

# Weighted Finite Automata over Strong Bimonoids

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**Abstract:** We investigate weighted finite automata over strings and strong bimonoids. Such algebraic structures satisfy the same laws as semirings except that no distributivity laws need to hold. We define two different behaviors and prove precise characterizations for them if the underlying strong bimonoid satisfies local finiteness conditions. Moreover, we show that in this case the given weighted automata can be determinized.

## 1 Introduction

In the seminal paper [31], Schützenberger extended Kleene’s classical result on the coincidence between recognizable and rational languages to the realm of weighted automata, their behaviors, and rational formal power series. Weighted finite automata are classical nondeterministic automata in which the transitions carry weights. These weights may model, e.g., the amount of resources needed for the execution of a transition, or the probability of its successful execution. The weights can be taken from any semiring, therefore weighted automata have both a rich structure theory [5, 11, 21, 29, 30, 34] as well as practical applications in digital image compression [1, 8, 12, 15, 17], natural language processing [7, 19, 26, 28], and probabilistic model checking [2]. In semirings, by definition, the multiplication operation is distributive over addition and this was crucial for almost all of the theory developed so far.

It is the goal of this paper to investigate automata with weights in strong bimonoids; these can be viewed as semirings where the distributivity assumption is dropped. Trivially, all semirings are bimonoids, but there are also many

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natural examples of bimonoids which are not semirings like, e.g., the tropical bimonoid and near-semirings (see Example 1), the interval  $[0, 1]$  with t-conorm and t-norm from multi-valued logic [18], the “string semiring” of all words over an alphabet arising in natural language processing [27], or the algebraic cost structure from algebraic path problems [22]. In fact, every bounded lattice is a strong bimonoid.

The main results of this paper are as follows. First, we define for every weighted finite automaton  $M$  over some bimonoid  $A$  two different kinds of behaviors, called the run semantics and the initial algebra semantics. Both of them are functions assigning to each word  $w$  an element of  $A$  as value, the weight obtained when executing  $M$  on  $w$ . We show that these two semantics coincide if and only if  $A$  is right distributive (cf. Lemma 4).

Secondly, if the addition and multiplication operation of  $A$  are each locally finite (meaning that finitely generated submonoids are finite), every weighted finite automaton over  $A$  assumes only finitely many weights as values; moreover, each value is assumed on a recognizable language of words (cf. Theorem 11).

A fundamental result in classical automata theory states that each non-deterministic finite automaton can be transformed into an equivalent deterministic one. Here we investigate weighted versions of this result. We show that, for each weighted finite automaton, its run semantics can be recognized by a crisp-deterministic weighted automaton if and only if  $A$  is additively and multiplicatively locally finite (cf. Theorem 14). A corresponding result holds also with respect to the initial algebra semantics provided that  $A$  is right distributive (cf. Theorem 21).

These results generalize several theorems from the literature [3, 16, 24, 25] derived for automata over lattice-ordered monoids or semiring-reducts of residuated lattices which are particular semirings (cf. Corollaries 13, 20, and 24). They also apply, for instance, to all bounded lattices, even without distributivity assumption, since these lattices are additively and multiplicatively locally finite.

## 2 Algebraic notions

Here we collect standard definitions concerning semirings, formal power series, and matrices. For a more detailed introduction to these concepts we refer the reader to [10, 11, 21, 30].

### 2.1 Strong bimonoids and semirings

A *bimonoid* is a structure  $(A, +, \cdot, 0, 1)$  consisting of a set  $A$ , two binary operations  $+$  and  $\cdot$  on  $A$  and two constants  $0, 1 \in A$  such that  $(A, +, 0)$  and  $(A, \cdot, 1)$  are monoids. As usual, we identify the structure  $(A, +, \cdot, 0, 1)$  with its carrier set  $A$ . We call  $A$  a *strong bimonoid* if the operation  $+$  is commutative and  $0$  acts as multiplicative zero, i.e.,  $a \cdot 0 = 0 = 0 \cdot a$  for every  $a \in A$ . We say that a strong bimonoid  $A$  is *right distributive*, if it satisfies  $(a + b) \cdot c = a \cdot c + b \cdot c$  for every  $a, b, c \in A$ ; we call  $A$  *left distributive*, if  $a \cdot (b + c) = a \cdot b + a \cdot c$  for every  $a, b, c \in A$ . Then a *semiring* is a strong bimonoid which is left and right distributive.

**Example 1.**

1. The *tropical bimonoid* is the strong bimonoid  $(\mathbb{N}_\infty, +, \min, 0, \infty)$  with  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$  and the usual extensions of  $+$  and  $\min$  from  $\mathbb{N}$  to  $\mathbb{N}_\infty$ . We note that it is not a semiring, because there are  $a, b, c \in \mathbb{N}_\infty$  with  $\min\{a, b + c\} \neq \min\{a, b\} + \min\{a, c\}$  (e.g., take  $a = b = c \neq 0$ ).
2. The *tropical semiring* is the semiring  $(\mathbb{N}_\infty, \min, +, \infty, 0)$ .
3. The algebra  $([0, 1], \oplus, \cdot, 0, 1)$  with the usual multiplication  $\cdot$  of real numbers is a strong bimonoid for, e.g., each of the following two definitions of  $\oplus$  for every  $a, b \in [0, 1]$ :
  - $a \oplus b = a + b - a \cdot b$  (called *algebraic sum* in [18]) and
  - $a \oplus b = \min\{a + b, 1\}$  (called *bounded sum* in [18]).

In neither of the two cases  $([0, 1], \oplus, \cdot, 0, 1)$  is a semiring.

4. Let  $(C, +, 0)$  be a commutative monoid and let  $A$  be the set of all mappings from  $C$  into itself with pointwise addition, composition of mappings, constant mapping zero, and the identity mapping. Then  $A$  constitutes a strong bimonoid satisfying only one distributivity law (which depends on the order used for defining the composition). Such structures are also called near-semirings [20, 33].
5. Let  $\Sigma$  be an alphabet. Consider the strong bimonoid  $(\Sigma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$  where  $\wedge$  is the longest common prefix operation,  $\cdot$  is the usual concatenation of words, and  $\infty$  is a new element such that  $w \wedge \infty = \infty \wedge w = w$  and  $w \cdot \infty = \infty \cdot w = \infty$  for every  $w \in \Sigma^* \cup \{\infty\}$ . This bimonoid occurs in investigations for natural language processing, see [27]. It is clear that  $(\Sigma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$  is left distributive but not right distributive.
6. The *Boolean semiring* is the semiring  $(\mathbb{B}, \vee, \wedge, 0, 1)$  with  $\mathbb{B}$  consisting of the truth values 0 and 1, and  $\vee$  and  $\wedge$  are disjunction and conjunction, respectively.
7. We note that there are only two strong bimonoids with exactly two elements: the field with two elements and the Boolean semiring (since addition is determined by whether  $1 + 1 = 0$  or  $1 + 1 = 1$ ). However, there are strong bimonoids with 3 elements which are not semirings, take, e.g.,  $(\{0, 1, 2\}, \max, \hat{\cdot}, 0, 1)$  where  $a \hat{\cdot} b = (a \cdot b) \bmod 3$  for every  $a, b \in \{0, 1, 2\}$ .
8. Bounded lattices (lattices containing a greatest element 1 and a smallest element 0) are strong bimonoids. As is well known, there are large classes of lattices that are not distributive [14].
9. Moreover, bounded distributive lattices, semiring-reducts of semi-lattice ordered monoids and of complete residuated lattices, and Brouwerian lattices are semirings.

*From now on and in the rest of the paper, we assume that  $\Sigma$  is an arbitrary alphabet, i.e., a finite non-empty set, and  $(A, +, \cdot, 0, 1)$  denotes an arbitrary strong bimonoid unless specified otherwise.*

## 2.2 Formal power series

A *formal power series*, for short, *series*, (over  $\Sigma$  and  $A$ ) is a mapping  $\varphi : \Sigma^* \rightarrow A$ . Instead of  $\varphi(w)$  we write  $(\varphi, w)$  for every  $w \in \Sigma^*$ . The set of all series over  $\Sigma$  and  $A$  is denoted by  $A\langle\langle \Sigma^* \rangle\rangle$ . The *image* of  $\varphi$  is the set  $\text{im}(\varphi) = \{(\varphi, w) \in A \mid w \in \Sigma^*\}$ . For every  $a \in A$ , we define  $\varphi_{=a} = \varphi^{-1}(a) = \{w \in \Sigma^* \mid (\varphi, w) = a\}$ .

Let  $a \in A$  and  $\varphi \in A\langle\langle \Sigma^* \rangle\rangle$ . The *scalar multiplication of  $a$  and  $\varphi$*  is the series  $a \cdot \varphi \in A\langle\langle \Sigma^* \rangle\rangle$  defined by  $(a \cdot \varphi, w) = a \cdot (\varphi, w)$  for every  $w \in \Sigma^*$ .

Let  $\varphi_1, \varphi_2 \in A\langle\langle \Sigma^* \rangle\rangle$ . The *sum of  $\varphi_1$  and  $\varphi_2$*  is the series  $\varphi_1 + \varphi_2 \in A\langle\langle \Sigma^* \rangle\rangle$  defined by  $(\varphi_1 + \varphi_2, w) = (\varphi_1, w) + (\varphi_2, w)$  for every  $w \in \Sigma^*$ . The commutativity and associativity of the addition of  $A$  carry over to the sum of series.

Let  $L \subseteq \Sigma^*$ . The *characteristic function  $\mathbf{1}_L \in A\langle\langle \Sigma^* \rangle\rangle$  of  $L$*  is for every  $w \in \Sigma^*$  defined as  $(\mathbf{1}_L, w) = 1$  if  $w \in L$  and  $(\mathbf{1}_L, w) = 0$  otherwise.

## 2.3 Matrices

Let  $P, Q$ , and  $R$  be sets. If  $f : P \rightarrow Q$  and  $g : Q \rightarrow R$  are functions, we denote their composition by  $f;g$  (i.e., apply first  $f$ , then  $g$ ). We let  $Q^P$  denote the set of all functions from  $P$  to  $Q$ .

Let  $Q$  be a finite non-empty set. A mapping  $M : Q \times Q \rightarrow A$  is called a  $Q \times Q$ -*matrix over  $A$* , and a mapping  $v : Q \rightarrow A$  is called a  $Q$ -*vector over  $A$* . For every  $M \in A^{Q \times Q}$ ,  $v \in A^Q$ , and  $q_1, q_2 \in Q$  we write  $M_{q_1, q_2}$  instead of  $M(q_1, q_2)$ , and  $v_{q_1}$  instead of  $v(q_1)$ . If  $A$  is a particular ordered set (e.g., the interval  $[0, 1]$ ), then matrices are called *fuzzy relations*, and vectors are called *fuzzy subsets* in the literature.

Now let  $M_1, M_2 \in A^{Q \times Q}$  and  $v_1, v_2 \in A^Q$ . Then we define the *matrix product*  $M_1 \cdot M_2 \in A^{Q \times Q}$ , the *matrix-vector products*  $v_1 \cdot M_1 \in A^Q$  and  $M_1 \cdot v_1 \in A^Q$ , and the *scalar product*  $v_1 \cdot v_2 \in A$  as follows for every  $q_1, q_2 \in Q$ :

$$\begin{aligned} (M_1 \cdot M_2)_{q_1, q_2} &= \sum_{q \in Q} (M_1)_{q_1, q} \cdot (M_2)_{q, q_2} , \\ (v_1 \cdot M_1)_{q_1} &= \sum_{q \in Q} (v_1)_q \cdot (M_1)_{q, q_1} , \\ (M_1 \cdot v_1)_{q_1} &= \sum_{q \in Q} (M_1)_{q_1, q} \cdot (v_1)_q , \\ v_1 \cdot v_2 &= \sum_{q \in Q} (v_1)_q \cdot (v_2)_q . \end{aligned}$$

Recall that the addition of  $A$  is commutative and that  $Q$  is non-empty; thus, the sums on the right-hand sides are well defined. We define the  $Q$ -*unit matrix*  $I_Q \in A^{Q \times Q}$  as follows for every  $q_1, q_2 \in Q$ :

$$(I_Q)_{q_1, q_2} = \begin{cases} 1, & \text{if } q_1 = q_2 ; \\ 0, & \text{otherwise.} \end{cases}$$

The following result is of fundamental importance in the theory of weighted automata over semirings; it is straightforward by elementary calculations.

**Lemma 2.** *If  $A$  is a semiring, then the matrix product and matrix-vector products (whenever defined) are associative, and  $(A^{Q \times Q}, \cdot, I_Q)$  is a monoid.*

Note that Lemma 2 fails in general, if  $A$  is not left or not right distributive.

### 3 Weighted finite automata

In this section, we introduce weighted finite automata over bimonoids and different definitions of their behaviors (semantics). Then we investigate conditions under which these behaviors coincide. Recall that  $(A, +, \cdot, 0, 1)$  is an arbitrary strong bimonoid.

A *weighted finite automaton* (for short: *wfa*) over  $\Sigma$  and  $A$  is a quadruple  $M = (Q, I, \tau, F)$  such that  $Q$  is a finite non-empty set (of *states*),  $I \in A^Q$  (initial weight vector),  $\tau : \Sigma \rightarrow A^{Q \times Q}$  (transition mapping), and  $F \in A^Q$  (final weight vector).

We define three different semantics for a wfa  $M$ , called *run semantics*, *initial algebra semantics*, and *free monoid semantics*.

The run semantics is the standard semantics of wfa (cf. Section VI.6 of [11]). Given a word  $w$  of length  $n$ , we consider all possible paths (= sequences of states) of length  $n + 1$ , determine their individual weights, then form the sum. Note that for various concrete strong bimonoids or semirings  $A$ , this run behavior has natural interpretations as counting the successful paths for  $w$ , determining their cost or reliability, etc., cf. e.g. [10].

Formally, the *r-behavior* of  $M$ , denoted by  $\llbracket M \rrbracket_r$ , is the series in  $A\langle\langle \Sigma^* \rangle\rangle$  defined for every  $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$  by letting

$$(\llbracket M \rrbracket_r, w) = \sum_{P \in Q^{n+1}} \text{weight}_M(P, w) ,$$

where for every  $P = (q_0, \dots, q_n) \in Q^{n+1}$  the *weight of  $P$  in  $M$  for  $w$*  is defined as

$$\text{weight}_M(P, w) = I_{q_0} \cdot \tau(\sigma_1)_{q_0, q_1} \cdot \dots \cdot \tau(\sigma_n)_{q_{n-1}, q_n} \cdot F_{q_n} .$$

For the initial algebra semantics, we need to introduce a preliminary notion. A *pointed  $\Sigma$ -algebra* is a triple  $(B, \theta, q)$  such that  $B$  is a set,  $\theta : \Sigma \rightarrow B^B$  is a mapping, and  $q \in B$ . We define a mapping  $h_\theta : \Sigma^* \rightarrow B$ , called the *successive evaluation of  $\theta$* , by letting  $h_\theta(\varepsilon) = q$  and  $h_\theta(w\sigma) = \theta(\sigma)(h_\theta(w))$  for every  $w \in \Sigma^*$  and  $\sigma \in \Sigma$ . In other words, if  $w = \sigma_1 \cdots \sigma_n$ , then  $h_\theta(w) = (\theta(\sigma_1); \dots; \theta(\sigma_n))(q)$ . We call  $(B, \theta, q)$  *finite* if  $B$  is finite.

In a pointed  $\Sigma$ -algebra, the elements  $\theta(\sigma)$  ( $\sigma \in \Sigma$ ) can be viewed as unary operations on  $B$ . In the initial algebra semantics, the letters  $\sigma \in \Sigma$  operate on  $Q$ -vectors by multiplication with a  $Q \times Q$ -matrix  $\theta(\sigma)$  from the right; this operation is extended to words by performing it successively letter after letter. To determine the initial algebra behavior of  $M$  for  $w \in \Sigma^*$ , we start with the initial vector  $I$ , execute  $w$ , at the end apply the final vector  $F$ .

Formally, we define the pointed  $\Sigma$ -algebra  $(A^Q, \theta_\tau, I)$  by letting  $\theta_\tau(\sigma)(v) = v \cdot \tau(\sigma)$  for every  $\sigma \in \Sigma$  and  $v \in A^Q$ . The *i-behavior* of  $M$ , denoted by  $\llbracket M \rrbracket_i$ , is the series in  $A\langle\langle \Sigma^* \rangle\rangle$  defined as follows for every  $w \in \Sigma^*$ :

$$(\llbracket M \rrbracket_i, w) = h_{\theta_\tau}(w) \cdot F .$$

So, if  $w = \sigma_1 \cdots \sigma_n$ , then  $h_{\theta_\tau}(w) = (\theta_\tau(\sigma_1); \dots; \theta_\tau(\sigma_n))(I) = (\dots((I \cdot \tau(\sigma_1)) \cdot \tau(\sigma_2)) \cdot \dots \cdot \tau(\sigma_{n-1})) \cdot \tau(\sigma_n)$ .

Finally, we define the free monoid semantics. Let  $A$  be a semiring and let  $h_\Sigma$  be the unique monoid-morphism from the free monoid  $(\Sigma^*, \cdot, \varepsilon)$  to the monoid

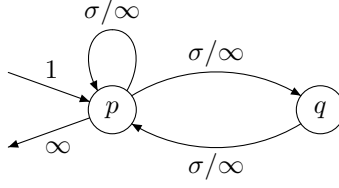


Figure 1: A wfa over the tropical bimonoid.

$(A^{Q \times Q}, \cdot, I_Q)$  extending  $\tau$  (cf. Lemma 2). Then the  $f$ -behavior of  $M$ , denoted by  $\llbracket M \rrbracket_f$ , is the series in  $A\langle\langle \Sigma^* \rangle\rangle$  defined as follows for every  $w \in \Sigma^*$ :

$$(\llbracket M \rrbracket_f, w) = I \cdot h_\Sigma(w) \cdot F.$$

The initial algebra semantics may be considered as more ‘abstract’ than the (combinatorial) run semantics, but has the advantage of permitting algebraic proofs. All kinds of semantics have been employed in the literature for semiring-weighted automata for different purposes, cf. [11, 13].

Let  $x \in \{i, r, f\}$ . A series  $\varphi \in A\langle\langle \Sigma^* \rangle\rangle$  is  $x$ -recognizable if there is a wfa  $M$  over  $\Sigma$  and  $A$  such that  $\llbracket M \rrbracket_x = \varphi$ . We say that two wfa  $M$  and  $M'$  over  $\Sigma$  and  $A$  are  $x$ -equivalent, if  $\llbracket M \rrbracket_x = \llbracket M' \rrbracket_x$ .

We note that we will use the free monoid semantics only in Lemma 5 and Corollary 20.

**Example 3.** Let  $\Sigma = \{\sigma\}$ . We consider the wfa  $M = (Q, I, \tau, F)$  over  $\Sigma$  and the tropical bimonoid  $(\mathbb{N}_\infty, +, \min, 0, \infty)$  with  $Q = \{q, p\}$ ,  $\tau(\sigma)_{p,p} = \tau(\sigma)_{p,q} = \tau(\sigma)_{q,p} = \infty$  and  $\tau(\sigma)_{q,q} = 0$ . Moreover, we define  $I_p = 1$ ,  $F_p = \infty$  and  $I_q = F_q = 0$ . If we neglect those transitions that have weight 0, then we can illustrate  $M$  as in Figure 1. Then,  $(\llbracket M \rrbracket_i, \sigma^n) = (\llbracket M \rrbracket_r, \sigma^n)$  is the  $n$ th Fibonacci-number, for every  $n \geq 0$ . We note that in [11] a similar automaton over the semiring of natural numbers has been used to define the Fibonacci-numbers.

We will see later that, in general, the initial algebra semantics and the run semantics differ (cf. Examples 25 and 26).

Next we obtain a simple characterization when for every wfa  $M$ , its initial algebra semantics and its run semantics coincide.

**Lemma 4.** The following two statements are equivalent:

1.  $A$  is right distributive.
2.  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r$  for every wfa  $M$  over  $\Sigma$  and  $A$ .

PROOF. (1)  $\Rightarrow$  (2): Let  $A$  be right distributive and let  $M = (Q, I, \tau, F)$  and  $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$ . We show that  $(\llbracket M \rrbracket_i, w) = (\llbracket M \rrbracket_r, w)$ . To this end we show by induction on the length  $n$  of  $w$  that for every  $q_n \in Q$  we have

$$h_{\theta_\tau}(w)_{q_n} = \sum_{(q_0, \dots, q_{n-1}) \in Q^n} I_{q_0} \cdot \tau(\sigma_1)_{q_0, q_1} \cdot \dots \cdot \tau(\sigma_n)_{q_{n-1}, q_n}. \quad (1)$$

Indeed, if  $n = 0$ , then  $w = \varepsilon$  and  $h_{\theta_\tau}(\varepsilon)_{q_0} = I_{q_0} = \sum_{() \in Q^0} I_{q_0}$ . Now let  $n > 0$  and  $w = w' \sigma_n$  for some  $w' \in \Sigma^*$  with  $|w'| = n - 1$ . Then

$$\begin{aligned}
h_{\theta_\tau}(w' \sigma_n)_{q_n} &= \theta_\tau(\sigma_n)(h_{\theta_\tau}(w'))_{q_n} \\
&= (h_{\theta_\tau}(w') \cdot \tau(\sigma_n))_{q_n} = \sum_{q_{n-1} \in Q} h_{\theta_\tau}(w')_{q_{n-1}} \cdot \tau(\sigma_n)_{q_{n-1}, q_n} \\
&= \sum_{q_{n-1} \in Q} \sum_{(q_0, \dots, q_{n-2}) \in Q^{n-1}} I_{q_0} \cdot \tau(\sigma_1)_{q_0, q_1} \cdot \dots \cdot \tau(\sigma_{n-1})_{q_{n-2}, q_{n-1}} \cdot \tau(\sigma_n)_{q_{n-1}, q_n} \\
&\hspace{15em} \text{(by induction hypothesis and right distributivity)} \\
&= \sum_{(q_0, \dots, q_{n-1}) \in Q^n} I_{q_0} \cdot \tau(\sigma_1)_{q_0, q_1} \cdot \dots \cdot \tau(\sigma_n)_{q_{n-1}, q_n} .
\end{aligned}$$

Now we have

$$\begin{aligned}
(\llbracket M \rrbracket_i, w) &= h_{\theta_\tau}(w) \cdot F = \sum_{q_n \in Q} h_{\theta_\tau}(w)_{q_n} \cdot F_{q_n} \\
&= \sum_{q_n \in Q} \sum_{(q_0, \dots, q_{n-1}) \in Q^n} I_{q_0} \cdot \tau(\sigma_1)_{q_0, q_1} \cdot \dots \cdot \tau(\sigma_n)_{q_{n-1}, q_n} \cdot F_{q_n} \\
&\hspace{15em} \text{(by Equation (1) and right distributivity)} \\
&= (\llbracket M \rrbracket_r, w) .
\end{aligned}$$

(2)  $\Rightarrow$  (1): Let  $a, b, c \in A$ . Let  $\sigma \in \Sigma$ . We construct the wfa  $M = (Q, I, \tau, F)$  over  $\Sigma$  and  $A$  by  $Q = \{p, q\}$ ,  $I_p = a$ ,  $I_q = b$ ,  $F_p = c$ , and  $F_q$  can be chosen arbitrarily. Moreover,  $\tau(\sigma)_{p,p} = \tau(\sigma)_{q,p} = 1$  and  $\tau(\sigma)_{p,q} = \tau(\sigma)_{q,q} = 0$ . Then  $(\llbracket M \rrbracket_i, \sigma) = (a + b) \cdot c$  and  $(\llbracket M \rrbracket_r, \sigma) = ac + bc$ . Hence  $(a + b) \cdot c = ac + bc$ . ■

For semirings  $A$ , we obtain as consequence the following well-known fact.

**Lemma 5.** *If  $A$  is a semiring, then  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r = \llbracket M \rrbracket_f$  for every wfa  $M$  over  $\Sigma$  and  $A$ .*

PROOF. Lemma 4 yields  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r$ . Moreover,  $\llbracket M \rrbracket_f = \llbracket M \rrbracket_i$  follows from  $(\llbracket M \rrbracket_f, w) = I \cdot h_\Sigma(\sigma_1 \dots \sigma_n) \cdot F = I \cdot \tau(\sigma_1) \cdot \dots \cdot \tau(\sigma_n) \cdot F$  and Lemma 2. ■

Next we introduce deterministic and crisp automata.

**Definition 6.** Let  $M = (Q, I, \tau, F)$  be a wfa over  $\Sigma$  and  $A$ .

- We call  $M$  *deterministic* if there is at most one  $q \in Q$  with  $I_q \neq 0$ , and for every  $\sigma \in \Sigma$  and  $q \in Q$  there is at most one  $q' \in Q$  with  $\tau(\sigma)_{q,q'} \neq 0$ .
- We call  $M$  *crisp* if  $I_q \in \{0, 1\}$  and  $\tau(\sigma)_{p,q} \in \{0, 1\}$  for every  $\sigma \in \Sigma$  and  $p, q \in Q$ .
- $M$  is *crisp-deterministic*, if  $M$  is crisp and deterministic.

If  $M$  is deterministic, then in every row of every matrix  $\tau(\sigma)$  there is at most one entry different from 0. Therefore it is easy to see by induction that for every string  $w = \sigma_1 \dots \sigma_n$  there is a sequence of states  $q_0, \dots, q_n \in Q$  such that for every  $j \in \{0, \dots, n\}$  the vector  $h_{\theta_\tau}(\sigma_1 \dots \sigma_j)$  might only have a non-zero entry at  $q_j$ . Moreover,  $(q_0, \dots, q_n)$  is the only run of  $M$  on  $w$  with possibly non-zero weight, which implies that its weight is equal to  $h_{\theta_\tau}(w) \cdot F$ . Therefore, the initial algebra semantics for  $M$  coincides with the run semantics for  $M$ . This shows the following result.

**Remark 7.** For every deterministic wfa  $M$  we have  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r$ , and if  $M$  is even crisp-deterministic, then  $\text{im}(\llbracket M \rrbracket_r) \subseteq \text{im}(F) \cup \{0\}$ , a finite set.

Finally, we note that we obtain the classical concept of an automaton in our context as follows. A *finite automaton* (for short: *fsa*) over  $\Sigma$  is a wfa  $M$  over  $\Sigma$  and the Boolean semiring  $\mathbb{B}$ . Clearly, then  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r = \llbracket M \rrbracket_f$  by Lemma 5. The *language recognized by  $M$*  is the set  $L(M) \subseteq \Sigma^*$  defined by  $L(M) = \{w \in \Sigma^* \mid (\llbracket M \rrbracket_r, w) = 1\}$ . A language  $L \subseteq \Sigma^*$  is *recognizable* if there is an fsa  $M$  over  $\Sigma$  such that  $L = L(M)$ . As is well-known, for every recognizable language there is a crisp-deterministic fsa  $M$  such that  $L = L(M)$ . Moreover,  $M = (Q, I, \tau, F)$  can be chosen to be *total*, i.e. there is a state  $q \in Q$  with  $I_q = 1$ , and for every  $\sigma \in \Sigma$  and  $q \in Q$  there exists (a unique)  $q' \in Q$  with  $\tau(\sigma)_{q,q'} = 1$ .

Next we characterize series that are i-recognizable and r-recognizable by crisp-deterministic wfa, in terms of recognizable step functions. A series  $\varphi \in A\langle\langle \Sigma^* \rangle\rangle$  over  $\Sigma$  and  $A$  is a *recognizable step function* if there are  $n \in \mathbb{N}$ , recognizable languages  $L_1, \dots, L_n \subseteq \Sigma^*$ , and  $a_1, \dots, a_n \in A$  such that  $\varphi = \sum_{i=1}^n a_i \cdot \mathbb{1}_{L_i}$ .

**Lemma 8.** Let  $\varphi \in A\langle\langle \Sigma^* \rangle\rangle$ . Then  $\varphi$  is a recognizable step function iff there exists a crisp-deterministic wfa  $M$  over  $\Sigma$  and  $A$  such that  $\varphi = \llbracket M \rrbracket_i = \llbracket M \rrbracket_r$ . In particular, if  $\varphi$  is a recognizable step function, then  $\varphi$  is i-recognizable and r-recognizable.

PROOF. “ $\Rightarrow$ ”: Let  $n \in \mathbb{N}$ ,  $L_1, \dots, L_n \subseteq \Sigma^*$ , and  $a_1, \dots, a_n \in A$  such that  $L_1, \dots, L_n$  are recognizable and  $\varphi = \sum_{i=1}^n a_i \cdot \mathbb{1}_{L_i}$ . For every  $i \in \{1, \dots, n\}$ , let  $M_i = (Q_i, I_i, \tau_i, F_i)$  be a deterministic and total fsa over  $\Sigma$  such that  $L(M_i) = L_i$ . We define the wfa  $M = (Q, I, \tau, F)$  as follows:  $Q = Q_1 \times \dots \times Q_n$  and for every  $\sigma \in \Sigma$  and  $(q_1, \dots, q_n), (q'_1, \dots, q'_n) \in Q$ :

$$\begin{aligned} I_{(q_1, \dots, q_n)} &= \begin{cases} 1, & \text{if } (I_i)_{q_i} = 1 \text{ for every } i \in \{1, \dots, n\}; \\ 0, & \text{otherwise,} \end{cases} \\ \tau(\sigma)_{(q_1, \dots, q_n), (q'_1, \dots, q'_n)} &= \begin{cases} 1, & \text{if } \tau_i(\sigma)_{q_i, q'_i} = 1 \text{ for every } i \in \{1, \dots, n\}; \\ 0, & \text{otherwise,} \end{cases} \\ F_{(q_1, \dots, q_n)} &= \sum_{\substack{i \in \{1, \dots, n\} \\ (F_i)_{q_i} = 1}} a_i. \end{aligned}$$

Clearly,  $M$  is crisp-deterministic. Let  $w \in \Sigma^*$  and  $(q_1, \dots, q_n)$  be the unique state in  $Q$  with  $h_{\theta_\tau}(w)_{(q_1, \dots, q_n)} = 1$ . Then  $w \in L_i$  iff  $(F_i)_{q_i} = 1$  for every  $i \in \{1, \dots, n\}$ . Let  $I_w = \{i \in \{1, \dots, n\} \mid w \in L_i\}$ . We obtain  $(\varphi, w) = \sum_{i \in I_w} a_i = F_{(q_1, \dots, q_n)} = (\llbracket M \rrbracket_i, w)$ , proving  $\varphi = \llbracket M \rrbracket_i$ . Remark 7 yields  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r$ .

“ $\Leftarrow$ ”: Let  $M = (Q, I, \tau, F)$  be a crisp-deterministic wfa over  $\Sigma$  and  $A$  such that  $\llbracket M \rrbracket_i = \llbracket M \rrbracket_r = \varphi$ . By Remark 7,  $\text{im}(\varphi)$  is finite. Let  $a \in \text{im}(\varphi)$ . We show that  $\varphi_{=a}$  is recognizable. We define an fsa  $M'_a = (Q, I', \tau', F'_a)$  over  $\Sigma$  by letting for every  $p, q \in Q$  and  $\sigma \in \Sigma$ :  $I'_q = 1$  (in  $\mathbb{B}$ ) iff  $I_q = 1$  (in  $A$ );  $\tau'(\sigma)_{p,q} = 1$  iff  $\tau(\sigma)_{p,q} = 1$ ; and  $(F'_a)_q = 1$  iff  $F_q = a$ . Then  $L(M'_a) = \varphi_{=a}$ . Hence  $\varphi = \sum_{a \in \text{im}(\varphi)} a \cdot \mathbb{1}_{\varphi_{=a}}$  is a recognizable step function. ■

We finish this section with an easy characterization of recognizable step functions.



**Proposition 9.** *Let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$ . Then  $\varphi$  is a recognizable step function iff  $\text{im}(\varphi)$  is finite and  $\varphi_{=a}$  is recognizable for every  $a \in A$ .*

PROOF. Let  $n \in \mathbb{N}$ ,  $L_1, \dots, L_n \subseteq \Sigma^*$  be recognizable, and  $a_1, \dots, a_n \in A$  such that  $\varphi = \sum_{i=1}^n a_i \cdot \mathbf{1}_{L_i}$ . Clearly,  $\text{im}(\varphi)$  is finite. For every  $I \subseteq \{1, \dots, n\}$  we let

$$L'_I = \bigcap_{i \in I} L_i \cap \bigcap_{i \in \{1, \dots, n\} \setminus I} \Sigma^* \setminus L_i \quad \text{and} \quad a_I = \sum_{i \in I} a_i.$$

It follows from the closure properties of the class of recognizable languages that every  $L'_I$  is recognizable. Furthermore, the languages  $L'_I$  ( $I \subseteq \{1, \dots, n\}$ ) form a partitioning of  $\Sigma^*$ . Hence,  $\varphi = \sum_{I \subseteq \{1, \dots, n\}} a_I \cdot \mathbf{1}_{L'_I}$ . So  $\varphi_{=a} = \bigcup \{L'_I \mid I \subseteq \{1, \dots, n\}, a_I = a\}$ , which is recognizable.

For the converse, note that  $\varphi = \sum_{a \in \text{im}(\varphi)} a \cdot \mathbf{1}_{\varphi_{=a}}$  which is a recognizable step function by assumption.  $\blacksquare$

## 4 Recognizable series and determinizability

In this section, we will investigate the relationships between i-recognizable series, r-recognizable series, and recognizable step functions. We also consider conditions under which for every wfa an i-equivalent crisp-deterministic one exists.

We let  $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1)$  be the semiring of natural numbers with the usual addition and multiplication. We will use the following lemma in which  $t + c\mathbb{N} = \{t + cn \mid n \in \mathbb{N}\}$ .

**Lemma 10** ([5], Cor. III.2.4,2.5). *Let  $\rho : \Sigma^* \rightarrow \mathbb{N}$  be an r-recognizable series over  $\Sigma$  and  $\mathcal{N}$ . Then for all  $c, t \in \mathbb{N}$ , the languages  $\rho^{-1}(t)$  and  $\rho^{-1}(t + c\mathbb{N})$  are recognizable.*

We call  $A$  *additively locally finite* (*multiplicatively locally finite*, respectively) if for every finite  $B \subseteq A$ , the smallest submonoid of  $(A, +, 0)$  (of  $(A, \cdot, 1)$ , respectively) containing  $B$  is finite. Now we will show the following important result.

**Theorem 11.** *Let  $A$  be additively and multiplicatively locally finite, and let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  be r-recognizable. Then  $\varphi$  is a recognizable step function.*

PROOF. Let  $M = (Q, I, \tau, F)$  be a wfa with  $\llbracket M \rrbracket_r = \varphi$ . We define  $B = \{I_q \mid q \in Q\} \cup \{\tau(\sigma)_{p,q} \mid p, q \in Q, \sigma \in \Sigma\} \cup \{F_q \mid q \in Q\}$ . Let  $Y$  comprise all finite products of elements from  $B$ . By assumption,  $Y$  is finite.

Note that  $\text{weight}_M(P, w) \in Y$  for every  $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$  and  $P \in Q^{n+1}$ . For each  $a \in Y$  we define a crisp wfa  $M'_a = (Q', I', \tau', F'_a)$  over  $\Sigma$  and  $\mathcal{N}$  as follows:  $Q' = Q \times Y$  and for every  $\sigma \in \Sigma$  and  $(q, y), (q', y') \in Q'$ , let

$$\tau'(\sigma)_{(q,y),(q',y')} = \begin{cases} 1, & \text{if } y' = y \cdot \tau(\sigma)_{q,q'}; \\ 0, & \text{otherwise,} \end{cases}$$

$$I'_{(q,y)} = \begin{cases} 1, & \text{if } y = I_q; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad (F'_a)_{(q,y)} = \begin{cases} 1, & \text{if } y \cdot F_q = a; \\ 0, & \text{otherwise.} \end{cases}$$

Let  $w = \sigma_1 \cdots \sigma_n \in \Sigma^*$  and  $a \in Y$ . We observe that for every path  $P = (q_0, \dots, q_n) \in Q^{n+1}$  with  $\text{weight}_M(P, w) = a$  there is a unique path  $P' = ((q_0, y_0), \dots, (q_n, y_n)) \in (Q')^{n+1}$  with  $\text{weight}_{M'_a}(P', w) = 1$ . Conversely, each path  $P'' \in (Q')^{n+1}$  with non-zero weight in  $M'_a$  for  $w$  arises in this form. It follows that  $(\llbracket M'_a \rrbracket_r, w) \in \mathbb{N}$  is precisely the number of all paths  $P$  in  $Q^{n+1}$  with  $\text{weight}_M(P, w) = a$ . Consequently, in  $A$  we have

$$(\varphi, w) = (\llbracket M \rrbracket_r, w) = \sum_{P \in Q^{n+1}} \text{weight}_M(P, w) = \sum_{a \in Y} (\llbracket M'_a \rrbracket_r, w) a, \quad (\star)$$

where, for every  $m \in \mathbb{N}$ , we write  $ma$  as a shorthand for  $a + a + \cdots + a$  ( $m$  summands). For every  $a \in Y$  we define  $\psi_a \in A\langle\langle \Sigma^* \rangle\rangle$  by letting  $(\psi_a, w) = (\llbracket M'_a \rrbracket_r, w) a$ . So  $\varphi = \sum_{a \in Y} \psi_a$ . It remains to show that each  $\psi_a$  ( $a \in Y$ ) is a recognizable step function. For this we use an argument similar to one used in [9], proof of Prop. 6.3. Choose  $a \in Y$ . The cyclic submonoid  $\langle a \rangle$  of  $(A, +, 0)$  is finite. Choose a minimal  $m_a \in \mathbb{N}$  such that  $m_a a = (m_a + y)a$  for some  $y > 0$ , and let  $c_a$  be the smallest such  $y > 0$ . We put  $d_a = m_a + c_a - 1$ ; note that then  $d_0 = 0$ . Then  $\langle a \rangle = \{0, a, 2a, \dots, d_a a\}$ . So for each  $s \in \mathbb{N}$  there is a uniquely determined  $t \in \{0, \dots, d_a\}$  such that  $sa = ta$ . Note that if  $0 \leq t < m_a$ , then  $sa = ta$  iff  $s = t$ , and if  $m_a \leq t \leq d_a$ , then  $sa = ta$  iff  $s \in t + c_a \mathbb{N}$ . Now let  $L_{a,t} = \{w \in \Sigma^* \mid (\llbracket M'_a \rrbracket_r, w) a = ta\}$ , for each  $t \in \mathbb{N}$  with  $0 \leq t \leq d_a$ ; note that  $L_{a,0} = \Sigma^*$ . We claim that  $L_{a,t}$  is recognizable. We have

- $L_{a,t} = \{w \in \Sigma^* \mid (\llbracket M'_a \rrbracket_r, w) = t\}$  if  $0 \leq t < m_a$ , and
- $L_{a,t} = \{w \in \Sigma^* \mid (\llbracket M'_a \rrbracket_r, w) \in t + c_a \mathbb{N}\}$  if  $m_a \leq t \leq d_a$ .

In each case,  $L_{a,t}$  is recognizable by Lemma 10.

Let  $w \in \Sigma^*$ . By the above, there is a unique number  $t \in \{0, \dots, d_a\}$  such that  $(\psi_a, w) = (\llbracket M'_a \rrbracket_r, w) a = ta$ , and so  $w \in L_{a,t}$ . Hence,

$$\psi_a = \sum_{0 \leq t \leq d_a} ta \cdot \mathbb{1}_{L_{a,t}}. \quad \blacksquare$$

Conversely, assuming that the image of every recognizable series over  $A$  is finite, we can deduce that  $A$  is additively and multiplicatively locally finite. Note that here our alphabet  $\Sigma$  is fixed.

**Lemma 12.** *Let  $|\Sigma| \geq 2$ . If for every wfa  $M$  over  $\Sigma$  and  $A$ ,  $\text{im}(\llbracket M \rrbracket_i)$  or  $\text{im}(\llbracket M \rrbracket_r)$  is finite, then  $A$  is additively and multiplicatively locally finite.*

PROOF. We show that the additive monoid  $(A, +, 0)$  and the multiplicative monoid  $(A, \cdot, 1)$  are locally finite.

For the additive monoid it suffices to show that for every  $a \in A$  the cyclic submonoid of  $(A, +, 0)$  generated by  $a$  is finite because  $+$  is commutative and associative. Let  $a \in A$  and construct the wfa  $M = (\{p, q\}, I, \tau, F)$  with  $I_p = 1$ ,  $I_q = F_p = 0$ , and  $F_q = 1$ . Moreover, for every  $\sigma \in \Sigma$  we define  $\tau(\sigma)_{p,p} = \tau(\sigma)_{q,q} = 1$ ,  $\tau(\sigma)_{p,q} = a$ , and  $\tau(\sigma)_{q,p} = 0$ . Then for every  $\sigma \in \Sigma$  and  $n \in \mathbb{N}$  we have  $(\llbracket M \rrbracket_i, \sigma^n) = (\llbracket M \rrbracket_r, \sigma^n) = a + \cdots + a$  ( $n$  times). Thus, the finite set  $\text{im}(\llbracket M \rrbracket_i) \cap \text{im}(\llbracket M \rrbracket_r)$  contains the cyclic submonoid of  $(A, +, 0)$  generated by  $a$ .

Next we show that the multiplicative monoid is locally finite. Let  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in A$ . We show that the set  $A' = \{a_{l_1} \cdots a_{l_k} \mid k \in \mathbb{N}, l_1, \dots, l_k \in$

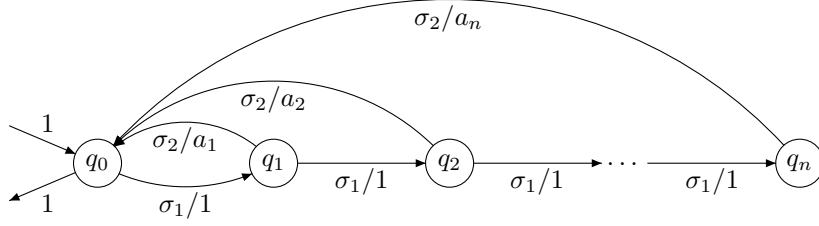


Figure 2: wfa  $M'$  from the proof of Lemma 12.

$\{1, \dots, n\}$  is finite. Let  $\sigma_1, \sigma_2 \in \Sigma$  be distinct symbols. We construct a wfa  $M' = (Q', I', \tau', F')$  over  $\Sigma$  and  $A$  with  $Q' = \{q_0, q_1, \dots, q_n\}$ ,  $I'_{q_0} = F'_{q_0} = 1$  and  $I'_q = F'_q = 0$  for every  $q \in Q' \setminus \{q_0\}$ , and  $\tau'$  is defined as follows (see Figure 2):

- $\tau'(\sigma_1)_{q_{i-1}, q_i} = 1$  for every  $i \in \{1, \dots, n\}$ ,
- $\tau'(\sigma_2)_{q_i, q_0} = a_i$  for every  $i \in \{1, \dots, n\}$ , and
- $\tau'(\sigma)_{q, q'} = 0$  for every other combination of  $\sigma \in \Sigma$  and  $q, q' \in Q'$ .

Then  $(\llbracket M' \rrbracket_i, \sigma_1^{l_1} \sigma_2 \sigma_1^{l_2} \sigma_2 \cdots \sigma_1^{l_k} \sigma_2) = (\llbracket M' \rrbracket_r, \sigma_1^{l_1} \sigma_2 \sigma_1^{l_2} \sigma_2 \cdots \sigma_1^{l_k} \sigma_2) = a_{l_1} \cdots a_{l_k}$  for every  $k \in \mathbb{N}$  and  $l_1, \dots, l_k \in \{1, \dots, n\}$ . Thus,  $A' \subseteq \text{im}(\llbracket M' \rrbracket_i) \cap \text{im}(\llbracket M' \rrbracket_r)$ , and therefore  $A'$  is finite.  $\blacksquare$

A bimonoid  $A$  is called *locally finite*, if for every finite  $B \subseteq A$ , the smallest subbimonoid of  $A$  containing  $B$  is finite. Now, Lemma 12 generalizes the following results of [16, 24].

**Corollary 13.** *Assume that for every wfa over  $A$  there is an  $i$ -equivalent crisp-deterministic wfa over  $A$ . Then  $A$  is locally finite provided that one of the following conditions holds.*

1.  $A$  is a lattice-ordered monoid (cf. only-if part of Theorem 3.4 of [24]).
2.  $A$  is the semiring-reduct of a residuated lattice (cf. only-if part of Theorem 4.2 of [16]).

PROOF. Since both, lattice-ordered monoids and semiring-reducts of residuated lattices are particular strong bimonoids, the two statements follow from Remark 7 and Lemma 12.  $\blacksquare$

The following summarizes our results.

**Theorem 14.** *Let  $|\Sigma| \geq 2$ . Then the following two statements are equivalent:*

1. *For every wfa  $M$  over  $\Sigma$  and  $A$  there is an  $r$ -equivalent crisp-deterministic wfa  $M'$  over  $\Sigma$  and  $A$ .*
2.  *$A$  is additively and multiplicatively locally finite.*

PROOF. (1)  $\Rightarrow$  (2): This follows from Remark 7 and Lemma 12.

(2)  $\Rightarrow$  (1): Immediate by Theorem 11 and Lemma 8.  $\blacksquare$

Next we will investigate properties of  $i$ -recognizable series. The idea of the following lemma is similar to Lemma 3.10 of [13]. It will be a useful tool for constructing weighted automata from finite pointed  $\Sigma$ -algebras.

**Lemma 15.** *Let  $(P, \theta, q)$  be a finite pointed  $\Sigma$ -algebra.*

1. *For every mapping  $f : P \rightarrow A$  there is a crisp-deterministic wfa  $M$  over  $\Sigma$  and  $A$  such that  $\llbracket M \rrbracket_i = h_\theta; f$ .*
2. *For every  $F \subseteq P$  the language  $h_\theta^{-1}(F) \subseteq \Sigma^*$  is recognizable.*

PROOF. To Statement 1: We define  $M = (P, I, \tau, F)$  as follows for every  $\sigma \in \Sigma$  and  $p, p' \in P$ :

$$I_p = \begin{cases} 1, & \text{if } p = q; \\ 0, & \text{otherwise,} \end{cases} \quad \tau(\sigma)_{p,p'} = \begin{cases} 1, & \text{if } p' = \theta(\sigma)(p); \\ 0, & \text{otherwise,} \end{cases}$$

$$F_p = f(p).$$

Observe that  $M$  is crisp-deterministic. It is easy to check that for every  $w \in \Sigma^*$  and  $p \in P$  we have  $h_{\theta_\tau}(w)_p = 1$  if  $h_\theta(w) = p$  and  $h_{\theta_\tau}(w)_p = 0$  otherwise. Thus, for every  $w \in \Sigma^*$  we have  $(\llbracket M \rrbracket_i, w) = h_{\theta_\tau}(w) \cdot F = \sum_{p \in P} h_{\theta_\tau}(w)_p \cdot F_p = F_{h_\theta(w)} = f(h_\theta(w)) = (h_\theta; f)(w)$ .

To Statement 2: Let  $f : P \rightarrow \mathbb{B}$  be defined for every  $p \in P$  by  $f(p) = 1$  iff  $p \in F$ . By Statement 1 there is an fsa  $M$  over  $\Sigma$  with  $\llbracket M \rrbracket_i = h_\theta; f$ . This yields  $L(M) = \{w \in \Sigma^* \mid f(h_\theta(w)) = 1\} = h_\theta^{-1}(F)$ . ■

For our subsequent results, we will need a particular finiteness condition on the strong bimonoid  $A$  (which may be infinite), which we introduce next.

**Definition 16.** For every  $B \subseteq A$  the *weak closure of  $B$* , denoted by  $\text{cl}(B)$ , is the smallest subset  $C \subseteq A$  such that  $B \subseteq C$  and  $c + c' \in C$  and  $c \cdot b \in C$  for all  $b \in B$  and  $c, c' \in C$ . We say that  $A$  is *weakly locally finite* if  $\text{cl}(B)$  is finite for every finite subset  $B \subseteq A$ .

Trivially, if  $A$  is locally finite, then  $A$  is weakly locally finite, and if  $A$  is weakly locally finite, then  $A$  is additively and multiplicatively locally finite. For example, if  $A$  is a bounded lattice, then  $A$  is a strong bimonoid which is additively and multiplicatively locally finite, but need not be locally finite. The following is easy to check.

**Remark 17.** *If  $A$  is right distributive, then  $A$  is additively and multiplicatively locally finite iff  $A$  is weakly locally finite. If  $A$  is left distributive, then  $A$  is weakly locally finite iff  $A$  is locally finite. In particular, if  $A$  is a semiring, then  $A$  is additively and multiplicatively locally finite iff  $A$  is locally finite.*

The idea of the following lemma is based on Lemma 3.14 of [13].

**Lemma 18.** *Let  $\varphi \in A\langle\langle \Sigma^* \rangle\rangle$  be  $i$ -recognizable. If  $A$  is weakly locally finite, then there is a finite pointed  $\Sigma$ -algebra  $(P, \theta, q)$  and a mapping  $f : P \rightarrow A$  such that  $\varphi = h_\theta; f$ .*

PROOF. Choose a wfa  $M = (Q, I, \tau, F)$  over  $\Sigma$  and  $A$  such that  $\varphi = \llbracket M \rrbracket_i$ . Let  $B = \{I_q, \tau(\sigma)_{p,q} \mid p, q \in Q, \sigma \in \Sigma\}$ . By assumption, the weak closure  $\text{cl}(B)$  is finite. Then  $P = \{h_{\theta_\tau}(w) \mid w \in \Sigma^*\} \subseteq \text{cl}(B)^Q$  is also finite, and  $(P, \theta_\tau, I)$  is a pointed  $\Sigma$ -algebra. Moreover, we define the mapping  $f$  by  $f(v) = v \cdot F$  for every  $v \in P$ . Then  $(\llbracket M \rrbracket_i, w) = h_{\theta_\tau}(w) \cdot F = f(h_{\theta_\tau}(w)) = (h_{\theta_\tau}; f)(w)$ . ■

The following result summarizes three important properties of  $i$ -recognizable series if  $A$  is weakly locally finite.

**Theorem 19.** *Let  $A$  be weakly locally finite and  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  be  $i$ -recognizable.*

1. *For any  $E \subseteq A$  the language  $\varphi^{-1}(E)$  is recognizable. In particular,  $\varphi$  is a recognizable step function.*
2. *For any mapping  $g : A \rightarrow A$ , the series  $\varphi;g$  is again  $i$ -recognizable.*
3. *(Determinization) There exists a crisp-deterministic wfa  $M$  over  $\Sigma$  and  $A$  such that  $\varphi = \llbracket M \rrbracket_i$ .*

PROOF. By Lemma 18 there are a finite pointed  $\Sigma$ -algebra  $(P, \theta, q)$  and a mapping  $f : P \rightarrow A$  such that  $\varphi = h_\theta; f$ . First we show Statement 1. Since  $\varphi^{-1}(E) = h_\theta^{-1}(f^{-1}(E))$  and  $f^{-1}(E) \subseteq P$ , it follows from Lemma 15(2) that  $\varphi^{-1}(E)$  is recognizable. In particular, for every  $a \in A$ , the language  $\varphi_{=a}$  is recognizable. Moreover,  $\text{im}(\varphi) \subseteq f(P)$  is finite. Hence,  $\varphi = \sum_{a \in \text{im}(\varphi)} a \cdot \mathbf{1}_{\varphi=a}$  is a recognizable step function.

Next we show Statements 2 and 3. Let  $g : A \rightarrow A$ . We have  $\varphi;g = (h_\theta; f);g = h_\theta;(f;g)$ . Then by Lemma 15(1) there is a crisp-deterministic wfa  $M$  over  $\Sigma$  and  $A$  with  $\varphi;g = \llbracket M \rrbracket_i$ . This proves Statement 2 and also Statement 3 (let  $g$  be the identity mapping). ■

We note that part 3 of Theorem 19 would also follow from part 1 and Lemma 8, but the present argument is simpler and leads to automata with smaller state sets. Theorem 19(3) generalizes the following results of [3, 25].

**Corollary 20.** *For every  $x$ -recognizable series  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  there is a crisp-deterministic wfa  $M$  such that  $\llbracket M \rrbracket_x = \varphi$  assuming that one of the following conditions holds.*

1.  *$A$  is a bounded, complete, distributive lattice and  $x = f$  (cf. Theorem 2.1 of [3]).*
2.  *$A$  is a bounded, distributive lattice and  $x = r$  (cf. Theorem 2.1 of [25]).*

PROOF. 1. In [3] Bělohlávek defined recognizable series by means of wfa over bounded, complete, locally finite lattices with the free monoid semantics; in fact, he also needed that the lattices are distributive [4]; then local finiteness follows from idempotence and distributivity. Now let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  be recognizable in this sense. Since every bounded, complete, distributive lattice is a locally finite semiring, it follows from Lemma 5 that  $\varphi$  is  $i$ -recognizable. Since every locally finite semiring is a weakly locally finite strong bimonoid, Theorem 19(3) shows Statement 1.

To 2: In [25] Li and Pedrycz defined recognizable series by means of wfa over bounded, distributive lattices with the run semantics. We can use the same arguments as for Statement 1. ■

Now we can give an analogue of Theorem 14 for the  $i$ -behavior of wfa.

**Theorem 21.** *Let  $|\Sigma| \geq 2$  and let  $A$  be right distributive. Then the following two statements are equivalent:*

1. For every wfa  $M$  over  $\Sigma$  and  $A$  there is an  $i$ -equivalent crisp-deterministic wfa  $M'$  over  $\Sigma$  and  $A$ .
2.  $A$  is additively and multiplicatively locally finite.

PROOF. This theorem is an immediate consequence of Remark 7, Lemma 12, Remark 17, and Theorem 19(3). ■

The following summarizes our results.

**Theorem 22.** *Let  $A$  be additively and multiplicatively locally finite, and let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$ . Then the following are equivalent:*

1.  $\varphi$  is  $r$ -recognizable.
2.  $\varphi$  is a recognizable step function.

Moreover, if  $A$  is weakly locally finite, these conditions are equivalent to

3.  $\varphi$  is  $i$ -recognizable.

PROOF. This follows immediately from Lemma 8 and Theorems 11 and 19(1). ■

Let  $\leq$  be a partial order on  $A$ . Moreover, let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  be a series and  $a \in A$ . Then the  $a$ -cut of  $\varphi$  is the set  $\varphi_{\geq a} = \{w \in \Sigma^* \mid (\varphi, w) \geq a\}$ . Several authors [6, 18, 23, 25, 32] have investigated these cuts of  $\varphi$ . The following is straightforward, but together with Theorem 22 it provides the connection to the work cited.

**Lemma 23.** *Let  $\leq$  be a partial order on  $A$ , and let  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$ . The following statements are equivalent.*

1.  $\varphi$  is a recognizable step function.
2.  $\text{im}(\varphi)$  is finite and  $\varphi_{\geq a}$  is recognizable for every  $a \in \text{im}(\varphi)$ .

PROOF. (1)  $\Rightarrow$  (2): Clearly,  $\text{im}(\varphi)$  is finite. By Proposition 9 we have that  $\varphi_{=b}$  is recognizable for every  $b \in \text{im}(\varphi)$ . Let  $a \in \text{im}(\varphi)$ . Then

$$\varphi_{\geq a} = \bigcup_{\substack{b \in \text{im}(\varphi) \\ b \geq a}} \varphi_{=b},$$

which is a finite union of recognizable languages and, hence, recognizable.

(2)  $\Rightarrow$  (1): Since  $\varphi = \sum_{a \in \text{im}(\varphi)} a \cdot \mathbb{1}_{\varphi_{=a}}$ , it suffices to show that  $\varphi_{=a}$  is recognizable for every  $a \in \text{im}(\varphi)$ . But

$$\varphi_{=a} = \varphi_{\geq a} \setminus \bigcup_{\substack{b \in \text{im}(\varphi) \\ b > a}} \varphi_{\geq b},$$

which implies our claim. ■

Now we can show that we have generalized a result of [23, 25].

**Corollary 24.** *For every  $\varphi \in A\langle\langle\Sigma^*\rangle\rangle$  the following conditions are equivalent*

- (i)  $\varphi$  is  $x$ -recognizable.

(ii)  $\text{im}(\varphi)$  is finite and  $\varphi_{=a} \in \Sigma^*$  is recognizable for every  $a \in A$ .

(iii)  $\text{im}(\varphi)$  is finite and  $\varphi_{\geq a} \in \Sigma^*$  is recognizable for every  $a \in A$ .

provided that one of the following properties holds.

1.  $A = ([0, 1], \max, *, 0, 1)$ , where  $*$  is a locally finite t-norm, and  $x = i$  (cf. Theorem 2.6 (2)-(4) of [23]).

2.  $A$  is a bounded, distributive lattice, and  $x = r$  (cf. Theorem 2.2 of [25]).

PROOF. 1. It is easy to prove that a t-norm [18] distributes from left and right over max. Since  $A = ([0, 1], \max, *, 0, 1)$  is distributive, additively locally finite and, by assumption, also multiplicatively locally finite, it follows from Remark 17, that  $A$  is a locally finite. Hence, in Theorem 2.6 of [23], Li considered particular locally finite, commutative semirings. Then the stated equivalence under Property 1 follows from Theorem 22 and Lemma 23.

2. Since every bounded, distributive lattice is a locally finite semiring, the statement follows from Lemma 5, Theorem 22, and Lemma 23. ■

Finally we show that for arbitrary strong bimonoids  $A$ , the concepts of i-recognizability and r-recognizability differ. Clearly, by Lemma 4 and Theorem 22,  $A$  should neither be right distributive nor weakly locally finite. We will give two examples. In the first example we construct an i-recognizable series that is not r-recognizable. In the second example we consider an r-recognizable series that is not i-recognizable.

**Example 25.** We consider the set  $\mathbb{N}$  of natural numbers and we define two new commutative operations  $\oplus$  and  $\odot$  on  $\mathbb{N}$  as follows. First, let  $0 \oplus a = a$ ,  $0 \odot a = 0$ , and  $1 \odot a = a$  for every  $a \in \mathbb{N}$ . If  $a, b \in \mathbb{N} \setminus \{0\}$  with  $a \leq b$ , we put (with  $+$  being the usual addition on  $\mathbb{N}$ )

$$a \oplus b = \begin{cases} b, & \text{if } b \text{ is even;} \\ b + 1, & \text{if } b \text{ is odd.} \end{cases}$$

If  $a, b \in \mathbb{N} \setminus \{0, 1\}$  with  $a \leq b$ , let

$$a \odot b = \begin{cases} b + 1, & \text{if } b \text{ is even;} \\ b, & \text{if } b \text{ is odd.} \end{cases}$$

Then  $A = (\mathbb{N}, \oplus, \odot, 0, 1)$  is a strong bimonoid. Note that  $A$  is neither right distributive (e.g.,  $5 = (3 \oplus 3) \odot 2 \neq (3 \odot 2) \oplus (3 \odot 2) = 4$ ) nor weakly locally finite (e.g.,  $a + 1 \in \{a \oplus a, a \odot 2\}$ , and hence  $a \in \text{cl}(2)$  for every  $a \geq 2$ ). But  $A$  is additively and multiplicatively locally finite. Hence, by Theorem 11 and Lemma 8, each r-recognizable series over  $\Sigma$  and  $A$  is also i-recognizable.

Now consider the wfa  $M = (Q, I, \tau, F)$  with two states and  $I_p = \tau(\sigma)_{p,q} = F_q = 2$  for every  $\sigma \in \Sigma$  and  $p, q \in Q$ . Then  $(\llbracket M \rrbracket_i, \sigma^n) = 2n + 4$  for every  $\sigma \in \Sigma$  and  $n \in \mathbb{N}$ , hence,  $\text{im}(\llbracket M \rrbracket_i)$  is infinite. Then, by Theorem 11, the series  $\llbracket M \rrbracket_i$  is not r-recognizable. Indeed, if  $M' = (Q', I', \tau', F')$  is any wfa over  $\Sigma$  and  $A$ , let  $m = \max\{I'_p, \tau'(\sigma)_{p,q}, F'_q \mid p, q \in Q', \sigma \in \Sigma\} \in \mathbb{N}$ . Then the definition of the r-behavior of  $M'$  shows that  $(\llbracket M' \rrbracket_r, w) \leq m + 2$  for each  $w \in \Sigma^*$ .

**Example 26.** We define two binary commutative operations  $\oplus$  and  $\odot$  on the set  $\mathbb{N}^2$  of pairs of natural numbers as follows. First, let  $(0, 0) \oplus (a, b) = (a, b)$ ,  $(0, 0) \odot (a, b) = (0, 0)$ , and  $(1, 0) \odot (a, b) = (a, b)$  for every  $(a, b) \in \mathbb{N}^2$ . If  $(a, b), (a', b') \in \mathbb{N}^2 \setminus \{(0, 0)\}$ , we put

$$(a, b) \oplus (a', b') = \begin{cases} (a, b + b'), & \text{if } a = a'; \\ (0, b + b'), & \text{otherwise.} \end{cases}$$

If  $(a, b), (a', b') \in \mathbb{N}^2 \setminus \{(0, 0), (1, 0)\}$ , let

$$(a, b) \odot (a', b') = \begin{cases} (a \cdot a', 1), & \text{if } b = b' = 1; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Then  $A = (\mathbb{N}^2, \oplus, \odot, (0, 0), (1, 0))$  is a strong bimonoid. Now consider the wfa  $M = (Q, I, \tau, F)$  with two states and  $I_p = \tau(\sigma)_{p,q} = F_q = (2, 1)$  for every  $\sigma \in \Sigma$  and  $p, q \in Q$ . Then  $(\llbracket M \rrbracket_r, \sigma^n) = (2^{n+2}, 2^{n+1})$  for every  $\sigma \in \Sigma$  and  $n \in \mathbb{N}$ .

We claim that  $\llbracket M \rrbracket_r$  is not i-recognizable. Suppose, contrary to our claim, that  $M' = (Q', I', \tau', F')$  is a wfa over  $\Sigma$  and  $A$  such that  $\llbracket M \rrbracket_r = \llbracket M' \rrbracket_i$ . For every  $(a, a') \in \mathbb{N}^2$  we let  $\pi_1(a, a') = a$  and  $\pi_2(a, a') = a'$ .

Consider a finite nonempty index set  $I$  and two families  $(a_i \mid i \in I)$  and  $(b_i \mid i \in I)$  with elements in  $\mathbb{N}^2$ . Let  $c = \max\{1, \pi_2(b_i) \mid i \in I\}$  and  $d = \sum_{i \in I} a_i \odot b_i$ . We claim that if  $\pi_1(d) > 0$  and  $\pi_2(d) > |I| \cdot c$ , then there is a  $j \in I$  such that  $\pi_1(a_j) = \pi_1(d)$  and  $\pi_2(d) \leq |I| \cdot c + |I| \cdot \pi_2(a_j)$ . First we prove this claim:

$$\begin{aligned} \pi_2(d) &= \pi_2\left(\sum_{i \in I} a_i \odot b_i\right) = \sum_{i \in I} \pi_2(a_i \odot b_i) \\ &= \sum_{\substack{i \in I \\ b_i \neq (1,0)}} \pi_2(a_i \odot b_i) + \sum_{b_i = (1,0)} \pi_2(a_i \odot b_i) \\ &\leq |I| \cdot c + \sum_{b_i = (1,0)} \pi_2(a_i). \end{aligned}$$

Since  $|I| \cdot c < \pi_2(d)$ , we obtain  $\sum_{i \in I: b_i = (1,0)} \pi_2(a_i) > 0$  and therefore the set  $\{i \in I \mid b_i = (1, 0)\}$  is nonempty. Choose  $j \in I$  such that  $b_j = (1, 0)$  and  $\pi_2(a_j) = \max\{\pi_2(a_i) \mid i \in I, b_i = (1, 0)\}$ . We obtain  $\pi_2(d) \leq |I| \cdot c + |I| \cdot \pi_2(a_j)$ . In particular, this implies  $a_j \neq (0, 0)$  and therefore  $a_j \odot b_j \neq (0, 0)$ . By the assumption  $\pi_1(d) > 0$  we obtain  $\pi_1(a_j) = \pi_1(d)$ . This proves our claim.

Now let  $c = \max\{1, \pi_2(F'_q) \mid q \in Q'\}$  and  $c' = \max\{1, \pi_2(\tau'(\sigma)_{q,p}) \mid q, p \in Q'\}$ . Choose  $N \geq |Q'|$  such that

$$2^{N+1} > c \cdot |Q'| + c' \cdot \sum_{i=2}^{|Q'|+1} |Q'|^i.$$

Let  $n \geq N$ . We show by induction that for every  $j$  with  $0 \leq j \leq |Q'|$ , there is a  $q_{n,j} \in Q'$  such that  $\pi_1(h_{\theta_{\tau'}}(\sigma^{n-j})_{q_{n,j}}) = 2^{n+2}$  and  $\pi_2(h_{\theta_{\tau'}}(\sigma^{n-j})_{q_{n,j}}) > c' \cdot \sum_{i=1}^{|Q'|-j} |Q'|^i$ . If  $j = 0$ , the above claim yields (by letting  $d = (\llbracket M' \rrbracket_i, \sigma^n) = (2^{n+2}, 2^{n+1})$ ) that there is a  $q_{n,0} \in Q'$  such that  $\pi_1(h_{\theta_{\tau'}}(\sigma^n)_{q_{n,0}}) = 2^{n+2}$  and  $2^{n+1} \leq |Q'| \cdot c + |Q'| \cdot \pi_2(h_{\theta_{\tau'}}(\sigma^n)_{q_{n,0}})$ , which implies  $\pi_2(h_{\theta_{\tau'}}(\sigma^n)_{q_{n,0}}) > c' \cdot \sum_{i=1}^{|Q'|} |Q'|^i$ . The induction step can be shown similarly by considering  $d = h_{\theta_{\tau'}}(\sigma^{n-j})_{q_{n,j}}$ .

Thus, for every  $k$  with  $0 \leq k \leq |Q'|$  we have  $\pi_1(h_{\theta_{\tau'}}(\sigma^N)_{q_{N+k,k}}) = 2^{N+k+2}$ , which implies that the  $|Q'| + 1$  states  $q_{N,0}, \dots, q_{N+|Q'|,|Q'|}$  are pairwise distinct, a contradiction. Hence,  $\llbracket M \rrbracket_r \neq \llbracket M' \rrbracket_i$ .



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