The normal subsemigroups of the monoid of injective maps *

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Abstract

We consider the monoid $\mathrm{Inj}(M)$ of injective self-maps of a set M and want to determine its normal subsemigroups by numerical invariants. This was established by Mesyan in 2012 if M is countable. Here we obtain an explicit description of all normal subsemigroups of $\mathrm{Inj}(M)$ for any uncountable set M.

1 Introduction

In two recent papers, Mesyan [11, 12] investigated the monoid $\mathrm{Inj}(M)$ of all injective self-maps of an infinite set M. A subsemigroup U of $\mathrm{Inj}(M)$ is normal if it is closed under conjugations by elements from $S(M)$, the symmetric group of all permutations of M. In a well-kown result, Schreier and Ulam [17] and Baer [1] showed that $S(M)$ has only few normal subgroups. Surprisingly, Mesyan [12] completely described the normal subsemigroups of $\mathrm{Inj}(M)$ if M is countable; there are uncountably many - determined by numerical invariants and subsemigroups of the monoid $(\mathbb{N}, +)$.

In this paper we will describe the normal subsemigroups of $\mathrm{Inj}(M)$ for all uncountable sets M. Due to the uncountability of M new classes of normal subsemigroups arise stemming from injections behaving on an uncountable subset of M like a permutation. Thus a combination of Mesyan's methods for injective functions and their conjugacy classes as well as results on permutation groups is needed.

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For the permutation group methods we employ a deep analysis of products of conjugacy classes in symmetric groups obtained by Moran [13, 14, 15]. In particular, he characterized products of conjugacy classes of the maximal factor group of $S(M)$. We extend his result to products of particular conjugacy classes of $S(M)$. Any normal subsemigroup G of $\mathrm{Inj}(M)$ decomposes into three parts: the group part G_{grp}, the subsemigroup G_{fin} comprising all elements with finite, non-trivial co-image and G_{inf}, the infinitary analog version of G_{fin}. The critical part of the characterization concerns G_{fin}. Here we need the results on products of conjugacy classes in symmetric groups of uncountable sets to obtain the description of G_{fin}.

Our main results are contained in Theorems 3.10, 5.1, 5.2 and 5.5. As a consequence we derive the precise number of normal subsemigroups of $\mathrm{Inj}(M)$, which is $2^{c(M)^{\aleph_0}}$, where $c(M) = |\{\mu \mid \mu \leq |M| \text{ is a cardinal}\}|$. In contrast we show that $\mathrm{Inj}(M)$ has only $|i| + 3$ maximal normal subsemigroups if $|M| = \aleph_i$. (i an ordinal).

We just note that the semigroup $\mathrm{Inj}_{\mathrm{fin}}(M)$ is also known as Baer-Levi semigroup, see [2, 9] for its importance in semigroup theory. In [7] its maximal subsemigroups were investigated, and in [10] it was shown not to have the Bergman property. Related results on products of conjugacy classes in the symmetric groups are contained in [3, 4, 6].

2 Definitions

Let M be an infinite set, $\mathrm{Inj}(M)$ the monoid of all injective maps of M and $S(M)$ the symmetric group of all permutations of M. If $f \in \mathrm{Inj}(M)$, we put $f^{S(M)} = \{g^{-1}fg \mid g \in S(M)\}$, the set of conjugates of f. We let $[f] = \{x \in M \mid xf \neq x\}$ denote the *support* of f. Moreover, we will write $|f| = |[f]|$ for the size of the support. Also $\mathrm{Fix}(f) = M \setminus [f]$ denotes the set of fixed points of f. If $x \in M$, the set $\{y \in M \mid y^{f^i} = x \text{ or } x^{f^i} = y \text{ for some } i \geq 0\}$ is called the *f-orbit of* x, or an *orbit* of f. If $x \notin Mf$, we call this orbit also a *forward orbit*. Observe that then the set $\{x^{f^i} \mid i \geq 0\}$, the f-orbit of x, is infinite; in particular, $[f]$ is infinite and $|M \setminus Mf| \leq |f|$. We call any orbit U of f with $U \subseteq Mf$ a *closed orbit*; then clearly $f \restriction U \in S(U)$.

We let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_\infty = \mathbb{N} \cup \{\aleph_0\}$. We let $\overline{f}$ be the map from $\mathbb{N}_\infty$ to the cardinals, where $\overline{f}(n)$ is the number of closed orbits of size n of f for each $n \in \mathbb{N}_\infty$.

Recall that κ^+ denotes the successor cardinal of κ. If $\aleph_0 \leq \nu \leq |M|^+$, then let $S^\nu(M) = \{f \in S(M) \mid |f| < \nu\}$ which is a normal subgroup of $S(M)$. We also write $\mathrm{Fin}(M) = S^{\aleph_0}(M)$, the group of finitary permutations of M, and we let $\mathrm{Alt}(M) = \{f \in \mathrm{Fin}(M) \mid f \restriction [f] \text{ is even}\}$, the infinite alternating group on M. Now let us consider $\mathrm{Inj}(M)$. We say that a subset $U \subseteq \mathrm{Inj}(M)$ is *normal* in $\mathrm{Inj}(M)$ if $U^f = f^{-1}Uf \subseteq U$ for all $f \in S(M)$. We will write $U \lhd \mathrm{Inj}(M)$ for normal subsemigroups.

For the rest of this paper, M will denote an infinite set.

3 Products of conjugates

In this section, we recall results about $S(M)$ and $\mathrm{Inj}(M)$ which will be needed later on. First we deal with the symmetric group. As is well-known, two permutations $f, g \in S(M)$ are conjugate if and only if $\overline{f} = \overline{g}$.

Lemma 3.1. ([5]) *Let $f, h \in S(M)$ with $|\,h\,| \leq |\,f\,|$ and let f have infinite support. Then $h \in (f^{S(M)})^4$.*

Next we turn to a sharpening of Lemma 3.1.

Definition 3.2. *Following [13, p. 325] we say that $f \in S(M)$ is of* type 0 *if the following holds.*

(i) $\overline{f}(1) = 0$ or $\overline{f}(1) = |\,M\,|$.

(ii) $\overline{f}(n) = 0$ or $\overline{f}(n) \geq \aleph_0$ for all $2 \leq n \in \mathbb{N}_\infty$.

Moreover, we say that $f \in S(M)$ is almost of type 0 *if $\overline{f}(n) = 0$ or $\overline{f}(n) \geq \aleph_0$ for all $n \in \mathbb{N}_\infty$.*

Lemma 3.3. (Moran [13, Theorem 2]) *Let $f, h \in S(M)$ be two permutations of type 0 with $|\,h\,| \leq |\,f\,|$. Then $h \in (f^{S(M)})^2$.*

Now we can derive the following strengthening of Lemmas 3.1 and 3.3 for uncountable sets M.

Proposition 3.4. *Let M be uncountable and let $f, h \in S(M)$ be two permutations which are almost of type 0 with $|\,h\,| \leq |\,f\,|$. Then $h \in (f^{S(M)})^2$.*

Proof. If h, f are of type 0, then the claim is straightforward by Lemma 3.3.

In the second case we assume that h is of type 0 but f is not. Hence $\aleph_0 \leq \overline{f}(1) < |\, M\,|$. Choose $m \in \mathbb{N}_\infty$ such that $\overline{f}(m) > \overline{f}(1)$. We can split $M = M_1 \dot{\cup} M_2$ such that M_1 consists of $\mathrm{Fix}(f)$ and $\overline{f}(1)$-many m-orbits of f. Hence $f_i = f \restriction M_i \in S(M_i)$ for $i = 1, 2$, and $|\, M_1\,| = |\,\mathrm{Fix}(f)\,| = |\, f_1\,|$, and f_1 and f_2 both are of type 0.

Choose $h_1 \in S(M_1)$, $h_2 \in S(M_2)$ of type 0 with $\overline{h}_1 + \overline{h}_2 = \overline{h}$, hence $\overline{h_1 \cup h_2} = \overline{h}$. Note that $|\, h_2\,| \leq |\, h\,| \leq |\, f\,| = |\, f_2\,|$. By the first case it follows that $h_1 \in (f_1^{S(M_1)})^2$ and similarly $h_2 \in (f_2^{S(M_2)})^2$. Thus $h \in (h_1 \cup h_2)^{S(M)} \subseteq (f^{S(M)})^2$ as required.

In the final case we assume that h is not of type 0, thus $\aleph_0 \leq \overline{h}(1) < |\, M\,|$. Choose $m \in \mathbb{N}_\infty$ such that $\overline{h}(m) > \overline{h}(1)$. We split $M = M_1 \dot{\cup} M_2$ such that M_1 consists of $\mathrm{Fix}(h)$ and $\overline{h}(1)$-many m-orbits of h. Then $h_i = h \restriction M_i \in S(M_i)$ for $i = 1, 2$ and $|\, M_1\,| = \left|\, \overline{h}(1)\,\right| = |\, h_1\,|$. Hence h_1 and h_2 are both of type 0. Now we choose $f_1 \in S(M_1)$, $f_2 \in S(M_2)$ both almost of type 0 with $\overline{f}_1 + \overline{f}_2 = \overline{f}$ and $|\, f_1\,| = |\, M_1\,|$; so $|\, f_2\,| = |\, f\,|$. By the first two cases it follows $h_1 \in (f_1^{S(M_1)})^2$ and similarly $h_2 \in (f_2^{S(M_2)})^2$, hence $h \in (f_1 \cup f_2^{S(M)})^2 = (f^{S(M)})^2$ as required. $\square$

Now we turn to $\mathrm{Inj}(M)$.

Observation 3.5. (Mesyan [11, Lemma 5]) *If $f, g \in \mathrm{Inj}(M)$, then*

$$|\, M \setminus Mfg\,| = |\, M \setminus Mf\,| + |\, M \setminus Mg\,|.$$

Let $U \subseteq M$ be a subset and $f \in \mathrm{Inj}(M)$. We say that U is $f^{\pm 1}$-invariant, if whenever $x \in U, y \in M$, and $y = x^{f^i}$ or $y^{f^i} = x$ for some $i \in \mathbb{N}$, then $y \in U$. That is, U is a union of f-orbits. If $f_i \in \mathrm{Inj}(M)$ $(i \in I)$, we say that U is $\{f_i^{\pm 1} \mid i \in I\}$-invariant if U is $f_i^{\pm 1}$-invariant for each $i \in I$. If $U' \subseteq M$ is a subset and U is the smallest $\{f_i^{\pm 1} \mid i \in I\}$-invariant subset of M with $U' \subseteq U$, we call U the $\{f_i^{\pm 1} \mid i \in I\}$*-closure* of U'. Note that then $|\, U\,| \leq \max\{|\, U'\,|, |\, I\,|, \aleph_0\}$. Trivially, $M \setminus U$ is also $\{f_i^{\pm 1} \mid i \in I\}$-invariant, and such splittings of M into invariant subsets will be very important for the rest of this paper. It can be used for the following basic result which describes conjugacy of elements of $\mathrm{Inj}(M)$.

Lemma 3.6. (Mesyan [11, Proposition 3]) *Let $f, g \in \mathrm{Inj}(M)$. Then*

$$g \in f^{S(M)} \iff (|\, M \setminus Mf\,| = |\, M \setminus Mg\,| \ \text{and}\ \overline{f} = \overline{g}).$$

Proof. The claim " $\Rightarrow$ " is trivial, and we only must show " $\Leftarrow$ " :

We indicate the proof for illustration. Let M_1 be the $f^{\pm 1}$-closure of $M \setminus Mf$, which is the union of all forward orbits of f, and $M_2 = M \setminus M_1$. Hence $|M_2| = \sum_{k \in \mathbb{N}_\infty} k\overline{f}(k)$. Similarly, we define $M = M_1' \dot\cup M_2'$ for g.

From $|M_1| = |\, M_1' \,|$ we have a bijection $h_1 : M_1 \longrightarrow M_1'$ such that $x^{h_1^{-1} f h_1} = x^g$ for all $x \in M_1'$. We can also choose $h_2 : M_2 \longrightarrow M_2'$ such that $h_2^{-1} f h_2 = g$ on M_2, thus $h = h_1 \cup h_2 \in S(M)$ satisfies $f^h = g$. $\square$

Next we consider products of conjugacy classes in $\mathrm{Inj}(M)$.

Lemma 3.7. (Mesyan [11, Corollary 10]) *Let M be countable and $f, g, h \in \mathrm{Inj}(M) \setminus S(M)$. Then $h \in f^{S(M)} \cdot g^{S(M)}$ if and only if $|\, M \setminus Mh \,| = |\, M \setminus Mf \,| + |\, M \setminus Mg \,|$.*

Lemma 3.8. (Mesyan [11, Corollary 13]). *If $f, h \in \mathrm{Inj}(M)$ and*

$$|\, M \setminus Mf \,| = |\, M \setminus Mh \,| = |\, f \,| = |\, h \,| \geq \aleph_0,$$

then $h \in (f^{S(M)})^2$.

Now we can show:

Lemma 3.9. *If $f, h \in \mathrm{Inj}(M)$, $|\, M \setminus Mf \,| = |\, M \setminus Mh \,| \geq \aleph_0$ and $|\, h \,| \leq |\, f \,|$, then $h \in (f^{S(M)})^2$.*

Proof. If $|\, M \setminus Mf \,| = |\, f \,|$, then Lemma 3.8 applies. Thus we may assume that $|\, M \setminus Mf \,| < |\, f \,|$. Let M_1' contain $(M \setminus Mf) \cup (M \setminus Mh)$ and all closed f-orbits and h-orbits of size $n \in \mathbb{N}_\infty$ for which $\overline{f}(n) \in \mathbb{N}$ or $\overline{h}(n) \in \mathbb{N}$. Let M_1 be the $\{f^{\pm 1}, h^{\pm 1}\}$-closure of M_1', and put $M_2 = M \setminus M_1$. Then $|\, M_1 \,| = |\, M \setminus Mf \,|$. Let $f_i = f \restriction M_i$ and $h_i = h \restriction M_i$ for $i = 1, 2$. Then $f_2, h_2 \in S(M_2)$. Also $\overline{h_2}(n) = 0$ or $\overline{h_2}(n) \geq \aleph_0$ and $\overline{f_2}(n) = 0$ or $\overline{f_2}(n) \geq \aleph_0$ for all $n \in \mathbb{N}_\infty$. Now we can apply Lemma 3.8 on M_1 and obtain $h_1 \in (f_1^{S(M_1)})^2$. On M_2 we apply Proposition 3.4 to get that $h_2 \in (f_2^{S(M_2)})^2$. Thus $h = h_1 \cup h_2 \in (f^{S(M)})^2$. $\square$

There are obvious normal subsemigroups of $\mathrm{Inj}(M)$: Let $\aleph_0 \leq \mu$, $\nu \leq |\, M \,|^+$, then $\mathrm{Inj}^\nu(M) = \{f \in \mathrm{Inj}(M) \mid |\, f \,| < \nu\}$ and $\mathrm{Inj}_\mu(M) = \{f \in \mathrm{Inj}(M) \mid |\, M \setminus Mf \,| = \mu\}$ are normal in $\mathrm{Inj}(M)$. If $\mu < \nu$, also $\mathrm{Inj}_\mu^\nu(M) = \mathrm{Inj}_\mu(M) \cap \mathrm{Inj}^\nu(M)$ is normal in $\mathrm{Inj}(M)$.

We also let

$$\mathrm{Inj}_{\mathrm{fin}}(M) = \{f \in \mathrm{Inj}(M) \mid M \neq Mf, |\, M \setminus Mf \,| \text{ is finite}\}. \tag{3.1}$$

If $G \subseteq \mathrm{Inj}(M)$, then let $G_{\mathrm{grp}} = G \cap S(M)$, $G_{\mathrm{fin}} = G \cap \mathrm{Inj}_{\mathrm{fin}}(M)$ and $G_\mu = G \cap \mathrm{Inj}_\mu(M)$ for any $\mu \leq |\, M \,|$. If $G \lhd \mathrm{Inj}(M)$, then also $G_{\mathrm{grp}}, G_{\mathrm{fin}}$ and G_μ are normal in $\mathrm{Inj}(M)$ because $S(M), \mathrm{Inj}_{\mathrm{fin}}(M), \mathrm{Inj}_\mu(M) \lhd \mathrm{Inj}(M)$. As noted in [12], G_{grp} is a group, since if $g \in G_{\mathrm{grp}}$, then $\overline{g} = \overline{g^{-1}}$, so $g^{-1} \in G$ by $G \lhd \mathrm{Inj}(M)$.

Now we can describe the structure of G_{grp} and G_μ for all $\mu \leq \kappa$.

Theorem 3.10. *Let $|\, M \,| = \kappa \geq \aleph_0$ and $G \lhd \mathrm{Inj}(M)$. Then*

$$G = G_{\mathrm{grp}} \,\dot\cup\, G_{\mathrm{fin}} \,\dot\cup\, \dot{\bigcup}_{\aleph_0 \leq \mu \leq \kappa} G_\mu .$$

(i) G_{grp} is either $S^\nu(M)$ for some $\nu \leq \kappa^+$ or $\mathrm{Alt}(M)$ or $\{1\}$ or $\emptyset$.

(ii) For all $\mu \leq \kappa$ either $G_\mu = \emptyset$ or $G_\mu = \mathrm{Inj}^\nu_\mu(M)$ for some $\mu < \nu \leq \kappa^+$.

(iii) For all $\mu < \mu' < \nu$ if $G_\mu = \mathrm{Inj}^\nu_\mu(M)$ and $G_{\mu'} \neq \emptyset$, then $\mathrm{Inj}^\nu_{\mu'}(M) \subseteq G_{\mu'}$.

(iv) For all $\mu' \leq \nu$ if $G_{\mathrm{fin}} \nsubseteq \mathrm{Inj}^\nu(M)$ and $G_{\mu'} \neq \emptyset$, then $\mathrm{Inj}^{\nu^+}_{\mu'}(M) \subseteq G_{\mu'}$.

All these combinations with a normal subsemigroup $G_{\mathrm{fin}} \subseteq \mathrm{Inj}_{\mathrm{fin}}(M)$ give rise to normal subsemigroups $G \lhd \mathrm{Inj}(M)$.

Proof. (i) Since G_{grp} is a normal subgroup of $S(M)$, this is the main result of [1]. It also follows from Lemma 3.1.

(ii) Let $\aleph_0 \leq \mu \leq \kappa$ and assume there is $f \in G_\mu$. Let $\nu = |\, f \,|$. Clearly $\mu \leq \nu$ and we claim that then $\mathrm{Inj}^{\nu^+}_\mu(M) \subseteq G_\mu$. For this, choose $h \in \mathrm{Inj}^{\nu^+}_\mu(M)$. Then $|\, M \setminus Mf \,| = \mu = |\, M \setminus Mh \,|$ and $|\, h \,| \leq \nu = |\, f \,|$. By Lemma 3.9, we obtain $h \in (f^{S(M)})^2 \in G$ and our claim. This implies the assertion of (ii) with $\nu = \sup\{|\, f \,|^+ \mid f \in G_\mu\}$.

(iii) Let $h \in \mathrm{Inj}^\nu_{\mu'}(M)$. Then $\alpha := |\, h \,| < \nu$. Choose any $f \in G_{\mu'}$. In case $|\, f \,| \geq \alpha$, by Lemma 3.9 we obtain $h \in (f^{S(M)})^2 \in G_{\mu'}$. Now assume that $|\, f \,| < \alpha < \nu$. By assumption, there is $g \in G_\mu$ with $|\, g \,| = \alpha$. Then $fg \in G_{\mu'}$ and $|\, fg \,| = \alpha$. Hence, by Lemma 3.9, we have $h \in ((fg)^{S(M)})^2 \subseteq G_{\mu'}$.

(iv) We proceed similarly to the argument for (iii). Let $h \in \mathrm{Inj}^{\nu^+}_{\mu'}(M)$. Choose any $f \in G_{\mu'}$. If $|\, h \,| \leq |\, f \,|$, again by Lemma 3.9 we have $h \in (f^{S(M)})^2 \in G_{\mu'}$. Therefore now assume that $|\, f \,| < |\, h \,|$. By assumption, there is $g \in G_{\mathrm{fin}}$ with $|\, g \,| \geq \nu$. Then $fg \in G_{\mu'}$ and $|\, h \,| \leq \nu \leq |\, g \,| = |\, fg \,|$. So by Lemma 3.9 we obtain $h \in ((fg)^{S(M)})^2 \subseteq G_{\mu'}$ and the result. ∎

Hence it remains to describe the structure of G_{fin}. As in Mesyan [12], this depends on the value of G_{grp}. Therefore we proceed by the case distinction given by Theorem 3.10(i).

4 Products of conjugacy classes for uncountable sets

Throughout this section we assume that $G \lhd \mathrm{Inj}(M)$. Let

$$N(G) = \{|\, M \setminus Mf \,|\ |\ f \in G_{\mathrm{fin}}\}. \tag{4.1}$$

It is clear by Observation 3.5 that $N(G) \subseteq \mathbb{N}$ is a subsemigroup.

Conversely, if N is a subsemigroup of $(\mathbb{N}, +)$, then following [12], we let

$$\mathrm{Inj}_N(M) = \{g \in \mathrm{Inj}(M) \mid |\, M \setminus Mg \,| \in N\} \lhd \mathrm{Inj}(M) \tag{4.2}$$

and

$$\mathrm{Inj}_N^{\alpha}(M) = \{g \in \mathrm{Inj}_N(M) \mid |\, g \,| < \alpha\} = \mathrm{Inj}_N(M) \cap \mathrm{Inj}^{\alpha}(M) \tag{4.3}$$

for any $\alpha \le |\, M \,|^{+}$. Clearly $\mathrm{Inj}_N(M) \subseteq \mathrm{Inj}_{\mathrm{fin}}(M)$. By Observation 3.5 it is easy to see that $\mathrm{Inj}_N(M) \lhd \mathrm{Inj}(M)$, cf. [12, p. 292].

Lemma 4.1. *Let $G \lhd \mathrm{Inj}(M)$ and $\alpha > \aleph_0$ with $G_{\mathrm{fin}} \subseteq \mathrm{Inj}^{\alpha}(M)$ and $S^{\alpha}(M) \subseteq G$. Then $G_{\mathrm{fin}} = \mathrm{Inj}_N^{\alpha}(M)$ for $N = N(G)$.*

Proof. If $h \in \mathrm{Inj}_N^{\alpha}(M)$, then by (4.2) and (4.1) there is some $f \in G_{\mathrm{fin}}$ such that $|\, M \setminus Mf \,| = |\, M \setminus Mh \,|$, so $|\, f \,| < \alpha$. Choose any bijection $k_1 : M \setminus Mf \to M \setminus Mh$, and define $k_2 : Mf \to Mh$ by $xf \mapsto xh$, which is also bijective. Then $k = k_1 \cup k_2 \in S(M)$. Since $[k] \setminus [f] \subseteq \mathrm{Fix}(f) \cap [h]$, we have $[k] \subseteq [f] \cup [h]$ and by $|\, f \,|, |\, h \,| < \alpha$ it also follows $k \in S^{\alpha}(M) \subseteq G$. So $h = fk \in G$ and $|\, M \setminus Mh \,| = |\, M \setminus Mf \,|$, hence also $h \in G_{\mathrm{fin}}$, so $\mathrm{Inj}_N^{\alpha}(M) \subseteq G_{\mathrm{fin}}$. □

In this context, we note that each subsemigroup of $\mathbb{N}$ is finitely generated, cf. [16]. Consequently, $\mathbb{N}$ contains precisely $\aleph_0$ subsemigroups.

For the remaining part of this section we consider the case that $G_{\mathrm{fin}} \not\subseteq \mathrm{Inj}^{\aleph_1}(M)$, i.e. there is $f \in G_{\mathrm{fin}}$ with $|\, f \,| \ge \aleph_1$.

4.1 Moran's characterization $\mathcal{P}$ and products of types

Our goal is to extend the crucial Lemma 3.7 to the uncountable case. For this, we will use Moran's property $\mathcal{P}$ which describes products of conjugacy classes.

Definition 4.2. *Let M be an uncountable set.*

(i) We call each function $T : \mathbb{N}_\infty \longrightarrow \{0\} \cup \{\mu \mid \aleph_1 \leq \mu \leq |M|\}$ such that $\sum_{n\in\mathbb{N}_\infty} T(n) = |M|$ a type *(of $S(M)$). We let $\mathfrak{T}_M$ be the collection of all types of $S(M)$.*

(ii) For a type T of $S(M)$ let $C_T = \{f \in S(M) \mid \overline{f} = T\}$, a conjugacy class in $S(M)$. We put $\mathcal{P}(T, T_1, T_2)$ if and only if $C_T \subseteq C_{T_1} \cdot C_{T_2}$ in $S(M)$.

Our aim is to characterize the relation $\mathcal{P}$. Moran [14] completely described all conjugacy classes C_1, C_2, C_3 with $\mathcal{P}(C_1, C_2, C_3)$ in the factor groups $H_\nu = S(M)/S^\nu(M)$ for $\nu = |M| \geq \aleph_1$. He reduced this to a description of the relation $\mathcal{P}$ for simple types; a type T of $S(M)$ is *simple*, if $T(n) \in \{0, |M|\}$ for each $n \in \mathbb{N}_\infty$. In our setting, if $|M| \geq \aleph_1$, we have to consider all possible values of T in $\{0\} \cup \{\mu \mid \aleph_1 \leq \mu \leq |M|\}$.

We recall Moran's results. For this we define a few particular types. If $F \subseteq \mathbb{N}_\infty$ is a subset, then let $\widehat{F}$ be the simple type satisfying $\widehat{F}(n) = |M|$ if and only if $n \in F$ (for $n \in \mathbb{N}_\infty$). If $k \in \mathbb{N}_\infty$, we put $\widehat{k} = \widehat{\{k\}}$. We let OD(M) be the set of all simple types T satisfying $T(n) = 0$ for any even $n \in \mathbb{N}$ and for $n = \aleph_0$. Clearly $\mathcal{P}(\widehat{1}, T_1, T_2)$ holds if and only if $T_1 = T_2$. Since the relation $\mathcal{P}$ is symmetric, it remains to characterize it for *non-unit* types T, T_1, T_2, i.e. for T, T_1, T_2 different from $\widehat{1}$. Following [14], we call a set $\{T, T_1, T_2\}$ of types a *non-$\mathcal{P}$-set* if $\mathcal{P}(T, T_1, T_2)$ does not hold. The following gives an explicit description of non-$\mathcal{P}$-sets.

Theorem 4.3. (Moran [14, Theorem 1]) *Let M be uncountable and T, T_1, T_2 non-unit simple types of $S(M)$. Then $\{T, T_1, T_2\}$ is a non-$\mathcal{P}$-set if and only if one of the following two mutually exclusive conditions holds:*

(i) $\{T, T_1, T_2\} = \{\widehat{2}, \widehat{\{1,2\}}, U\}$ for some $U \in$ OD(M).

(ii) $\{T, T_1, T_2\}$ is one of the sets $\{\widehat{3}, \widehat{\{1,3\}}, \widehat{2}\}$ or $\{\widehat{3}, \widehat{\{1,3\}}, \widehat{\{1,2\}}\}$ or $\{\widehat{2}, \widehat{3}, \widehat{\{1,2,3\}}\}$.

Let α be a cardinal with $\aleph_1 \leq \alpha \leq |M|$. We let cf$(\alpha)$ denote the cofinality of α. Now let T be a type of $S(M)$. We define a function $T^\alpha : \mathbb{N}_\infty \to \{0, \alpha\}$ by letting (for $n \in \mathbb{N}_\infty$)

$$T^\alpha(n) = \begin{cases} \alpha & \text{if } T(n) \geq \alpha \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha < |M|$ or cf$(\alpha) \neq \omega$, the condition $\sum_{n\in\mathbb{N}_\infty} T(n) = |M|$ implies that $T(n) \geq \alpha$ for some $n \in \mathbb{N}_\infty$, hence T^α is a simple type of $S(M_\alpha)$ where $|M_\alpha| = \alpha$. Now we show:

Theorem 4.4. *Let M be uncountable and T, T_1, T_2 types of $S(M)$. Then $\mathcal{P}(T, T_1, T_2)$ if and only if*

$$\mathcal{P}(T^\alpha, T_1^\alpha, T_2^\alpha) \text{ for each cardinal } \alpha \text{ with } \aleph_1 \leq \alpha \leq |\, M \,| \text{ and } \mathrm{cf}(\alpha) \neq \omega \tag{4.4}$$

Proof. "$\Rightarrow$" Let $\mathcal{P}(T, T_1, T_2)$ and α a cardinal with $\aleph_1 \leq \alpha \leq |\, M \,|$ and $\mathrm{cf}(\alpha) \neq \omega$. We choose $h, f, g \in S(M)$ with $\overline{h} = T, \overline{f} = T_1, \overline{g} = T_2$ and $h = fg$. Let M_1' be the union of all orbits of h, f and g of length $n \in \mathbb{N}_\infty$ for which $\overline{h}(n) < \alpha$ resp. $\overline{f}(n) < \alpha$ or $\overline{g}(n) < \alpha$, and let M_1 be the $\{h^{\pm 1}, f^{\pm 1}, g^{\pm 1}\}$-closure of M_1'. Since $\mathrm{cf}(\alpha) \neq \omega$, α is not the sum of countably many smaller cardinals. Hence $|\, M_1 \,| < \alpha$. Put $M_2 = M \setminus M_1$, so $|\, M_2 \,| = |\, M \,|$.

Choose $M' \subseteq M_2$ such that M' is $\{h^{\pm 1}, f^{\pm 1}, g^{\pm 1}\}$-invariant and $|\, M' \,| = \alpha$. Let $h' = h \restriction M', f' = f \restriction M'$ and $g' = g \restriction M'$. Then $h' = f'g'$ and $\overline{h'}(n) = 0$ or $\overline{h'}(n) = \alpha$ for each $n \in \mathbb{N}_\infty$, and similarly for f' and g'. Hence $\overline{h'}, \overline{f'}$, and $\overline{g'}$ are simple types of $S(M')$ and $\overline{h'} = T^\alpha, \overline{f'} = T_1^\alpha, \overline{g'} = T_2^\alpha$, proving $\mathcal{P}(T^\alpha, T_1^\alpha, T_2^\alpha)$.

"$\Leftarrow$" Assume (4.4). We wish to construct $h, f, g \in S(M)$ such that $h = fg$ and $\overline{h} = T, \overline{f} = T_1, \overline{g} = T_2$; then $\mathcal{P}(T, T_1, T_2)$. We decompose $M = \dot{\bigcup}_{\alpha \in D} M_\alpha$ (with $D = \{\alpha \mid \aleph_1 \leq \alpha \leq |\, M \,| \text{ and } \mathrm{cf}(\alpha) \neq \omega\}$ into pairwise disjoint sets M_α of cardinality $|\, M_\alpha \,| = \alpha$. By assumption, for each $\alpha \in D$ we have $\mathcal{P}(T^\alpha, T_1^\alpha, T_2^\alpha)$, hence there are $h_\alpha, f_\alpha, g_\alpha \in S(M_\alpha)$ such that $h_\alpha = f_\alpha g_\alpha$ and $\overline{h_\alpha} = T^\alpha, \overline{f_\alpha} = T_1^\alpha, \overline{g_\alpha} = T_2^\alpha$. Put $h = \dot{\bigcup}_{\alpha \in D} h_\alpha, f = \dot{\bigcup}_{\alpha \in D} f_\alpha$ and $g = \dot{\bigcup}_{\alpha \in D} g_\alpha$. Clearly $h = fg$, and for each $n \in \mathbb{N}_\infty$ we have $\overline{h}(n) = \sum_{\alpha \in D} \overline{h_\alpha}(n) = \sum_{\alpha \in D} T^\alpha(n) = \sup_{\alpha \in D} T^\alpha(n)$. Note that $T^\alpha(n) = 0$ if $\alpha > T(n)$. If $\mathrm{cf}(T(n)) \neq \omega$ and $\alpha = T(n)$, we have $T^\alpha(n) = \alpha = T(n)$. If $\mathrm{cf}(T(n)) = \omega$, we have $T^\alpha(n) = \alpha$ for each $\alpha \in D$ with $\alpha < T(n)$, and the supremum of all these α equals $T(n)$. Hence in any case $\overline{h}(n) = T(n)$, showing $\overline{h} = T$. Similarly, $\overline{f} = T_1$ and $\overline{g} = T_2$. $\square$

Observe that condition (4.4) for each α is characterized by Theorem 4.3. Hence Theorem 4.3 and Theorem 4.4 together give a complete description of the relation $\mathcal{P}$ on $\mathfrak{T}_M$.

Now we turn $\mathrm{Inj}(M)$ for uncountable sets M. If $f \in \mathrm{Inj}(M)$ with $M \setminus Mf$ countable, we define $T_f : \mathbb{N}_\infty \longrightarrow \{0\} \cup \{\mu \mid \aleph_1 \leq \mu \leq |\, M \,|\}$ by letting

$$T_f(n) = \begin{cases} \overline{f}(n) & \text{if } \overline{f}(n) \geq \aleph_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then T_f is a type of $S(M)$, the *type of* f. Observe that if $f \in \mathrm{Inj}(M)$ satisfies $|\, f \,| \leq \aleph_0$, then $T_f(1) = |\, M \,|$ and $T_f(n) = 0$ for all $n \geq 2$. Also, for $f \in \mathrm{Inj}(M)$, we let

$$\mathrm{orb}(f, \omega) = \bigcup \{ \text{ closed orbits of } f \text{ of size } n \mid n \in \mathbb{N}_\infty \text{ with } \overline{f}(n) \leq \aleph_0 \}.$$

Now we state our extension of Lemma 3.7 to the uncountable case.

Proposition 4.5. *Let M be uncountable. For $f, g, h \in \mathrm{Inj}(M) \setminus S(M)$ with $M \setminus Mf, M \setminus Mg, M \setminus Mh$ countable, the following conditions are equivalent.*

(i) $h \in f^{S(M)} \cdot g^{S(M)}$

(ii) $|\, M \setminus Mh \,| = |\, M \setminus Mf \,| + |\, M \setminus Mg \,|$ *and* $\mathcal{P}(T_h, T_f, T_g)$ *holds.*

Proof. Let

$$M_1' = (M \setminus Mf) \cup (M \setminus Mg) \cup (M \setminus Mh) \cup \mathrm{orb}(f, \omega) \cup \mathrm{orb}(g, \omega) \cup \mathrm{orb}(h, \omega).$$

"$(ii) \Rightarrow (i)$": Let M_1 be the $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$-closure of M_1' and $M_2 = M \setminus M_1$. Then $|\, M_1 \,| = \aleph_0$ and $|\, M_2 \,| = |\, M \,|$. Put $f_i = f \restriction M_i, g_i = g \restriction M_i, h_i = h \restriction M_i$ for $i = 1, 2$. Clearly $f_2, g_2, h_2 \in S(M_2)$, and (by definition of types T), we have $T_{f_2} = T_f, T_{g_2} = T_g$ and $T_{h_2} = T_h$.

By Lemma 3.7 it follows that $h_1 \in f_1^{S(M_1)} \cdot g_1^{S(M_1)}$ and the property $\mathcal{P}(T_h, T_f, T_g)$ implies that $h_2 \in f_2^{S(M_2)} \cdot g_2^{S(M_2)}$. Patching the components together we get $h \in f^{S(M)} \cdot g^{S(M)}$.

"$(i) \Rightarrow (ii)$": Let $h = f^k \cdot g^{k'}$ with $k, k' \in S(M)$ be chosen by (i). Then by Observation 3.5 the first claim in (ii) follows immediately. Now let M_1 be the $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}, k^{\pm 1}, (k')^{\pm 1}\}$-closure of M_1', which is countable, and put $M_2 = M \setminus M_1$. Thus $|\, M_2 \,| = |\, M \,|$. Again, let $f_2 = f \restriction M_2$, $g_2 = g \restriction M_2$, $h_2 = h \restriction M_2$, $k_2 = k \restriction M_2$ and $k_2' = k' \restriction M_2$. Then $h_2 = f_2^{k_2} g_2^{k_2'}$ on M_2 and $\overline{h}_2(n) = 0$ or $\overline{h}_2(n) = \overline{h}(n) \geq \aleph_1$ for all $n \in \mathbb{N}_\infty$, so $T_{h_2} = T_h$ and the same holds for f_2, g_2. Thus we have $\mathcal{P}(T_{h_2}, T_{f_2}, T_{g_2}) = \mathcal{P}(T_h, T_f, T_g)$, so $\mathcal{P}$ holds in (ii). □

4.2 Normal subsemigroups of $\mathrm{Inj}(M)$

Throughout this section, let M be uncountable. Let $N \subseteq \mathbb{N}$ be a subsemigroup. Now we want to refine the correspondence between the subsemigroup N and the normal subsemigroup $\mathrm{Inj}_N(M)$ obtained in (4.1) and (4.2). We fix the family of pairs

$$\mathfrak{P} = \mathbb{N} \times \mathfrak{T}_M. \tag{4.5}$$

We call a subset P of $\mathfrak{P}$ an *N-type set* if it satisfies the following conditions:

(i) If $(n, T) \in P$, then $n \in N$.

(ii) For each $n \in N$ there is $T \in \mathfrak{T}_M$ such that $(n, T) \in P$.

(iii) If $(n_1, T_1), (n_2, T_2) \in P$ and $T \in \mathfrak{T}_M$ satisfies $\mathcal{P}(T, T_1, T_2)$, then $(n_1 + n_2, T) \in P$.

As an illustration we just note:

Corollary 4.6. *Let N be a subsemigroup of $\mathbb{N}$ and P an N-type set. If $(n, T) \in P$ with $\sum_{n \geq 2} T(n) = |\, M \,|$, then $(2n, T') \in P$ for any $T' \in \mathfrak{T}_M$.*

Proof. Let $T' \in \mathfrak{T}_M$. We claim that $P(T', T, T)$ holds. This follows from Theorem 4.4, since condition (4.4) is satisfied by Theorem 4.3. Now condition (iii) implies $(2n, T') \in P$. $\square$

We observe that for a normal subsemigroup $G \lhd \mathrm{Inj}(M)$ the family

$$P(G) = \{(|\, M \setminus Mf \,|, T_f) \mid f \in G_{\mathrm{fin}}\} \tag{4.6}$$

satisfies the

Remark 4.7. *$P(G)$ is an $N(G)$-type set.*

Proof. Conditions (i) and (ii) are obvious. Now let $f, g \in G_{\mathrm{fin}}$ and $m = |\, M \setminus Mf \,|, n = |\, M \setminus Mg \,|$ and let $T \in \mathfrak{T}_M$ with $\mathcal{P}(T, T_f, T_g)$. Choose $h \in \mathrm{Inj}(M)$ with $|\, M \setminus Mh \,| = m + n$ and $T_h = T$. By Proposition 4.5, we obtain $h \in f^{S(M)} g^{S(M)} \subseteq G$. So, $(m + n, T) \in P(G)$, proving condition (iii). $\square$

Now let N be a subsemigroup of $\mathbb{N}$ and let P be an N-type set. We put

$$\mathrm{Inj}_P(M) = \{f \in \mathrm{Inj}(M) \mid (|\, M \setminus Mf \,|, T_f) \in P\}.$$

We have an easy

Observation 4.8. $\mathrm{Inj}_P(M)$ *is a normal subsemigroup of* $\mathrm{Inj}(M)$*, and* $\mathrm{Inj}_P(M) \subseteq \mathrm{Inj}_N(M)$.

Proof. Let $f, g \in \mathrm{Inj}_P(M), h = fg$, and $m = |\, M \setminus Mf\,|, n = |\, M \setminus Mg\,|$. So $(m, T_f), (n, T_g) \in P$. By Proposition 4.5 we have $\mathcal{P}(T_h, T_f, T_g)$ and so $(m+n, T_h) \in P$ by condition (iii) for P. Since $|\, M \setminus Mh\,| = m+n$, we obtain $h \in \mathrm{Inj}_P(M)$. Clearly, $\mathrm{Inj}_P(M)$ is normal and $\mathrm{Inj}_P(M) \subseteq \mathrm{Inj}_N(M)$. $\square$

We give two examples of extreme cases. Let $N = N(G)$ and $P = P(G)$.

First assume that $G_{\mathrm{fin}} \subseteq \mathrm{Inj}^{\aleph_1}(M)$. Then $T_f = \widehat{1}$ for each $f \in G_{\mathrm{fin}}$, so $P = N \times \{\widehat{1}\}$ and $\mathrm{Inj}_P(M) = \mathrm{Inj}_N^{\aleph_1}(M)$.

Secondly, assume that $G_{\mathrm{grp}} = S(M)$. We claim that then $P = N \times \mathfrak{T}_M$. Indeed, choose any $n \in N$ and $T \in \mathfrak{T}_M$. There is $f \in G_{\mathrm{fin}}$ with $|\, M \setminus Mf\,| = n$. Let M_1 be the $f^{\pm 1}$-closure of $M \setminus Mf$, which is countable. Put $M_2 = M \setminus M_1$ and let $f_i = f \restriction M_i$ $(i = 1, 2)$. Then $f_2 \in S(M_2)$. We put $f' = \mathrm{id}_{M_1} \dot{\cup} f_2^{-1} \in S(M)$. Also, there is $g \in S(M)$ with $T_g = T$. Then $f'g \in S(M) = G_{\mathrm{grp}}$, so $ff'g \in G$, $|\, M \setminus Mff'g\,| = |\, M \setminus Mf\,| = n$, and $T_{ff'g} = T_g = T$, showing $(n, T) \in P$ and our claim. Hence $\mathrm{Inj}_P(M) = \mathrm{Inj}_N(M)$.

Lemma 4.9. *If $f, g \in \mathrm{Inj}_{\mathrm{fin}}(M) \cup \mathrm{Inj}_{\aleph_0}(M)$, then*

$$f \in g^{S(M)} S^{\aleph_1}(M) \iff \big(|\, M \setminus Mf\,| = |\, M \setminus Mg\,| \text{ and } T_f = T_g\big).$$

Proof. " $\Rightarrow$ " : The first condition $|\, M \setminus Mf\,| = |\, M \setminus Mg\,|$ is clear.

If $f = g^h \cdot k$ for $h \in S(M)$ and $k \in S^{\aleph_1}(M)$, then let

$$M_1' = [k] \cup (M \setminus Mf) \cup (M \setminus Mg) \cup \mathrm{orb}(f, \omega) \cup \mathrm{orb}(g, \omega),$$

and let M_1 be the $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}, k^{\pm 1}\}$-closure of M_1', which is countable. Set $M_2 = M \setminus M_1$ and consider the restrictions f_2, g_2, k_2, h_2 of f, g, k, h to M_2, respectively. It follows that $k_2 = \mathrm{id}_{M_2}, f_2 = g_2^{h_2}$, and $T_f = T_{f_2} = T_{g_2} = T_g$, as required.

" $\Leftarrow$ " : We let M_1 be the $\{f^{\pm 1}, g^{\pm 1}\}$-closure of

$$M_1' = (M \setminus Mf) \cup (M \setminus Mg) \cup \mathrm{orb}(f, \omega) \cup \mathrm{orb}(g, \omega),$$

which is countable, and let $M_2 = M \setminus M_1$. Let f_i, g_i be the restrictions of f, g on M_i $(i = 1, 2)$. Then $M_1 \setminus M_1 f_1 = M \setminus Mf$ and $M_1 \setminus M_1 g_1 = M \setminus Mg$. We define $h_1 \in S(M_1)$ such that

$$h_1 \restriction (M_1 \setminus M_1 g_1) : M_1 \setminus M_1 g_1 \longrightarrow M_1 \setminus M_1 f_1$$

is any bijection and

$$h_1 \restriction M_1 g_1 : M_1 g_1 \longrightarrow M_1 f_1 \ (x g_1 \mapsto x f_1)$$

which is also bijective. Then $f_1 = g_1 h_1$. Also, $f_2, g_2 \in S(M_2)$ which are conjugates by $T_{f_2} = T_f = T_g = T_{g_2}$. We write $f_2 = (g_2)^{h_2}$, $h'_2 = \mathrm{id}_{M_1} \dot{\cup} h_2$ and $h'_1 = h_1 \dot{\cup} \mathrm{id}_{M_2}$. Thus $|\, h'_1 \,| \leq \aleph_0$ and $f = g^{h'_2} h'_1$ is as required. $\square$

Lemma 4.10. *Let $G_{\mathrm{grp}} \supseteq S^{\aleph_1}(M)$ and $G_{\mathrm{fin}} \not\subseteq \mathrm{Inj}^{\aleph_1}(M)$. Then $G_{\mathrm{fin}} = \mathrm{Inj}_{P(G)}(M)$.*

Proof. The inclusion $G_{\mathrm{fin}} \subseteq \mathrm{Inj}_{P(G)}(M)$ is trivial. For the converse, let $f \in \mathrm{Inj}_{P(G)}(M)$. So there is $g \in G_{\mathrm{fin}}$ with $|\, M \setminus Mf \,| = |\, M \setminus Mg \,|$ and $T_f = T_g$. By Lemma 4.9 and the assumption on G_{grp}, we obtain $f \in g^{S(M)} S^{\aleph_1}(M) \subseteq G$. Hence $f \in G_{\mathrm{fin}}$. $\square$

5 Characterizing G_{fin} for uncountable sets M

5.1 The case: G contains a permutation with infinite support

We are ready to characterize the normal subsemigroups of $\mathrm{Inj}(M)$ under the restrictions of this section. By Theorem 3.10 we have $S^{\aleph_1}(M) \subseteq G_{\mathrm{grp}}$.

Theorem 5.1. *Let M be uncountable, $G \lhd \mathrm{Inj}(M)$ with $G_{\mathrm{fin}} \neq \emptyset$ and $S^{\aleph_1}(M) \subseteq G_{\mathrm{grp}}$.*
Then there is a subsemigroup $N \subseteq (\mathbb{N}, +)$ such that $G_{\mathrm{fin}} \subseteq \mathrm{Inj}_N(M)$. Moreover, we have:

(i) If $G_{\mathrm{fin}} \subseteq \mathrm{Inj}^{\aleph_1}(M)$, then $G_{\mathrm{fin}} = \mathrm{Inj}^{\aleph_1}_N(M)$.

(ii) If $G_{\mathrm{fin}} \not\subseteq \mathrm{Inj}^{\aleph_1}(M)$, there is an N-type set P such that $G_{\mathrm{fin}} = \mathrm{Inj}_P(M)$.

(iii) If $G_{\mathrm{grp}} = S(M)$, then $G_{\mathrm{fin}} = \mathrm{Inj}_N(M)$.

All these combinations give rise to normal subsemigroups $G \lhd \mathrm{Inj}(M)$.

Proof. Let $N = N(G)$. The descriptions of G_{fin} in (i) and (iii) follow from Lemma 4.1 (with $\alpha = \aleph_1$, respectively $\alpha = |\, M \,|^+$) and in (ii) from Lemma 4.10. The last statement is immediate by Observation 4.8. $\square$

5.2 The case $G \cap S(M) = 1$

In this section, we assume that $G \cap S(M) = 1$. Then it may be the case that there are $f \in G_{\text{fin}}$ and $g \in \operatorname{Inj}(M)$ with $|\, M \setminus Mf\,| = |\, M \setminus Mg\,|$ and $T_f = T_g$, but $\overline{f} \neq \overline{g}$ and $g \notin G_{\text{fin}}$. (For instance, we may choose any $f \in \operatorname{Inj}(M) \setminus S(M)$ with $\overline{f}(n) \leq \aleph_0$ for some $n \in \mathbb{N}$. Let G be the normal subsemigroup of $\operatorname{Inj}(M)$ generated by f. If $g \in \operatorname{Inj}(M)$ with $|\, M \setminus Mf\,| = |\, M \setminus Mg\,|$ and $\overline{g}(n) \neq \overline{f}(n)$, then $g \notin G$ by Lemma 3.6 and Observation 3.5.) Then $G_{\text{fin}} \neq \operatorname{Inj}_{P(G)}(M)$, so we do not have the characterization of Theorem 5.1.

For any subset $B \subseteq \operatorname{Inj}_{\text{fin}}(M)$ let

$$P(B) = \{(|\, M \setminus Mf\,|, T_f) \mid f \in B\} \subseteq \mathfrak{P} \tag{5.1}$$

We also say that $(k, T) \in P$ is *reducible* if and only if there are two types (n_1, T_1), $(n_2, T_2) \in P$ such that $k = n_1 + n_2$ and $P(T, T_1, T_2)$ holds. Otherwise, (k, T) is called *irreducible.*

Theorem 5.2. *If M is uncountable, $G \lhd \operatorname{Inj}(M)$, $G \cap S(M) = 1$ and $P = P(G)$, then*

$$G_{\text{fin}} = B \,\dot{\cup}\, \{f \in \operatorname{Inj}(M) \mid (|\, M \setminus Mf\,|, T_f) \in P \textit{ is reducible}\} \tag{5.2}$$

and $B \subseteq \operatorname{Inj}_{\text{fin}}(M)$ is a normal subset which satisfies

$$P(B) = \{(n, T) \in P \mid (n, T) \textit{ is irreducible}\}. \tag{5.3}$$

Conversely, each righthand side of the displayed equation (5.2) is a subsemigroup of $\operatorname{Inj}_{\text{fin}}(M)$ normal in $\operatorname{Inj}(M)$. Moreover, in this case G_{fin} is the subsemigroup generated by B.

Proof. Put $B = \{f \in G_{\text{fin}} \mid (|\, M \setminus Mf\,|, T_f) \text{ irreducible in } P\}$. Then B is normal in $\operatorname{Inj}(M)$ and we claim that (5.2) holds.

If $h \in \operatorname{Inj}(M)$ and $(|\, M \setminus Mh\,|, T_h) \in P(G)$ is reducible, then there are $f, g \in G_{\text{fin}}$ such that $|\, M \setminus Mh\,| = |\, M \setminus Mf\,| + |\, M \setminus Mg\,|$ and $\mathcal{P}(T_h, T_f, T_g)$ holds. By Proposition 4.5 it follows that $h \in f^{S(M)} g^{S(M)} \subseteq G_{\text{fin}}$. This is one inclusion of (5.2), and the converse inclusion holds trivially.

To verify (5.3), let $(n, T) \in P(G)$ be irreducible. So there is $f \in G_{\text{fin}}$ with $|\, M \setminus Mf\,| = n$ and $T_f = T$. By definition, then $f \in B$, proving (5.3).

Given a subsemigroup $N \subseteq \mathbb{N}$ and an N-type set P, then the corresponding righthand side of (5.2) is a normal subsemigroup of $\operatorname{Inj}(M)$ as seen by the proof of Observation 4.8.

It remains to show that G_{fin} is generated by B. Let $h \in G_{\text{fin}} \setminus B$. First assume that there are two irreducible types $(n_1, T_1), (n_2, T_2) \in P$ such that $|\, M \setminus Mh \,| = n_1 + n_2$ and $P(T_h, T_1, T_2)$. Then $(n_1, T_1), (n_2, T_2) \in P(B)$, so there are $f, g \in B$ with $n_1 = |\, M \setminus M_f \,|, T_1 = T_f, n_2 = |\, M \setminus M_g \,|, T_2 = T_g$. By Proposition 4.5, we obtain $h \in f^{S(M)} g^{S(M)} \subseteq B \cdot B$. In the general case, an induction shows that h is a finite product of elements from B. $\square$

5.3 The cases $G \cap S(M) = \mathrm{Fin}(M)$ and $G \cap S(M) = \mathrm{Alt}(M)$

In this case we adopt an equivalence relation $\approx$ on $\mathrm{Inj}(M)$ from Mesyan [12, Definition 19] and say that $f \approx g$ for $f, g \in \mathrm{Inj}(M)$ if the following conditions hold:

(i) $F = \{n \in \mathbb{N} \mid \overline{f}(n) \neq \overline{g}(n)\}$ is finite, and if $n \in F$, then $\overline{f}(n), \overline{g}(n) \in \mathbb{N}$.

(ii) $\overline{f}(\aleph_0) = \overline{g}(\aleph_0)$

(iii) $|\, M \setminus Mf \,| = |\, M \setminus Mg \,|$

A set $B \subseteq \mathrm{Inj}(M)$ is $\approx$*-closed* if for any $f \in B$, $f \approx g \in \mathrm{Inj}(M)$ implies $g \in B$. Clearly, then B is normal in $\mathrm{Inj}(M)$. The following characterization of $\approx$ can be shown just as in [12]. It rests on the effect of multiplying one or two infinite orbits by a transposition.

Proposition 5.3. (Mesyan [12, Proposition 24]) *Let M be any infinite set and $f, g \in \mathrm{Inj}(M) \setminus S(M)$. Then $f \approx g$ if and only if $f \in \mathrm{Fin}(M)(g^{S(M)})\,\mathrm{Fin}(M)$.*

For uncountable sets M, we can strengthen this result as follows.

Proposition 5.4. *Let M be any uncountable set and $f, g \in \mathrm{Inj}_{\text{fin}}(M) \setminus S(M)$. Then $f \approx g$ if and only if $f \in \mathrm{Alt}(M)(g^{S(M)})\,\mathrm{Alt}(M)$.*

Proof. The 'if'-direction is immediate by Proposition 5.3. Hence we may assume $f \approx g$. Choose $n \in \mathbb{N}_\infty$ such that $\overline{f}(n)$ is uncountable, hence $\overline{g}(n) = \overline{f}(n)$. Let A (resp. B) be the union of countably-infinitely many n-orbits of f (resp. g). Let M_1 be the $\{f^{\pm 1}, g^{\pm 1}\}$-closure of the set

$$(M \setminus Mf) \cup (M \setminus Mg) \cup \mathrm{orb}(f, \omega) \cup \mathrm{orb}(g, \omega) \cup A \cup B,$$

which is countable, and $M_2 = M \setminus M_1$. Let $f_i = f \restriction M_i$, $g_i = g \restriction M_i$ for $i = 1, 2$. Then $f_1 \approx g_1$ in $\mathrm{Inj}(M_1)$ and $\overline{f_1}(n) = \overline{g_1}(n) = \aleph_0$. Applying Proposition 5.3 we obtain

$f_1 \in \mathrm{Fin}(M_1)(g_1^{S(M_1)})\,\mathrm{Fin}(M_1)$, and [12, Lemma 26] using that $\overline{f_1}(n) = \overline{g_1}(n) = \aleph_0$ implies $f_1 \in \mathrm{Alt}(M_1)(g_1^{S(M_1)})\,\mathrm{Alt}(M_1)$. Also $f_2, g_2 \in S(M_2)$ and $f_2 \approx g_2$, so $\overline{f_2} = \overline{g_2}$. Hence $f_2 \in g_2^{S(M_2)}$. Thus $f \in \mathrm{Alt}(M)(g^{S(M)})\,\mathrm{Alt}(M)$ as needed. $\square$

This result will enable us to use in Theorem 5.5 the same relation $\approx$ for both cases $G_{\mathrm{grp}} = \mathrm{Fin}(M)$ and $G_{\mathrm{grp}} = \mathrm{Alt}(M)$, which provides a contrast to the result for countable sets M, cf. [12, Theorem 34].

Theorem 5.5. *Let M be uncountable, $G \lhd \mathrm{Inj}(M)$, $G_{\mathrm{grp}} = \mathrm{Fin}(M)$ or $G_{\mathrm{grp}} = \mathrm{Alt}(M)$, $G_{\mathrm{fin}} \neq \emptyset$ and $P = P(G)$. Then*

$$G_{\mathrm{fin}} = B \mathbin{\dot\cup} \{f \in \mathrm{Inj}(M) \mid (|\, M \setminus Mf \,|, T_f) \in P \text{ is reducible}\} \tag{5.4}$$

and $B \subseteq \mathrm{Inj}_{\mathrm{fin}}(M)$ is a $\approx$-closed subset which satisfies

$$P(B) = \{(n, T) \in P \mid (n, T) \text{ is irreducible}\}. \tag{5.5}$$

Conversely, each righthand side of the displayed equation (5.5) is a normal subsemigroup of $\mathrm{Inj}_{\mathrm{fin}}(M)$. Moreover, in this case G_{fin} is the subsemigroup generated by B.

Proof. Assume $G_{\mathrm{grp}} = \mathrm{Fin}(M)$. Put

$$B = \{f \in G_{\mathrm{fin}} \mid (|\, M \setminus Mf \,|, T_f) \text{ irreducible in } P\}.$$

Then B is $\approx$-closed by Proposition 5.3 and $\mathrm{Fin}(M) \subseteq G$. The remaining arguments for the theorem are the same as in Theorem 5.2.

Now let $G_{\mathrm{grp}} = \mathrm{Alt}(M)$. We can follow the above argument, but we use Proposition 5.4 to get that B is $\approx$-closed to obtain the result. $\square$

6 Maximal normal subsemigroups of $\mathrm{Inj}(M)$

We determine the maximal normal subsemigroups of $\mathrm{Inj}(M)$.

Theorem 6.1. *The following constitute all the maximal normal subsemigroups of $\mathrm{Inj}(M)$ where $\kappa = |\, M \,|$:*

(i) $S^{\kappa}(M) \mathbin{\dot\cup} \mathrm{Inj}_{\mathrm{fin}}(M) \mathbin{\dot\cup} \dot\bigcup_{\aleph_0 \leq \mu \leq \kappa} \mathrm{Inj}_{\mu}^{\kappa^+}(M)$.

(ii) $S(M) \mathbin{\dot\cup} \mathrm{Inj}_{\mathbb{N}\setminus\{1\}}(M) \mathbin{\dot\cup} \dot\bigcup_{\aleph_0 \leq \mu \leq \kappa} \mathrm{Inj}_{\mu}^{\kappa^+}(M)$.

(iii) $S(M)\dot\cup\,\mathrm{Inj}_{\mathrm{fin}}(M)\dot\cup\,\dot\bigcup_{\mu\in X}\mathrm{Inj}_{\mu}^{\kappa^+}(M)$*, for some* $\aleph_0\le\mu'\le\kappa$ *and* $X=\{\mu\mid\mu\ne\mu',\aleph_0\le\mu\le\kappa\}$.

Each proper normal subsemigroup of $\mathrm{Inj}(M)$ *is contained in a maximal one.*

Proof. By Theorems 3.10 and 5.1, the above sets in (i)-(iii) are normal subsemigroups of $\mathrm{Inj}(M)$. Clearly, $\mathbb{N}\setminus\{1\}$ is the greatest proper subsemigroup of $\mathbb{N}$. Hence, by Theorems 3.10 and 5.1, each proper, normal subsemigroup of $\mathrm{Inj}(M)$ is contained in one of the subsemigroups of (i)-(iii). Hence these are maximal. □

Consequently, if $|\,M\,|=\aleph_i$ (i an ordinal), then $\mathrm{Inj}(M)$ has precisely $|\,i\,|+3$ maximal normal subsemigroups. In the contrast we note:

Corollary 6.2. $\mathrm{Inj}(M)$ *contains precisely* $2^{c(M)^{\aleph_0}}$ *normal subsemigroups, where* $c(M)=|\,\{\mu\mid\mu$ *cardinal*$,\mu\le|\,M\,|\}\,|$.

For instance, if $|\,M\,|=\aleph_0$ or $|\,M\,|=\aleph_1$, we have $c(M)=\aleph_0$ and $2^{c(M)^{\aleph_0}}=2^{2^{\aleph_0}}$.

Proof. We can obtain $2^{c(M)^{\aleph_0}}$ normal subsemigroups as follows. For any set X of functions $T:\mathbb{N}_\infty\longrightarrow c(M)$ put

$$B_X=\{f\in\mathrm{Inj}(M)\mid|\,M\setminus Mf\,|=1,\overline{f}\in X\},$$

and let

$$G_X=B_X\dot\cup\{f\in\mathrm{Inj}_{\mathrm{fin}}(M)\mid|\,M\setminus Mf\,|\ge 2\}.$$

By Theorem 5.2, G_X is a normal subsemigroup, and $G_X\subseteq G_Y$ if and only if $X\subseteq Y$. Since the powerset of a set of size $c(M)^{\aleph_0}$ contains an antichain of subsets of size $2^{c(M)^{\aleph_0}}$, we even obtain such a large antichain in the lattice of normal subsemigroups of $\mathrm{Inj}(M)$.

It remains to show that $2^{c(M)^{\aleph_0}}$ is the maximal number of normal subsemigroups G of $\mathrm{Inj}(M)$. If M is countable, this is clear since $|\,\mathrm{Inj}(M)|=2^{\aleph_0}$. Hence we may assume that M is uncountable. By Theorem 3.10, we have to show that there are no more than $2^{c(M)^{\aleph_0}}$ choices for G_{fin}. Recall that $\mathbb{N}$ contains only countably many subsemigroups, cf. [16]. Hence it suffices to consider the possibilities for Theorems 5.1(ii), 5.2 and 5.5. For Theorem 5.1(ii), let N be any subsemigroup of $\mathbb{N}$. Any N-type set P contains for each $n\in\mathbb{N}$ at most $c(M)^{\aleph_0}$ pairs (n,T) with $T\in\mathfrak{T}_M$. Hence there are at most $2^{c(M)^{\aleph_0}}$ distinct N-type sets P, and consequently the number of possibilities for G_{fin} as in Theorem 5.1(ii) has the same upper bound. In the situation of Theorems 5.2 and 5.5, G_{fin} is generated by a normal s u bset $B\subseteq\mathrm{Inj}_{\mathrm{fin}}(M)$. Any such B is the union of conjugacy classes $f^{S(M)}$ with $f\in\mathrm{Inj}_{\mathrm{fin}}(M)$. There are $c(M)^{\aleph_0}$ possible choices for $\overline{f}$ and thus, by Lemma 3.6, the same number of choices for $f^{S(M)}$. Consequently, the number of possibilities for B and hence for G_{fin} is bounded by $2^{c(M)^{\aleph_0}}$. □

References

[1] R. Baer, *Die Kompositionsreihe der Gruppe aller eineindeutigen Abbildungen einer unendlichen Menge auf sich*, Stud. Math. **5** (1934), 15 – 17.

[2] A. H. Clifford, G. P. Preston, *Algebraic Theory of Semigroups*, Vol. II, Math. Surveys No. **7**, Amer. Math. Soc., Providence, R.I. (1967).

[3] M. Droste, *Cubes of conjugacy classes covering the infinite symmeric groups*, Trans. Amer. Math. Soc. **288** (1985), 381 – 393.

[4] M. Droste, *Squares of conjugacy classes in the infinite symmeric groups*, Trans. Amer. Math. Soc. **303** (1987), 503 – 515.

[5] M. Droste, R. Göbel, *On a theorem of Baer, Schreier, and Ulam for permutations*, Journal of Algebra **58** (1979), 282 – 290.

[6] M. Droste, R. Göbel, *Products of conjugate permutations*, Pacific Journal **94** (1981), 47 – 60.

[7] E. Hotzel, *Maximality properties of some subsemigroups of Baer-Levi semigroups*, Semigroup Forum **51** (1995), 153 – 190.

[8] T. Jech, *Set Theory*, Springer Verlag, Berlin, Heidelberg, New York (2000).

[9] D. Lindsey, B. Madison, *Conjugation of injections by permutations*, Semigroup Forum **12** (1976), 63 – 70.

[10] V. Maltcev, J. D. Mitchell, N. Ruškuc, *The Bergman property for semigroups*, J. London Math. Soc. **80** (2009), 212 – 232.

[11] Z. Mesyan, *Conjugation of injections by permutations*, Semigroup Forum **81** (2010), 297–324.

[12] Z. Mesyan, *Monoids of injective maps closed under conjugation by permutations*, Israel J. Math. **189** (2012), 287-305.

[13] G. Moran, *On planar eulerian graphs and permutations*, Trans. Amer. Math. Soc. **287** (1985), 323 – 341.

[14] G. Moran, *The products of conjugacy classes in some infinite groups*, Israel Journ. Math. **50** (1985), 54 – 74.

[15] G. Moran, *Conjugacy classes whose squares are infinite symmetric groups*, Trans. Amer. Math. Soc. **316** (1989), 493 – 521.

[16] W. Y. Sit, M-K. Sui, *On the subsemigroups of* $\mathbb{N}$, Mathematics Magazine **48** (1975), 225 - 227.

[17] J. Schreier and S. Ulam, *Über die Permutationsgruppe der natürlichen Zahlenfolge*, Studia Math. **4** (1933), 134 – 141.

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