Weighted Automata and Logics on Graphs

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Abstract. Weighted automata model quantitative features of the behavior of systems and have been investigated for various structures like words, trees, traces, pictures, and nested words. In this paper, we introduce a general model of weighted automata acting on graphs, which form a quantitative version of Thomas' unweighted model of graph acceptors. We derive a Nivat theorem for weighted graph automata which shows that their behaviors are precisely those obtainable from very particular weighted graph automata and unweighted graph acceptors with a few simple operations. We also show that a suitable weighted MSO logic is expressively equivalent to weighted graph automata. As a consequence, we obtain corresponding Büchi-type equivalence results known from the recent literature for weighted automata and weighted logics on words, trees, pictures, and nested words. Establishing such a general result has been an open problem for weighted logic for some time.

Keywords: quantitative automata, graphs, quantitative logic, weighted automata, Büchi, Nivat

1 Introduction

In automata theory, the fundamental Büchi-Elgot-Trakhtenbrot theorem [6, 16, 35] established the coincidence of regular languages with languages definable in monadic second order logic. This led both to practical applications, e.g. in verification of finite-state programs, and to important extensions covering, for example, trees [7, 28, 31], traces [32], pictures [18], nested words [1], and texts [21]; a general result for graphs was given in [33].

At the same time as Büchi, Schützenberger [30] introduced weighted finite automata and characterized their quantitative behaviors as rational formal power series. This model also quickly developed a flourishing theory (see [15, 29, 22, 3, 11]). Recently, Droste and Gastin [9] introduced a weighted MSO logic and showed its expressive equivalence to weighted automata. Soon, different authors extended this result to weighted automata and weighted logics on trees [14], traces [25], pictures [17], nested words [24, 8], and texts [23].

However, a general result for weighted automata and weighted logics covering graphs and linking the previous results remained, up to now, open. The main contributions of this paper are the following.

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- We establish a model of weighted automata on graphs, which extends both Thomas' graph acceptors [34] and the previous weighted automata models for words, trees, pictures, and others. We show that this enables us to model new quantitative properties of graphs which could not be expressed by the previous models.
- To show the robustness of our model, we extend a classical result of Nivat [27] to weighted automata on graphs, showing that their behaviors are exactly those which can be constructed from very particular weighted graph automata and recognizable graph languages using operations like morphisms and intersections.
- We derive a Büchi-type equivalence result for the expressive power of weighted automata and a suitable weighted logic on graphs. We obtain corresponding equivalence results for structures like words, trees, pictures, and nested words as a consequence.
- We show that if the underlying semiring of weights is idempotent, then both our Nivat and Büchi type equivalence results can be sharpened, but not in general.

We note that an interesting approach connecting pebble navigating weighted automata and weighted first-order logic was given in [5, 26]. The present automata model is different, using tiles of graphs.

In our proofs, in comparison to the situation for words, trees, or pictures, several difficulties arise. The crucial difference to previous approaches is a global acceptance condition in form of a check of occurrence numbers of finite tiles in runs of our automata on graphs; we need this condition to connect logic and automata. Furthermore, since we are dealing with graphs, the underlying unweighted automata model cannot be determinized. Accordingly, the unweighted logic subfragment covers only existential MSO. Also, the closure under weighted universal quantifications requires new methods; here we employ a theorem of Thomas [34] whose proof in turn relied on Hanf's theorem [20]. Finally, in contrast to words, trees, pictures, or nested words, our general graphs do not have a distinguished vertex, which yields technical complications.

All our constructions of weighted graph automata are effective. For technical simplicity, here we assume that the weights stem from a commutative semiring. Several constructions would also work for general semirings or for e.g. average computations of weights; this will be later work. It is also tempting to investigate now how the rich field of weighted graph algorithms could be utilized, e.g., for developing decision procedures for weighted graph automata.

2 Graph Acceptors

In this section, we introduce the basic concepts around graphs and graph acceptors. Following [34], we define a *(directed) graph* as a relational structure $G = (V, (P_a)_{a \in A}, (E_b)_{b \in B})$ over two finite alphabets A and B, where V is the set of *vertices*, the sets P_a ($a \in A$) form a partition of V, and the sets E_b ($b \in B$) are disjoint irreflexive binary relations on V, called *edges*. We denote with $E = \bigcup_{b\in B} E_b$ the set of all edges. A graph is *bounded by* t if every vertex has an (in- plus out-) degree smaller than or equal to t. We denote with $\mathrm{DG}_t(A,B)$ the class of all finite directed graphs over A and B bounded by t.

We call a class of graphs *pointed* if every graph G of this class is *pointed with a vertex* v, i.e. it has a unique designated vertex $v \in V$. Formally, this assumption can be defined by adding a unary relation *center* to G with center $= \{v\}$.

Let $r \geq 0$. We say the distance of u and v is at most r if there exists a path $(u = u_0, u_1, ..., u_j = v)$ with $j \leq r$ and $(u_i, u_{i+1}) \in E$ or $(u_{i+1}, u_i) \in E$ for all $i < j$. We denote with $\mathrm{sph}^r(G,v)$, the *r-sphere of* G *around the vertex* v, the unique subgraph of G pointed with v and consisting of all vertices with a distance to v of at most r, together with their edges. We call $\tau = (G,v)$ an *r-tile* if $(G,v) = \mathrm{sph}^r(G,v)$, and may omit the r if the context is clear. The bound t ensures that there exists only finitely many pairwise non-isomorphic r-tiles.

Definition 1 ([33, 34]). *A* graph acceptor (GA) $\mathcal{A}$ *over the alphabets* A *and* B *is defined as a quadruple* $\mathcal{A} = (Q, \Delta, \mathrm{Occ}, r)$ *where*

- Q *is a finite set of states,*
- $r \in \mathbb{N}_0$*, the* tile-size,
- Δ *is a finite set of pairwise non-isomorphic* r*-tiles over* $A \times Q$ *and* B*,*
- Occ*, the* occurrence constraint*, is a boolean combination of formulas "*$\mathrm{occ}(\tau) \geq n$*", where* $n \in \mathbb{N}$ *and* $\tau \in \Delta$

Given a graph G of $\mathrm{DG}_t(A,B)$ and a mapping $\rho : V \to Q$, we consider the Q-labeled graph $G_\rho \in \mathrm{DG}_t(A \times Q, B)$, obtained by labeling every vertex $v \in V$ also with $\rho(v)$. We call ρ a *run (or tiling) of* $\mathcal{A}$ *on* G if for every $v \in V$, $\mathrm{sph}^r(G_\rho, v)$ is isomorphic to a tile in Δ.

We say G_ρ *satisfies* $\mathrm{occ}(\tau) \geq n$ if there exist at least n vertices $v \in V$ such that $\mathrm{sph}^r(G_\rho, v)$ is isomorphic to τ. The semantics of G_ρ *satisfies* Occ is then defined in the usual way. We call a run ρ *accepting* if G_ρ satisfies the constraint Occ. We let $L(\mathcal{A}) = \{G \in \mathrm{DG}_t(A,B) \mid \text{there exists an accepting run } \rho : V \to Q \text{ of } \mathcal{A} \text{ on } G\}$, the *language accepted by* $\mathcal{A}$. We call a language $L \subseteq \mathrm{DG}_t(A,B)$ *recognizable* if $L = L(\mathcal{A})$ for some GA $\mathcal{A}$.

Next, we introduce the logic $\mathrm{MSO}(\mathrm{DG}_t(A,B))$, short MSO, cf. [34]. We denote with $x, y, ...$ and $X, Y, ...$ first- and second-order variables ranging over vertices, resp. over sets of vertices. The formulas of MSO are defined inductively by

$$\varphi ::= P_a(x) \mid E_b(x,y) \mid x = y \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where $a \in A$ and $b \in B$. An *FO-formula* is a formula of MSO without $\exists X$. An *EMSO-formula* is a formula of the form $\exists X_1 ... \exists X_k.\varphi$ where φ is an FO-formula.

The satisfaction relation $\models$ for graphs and MSO-sentences is defined in the natural way. For a sentence $\varphi \in$ MSO, we define the *language of* φ as $L(\varphi) = \{G \in \mathrm{DG}_t(A,B) \mid G \models \varphi\}$. We call a language $L \subseteq \mathrm{DG}_t(A,B)$ *MSO-* (resp. *FO-*) *definable* if $L = L(\varphi)$ for some MSO- (resp. FO-) sentence φ.

Theorem 2 ([34]). *Let* $L \subseteq \mathrm{DG}_t(A,B)$ *be a set of graphs. Then:*

1. L *is recognizable by a one-state GA iff* L *is definable by an FO-sentence.*
2. L *is recognizable iff* L *is definable by an EMSO-sentence.*

3 Weighted Graph Automata

In this section, we introduce and investigate a quantitative version of graph acceptors. In the following, *let* $\mathbb{K} = (K, +, \cdot, 0, 1)$ *be a commutative semiring*, i.e. $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids, $(x+y) \cdot z = x \cdot z + y \cdot z$ and $0 \cdot x = x \cdot 0 = 0$ for all $x, y, z \in K$. Important examples of commutative semirings are the Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$, the semiring of the natural numbers $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$, and the tropical semirings $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ and $\mathbb{R}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.

We say $\mathbb{K}$ is *idempotent* if the addition is idempotent, i.e. $x + x = x$ for all $x \in K$. The semirings $\mathbb{B}$, $\mathbb{R}_{\max}$, and $\mathbb{R}_{\min}$ are idempotent. For a general introduction into the theory of semirings and extensive examples, see [11, 19].

Definition 3. *A* weighted graph automaton (or weighted graph acceptor; wGA) *over A, B, and $\mathbb{K}$ is a tuple $\mathcal{A} = (Q, \Delta, \mathrm{wt}, \mathrm{Occ}, r)$ where*

- $\mathcal{A}' = (Q, \Delta, \mathrm{Occ}, r)$ *is a graph acceptor over the alphabets A and B,*
- $\mathrm{wt} : \Delta \to K$ *is the weight function assigning to every tile of Δ a value of K.*

An *accepting run* $\rho : V \to Q$ *of $\mathcal{A}$ on G* is defined as an accepting run of $\mathcal{A}'$ on G. Let $\mathrm{sph}^r_{\mathcal{A}}(G_\rho, v)$ be the tile of Δ which is isomorphic to $\mathrm{sph}^r(G_\rho, v)$. We let

$$\mathrm{wt}_{\mathcal{A},G,\rho}(v) = \mathrm{wt}(\mathrm{sph}^r_{\mathcal{A}}(G_\rho, v)) \ .$$

We define the weight $\mathrm{wt}_{\mathcal{A},G}(\rho)$ of the run ρ of $\mathcal{A}$ on G as

$$\mathrm{wt}_{\mathcal{A},G}(\rho) = \prod_{v \in V} \mathrm{wt}_{\mathcal{A},G,\rho}(v) \ .$$

The behavior $\|\mathcal{A}\| : \mathrm{DG}_t(A, B) \to K$ of $\mathcal{A}$ is defined, for each $G \in \mathrm{DG}_t(A, B)$, as

$$\|\mathcal{A}\|(G) = \sum_{\rho \text{ accepting run of } \mathcal{A} \text{ on } G} \mathrm{wt}_{\mathcal{A},G}(\rho) \ .$$

We call any function $S : \mathrm{DG}_t(A, B) \to K$ a *series*. Then S is *recognizable* if $S = \|\mathcal{A}\|$ for some wGA $\mathcal{A}$.

By the usual identification of languages with functions assuming values in $\{0, 1\}$, we see that graph acceptors are expressively equivalent to weighted graph automata over the Boolean semiring $\mathbb{B}$.

Example 4. The following wGA $\mathcal{A}$ counts the number of connected components as exponent of 2. We define $\mathcal{A} = (Q, \Delta, \mathrm{wt}, \mathrm{Occ}, r)$ over arbitrary alphabets A and B, and the semiring $\mathbb{K} = (\mathbb{N}, +, \cdot, 0, 1)$. We set $r = 1$, $\mathrm{Occ} = \mathit{true}$, $\mathrm{wt} \equiv 1$, and $Q = \{q_1, q_2\}$. The set of tiles is defined as $\Delta = \Delta_1 \cup \Delta_2$, where

$$\Delta_i = \{\tau \mid \text{every vertex of } \tau \text{ is labeled with some } (a, q_i), a \in A\}, \ i \in \{1, 2\} \ .$$

Then every connected component of a given graph G is tiled either completely with q_1 or completely with q_2, thus $\|\mathcal{A}\|(G) = 2^{m(G)}$, where $m(G)$ is the number of connected components of G.

To count occurrences of tiles with a certain pattern, we introduce the following notation. Let τ^* be a finite set of tiles enumerated by $\tau^* = \{\tau_1, ..., \tau_m\}$. For $N \in \mathbb{N}$, we define the formula

$$\big(\sum_{\tau \in \tau^*} \mathrm{occ}(\tau)\big) \geq N \quad \text{as} \quad \bigvee_{\substack{\sum_{i=1}^m n_i = N \\ n_i \in \{0,...,N\}}} \; \bigwedge_{i=1,...,m} \mathrm{occ}(\tau_i) \geq n_i \ . \tag{1}$$

Using this formula, we can prove the following.

Lemma 5. *Let $S : \mathrm{DG}_t(A, B) \to K$ be a series recognizable by a wGA $\mathcal{A}$ with tile-size s. Then for all $r \geq s$, S is recognizable by a wGA $\mathcal{B}$ with tile-size r.*

Example 6. Using a weight function which applies the degree of the center of a tile as weight, we can construct a wGA $\mathcal{A}_1$ satisfying $\|\mathcal{A}_1\|(G) = \prod_{v \in V} \mathrm{degree}(v)$. Using formula (1), we can also construct a wGA $\mathcal{A}_2$ with $\|\mathcal{A}_2\|(G) = \sum_{v \in V} \mathrm{degree}(v)$. In both cases, we are free to choose a semiring with the desired product or summation, and adjusting wt, we are able to only multiply or sum over vertices of a certain form, e.g., only over vertices labeled with a.

Example 7. The following wGA computes the 'weighted diameter' (i.e. the maximal distance between two vertices) of edge-weighted graphs up to a threshold $N \in \mathbb{N}$. Let A be a finite set and $B = \{1, ..., N\}$ be the edge labels. Note that, here, we sum over the edge labels when computing shortest paths between vertices.

We define $\mathcal{A} = (Q, \Delta, \mathrm{wt}, \mathrm{Occ}, 1)$ over A, B, and $\mathbb{R}_{\max}$ as follows. We set $Q = \{0, ..., N\} \times \{0, 1\}$. For a vertex v labeled also with $(q, m) \in Q$, we refer to q as the *state* of v, and say v is *marked* if $m = 1$.

Let $\tau = (H, v)$ be a 1-tile with state k at the center v. Then Δ checks that every vertex connected to v by an edge i has a state between $\max\{0, k - i\}$ and $\min\{k + i, N\}$. Furthermore, Δ checks that whenever $k \notin \{0, N\}$, then there has to be at least one vertex in τ which has state $k - i$. The weight function wt assigns to τ the weight k if v is marked, and 0 otherwise. Finally, Occ checks that we have exactly one vertex y with state 0 and exactly one marked vertex z.

Then we can show by induction that every state of a vertex has to be equal to the distance of this vertex to y up to the threshold N. Furthermore, every run has the weight of the vertex z. Thus, $\mathcal{A}$ computes the maximum distance between two vertices up to the threshold N, which is our desired property. Here, the distance takes the edge-labels into account, whereas setting $B = \{1\}$ yields the unweighted setting and the *geodesic distance* to y up to the threshold N.

For a class $\mathcal{C}$ of graphs, we call the semiring $\mathbb{K}$ *$\mathcal{C}$-regular* if for every $k \in K$, there exists a wGA $\mathcal{A}_k$ with $\|\mathcal{A}_k\|(G) = k$ for every $G \in \mathcal{C}$. We call $\mathbb{K}$ *regular* if $\mathbb{K}$ is $\mathrm{DG}_t(A, B)$-regular. We give two easy conditions which ensure regularity of $\mathbb{K}$.

Lemma 8. *If $\mathbb{K}$ is idempotent, then $\mathbb{K}$ is regular. If $\mathcal{C}$ is a class of pointed graphs, then $\mathbb{K}$ is $\mathcal{C}$-regular.*

We extend the operations $+$ and $\cdot$ of our semiring to series by defining point-wise $(S+T)(G) = S(G)+T(G)$ and $(S \odot T)(G) = S(G) \cdot T(G)$ for each $G \in \mathrm{DG}_t(A, B)$.

Proposition 9. *The class of recognizable series is closed under* $+$ *and* $\odot$.

In the following, we show that recognizable series are closed under projection. Let $h : A' \to A$ be a mapping between two alphabets. Then h defines naturally a relabeling of graphs from $\mathrm{DG}_t(A', B)$ into graphs from $\mathrm{DG}_t(A, B)$, also denoted by h. Let $S : \mathrm{DG}_t(A', B) \to K$ be a series. We define $h(S) : \mathrm{DG}_t(A, B) \to K$ by

$$h(S)(G) = \sum_{G' \in \mathrm{DG}_t(A,B),\ h(G')=G} S(G') \ . \tag{2}$$

Proposition 10. *Let* $S : \mathrm{DG}_t(A', B) \to K$ *be a recognizable series and* $h : A' \to A$*. Then* $h(S) : \mathrm{DG}_t(A, B) \to K$ *is recognizable.*

4 A Nivat Theorem for Weighted Graph Automata

In this section, we establish a connection between unweighted recognizable languages and recognizable graph series. Note that a corresponding result for weighted automata on words (cf. [12]) makes crucial use of the possible determinization of every unweighted word automaton. Unfortunately, this is not the case for graph languages. To deal with this problem, we require either the underlying semiring to be idempotent or the considered languages to be recognizable by a one-state graph acceptor. For a similar distinction, see [13].

Let $S : \mathrm{DG}_t(A', B) \to K$ and $L \subseteq \mathrm{DG}_t(A', B)$. We consider $h : A' \to A$, and the induced mappings $h : \mathrm{DG}_t(A', B) \to \mathrm{DG}_t(A, B)$, and $h(S) : \mathrm{DG}_t(A, B) \to K$, as before, see formula (2). We define the *restriction* $S \cap L : \mathrm{DG}_t(A', B) \to K$ by letting $(S \cap L)(G) = S(G)$ if $G \in L$ and $(S \cap L)(G) = 0$, otherwise.

Let $g : A' \to K$ be a map. Let $G \in \mathrm{DG}_t(A', B)$ and let $\mathrm{Lab}_G(v) \in A'$ be the label of a vertex v of G. We define the map $\mathrm{prod} \circ g : \mathrm{DG}_t(A', B) \to K$ by $(\mathrm{prod} \circ g)(G) = \prod_{v \in V} g(\mathrm{Lab}_G(v))$. So, $\mathrm{prod} \circ g : \mathrm{DG}_t(A', B) \to K$ is a very particular series obtained by assigning, for a graph $G \in \mathrm{DG}_t(A', B)$, to each vertex a weight (depending only on its label) and then multiplying all these weights.

Let $\mathcal{N}_t(A, B, \mathbb{K})$ comprise all series $S : \mathrm{DG}_t(A, B) \to K$ for which there exist an alphabet A', a map $g : A' \to K$, a map $h : A' \to A$, and a recognizable language $L \subseteq \mathrm{DG}_t(A', B)$ such that $S = h((\mathrm{prod} \circ g) \cap L)$. We denote by $\mathcal{N}_t^{\mathrm{one}}(A, B, \mathbb{K})$ the set of series defined similarly but with a language L which is recognizable by a one-state GA. Trivially, $\mathcal{N}_t^{\mathrm{one}}(A, B, \mathbb{K}) \subseteq \mathcal{N}_t(A, B, \mathbb{K})$.

Using closure properties of series, we get the following Nivat-Theorem for weighted graph automata.

Theorem 11. *Let* $\mathbb{K}$ *be a commutative semiring and* $S : \mathrm{DG}_t(A, B) \to K$ *be a series. Then* S *is recognizable if and only if* $S \in \mathcal{N}_t^{one}(A, B, \mathbb{K})$*. If* $\mathbb{K}$ *is idempotent, then* S *is recognizable if and only if* $S \in \mathcal{N}_t(A, B, \mathbb{K})$*.*

Proof (sketch). First, let S be recognizable by the wGA $\mathcal{A} = (Q, \Delta, \mathrm{wt}, \mathrm{Occ}, r)$ over A, B, and $\mathbb{K}$. We set $A' = A \times Q \times \mathrm{wt}(\Delta)$. Let h be the projection of A' to A and let g be the projection of A' to $\mathrm{wt}(\Delta)$.

Let $L \subseteq \mathrm{DG}_t(A', B)$ be the language consisting of all graphs G' over A' and B such that assigning to every vertex v' of G' the second component of the label of v' defines an accepting run of $\mathcal{A}$ on $h(G')$ and the added weights are consistent with the weight function wt of $\mathcal{A}$. We can construct a one-state GA accepting L. It follows that $S = h((\mathrm{prod} \circ g) \cap L) \in \mathcal{N}_t^{\mathrm{one}}(A, B, \mathbb{K})$.

For the converse, let A' be an alphabet, $g : A' \to K$, $h : A' \to A$, $L \subseteq \mathrm{DG}_t(A', B)$ be a recognizable language, and $S = h((\mathrm{prod} \circ g) \cap L)$. We can construct a wGA $\mathcal{C}$ which simulates $\mathrm{prod} \circ g$. Using that L is recognizable by a one-state GA or that $\mathbb{K}$ is idempotent, we construct a wGA $\mathcal{B}$ with $\|\mathcal{B}\| = \mathbb{1}_L$. Then Propositions 9 and 10 yield the result. $\square$

The following lemma and example show that in the general setting there exist series in $\mathcal{N}_t(A, B, \mathbb{K})$ which are not recognizable. For this purpose, we say that a GA $\mathcal{A}$ is *unambiguous* if for every graph G, $\mathcal{A}$ has at most one accepting run on G. We call a graph language L *unambiguously recognizable* if there exists an unambiguous GA accepting L.

We can show that the set of unconnected graphs is a recognizable language which is not unambiguously recognizable. Showing that for every language which is not unambiguously recognizable, the series $\mathbb{1}_L$ is not recognizable over $\mathbb{N}$, we obtain:

Lemma 12. *1. The class of unambiguously recognizable languages is a proper subclass of all recognizable languages.*
2. There exists a recognizable language L such that $\mathbb{1}_L$ is not recognizable over the semiring of the natural numbers $\mathbb{N}$.

Example 13. Using ideas of [13], we construct a series in $\mathcal{N}_t(A, B, \mathbb{K})$ which is not recognizable, as follows. We let $\mathbb{K} = (\mathbb{N}, +, \cdot, 0, 1)$, $A' = A$, h be the identity function, and $g \equiv 1$ be the constant function to 1. By Part 2 of Lemma 12, let L be a recognizable language such that $\mathbb{1}_L$ is not recognizable. Then $\mathbb{1}_L = h(\mathrm{prod} \circ g \cap L) \in \mathcal{N}_t(A, B, \mathbb{K})$.

There also exist recognizable but not unambiguously recognizable languages if we consider connected graphs. Over the class of pictures, the existence of such a language was shown in [2].

5 Weighted Logics for Graphs

In the following, we introduce a weighted MSO logic for graphs, following the approach of Droste and Gastin [9] for words. We also incorporate an idea of Bollig and Gastin [4] to consider Boolean formulas.

Definition 14. *We define the weighted logic* $\mathrm{MSO}(\mathbb{K}, \mathrm{DG}_t(A, B))$*, short* $\mathrm{MSO}(\mathbb{K})$*, as*

$$\beta ::= P_a(x) \mid E_b(x, y) \mid x = y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$
$$\varphi ::= \beta \mid k \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus_x \varphi \mid \bigoplus_X \varphi \mid \bigotimes_x \varphi$$

where $k \in K$; x, y are first-order variables; and X is a second order variable.

In [9], the weighted connectors were also denoted by $\vee$, $\wedge$, $\exists x$, $\exists X$, and $\forall x$. We employ this symbolic change to stress the quantitative evaluation of formulas.

Let $G \in \mathrm{DG}_t(A, B)$ and $\varphi \in \mathrm{MSO}(\mathbb{K})$. We follow classical approaches for logics and semantics. Let $\mathrm{free}(\varphi)$ be the set of all free variables in φ, and let $\mathcal{V}$ be a finite set of variables containing $\mathrm{free}(\varphi)$. A $(\mathcal{V}, G)$-*assignment* σ is a function assigning to every first-order variable of $\mathcal{V}$ an element of V and to every second order variable a subset of V. We define $\sigma[x \to v]$ as the $(\mathcal{V} \cup \{x\}, G)$-assignment mapping x to v and equaling σ everywhere else. The assignment $\sigma[X \to I]$ is defined analogously.

We represent the graph G together with the assignment σ as a graph (G, σ) over the vertex alphabet $A_{\mathcal{V}} = A \times \{0,1\}^{\mathcal{V}}$ where 1 denotes every position where x resp. X holds. A graph over $\mathcal{A}_{\mathcal{V}}$ is called *valid*, if every first-order variable is assigned to exactly one position.

We define the *semantics* of $\varphi \in \mathrm{MSO}(\mathbb{K})$ as a function $[\![\varphi]\!]_{\mathcal{V}} : \mathrm{DG}_t(A_{\mathcal{V}}, B) \to K$ inductively for all valid $(G, \sigma) \in \mathrm{DG}_t(A_{\mathcal{V}}, B)$, as seen in Figure 1. For not valid (G, σ), we set $[\![\varphi]\!]_{\mathcal{V}}(G, \sigma) = 0$. We write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{free}(\varphi)}$.

$$
\begin{aligned}
[\![\beta]\!]_{\mathcal{V}}(G,\sigma) &= \begin{cases} 1\ , & \text{if } (G,\sigma) \models \beta \\ 0\ , & \text{otherwise} \end{cases} \qquad\qquad [\![k]\!]_{\mathcal{V}}(G,\sigma) = k \quad \text{for all } k \in K \\
[\![\varphi \oplus \psi]\!]_{\mathcal{V}}(G,\sigma) &= [\![\varphi]\!]_{\mathcal{V}}(G,\sigma) + [\![\psi]\!]_{\mathcal{V}}(G,\sigma) \\
[\![\varphi \otimes \psi]\!]_{\mathcal{V}}(G,\sigma) &= [\![\varphi]\!]_{\mathcal{V}}(G,\sigma) \cdot [\![\psi]\!]_{\mathcal{V}}(G,\sigma) \\
[\![\textstyle\bigoplus_x \varphi]\!]_{\mathcal{V}}(G,\sigma) &= \sum_{v \in V} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(G, \sigma[x \to v]) \\
[\![\textstyle\bigoplus_X \varphi]\!]_{\mathcal{V}}(G,\sigma) &= \sum_{I \subseteq V} [\![\varphi]\!]_{\mathcal{V} \cup \{X\}}(G, \sigma[X \to I]) \\
[\![\textstyle\bigotimes_x \varphi]\!]_{\mathcal{V}}(G,\sigma) &= \prod_{v \in V} [\![\varphi]\!]_{\mathcal{V} \cup \{x\}}(G, \sigma[x \to v])
\end{aligned}
$$

Fig. 1. Semantics

Whether a graph is valid can be checked by an FO-formula, hence the language of all valid graphs over $A_{\mathcal{V}}$ is recognizable. For the Boolean semiring $\mathbb{B}$, the unweighted MSO is expressively equivalent to $\mathrm{MSO}(\mathbb{B})$.

Lemma 15. *Let $\varphi \in \mathrm{MSO}(\mathbb{K})$ and $\mathcal{V}$ a finite set of variables with $\mathcal{V} \supseteq \mathrm{free}(\varphi)$. Then $[\![\varphi]\!]_{\mathcal{V}}(G, \sigma) = [\![\varphi]\!](G, \sigma \upharpoonright \mathrm{free}(\varphi))$ for each valid $(G, \sigma) \in \mathrm{DG}_t(A_{\mathcal{V}}, B)$. Furthermore, if $[\![\varphi]\!]$ is recognizable, then $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable.*

We note that, in contrast to all previous papers of the literature on weighted logic, in general, the converse of the second statement of Lemma 15 is not true. If we restrict ourselves to pointed graphs or to an idempotent semiring, we can show that $[\![\varphi]\!]$ is recognizable if and only if $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable. However, to prove the following statements, the implication above together with Proposition 10 suffices.

Lemma 16. *Let $[\![\varphi]\!]$ be recognizable. Then $[\![\bigoplus_x \varphi]\!]$ and $[\![\bigoplus_X \varphi]\!]$ are recognizable.*

The difficult case is the $\bigotimes_x$-quantification (previously the universal quantification [9]). Similarly to [9], our unrestricted logic is strictly more powerful than our automata model. Therefore, we introduce the following fragment.

We call a formula $\varphi \in \mathrm{MSO}(\mathbb{K})$ *almost FO-boolean* if φ is built up inductively from unweighted FO-formulas β and constants k using the connectives $\oplus$ and $\otimes$.

Proposition 17. *Let φ be almost FO-boolean. Then $[\![\bigotimes_x \varphi]\!]$ is recognizable.*

Proof (sketch). We follow the proof for the universal quantification for words [9] with the crucial difference that we cannot determinize a GA. Let φ be almost FO-boolean. We can show that φ is semantically equivalent to $(k_1 \otimes \varphi_1) \oplus ... \oplus (k_m \otimes \varphi_m)$, where $m \in \mathbb{N}$, $k_i \in K$, and φ_i are unweighted FO-formulas. Using an extended alphabet $\tilde{A} = A \times \{1, ..., m\}$, we can encode the finitary structure of $[\![\varphi]\!]$ into a language $\tilde{L}$. Since all φ_i are FO-formulas, we can show that $\tilde{L}$ is FO-definable.

Applying Part 1 of Theorem 2, we get a one-state (in particular an unambiguous) GA $\tilde{\mathcal{A}}$, accepting $\tilde{L}$. We transform $\tilde{\mathcal{A}}$ into a wGA $\mathcal{A}$ which adds the weight defined by the second component of $\tilde{A}$ to the tiles. Then we can show that $\|\mathcal{A}\| = [\![\bigotimes_x \varphi]\!]$. $\square$

Let $\varphi \in \mathrm{MSO}(\mathbb{K})$. We call φ *restricted* if all unweighted subformulas β are EMSO-formulas and for all subformulas $\bigotimes_x \psi$ of φ, ψ is almost FO-boolean. We call φ *FO-restricted*, if φ is restricted and all unweighted subformulas β are FO-formulas. For the Boolean semiring $\mathbb{B}$, the unweighted EMSO is expressively equivalent to restricted $\mathrm{MSO}(\mathbb{B})$.

Note that, contrary to [9], we cannot relax our restriction to include unweighted EMSO-formulas after $\bigotimes_x$ because then our restricted weighted logic would still be strictly stronger than EMSO, even for the Boolean semiring.

We can show that every weighted graph automaton can be simulated by an FO-restricted $\mathrm{MSO}(\mathbb{K})$-sentence. Together with the closure properties of this section and Section 3, we obtain our main result.

Theorem 18. *Let $\mathbb{K} = (K, +, \cdot, 0, 1)$ be a commutative and regular semiring, and let $S : \mathrm{DG}_t(A, B) \to K$ be a series. Then the following are equivalent:*

1. *S is recognizable.*
2. *S is definable by an FO-restricted $\mathrm{MSO}(\mathbb{K})$-sentence.*

If $\mathbb{K}$ is idempotent, then 1. and 2. are equivalent to

3. *S is definable by a restricted $\mathrm{MSO}(\mathbb{K})$-sentence.*

Note that to ensure regularity of $\mathbb{K}$, it suffices to assume $\mathbb{K}$ to be idempotent (for instance as in the case of the tropical semirings) or to consider only pointed graphs (cf. Lemma 8). Words, trees, pictures, and nested words can be seen as pointed structures.

6 Words, Trees, and other Structures

In this section, using ideas from Thomas [34], we show that existing quantitative automata models over words [9], trees [14], pictures [17], and nested words [24] can be seen as special incidences of weighted graph automata with the same expressive power. Hence, we get previous equivalence results connecting weighted logic and weighted automata over these structures as a consequence (in a slightly modified version).

As already stated by [34], two significant differences to the previous models are the occurrence constraint and the possibly bigger tile-size. As first step, following [34], we give a sufficient condition to drop the occurrence constraint. We say that a weighted graph automaton $\mathcal{A} = (Q, \Delta, \mathrm{wt}, \mathrm{Occ}, r)$ *is without occurrence constraint*, if $\mathrm{Occ} = \mathit{true}$.

We call a class of graphs $\mathcal{C} \subseteq \mathrm{DG}_t(A, B)$ *partial sortable* if there exists an element $b_0 \in B$ such that for every graph $G = (V, (P_a)_{a \in A}, (E_b)_{b \in B})$ of $\mathcal{C}$, the subgraph $G' = (V, (P_a)_{a \in A}, E_{b_0})$ is acyclic, connected, and every vertex of G' has at most one outgoing edge.

Note that words, trees, pictures, and nested words can be seen as graph classes of this type. The following result is a weighted (and slightly more general) version of Proposition 5.3 in [34].

Lemma 19. *Let $\mathcal{C} \subseteq \mathrm{DG}_t(A, B)$ be partial sortable and $S : \mathcal{C} \to K$ a recognizable series. Then there exists a wGA $\mathcal{B}$ without occurrence constraint with $\|\mathcal{B}\| = S$.*

Definition 20. *A* weighted finite automata (wFA) $\mathcal{A} = (Q, I, F, \delta, \mu)$ *consists of a finite set of states Q, a set of initial states $I \subseteq Q$, a set of final states $F \subseteq Q$, a set the transitions $\delta \subseteq Q \times A \times Q$, and a weight function $\mu : \delta \to K$. We define an* accepting run*, the* language of $\mathcal{A}$*, and* recognizable word series *as usual (cf. [9, 11, 30]).*

Now we consider words (trees, pictures, nested words, respectively) as relational structures, and hence also as graphs. Using Lemma 19, we can prove the following.

Proposition 21. *Let $S : A^* \to K$ be a word series. Then S is recognizable by a weighted graph automaton iff S is recognizable by a weighted finite automaton.*

A similar result can be proved for trees, pictures, and nested words, and their respective automata models defining recognizable series.

Note that this is possible because, essentially, we can reduce the tile-size of a weighted graph automaton over these specific graphs from r to 1. In general, this is not possible, and it is already stated by Thomas [34] as an open question to precisely describe the class of graphs where the use of 1-tiles suffices.

We obtain the following consequence of Theorem 18 and Proposition 21.

Corollary 22 ([9]). *For a word series $S : A^* \to K$ the following are equivalent:*

1. *S is recognizable by a weighted finite automata.*
2. *S is definable by an FO-restricted* $\mathrm{MSO}(\mathbb{K})$*-sentence.*

If we replace "word" by "tree", "picture", or "nested word", we get a similar result for all four automata models and and their respective logics.

Note that our implication (2) $\Rightarrow$ (1) is a slightly weaker version of the results in [9] since in our logic, we can apply $\bigotimes_x$-quantification (resp. universal quantification) only to almost FO-boolean formulas, whereas in [9, 10], we can use almost MSO-boolean formulas.

As shown before, this difference originates from the fact that in the word case EMSO = MSO, which is not true for pictures or graphs. In this sense, our result captures the 'biggest logic possible', if we want to include pictures.

7 Conclusion

We introduced a weighted generalization of Thomas' graph acceptors [33, 34] and a suitable weighted logic in the sense of Droste and Gastin [9]. For commutative semirings, we proved a Nivat-like and a Büchi-like characterization of the expressive power of weighted graph automata. We showed slightly stronger results in the case of an idempotent semiring.

We showed that weighted word, tree, picture, or nested word automata are special instances of these general weighted graph automata, which gives us results of [9], [14], [17], and [24] as corollaries (under the appropriate restrictions to the underlying logic). Although not considered explicitly, we conjecture that similar equivalence results also hold for other finite structures like traces [25] or texts [23] and their respective automata models. We gave several examples that our weighted graph automata can recognize particular quantitative properties of graphs which were not covered by previous automata models.

Most statements in this paper can also be proven for general semirings. In this case, however, we would have to enforce an ordering of our graphs to get a well-defined product of weights. Furthermore, we would need some restrictions on the conjunction of our logic (similar to, e.g., [10]). Here, we restricted ourselves to commutative semirings in order to avoid technical conditions.

All the constructions given in this paper are effective. Subsequent research could investigate applications of weighted graph algorithms for developing decision procedures for weighted graph automata.

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