A Feferman-Vaught Decomposition Theorem for Weighted MSO Logic

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QuantLA

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Feferman-Vaught theorem

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$\Uparrow$

combine answers

$\nearrow$ $\nwarrow$

questions about $\mathcal{A}$ questions about $\mathcal{B}$

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$\mathcal{A} = (A, \mathcal{I})$	σ-structure
A	finite universe
$\mathcal{I}(R) \subseteq A^{\mathsf{ar}(R)}$ $(R \in \mathsf{Rel})$	interpretation

Disjoint union $\mathcal{A} \sqcup \mathcal{B}$ of σ-structures

$A \sqcup B$	universe
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MSO(σ) logic

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Propositional formulas Prop

$$P ::= x_i \mid y_i \mid P \vee P \mid P \wedge P$$

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such that for all structures $\mathcal{A}$,$\mathcal{B}$

$\mathcal{A} \sqcup \mathcal{B} \models \beta$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \mathtt{true}$

where $x_i = \mathtt{true}$ iff $\beta_i^{\mathbb{1}} \models \mathcal{A}$

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$[\![\bigoplus x. \bigoplus y.\mathrm{edge}(x, y)]\!]$ $=$ number of edges

Weighted Logics and Expressions

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$\langle\!\langle E \rangle\!\rangle : S^n \times S^n \to S$

$$\langle\!\langle x_1 \oplus y_2 \rangle\!\rangle(\bar{s}, \bar{t}) = s_1 + t_2$$

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$$\bar{\varphi}^{1} = \bar{\varphi}^{2} = (\varphi_{|b-b|}, \varphi_{|a|}) \qquad E = (x_1 \oplus y_1) \otimes (x_2 \oplus y_2)$$

Results of the paper: overview

- restrictions on product quantifiers necessary

  counterexamples exist

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- specific semirings no restrictions

  De Morgan algebras, locally finite semirings

Restriction

$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \otimes \psi$$
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fails for

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Ramsey Theorem

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$$\left[\frac{X}{2} \right] = \{\text{subsets } T \subseteq X \text{ with } |T| = 2\}$$

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Then

$$\exists Y \subseteq X \text{ infinite} \quad \text{with} \quad f\restriction_{\left[{Y \atop 2}\right]} \equiv \text{constant}$$

assume

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$$E = \bigoplus_{i=1}^{k} \left(x_1^{g_{1,i}} \otimes \ldots \otimes x_n^{g_{n,i}} \otimes y_1^{h_{1,i}} \otimes \ldots \otimes y_n^{h_{n,i}} \right) \qquad \text{wlog}$$

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$$a_{li} = ([\![\bar{\varphi}_1^1]\!](\mathcal{S}_l))^{g_{1,i}} \otimes \dots \otimes ([\![\bar{\varphi}_n^1]\!](\mathcal{S}_l))^{g_{n,i}}$$

$$b_{mi} = ([\![\bar{\varphi}_1^2]\!](\mathcal{S}_m))^{h_{1,i}} \otimes \dots \otimes ([\![\bar{\varphi}_n^2]\!](\mathcal{S}_m))^{h_{n,i}}$$

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Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

assume

$$[\![\bigotimes x.\bigotimes y.1]\!](\mathcal{S}_l \sqcup \mathcal{S}_m) = \langle\!\langle E\rangle\!\rangle([\![\bar{\varphi}^{1}]\!](\mathcal{S}_l), [\![\bar{\varphi}^{2}]\!](\mathcal{S}_m)) \qquad \forall l, m$$

$$\mathcal{S}_l = (\{1, \ldots, l\}, \emptyset)$$

$$E = \bigoplus_{i=1}^{k} \left(x_1^{g_{1,i}} \otimes \ldots \otimes x_n^{g_{n,i}} \otimes y_1^{h_{1,i}} \otimes \ldots \otimes y_n^{h_{n,i}}\right) \qquad \text{wlog}$$

$$a_{li} = ([\![\bar{\varphi}_1^{1}]\!](\mathcal{S}_l))^{g_{1,i}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^{1}]\!](\mathcal{S}_l))^{g_{n,i}}$$

$$b_{mi} = ([\![\bar{\varphi}_1^{2}]\!](\mathcal{S}_m))^{h_{1,i}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^{2}]\!](\mathcal{S}_m))^{h_{n,i}}$$

$$(l+m)^2 = [\![\bigotimes x.\bigotimes y.1]\!](\mathcal{S}_l \sqcup \mathcal{S}_m) = \min_{i=1}^{k} a_{li} + b_{mi}$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

assume

$$[\![\bigotimes x. \bigotimes y.1]\!](\mathcal{S}_l \sqcup \mathcal{S}_m) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{S}_l), [\![\bar{\varphi}^2]\!](\mathcal{S}_m)) \qquad \forall l, m$$

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$$a_{li} = ([\![\bar{\varphi}_1^1]\!](\mathcal{S}_l))^{g_{1,i}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^1]\!](\mathcal{S}_l))^{g_{n,i}}$$
$$b_{mi} = ([\![\bar{\varphi}_1^2]\!](\mathcal{S}_m))^{h_{1,i}} \otimes \ldots \otimes ([\![\bar{\varphi}_n^2]\!](\mathcal{S}_m))^{h_{n,i}}$$

$$(l+m)^2 = [\![\bigotimes x. \bigotimes y.1]\!](\mathcal{S}_l \sqcup \mathcal{S}_m) = \min_{i=1}^{k} a_{li} + b_{mi}$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \textsc{min}, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$

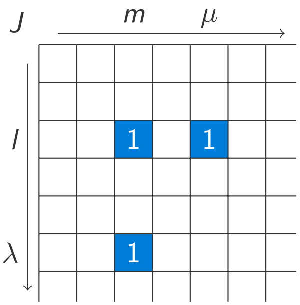
J (m →, l ↓)

2	2	1	1	3	1
2	1	2	3	2	2
1	3	1	2	1	2
3	2	1	2	1	3
1	2	3	2	2	1
1	2	1	3	1	2

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} \; a_{li} + b_{mi} \qquad \forall l, m$$

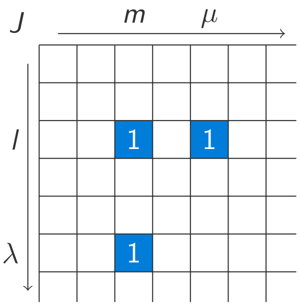
choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

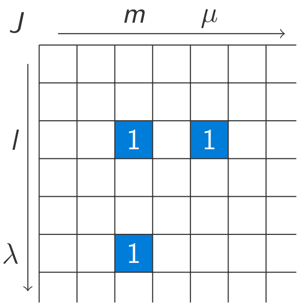
choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$

$$(l+m)^2 = a_{l1} + b_{m1}$$



$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



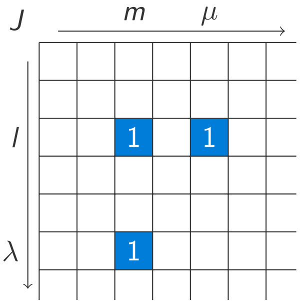
$$(l+m)^2 = a_{l1} + b_{m1}$$

$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



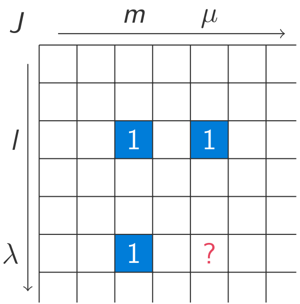
$$(l+m)^2 = a_{l1} + b_{m1}$$

$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



$$(l+m)^2 = a_{l1} + b_{m1}$$

$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

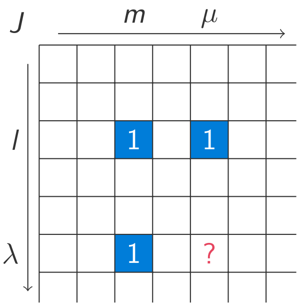
$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$(\lambda+\mu)^2$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} \; a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



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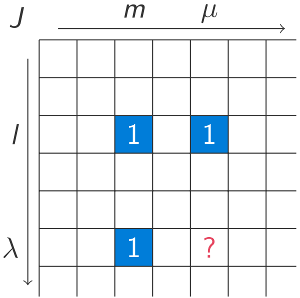
$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$(\lambda+\mu)^2 \leq a_{\lambda 1} + b_{\mu 1}$$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



$$(l+m)^2 = a_{l1} + b_{m1}$$

$$(\lambda+m)^2 = a_{\lambda 1} + b_{m1}$$

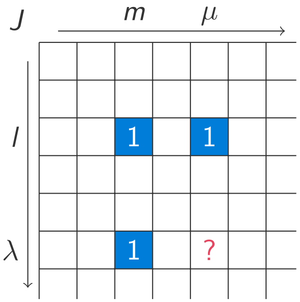
$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$(\lambda+\mu)^2 \le a_{\lambda 1} + b_{\mu 1}$$
$$= (\lambda+m)^2 - b_{m1} + (l+\mu)^2 - a_{l1}$$

RESTRICTION: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



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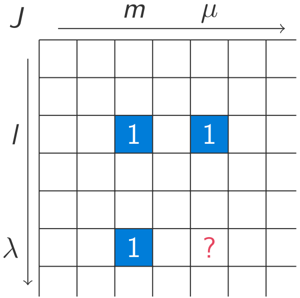
$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$\begin{aligned}(\lambda+\mu)^2 &\leq a_{\lambda 1} + b_{\mu 1} \\ &= (\lambda+m)^2 - b_{m1} + (l+\mu)^2 - a_{l1} \\ &= (\lambda+m)^2 + (l+\mu)^2 - (l+m)^2\end{aligned}$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} \; a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



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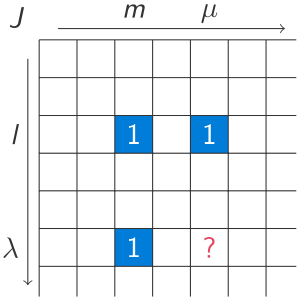
$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$\begin{aligned}(\lambda+\mu)^2 &\leq a_{\lambda 1} + b_{\mu 1} \\ &= (\lambda+m)^2 - b_{m1} + (l+\mu)^2 - a_{l1} \\ &= (\lambda+m)^2 + (l+\mu)^2 - (l+m)^2 \\ &\vdots \\ &= (\lambda+\mu)^2 - 2(\lambda-l)(\mu-m)\end{aligned}$$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

$$(l+m)^2 = \min_{i=1}^{k} a_{li} + b_{mi} \qquad \forall l, m$$

choose j_{lm} with $(l+m)^2 = a_{l j_{lm}} + b_{m j_{lm}}$



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$$(l+\mu)^2 = a_{l1} + b_{\mu 1}$$

$$\begin{aligned}
(\lambda+\mu)^2 &\leq a_{\lambda 1} + b_{\mu 1} \\
&= (\lambda+m)^2 - b_{m1} + (l+\mu)^2 - a_{l1} \\
&= (\lambda+m)^2 + (l+\mu)^2 - (l+m)^2 \\
&\vdots \\
&< (\lambda+\mu)^2
\end{aligned}$$

$$(l+m)^2 = \min_{i=1}^{k} \; a_{li} + b_{mi} \qquad \forall l, m$$

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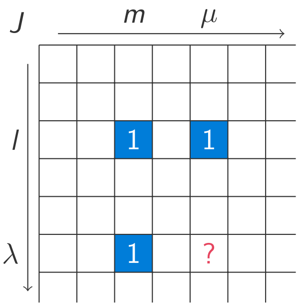

Ramsey: define $f : \binom{\mathbb{N}}{2} \to \{1, \ldots, k\}$

$\{l, m\} \mapsto j_{lm}$ for $l < m$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

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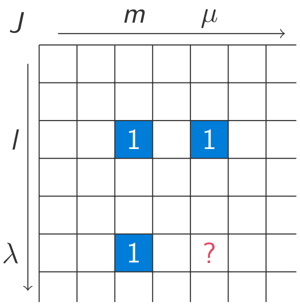

Ramsey: define $f\colon \binom{\mathbb{N}}{2} \to \{1, \dots, k\}$

$\{l, m\} \mapsto j_{lm}$ for $l < m$

Let

$Y \subseteq \mathbb{N}$ infinite with $f\restriction_{\binom{Y}{2}} \equiv j$

Restriction: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$

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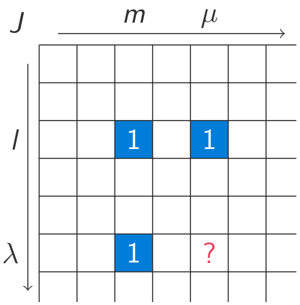

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$l < \lambda < m < \mu \in Y$

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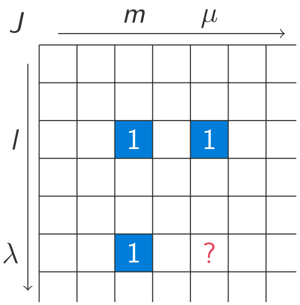

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$\implies$ contradiction