Weighted Simple Reset Pushdown Automata

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Abstract

We define a new normal form for weighted pushdown automata. The new type of automaton uses a stack but has only limited access to it. Only three stack commands are available: popping a symbol, pushing a symbol or leaving the stack unaltered. Additionally, ϵ-transitions are not used. We prove that this automaton model can recognize all weighted context-free languages (i.e., generates all algebraic power series).

Keywords: context-free languages, pushdown automata, weighted automata

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1. Introduction

Context-free languages model many aspects of programming languages. Adding weights allows a quantitative view of these aspects. Weighted pushdown automata were introduced by Kuich, Salomaa [11] where many fundamental results are established. A survey on weighted pushdown automata and their series is given in Petre, Salomaa [12]. In [7], a Chomsky-Schützenberger type result for weighted pushdown automata was established. Recently, in [6], a weighted logic with the same expressive power as weighted pushdown automata was developed.

Some applications need a specialized automaton model. In [4], to prove the equivalence of ω-context-free languages and a logical formalism, we use

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a restricted pushdown automaton. This restricted version recognizes all ω-context-free languages and behaves like some kind of normal form for pushdown automata. These restricted pushdown automata do not allow ϵ-transitions and use the stack differently. Computations start and end with an empty pushdown tape. Additionally, only three stack commands are used: popping a symbol, pushing a symbol or leaving the stack unaltered. Note that therefore, it is only possible to read the topmost stack symbol by popping it.

Even though an equivalent idea is used in [2] to prove that pushdown automata can recognize the projections of nested words, it seems that this restricted model has not attracted the attention it deserves.

The goal of this paper is to establish a weighted model of such pushdown automata on finite words. This extension shows that the basic model is very natural. Furthermore, it will be needed for an equivalence result between weighted ω-context-free languages and weighted logical formalisms for infinite words (which is currently work in progress).

The paper is structured as follows. Section 2 explains some basics and introduces pushdown matrices that will later be used similarly to an adjacency matrix of the graph representing an automaton.

For weighted pushdown automata there exists already the notion of a reset pushdown automaton (cf. [11]) that starts and ends with an empty pushdown tape and that naturally allows to push onto an empty tape. We define the corresponding reset pushdown matrices in Section 3.

The restrictions we discussed above are defined as simple reset pushdown matrices in Section 4. Here we also prove some basic properties for these matrices.

The last section, Section 5, defines how the matrix is used in a simple reset pushdown automaton. Afterwards, we prove that simple reset pushdown automata generate all algebraic power series (i.e., weighted context-free languages, cf. [11]). The proof starts with algebraic systems (cf. [11]) in Greibach normal form and constructs for every such system an equivalent simple reset pushdown automaton. Additionally, we introduce a new normal form for algebraic power series.

2. Preliminaries

For the convenience of the reader, we quote definitions and results from Ésik, Kuich [9].

A semiring S is called *complete* if it has sum operations for all families $(a_i \mid i \in I)$ of elements of S, where I is an arbitrary index set, such that the following conditions are satisfied (see Conway [3], Eilenberg [8], Kuich [10]):

(i) $\sum_{i\in\emptyset} a_i = 0, \qquad \sum_{i\in\{j\}} a_i = a_j, \qquad \sum_{i\in\{j,k\}} a_i = a_j + a_k \text{ for } j \neq k\,,$

(ii) $\sum_{j\in J} \big(\sum_{i\in I_j} a_i\big) = \sum_{i\in I} a_i\,, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j'\,,$

(iii) $\sum_{i\in I} (c\cdot a_i) = c\cdot\big(\sum_{i\in I} a_i\big), \qquad \sum_{i\in I}(a_i\cdot c) = \big(\sum_{i\in I} a_i\big)\cdot c\,.$

This means that a semiring S is complete if it has "infinite sums" (i) that are an extension of the finite sums, (ii) that are associative and commutative and (iii) that satisfy the distribution laws.

A semiring S equipped with an additional unary star operation $^* : S \to S$ is called a *starsemiring*. In complete semirings for each element a, the *star* a^* of a is defined by

$$a^* = \sum_{j\geq 0} a^j\,.$$

Hence, each complete semiring is a starsemiring, called a *complete starsemiring*.

Following Kuich, Salomaa [11] and Kuich [10], we introduce pushdown transitions matrices (see also Ésik, Kuich [9]). Let Γ be an alphabet, called *pushdown alphabet* and let $n \geq 1$. A matrix $\bar{M} \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is termed a *pushdown matrix* (with *pushdown alphabet* Γ and *state set* $\{1,\dots,n\}$) if

(i) for each $p \in \Gamma$ there exist only finitely many blocks $\bar{M}_{p,\pi}$, $\pi \in \Gamma^*$, that are unequal to 0;

(ii) for all $\pi_1, \pi_2 \in \Gamma^*$,

$$\bar{M}_{\pi_1,\pi_2} = \begin{cases} \bar{M}_{p,\pi}, & \text{if there exist } p \in \Gamma, \pi, \pi' \in \Gamma^* \text{ with} \\ & \qquad\qquad \pi_1 = p\pi' \text{ and } \pi_2 = \pi\pi', \\ 0, & \text{otherwise.} \end{cases}$$

A matrix $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is called *row-finite* if $\{\pi' \mid M_{\pi,\pi'} \neq 0\}$ is finite for all $\pi \in \Gamma^*$. We denote the *identity matrix* by E.

Mappings r of Σ^* into S are called *series*. The collection of all such series r is denoted by $S\langle\langle\Sigma^*\rangle\rangle$. We denote by $S\langle\Sigma\rangle$, $S\langle\{\epsilon\}\rangle$ and $S\langle\Sigma \cup \{\epsilon\}\rangle$ the polynomials with support in Σ, $\{\epsilon\}$ and $\Sigma \cup \{\epsilon\}$, respectively. See [11], p. 7, for more information.

3. Reset Pushdown Matrices

In the sequel, *S is assumed to be a complete starsemiring.*

Let Γ be a pushdown alphabet and $\{1, \dots, n\}$ for $n \geq 1$ be a set of states.

A *reset matrix* $M_R \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is a row-finite matrix such that

$$(M_R)_{\pi_1,\pi_2} = 0 \qquad \text{for } \pi_1, \pi_2 \in \Gamma^* \text{ with } \pi_1 \neq \epsilon\,.$$

A *reset pushdown matrix* $M \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ is the sum of a reset matrix M_R and a pushdown matrix $\bar{M}$,

$$M = M_R + \bar{M}\,.$$

Intuitively, the reset pushdown matrix behaves like a pushdown matrix (i.e., it models the LIFO property of the pushdown tape) but additionally inherits from reset matrices the ability to push symbols onto the empty pushdown tape.

For the convenience of the reader, we recall the following result.

Theorem 1 (Corollary 2 of [5])**.** *Let $\bar{M} \in (S^{n\times n})^{\Gamma^*\times\Gamma^*}$ be a pushdown matrix. Then, for all $\pi_1, \pi_2 \in \Gamma^*$,*

$$(\bar{M}^*)_{\pi_1\pi_2,\epsilon} = (\bar{M}^*)_{\pi_1,\epsilon}(\bar{M}^*)_{\pi_2,\epsilon}\,.$$

Now we show

Theorem 2. *Let $M = M_R + \bar{M}$ be a reset pushdown matrix. Then*

$$(M^*)_{\pi,\epsilon} = (\bar{M}^*)_{\pi,\epsilon}(M^*)_{\epsilon,\epsilon} \quad \textit{for } \pi \in \Gamma^*\,.$$

Proof. The proof is by induction on the length of π. The case $\pi = \epsilon$ is trivial. We assume that Theorem 2 is proven for $\pi \in \Gamma^*$ and derive it for $p\pi$ with

$p \in \Gamma$ as follows, where for $t = 1$ we have $M_{p\pi,\pi_1\pi} = M_{p\pi,\pi}$:

$$\begin{aligned}
(M^*)_{p\pi,\epsilon} &= \sum_{t\geq 1} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^+} M_{p\pi,\pi_1\pi} M_{\pi_1\pi,\pi_2\pi} \cdots M_{\pi_{t-1}\pi,\pi} (M^*)_{\pi,\epsilon} \\
&= \sum_{t\geq 1} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^+} M_{p,\pi_1} M_{\pi_1,\pi_2} \cdots M_{\pi_{t-1},\epsilon} (M^*)_{\pi,\epsilon} \\
&= \Big(\sum_{t\geq 1} (\bar{M}^t)_{p,\epsilon}\Big)(\bar{M}^*)_{\pi,\epsilon}(M^*)_{\epsilon,\epsilon} \\
&= (\bar{M}^*)_{p,\epsilon}(\bar{M}^*)_{\pi,\epsilon}(M^*)_{\epsilon,\epsilon} \\
&= (\bar{M}^*)_{p\pi,\epsilon}(M^*)_{\epsilon,\epsilon} \,.
\end{aligned}$$

The last equality above is implied by Theorem 1. □

Corollary 3. *Let $M = M_R + \bar{M}$ be a reset pushdown matrix. Then*

$$(M^*)_{p_1\ldots p_k,\epsilon} = (\bar{M}^*)_{p_1,\epsilon} \cdots (\bar{M}^*)_{p_k,\epsilon}(M^*)_{\epsilon,\epsilon} \,,$$

for $p_1, \ldots, p_k \in \Gamma$ ($k \geq 0$).

Theorem 4. *Let $M = M_R + \bar{M}$ be a reset pushdown matrix. Then the $S^{n\times n}$-algebraic system with variables $x_\epsilon, \bar{x}_p$ ($p \in \Gamma$)*

$$\begin{aligned}
x_\epsilon &= \sum_{\pi\in\Gamma^*} M_{\epsilon,\pi}\bar{x}_\pi x_\epsilon + E \,, \\
\bar{x}_p &= \sum_{\pi\in\Gamma^*} \bar{M}_{p,\pi}\bar{x}_\pi, \quad p \in \Gamma \,,
\end{aligned}$$

where $\bar{x}_\epsilon = E, \bar{x}_{p\pi} = \bar{x}_p\bar{x}_\pi$ for $p \in \Gamma$ and $\pi \in \Gamma^*$,
has a solution

$$x_\epsilon = (M^*)_{\epsilon,\epsilon}, \bar{x}_p = (\bar{M}^*)_{p,\epsilon}, \quad p \in \Gamma \,.$$

Proof. By Theorem 2, we obtain

$$\begin{aligned}
(M^*)_{\epsilon,\epsilon} &= \sum_{\pi\in\Gamma^*} M_{\epsilon,\pi}(M^*)_{\pi,\epsilon} + E \\
&= \sum_{\pi\in\Gamma^*} M_{\epsilon,\pi}(\bar{M}^*)_{\pi,\epsilon}(M^*)_{\epsilon,\epsilon} + E \,,
\end{aligned}$$

and

$$(\bar{M}^*)_{p,\epsilon} = \sum_{\pi\in\Gamma^*} \bar{M}_{p,\pi}(\bar{M}^*)_{\pi,\epsilon} \,.$$

By Theorem 1, we have $\bar{x}_\pi = (\bar{M}^*)_{\pi,\epsilon}$ for each $\pi \in \Gamma^*$. The result follows. □

Corollary 5. *Let S be a commutative complete starsemiring and Σ be an alphabet. If $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ is a reset pushdown matrix, then the algebraic system of Theorem 4 has a unique solution.*

Proof. The algebraic system is strict and thus has a unique solution; see [11], p. 302, for details. □

We denote by $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$ the collection of *algebraic series*. See [10], pp. 622-623, for details.

Corollary 6. *Let S be a commutative complete starsemiring and Σ be an alphabet. If $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ is a reset pushdown matrix, then the components of the unique solution of the algebraic system of Theorem 4*

$$(M^*)_{\epsilon,\epsilon}, (\bar{M}^*)_{p,\epsilon}, \quad p \in \Gamma\,,$$

are in $(S^{alg}\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$.

Proof. This follows from the definition of $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$, see [10], pp. 622-623, for more information. □

4. Simple Reset Pushdown Matrices

For the rest of this paper, *the complete starsemiring S is additionally assumed to be commutative*; and Σ denotes an alphabet.

A reset pushdown matrix M is called *simple* if $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ for some $n \geq 1$ and for all $p, p_1 \in \Gamma$,

$$M_{\epsilon,p},\ M_{p,\epsilon},\ M_{p,p} = M_{\epsilon,\epsilon} \text{ and } M_{p,p_1p} = M_{\epsilon,p_1},$$

are the only blocks $M_{\pi,\pi'}$, where $\pi \in \{\epsilon, p\}$ and $\pi' \in \Gamma^*$, that may be unequal to the zero matrix 0. Hence, a simple reset pushdown matrix M is defined by its blocks $M_{\epsilon,\epsilon}$ and $M_{p,\epsilon}$, $M_{\epsilon,p}$ $(p \in \Gamma)$.

If M is a simple reset pushdown matrix then the algebraic system of Theorem 4 has the form (1)

$$\begin{aligned} x_\epsilon &= M_{\epsilon,\epsilon}x_\epsilon + \sum_{p\in\Gamma} M_{\epsilon,p}\bar{x}_p x_\epsilon + E\,,\\ \bar{x}_p &= \bar{M}_{p,\epsilon} + \bar{M}_{p,p}\bar{x}_p + \sum_{p_1\in\Gamma} \bar{M}_{p,p_1p}\bar{x}_{p_1}\bar{x}_p\,,\ \ p \in \Gamma. \end{aligned} \tag{1}$$

The variables of this system are $x_\epsilon, \bar{x}_p$ ($p \in \Gamma$). They are variables for matrices in $(S\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$.

Our next lemma states that for simple reset pushdown matrices, emptying the pushdown tape with contents p (i.e., applying $(\bar{M}^*)_{p,\epsilon}$) has the same effect as emptying first the pushdown tape with contents ϵ (i.e., applying $(M^*)_{\epsilon,\epsilon}$) and then reading p in a single move (i.e., applying $M_{p,\epsilon}$). This is due to the fact that p can not be replaced by any other pushdown symbol, but can only be erased. Note that the pushdown matrix $\bar{M}$ cannot continue calculations from the pushdown tape ϵ.

Lemma 7. *Let $M = M_R + \bar{M}$ be a simple reset pushdown matrix. Then*

$$(\bar{M}^*)_{p,\epsilon} = (M^*)_{\epsilon,\epsilon} M_{p,\epsilon} \,.$$

Proof. We have

$$\begin{aligned}
(\bar{M}^*)_{p,\epsilon} &= \sum_{t\geq 0} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} \bar{M}_{p,\pi_1 p} \cdots \bar{M}_{\pi_{t-1}p,p} \bar{M}_{p,\epsilon} \\
&= \sum_{t\geq 0} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{p,\pi_1 p} \cdots M_{\pi_{t-1}p,p} M_{p,\epsilon} \\
&= \Big(\sum_{t\geq 0} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{\epsilon,\pi_1} \cdots M_{\pi_{t-1},\epsilon}\Big) M_{p,\epsilon} \\
&= \sum_{t\geq 0} (M^t)_{\epsilon,\epsilon} M_{p,\epsilon} = (M^*)_{\epsilon,\epsilon} M_{p,\epsilon} \,.
\end{aligned}$$

For $t = 0$ and $t = 1$ the respective summands are $M_{p,\epsilon}$ and $M_{p,p}M_{p,\epsilon}$.

Observe that the bottom p can be never replaced by another pushdown symbol $p_1 \neq p$; it can only be emptied. Also observe that we use $M_{p,p} = M_{\epsilon,\epsilon}$ in the third equality. □

Our next lemma is similar to Lemma 7. This time, a simple reset pushdown matrix $(M^*)_{p,\epsilon})$ is considered. Therefore, in the end, it is possible to empty the pushdown tape with contents ϵ (i.e., apply $(M^*)_{\epsilon,\epsilon}$).

Lemma 8. *Let M be a simple reset pushdown matrix. Then*

$$(M^*)_{p,\epsilon} = (M^*)_{\epsilon,\epsilon} M_{p,\epsilon} (M^*)_{\epsilon,\epsilon}$$

Proof. We obtain

$$(M^*)_{p,\epsilon} = \sum_{t\geq 0} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{p,\pi_1 p} \cdots M_{\pi_{t-1}p,p} M_{p,\epsilon}(M^*)_{\epsilon,\epsilon}$$
$$= \Big(\sum_{t\geq 0} \sum_{\pi_1,\ldots,\pi_{t-1}\in\Gamma^*} M_{\epsilon,\pi_1} \cdots M_{\pi_{t-1},\epsilon}\Big) M_{p,\epsilon}(M^*)_{\epsilon,\epsilon}$$
$$= (M^*)_{\epsilon,\epsilon} M_{p,\epsilon}(M^*)_{\epsilon,\epsilon}\,. \qquad \square$$

Theorem 9. *Let M be a simple reset pushdown matrix. Then $(M^*)_{\epsilon,\epsilon}$ is the unique solution of*

$$x = M_{\epsilon,\epsilon}x + \sum_{p\in\Gamma} M_{\epsilon,p} x M_{p,\epsilon} x + E\,.$$

Proof. By Theorem 4, $((M^*)_{\epsilon,\epsilon}, ((\bar{M}^*)_{p,\epsilon})_{p\in\Gamma})$ is the solution of (1). Hence, we obtain by Lemma 7

$$(M^*)_{\epsilon,\epsilon} = M_{\epsilon,\epsilon}(M^*)_{\epsilon,\epsilon} + \sum_{p\in\Gamma} M_{\epsilon,p}(\bar{M}^*)_{p,\epsilon}(M^*)_{\epsilon,\epsilon} + E$$
$$= M_{\epsilon,\epsilon}(M^*)_{\epsilon,\epsilon} + \sum_{p\in\Gamma} M_{\epsilon,p}(M^*)_{\epsilon,\epsilon} M_{p,\epsilon}(M^*)_{\epsilon,\epsilon} + E$$

This proves that $(M^*)_{\epsilon,\epsilon}$ is a solution of the equation of our theorem. Since $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$, this equation is strict and thus has a unique solution. $\square$

Now we consider the $(S\langle\Sigma \cup \{\epsilon\}\rangle)^{n\times n}$-algebraic system of Theorem 9. Recall that the variable x is a variable for $(S\langle\langle\Sigma^*\rangle\rangle)^{n\times n}$. So we substitute the $n \times n$-matrix $X = (x_{i,j})_{1\leq i,j\leq n}$ of variables for $S\langle\langle\Sigma^*\rangle\rangle$ for the variable x and we get the strict $S\langle\Sigma \cup \{\epsilon\}\rangle$-algebraic system

$$X = M_{\epsilon,\epsilon}X + \sum_{p\in\Gamma} M_{\epsilon,p} X M_{p,\epsilon} X + E\,. \tag{2}$$

Let $Y = \{x_{i,j} \mid 1 \leq i,j \leq n\}$ be the set of the variables of (2). Then the support of the right sides of equations of (2) is contained in $\{\epsilon\}\cup\Sigma Y\cup\Sigma Y\Sigma Y$. Hence, this system is of Greibach normal form type and at the same time of operator normal form type (see Ésik, Kuich [9], Section 2.2.4).

5. Simple Reset Pushdown Automata

A reset pushdown automaton starts its computations with empty tape and finishes them with empty tape and final states.

A *reset pushdown automaton* (with input alphabet Σ) $\mathfrak{A} = (n, \Gamma, I, M, P)$ is given by

- a *set of states* $\{1, \ldots, n\}$, $n \geq 1$,
- a *pushdown alphabet* Γ,
- a reset pushdown matrix $M \in ((S\langle \Sigma \cup \{\epsilon\}\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$ called *transition matrix*,
- a row vector $I \in (S\langle\{\epsilon\}\rangle)^{1\times n}$, called *initial state vector*,
- a column vector $P \in (S\langle\{\epsilon\}\rangle)^{n\times 1}$, called *final state vector*.

The *behavior* $\|\mathfrak{A}\|$ of a reset pushdown automaton $\mathfrak{A}$ is defined by

$$\|\mathfrak{A}\| = I(M^*)_{\epsilon,\epsilon}P\,.$$

A reset pushdown automaton $\mathfrak{A} = (n, \Gamma, I, M, P)$ is called *simple* if M is a simple reset pushdown matrix.

Given a series $r \in S^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$, we want to construct a simple reset pushdown automaton with behavior r. By Theorems 5.10 and 5.4 of [10], r is a component of the unique solution of a strict algebraic system in binary Greibach normal form.

We first consider the algebraic power series r to have $(r, \epsilon) = 0$. So we assume without loss of generality that r is the x_1-component of the unique solution of the algebraic system (3) with variables $x_1, \ldots, x_n$

$$x_i = p_i, \quad 1 \leq i \leq n,$$

of the form

$$\begin{aligned} x_i = &\sum_{1\leq j,k\leq n}\sum_{a\in\Sigma}(p_i, ax_jx_k)ax_jx_k + \\ &\sum_{1\leq j\leq n}\sum_{a\in\Sigma}(p_i, ax_j)ax_j + \\ &\sum_{a\in\Sigma}(p_i, a)a\,. \end{aligned} \tag{3}$$

We now show the construction of the simple reset pushdown automaton $\mathfrak{A}_s = (n{+}1, \Gamma, I_s, M, P)$ for $1 \leq s \leq n$ with $r = \|\mathfrak{A}_1\|$:
We let $\Gamma = \{x_1, \ldots, x_n\}$; we also denote the state $n{+}1$ by f; the entries of M of the form $(M_{x_k,x_k})_{i,j}$, $(M_{x_k,\epsilon})_{i,j}$, $(M_{\epsilon,x_k})_{i,j}$, $(M_{\epsilon,\epsilon})_{i,j}$, $(M_{\epsilon,\epsilon})_{i,f}$ for $1 \leq i, j, k \leq n$, that may be unequal to 0 are

$$(M_{\epsilon,x_k})_{i,j} = \sum_{a\in\Sigma}(p_i, ax_jx_k)a\,,$$
$$(M_{x_k,x_k})_{i,j} = (M_{\epsilon,\epsilon})_{i,j} = \sum_{a\in\Sigma}(p_i, ax_j)a\,,$$
$$(M_{x_k,\epsilon})_{i,k} = (M_{x_k,x_k})_{i,f} = (M_{\epsilon,\epsilon})_{i,f} = \sum_{a\in\Sigma}(p_i, a)a\,;$$

we further put $(I_s)_s = \epsilon, (I_s)_i = 0$, for $1 \leq i \leq s-1$ and $s+1 \leq i \leq n+1$; finally let $P_f = \epsilon$ and $P_j = 0$ for $1 \leq j \leq n$.

Intuitively, the variables in the algebraic system are simulated by states in the simple reset pushdown automaton $\mathfrak{A}_s$. By the binary Greibach normal form, only two variables on the right-hand side are allowed. The first is modeled directly by changing the state, the second is pushed to the pushdown tape and the state is changed to it later when the variable is popped again. The special final state f will only be used as the last state.

Note that $(M_{x_k,x_k})_{i,f}$ allows to change to the final state with a non-empty pushdown tape. This is an artificial addition to fit the definition of simple reset pushdown matrices. Even though the automaton can enter the final state too early, it can not continue from there as it is a sink.

Observe that $\|\mathfrak{A}_s\| = I_s(M^*)_{\epsilon,\epsilon}P = ((M^*)_{\epsilon,\epsilon})_{s,f}$ for $1 \leq s \leq n$. Subsequently we will show that $\|\mathfrak{A}_1\| = r$.

This simple reset pushdown matrix M is called the simple pushdown matrix *induced* by the binary Greibach normal form (3). The simple reset pushdown automata $\mathfrak{A}_s$, $1 \leq s \leq n$, are called the simple reset pushdown automata *induced* by the binary Greibach normal form (3).

The next lemma formalizes the meaning of the pushdown tape for induced simple reset pushdown matrices. Intuitively, going from state j to the final state f and erasing the variable x_k from the pushdown tape on the way (i.e., applying $((M^*)_{x_k,\epsilon})_{j,f}$) has the same effect as first going from state j to the final state f without changing the pushdown tape (i.e., applying $((M^*)_{\epsilon,\epsilon})_{j,f}$) and then restarting in state k (i.e., applying $((M^*)_{\epsilon,\epsilon})_{k,f}$). This results from the definition of $\mathfrak{A}_s$: popping a variable from the pushdown tape and changing to its state has the same weight as changing to the final state instead. It allows the automaton to process the variables in the algebraic system individually.

Lemma 10. *Let M be a simple reset pushdown matrix induced by a binary Greibach normal form* (3)*. Then, for all $1 \leq j, k \leq n$,*

$$((M^*)_{x_k,\epsilon})_{j,f} = ((M^*)_{\epsilon,\epsilon})_{j,f}((M^*)_{\epsilon,\epsilon})_{k,f} \,.$$

Proof. We obtain

$$\begin{aligned} ((M^*)_{\epsilon,\epsilon})_{j,f} &= ((M^+)_{\epsilon,\epsilon})_{j,f} = ((M^*M)_{\epsilon,\epsilon})_{j,f} \\ &= \sum_{1\leq t_1\leq f} ((M^*)_{\epsilon,\epsilon})_{j,t_1}(M_{\epsilon,\epsilon})_{t_1,f} + \\ &\qquad\qquad \sum_{1\leq t_1\leq f}\sum_{1\leq t\leq n} ((M^*)_{\epsilon,x_t})_{j,t_1}(M_{x_t,\epsilon})_{t_1,f} \\ &= ((M^*)_{\epsilon,\epsilon}M_{\epsilon,\epsilon})_{j,f} \end{aligned}$$

since $(M_{x_t,\epsilon})_{t_1,f} = 0$ for all $1 \leq t_1 \leq f$ and $1 \leq t \leq n$ by our construction. We now obtain, by Lemma 8,

$$\begin{aligned} ((M^*)_{x_k,\epsilon})_{j,f} &= \sum_{1\leq t_1,t_2\leq f} ((M^*)_{\epsilon,\epsilon})_{j,t_1}(M_{x_k,\epsilon})_{t_1,t_2}((M^*)_{\epsilon,\epsilon})_{t_2,f} \\ &= \sum_{1\leq t_1\leq f} ((M^*)_{\epsilon,\epsilon})_{j,t_1}(M_{\epsilon,\epsilon})_{t_1,f}((M^*)_{\epsilon,\epsilon})_{k,f} \\ &= ((M^*)_{\epsilon,\epsilon}M_{\epsilon,\epsilon})_{j,f}((M^*)_{\epsilon,\epsilon})_{k,f} \\ &= ((M^*)_{\epsilon,\epsilon})_{j,f}((M^*)_{\epsilon,\epsilon})_{k,f} \,. \end{aligned}$$

The second equality is implied by the fact that

$$\begin{aligned} &(M_{x_k,\epsilon})_{t_1,k} = (M_{\epsilon,\epsilon})_{t_1,f} \text{ and} \\ &(M_{x_k,\epsilon})_{t_1,t_2} = 0 \text{ for } t_2 \neq k \,. \end{aligned}$$

□

Now we show that the constructed automata realize the algebraic system (3).

Theorem 11.

$$(\|\mathfrak{A}_1\|, \ldots, \|\mathfrak{A}_n\|) = (((M^*)_{\epsilon,\epsilon})_{1,f}, \ldots, ((M^*)_{\epsilon,\epsilon})_{n,f})$$

is the unique solution of the algebraic system (3)*. In particular, $r = \|\mathfrak{A}_1\|$.*

Proof. We obtain, for $1 \le i \le n$, by substituting into the right sides of (3) and by Lemma 10,

$$\begin{aligned}
&\sum_{1\le j,k\le n}\sum_{a\in\Sigma}(p_i a x_j x_k)a((M^*)_{\epsilon,\epsilon})_{j,f}((M^*)_{\epsilon,\epsilon})_{k,f} + \\
&\qquad\qquad \sum_{1\le j\le n}\sum_{a\in\Sigma}(p_i, a x_j)a((M^*)_{\epsilon,\epsilon})_{j,f} + \sum_{a\in\Sigma}(p_i, a)a \\
&= \sum_{1\le j,k\le n}(M_{\epsilon,x_k})_{i,j}((M^*)_{x_k,\epsilon})_{j,f} + \sum_{1\le j\le n}(M_{\epsilon,\epsilon})_{i,j}((M^*)_{\epsilon,\epsilon})_{j,f} + (M_{\epsilon,\epsilon})_{i,f} \\
&= \sum_{1\le k\le n}(M_{\epsilon,x_k}(M^*)_{x_k,\epsilon})_{i,f} + (M_{\epsilon,\epsilon}(M^*)_{\epsilon,\epsilon})_{i,f} \\
&= ((M^+)_{\epsilon,\epsilon})_{i,f} = ((M^*)_{\epsilon,\epsilon})_{i,f}\,.
\end{aligned}$$

Here in the second equality, we have replaced $(M_{\epsilon,\epsilon})_{i,f}$ by $(M_{\epsilon,\epsilon})_{i,f}((M^*)_{\epsilon,\epsilon})_{f,f}$, since $((M^*)_{\epsilon,\epsilon})_{f,f} = \epsilon$; also note that $(M_{\epsilon,x_t})_{i,f} = 0$. Since the algebraic system (3) is strict, it has a unique solution. In particular, $r = \|\mathfrak{A}_1\|$. $\square$

Note that the automaton $\mathfrak{A}_s$ used in Theorem 11 is induced by the Greibach normal form (3) for the series r with $(r,\epsilon) = 0$. We now consider the second case.

If we are given a series $r \in S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$, where $(r,\epsilon) \ne 0$, then we modify the reset pushdown automaton $\mathfrak{A}_1$ to obtain $\mathfrak{A}' = (n+2, \Gamma, I', M', P')$. The new state $n+2$ is an isolated state, i.e., no moves to $n+2$ or from $n+2$ are possible. This means that, for all $\pi_1, \pi_2 \in \Gamma^*$,

$$M'_{\pi_1,\pi_2} = \begin{pmatrix} M_{\pi_1,\pi_2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(M'^*)_{\pi_1,\pi_2} = \begin{pmatrix} (M^*)_{\pi_1,\pi_2} & 0 \\ 0 & \delta_{\pi_1,\pi_2} \end{pmatrix},$$

where δ_{π_1,π_2} is the Kronecker delta. Moreover let $I' = \begin{pmatrix} I_1 & \epsilon \end{pmatrix}$ and $P' = \begin{pmatrix} P \\ (r,\epsilon)\epsilon \end{pmatrix}$. Hence, we obtain

$$\|\mathfrak{A}'\| = I'(M'^*)_{\epsilon,\epsilon}P' = I_1(M^*)_{\epsilon,\epsilon}P + (r,\epsilon)\epsilon = r\,.$$

This proves

Corollary 12. *Let $r \in S^{alg}\langle\langle \Sigma^* \rangle\rangle$. Then there exists a simple reset pushdown automaton with behavior r.*

Theorem 11 in connection with Theorem 9 yields a new normal form for algebraic power series.

Theorem 13. *Let $r \in S^{alg}\langle\langle \Sigma^* \rangle\rangle$ with $(r, \epsilon) = 0$. Then for some $n \geq 2$, there exist matrices $M_0, M_{1,t}, M_{2,t} \in (S\langle\Sigma\rangle)^{n\times n}$, $1 \leq t \leq n-1$ such that r is the $(1,n)$-component of the unique solution of the algebraic system*

$$X = M_0 X + \sum_{1 \leq t \leq n-1} M_{1,t} X M_{2,t} X + E\,,$$

where X is an $n \times n$-matrix of variables.

Proof. Assume that r equals the first component of the unique solution of the algebraic system (3) with $n-1$ variables $x_1, \ldots, x_{n-1}$. Let $M \in ((S\langle\Sigma\rangle)^{n\times n})^{\Gamma^*\times\Gamma^*}$, with $\Gamma = \{x_1, \ldots, x_{n-1}\}$, be the simple pushdown matrix induced by (3). Then by Theorem 11, $((M^*)_{\epsilon,\epsilon})_{1,n}$ is the first component of the solution of (3) and $r = ((M^*)_{\epsilon,\epsilon})_{1,n}$.

By Theorem 9, $(M^*)_{\epsilon,\epsilon}$ is the solution of equation (2). Let now $M_0 = M_{\epsilon,\epsilon}$, $M_{1,t} = M_{\epsilon,x_t}$ and $M_{2,t} = M_{x_t,\epsilon}$ for $1 \leq t \leq n-1$.

Then equation (2) now reads

$$X = M_0 X + \sum_{1 \leq t \leq n-1} M_{1,t} X M_{2,t} X + E \tag{4}$$

and r is the $(1,n)$-component of its unique solution. □

In language theory, the *restricted Dyck languages* $D_n'^*$ $(n \geq 1)$ are formed of the words over n pairs of associated parentheses which are "well-formed" in the usual sense. Here a word is considered to be "well-formed" iff successive deletions of subwords of associated parentheses, say (,), [,], ... yield the empty word. By Theorem II. 3.7. of Berstel [1], $D_n'^*$ $(n \geq 1)$ is generated by the context-free grammar with productions

$$x \to \epsilon, \quad x \to a_k x \bar{a}_k x, \qquad 1 \leq k \leq n\,.$$

Here a_k and $\bar{a}_k$ are the pairs of associated parentheses. By Theorem VII. 1.2. of Berstel [1], any of the languages $D_n'^*$ $(n \geq 2)$ is a cone generator of the principal cone of context-free languages.

These results were transferred in Kuich, Salomaa [11] to algebraic power series over commutative semirings S. The *restricted Dyck series* $D_n'^*$ ($n \geq 1$) are now the unique solutions of the strict algebraic systems

$$x = \sum_{1 \leq k \leq n} a_k x \bar{a}_k x + \epsilon$$

and $D_2'^*$, and hence $D_n'^*$ for $n \geq 2$, are cone generators of the principal cone $S^{\mathrm{alg}}\langle\langle \Sigma_\infty^* \rangle\rangle$ of algebraic power series. (See Theorem 13.15 of Kuich, Salomaa [11].)

In Example 14.3 of Kuich, Salomaa [11], it is described how a "master system" generates a normal form for algebraic system. The "master system" generating equation (4) reads now, for a given $n \geq 2$,

$$x = ax + \sum_{1 \leq k \leq n-1} a_k x \bar{a}_k x + \epsilon \, .$$

The important difference to the normal form given in this Example 14.3 is that now all M-matrices contain no ϵ-terms.

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