{U}}, (\psi_w)_{w \in \tau * k})$ be a k-σ-τ-transduction over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \texttt{def}\text{-wMSO}^{\bigotimes\text{-res}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in \texttt{def}\text{-wMSO}^{\bigotimes\text{-res}}(\sigma, K)^n$ with $\text{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \text{Free}(\bar{\varphi}^2) \subseteq \text{Free}(\varphi)^{\sqcup k} \cup \text{Free}(\Psi)$, and an expression $E_\varphi \in \text{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \text{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \text{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A} \sqcup \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Psi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Psi^\star(\mathfrak{A} \sqcup \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho^\#)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, (\varsigma \cup \rho^\#)\restriction_{\mathfrak{B}})).$$

Theorem 46. *Let K be a commutative semiring. Let $\Psi = (\psi^1_{\mathcal{U}}, \dots, \psi^k_{\mathcal{U}}, (\psi_w)_{w \in \tau * k})$ be a k-σ-τ-transduction over $\mathcal{W}$ and $\mathcal{Z}$, $\mathcal{V}$ be a set of first and second order variables such that $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Z}$ are pairwise disjoint, and $\varphi \in \texttt{def}\text{-wFO}^{\bigotimes\text{-free}}(\tau, K)$ with variables from $\mathcal{V}$. Then there exist $n \geq 1$, tuples of formulas $\bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^2 \in \texttt{def}\text{-wFO}^{\bigotimes\text{-free}}(\sigma, K)^n$ with $\text{Free}(\bar{\varphi}^{\mathbb{1}}) \cup \text{Free}(\bar{\varphi}^2) \subseteq \text{Free}(\varphi)^{\sqcup k} \cup \text{Free}(\Psi)$, and an expression $E_\varphi \in \text{Exp}_n(K)$ such that the following holds. For all finite structures $\mathfrak{A}, \mathfrak{B} \in \text{Str}(\sigma)$, or, for all structures $\mathfrak{A}, \mathfrak{B} \in \text{Str}(\sigma)$ if K is bicomplete, all $(\mathcal{W}, \mathfrak{A} \times \mathfrak{B})$-assignments ς, and all $(\mathcal{V}, \Psi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma))$-assignments ρ we have*

$$[\![\varphi]\!](\Psi^\star(\mathfrak{A} \times \mathfrak{B}, \varsigma), \rho) = \langle\!\langle E_\varphi \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathfrak{A}, (\varsigma \cup \rho^\#)\restriction_{\mathfrak{A}}), [\![\bar{\varphi}^2]\!](\mathfrak{B}, (\varsigma \cup \rho^\#)\restriction_{\mathfrak{B}})).$$

Proof. Theorems 45 and 46 are immediate by first applying Lemma 44 and then Theorem 20 or Theorem 21, respectively, while treating $\texttt{def}$ as a relation symbol. $\square$

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