Weighted automata and weighted logics on infinite words

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Abstract

We introduce weighted automata over infinite words with Muller acceptance condition and we show that their behaviors coincide with the semantics of weighted restricted MSO-sentences. Furthermore, we establish an equivalence property of weighted Muller and weighted Büchi automata over certain semirings.

Keywords: Weighted logics, Weighted Muller automata, Infinitary formal power series, Weighted Büchi automata.

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1 Introduction

One of the cornerstones of automata theory is Büchi's theorem [6] on the coincidence of the class of regular languages of infinite words with the family of languages definable by monadic second order logic. This led to the development of several models of automata acting on infinite words, like Büchi, Muller, Rabin and Streett, cf. [33, 37, 38] for surveys; it also led to practical applications in model checking and for non-terminating processes, cf. [1, 28, 30].

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On the other hand, Schützenberger [36] introduced finite automata with weights which can model quantitative aspects of transitions like use of resources, reliability or capacity. Schützenberger characterized the behavior of such automata as rational formal power series. For the theory of weighted automata, see [3, 24, 27, 35] for surveys. Recently, weighted automata were applied in digital image compression [7, 19, 20, 21] as well as in speech-to-text processing [22, 32].

It is the goal of this paper to extend Büchi's theorem mentioned above into the context of weighted automata, thereby obtaining a quantitative version. The last few years weighted automata over infinite words have attracted the interest of several researchers. This effort is not a simple generalization of the finitary case since convergence problems arise depending on the underlying semiring. This issue is dealt with either by considering special classes of automata [8, 10] or by restricting the underlying semirings so that convergence problems can be solved [11, 17, 18, 23, 26, 34].

Very recently, Droste and Gastin [9] extended the result of Büchi and Elgot [5, 15] to weighted automata over finite words. They introduced a monadic second order logic with weights and described the semantics of the formulas obtained as formal power series. The main result of their paper states that the recognizable formal power series over commutative semirings coincide with the series definable by certain weighted MSO-sentences.

In this paper, we will introduce weighted Muller automata acting on infinite words, and we will extend the weighted monadic second order logic of [9] to infinite words. We describe the behavior of weighted Muller automata as formal power series on infinite words. Our first main result states the coincidence of these ω-Muller-recognizable series with the semantics of a restricted weighted MSO logic and also with the semantics of a restricted existential MSO logic. Furthermore, we prove an equivalence to the important model of weighted Büchi automata investigated in Ésik and Kuich [17, 18]. They have characterized the behaviors of weighted Büchi automata precisely as the ω-rational formal power series; for further work on this model, see [25, 26]. Combining these results, we thus obtain a robust notion of weighted automata, logics and rational series on infinite words. As in [17, 18], we assume our semiring of weights to permit infinite sum and product operations. Such "complete" semirings have been investigated in detail in the literature, cf. [4, 14, 24]. In particular, when considering the Boolean semiring, we obtain Büchi's result as a very special consequence.

Next we describe the structure of our paper. In Section 2, we introduce the notion of totally commutative complete semirings. Furthermore, we define our concept of weighted Muller automata and prove their basic properties. Section 3 introduces weighted MSO logic [9]. As in [9], we use a restricted MSO logic without universal set quantifications and with negations applied only to atomic formulas and universal first order quantifications only to Muller recognizable step functions. The main result of the paper in Section 4 states that a formal power series is Muller recognizable iff it is definable in our restricted weighted MSO logic iff it is definable in restricted existential weighted MSO logic. Büchi's classical theorem follows as a very special case. Then in Section 5, we relate our weighted Muller automata to the weighted Büchi automata of Ésik and Kuich [17, 18], and we show that these two models are equivalent. Finally in the conclusion, we expose some ideas for further research.

An extended abstract of this paper has appeared in [12]. In the meantime weighted MSO logics has been investigated for tree automata [13], for picture automata [29] and for automata on traces [31].

2 Semirings and weighted Muller automata

A *semiring* $(K, +, \cdot, 0, 1)$ consists of a set K, two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that

(i) $\langle K, +, 0\rangle$ is a commutative monoid,

(ii) $\langle K, \cdot, 1\rangle$ is a monoid,

(iii) the distributivity laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for all a, b, $c \in K$, and

(iv) $0 \cdot a = a \cdot 0 = 0$ for all $a \in K$.

The semiring is denoted simply by K if the operations and the constant elements are understood.

The semiring K is called *commutative* iff $a \cdot b = b \cdot a$ for all a,$b \in K$. Now assume that the semiring K is equipped with infinitary sum operations $\sum_I : K^I \to K$, for any index set I, such that for all I and all families $(a_i \mid i \in I)$ of elements of K the following hold:

$$\sum_{i\in\emptyset} a_i = 0, \quad \sum_{i\in\{j\}} a_i = a_j, \quad \sum_{i\in\{j,k\}} a_i = a_j + a_k \text{ for } j \neq k, \tag{1}$$

$$\sum_{j\in J}\Big(\sum_{i\in I_j} a_i\Big) = \sum_{i\in I} a_i, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j', \tag{2}$$

$$\sum_{i\in I}(c \cdot a_i) = c \cdot \Big(\sum_{i\in I} a_i\Big), \quad \sum_{i\in I}(a_i \cdot c) = \Big(\sum_{i\in I} a_i\Big) \cdot c. \tag{3}$$

Then K together with the operations $\sum_I$ is called *complete* [14, 24].

A complete semiring is said to be *totally complete*[16], if it is endowed with a countably infinite product operation satisfying for all sequences $(a_i \mid i \geq 0)$ of elements of K the following conditions:

$$\prod_{i\geq 0} 1 = 1 \tag{4}$$

$$\prod_{i\geq 0} a_i = \prod_{i\geq 0} a'_i \tag{5}$$

$$a_0 \cdot \prod_{i\geq 0} a_{i+1} = \prod_{i\geq 0} a_i \tag{6}$$

$$\prod_{j\geq 1} \sum_{i\in I_j} a_i = \sum_{(i_1,i_2,\dots)\in I_1\times I_2\times\dots} \prod_{j\geq 1} a_{i_j}, \tag{7}$$

where in the second equation $a'_0 = a_0 \cdot \ldots \cdot a_{n_1}, a'_2 = a_{n_1+1} \cdot \ldots \cdot a_{n_2}, \ldots$ for an increasing sequence $0 < n_1 < n_2 < \ldots$, and in the last equation $I_1, I_2, \ldots$ are arbitrary index sets.

Furthermore, we will call a totally complete semiring *totally commutative complete* if it is commutative and satisfies the statement:

$$\prod_{i\geq 0} (a_i \cdot b_i) = \left(\prod_{i\geq 0} a_i\right) \cdot \left(\prod_{i\geq 0} b_i\right). \tag{8}$$

Example 1 *The following semirings are totally commutative complete:*

- *the semiring* $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ *of* extended natural numbers *[17],*
- *the* tropical *or* min-plus semiring $(\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$ *where* $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$,
- *the* arctical semiring *or* max-plus semiring $(\mathbb{R}_+ \cup \{-\infty\}, \max, +, -\infty, 0)$,
- *each completely distributive complete lattice (cf. [2]) with the operations supremum and infimum, in particular the* fuzzy semiring $F = ([0, 1], \sup, \inf, 0, 1)$ *[26, 34],*

In the rest of this section, we introduce the notion of weighted Muller automata and we establish closure properties of the class of their behaviors. First, we need some preparations.

Let A be a finite alphabet. A finite word $w = a_0a_1 \ldots a_{n-1} \in A^*$, $(a_i \in A, 0 \leq i \leq n-1)$ will also be written as $w = w(0)w(1) \ldots w(n-1) = w(i)_{0\leq i\leq n-1}$ with $w(i) = a_i$, for $0 \leq i \leq n-1$. In the same manner an infinite word $w = a_0a_1 \ldots \in A^\omega$ is written as $w = w(0)w(1) \ldots = w(i)_{i\geq 0}$ with $w(i) = a_i$ $(i \geq 0)$. For any $w \in A^\omega$ and any $m \geq 0$ the infinite suffix $w(m+1)w(m+2)\ldots$ will be denoted by $w|_{>m}$. We shall write also ω for the set of natural numbers $\mathbb{N}$.

The quantifier "there exists infinitely many times" will be denoted by $\exists^\omega$.

A *(nondeterministic) Muller automaton over* A is a system $\mathcal{A} = (Q, q_0, \Delta, \mathcal{F})$ with *finite state set* Q, *initial state* q_0, *transitions* $\Delta \subseteq Q \times A \times Q$, and *final state sets* $\mathcal{F} \subseteq \mathcal{P}(Q)$.

Let $a_0 a_1 \ldots \in A^\omega$ be an infinite word. A *path of* $\mathcal{A}$ *over* w is an infinite sequence of transitions

$$P_w := (t_i)_{i \geq 0}$$

so that $t_i = (q_i, a_i, q_{i+1}) \in \Delta$ for all $i \geq 0$.

We denote by $In^Q(P_w)$ the set of states which appear infinitely many times in P_w, i.e.,

$$In^Q(P_w) = \{q \in Q \mid \exists^\omega i : t_i = (q, a_i, q_{i+1})\}.$$

A path P_w of $\mathcal{A}$ over $w \in A^\omega$ is called *successful* if $In^Q(P_w) \in \mathcal{F}$, that is, the states which appear infinitely often in P_w form a set in $\mathcal{F}$. Then we say that w is *recognized (accepted) by* $\mathcal{A}$.

The *infinitary language* of $\mathcal{A}$ consists of all infinite words accepted by $\mathcal{A}$ and is denoted by $\mathcal{L}_\omega(\mathcal{A})$.

A Muller automaton $\mathcal{A} = (Q, q_0, \Delta, \mathcal{F})$ is *deterministic* if the transition set Δ is replaced by a mapping $\delta : Q \times A \to Q$. It is well known and will be important for us that for each Muller automaton $\mathcal{A}$, we can effectively construct a deterministic Muller automaton with the same language [37, 38].

A language $L \subseteq A^\omega$ is called ω*-recognizable* if there is a Muller automaton $\mathcal{A}$ so that $L = \mathcal{L}_\omega(\mathcal{A})$.

Given a finite alphabet A and a semiring K, an *infinitary formal power series* or *series* for short, is a mapping $S : A^\omega \to K$. We usually write (S, w) instead of $S(w)$ for $w \in A^\omega$. The *support of* S is the set $Supp(S) = \{w \in A^\omega \mid (S, w) \neq 0\}$. The class of all power series over A and K is denoted by $K\langle\langle A^\omega \rangle\rangle$.

For any $L \subseteq A^\omega$, the *characteristic series of* L, $1_L : A^\omega \to K$ is determined by

$$(1_L, w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise} \end{cases}$$

for $w \in A^\omega$.

Let $S, T \in K\langle\langle A^\omega \rangle\rangle$ and $k \in K$. The sum $S + T$, the scalar products kS and Sk as well as the Hadamard product $S \odot T$ are defined elementwise

$$\begin{aligned} (S + T, w) &= (S, w) + (T, w), \\ (kS, w) &= k \cdot (S, w), \quad (Sk, w) = (S, w) \cdot k, \\ (S \odot T, w) &= (S, w) \cdot (T, w) \end{aligned}$$

for any $w \in A^\omega$.

Consider two alphabets A, B and a non-deleting homomorphism $h : A^* \to B^*$, i.e., $h(a) \neq \varepsilon$ for each $a \in A$. Then h can be extended to a mapping $h : A^\omega \to B^\omega$ by letting $h(w) = (h(w(i)))_{i \geq 0}$ for each $w \in A^\omega$. If K is complete, for any power series $S \in K\langle\langle A^\omega \rangle\rangle$ the series $h(S) \in K\langle\langle B^\omega \rangle\rangle$ is defined by

$$(h(S), w) = \sum_{u \in h^{-1}(w)} (S, u)$$

for $w \in B^\omega$.

Furthermore, if $h : A^\omega \to B^\omega$ is an homomorphism and $T \in K\langle\langle B^\omega\rangle\rangle$, then the series $h^{-1}(T) \in K\langle\langle A^\omega\rangle\rangle$ is specified by

$$(h^{-1}(T), u) = (T, h(u))$$

for $u \in A^\omega$, that is, $h^{-1}(T) = T \circ h$.

For the rest of this section, let A be a finite alphabet and K be a totally complete semiring. We shall simply denote the operation $\cdot$ by concatenation.

Definition 2 *A* weighted Muller automaton *(WMA for short)* over A and K *is a quadruple $\mathcal{A} = (Q, in, wt, \mathcal{F})$, where Q is the* finite state set, *$in : Q \to K$ is the* initial distribution, *$wt : Q \times A \times Q \to K$ is a* mapping assigning weights to the transitions *of the automaton, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the* family of final state sets.

Let $w = a_0 a_1 \ldots \in A^\omega$. A *path of $\mathcal{A}$ over w* is an infinite sequence of transitions $P_w := (t_i)_{i\geq 0}$, so that $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. The *weight of* P_w is defined by

$$weight(P_w) := in(q_0) \cdot \prod_{i\geq 0} wt(t_i).$$

The path P_w is called *successful* if the set of states that appear infinitely often along P_w constitute a final state set, i.e.,

$$In^Q(P_w) \in \mathcal{F}.$$

The *behavior* of $\mathcal{A}$ is the formal power series

$$\|\mathcal{A}\| : A^\omega \to K$$

whose coefficients are determined by

$$(\|\mathcal{A}\|, w) = \sum_{P_w} weight(P_w)$$

for $w \in A^\omega$, where the sum is taken over all successful paths P_w of $\mathcal{A}$ over w.

A series $S : A^\omega \to K$ is said to be *Muller recognizable* if there is a WMA $\mathcal{A}$ so that $S = \|\mathcal{A}\|$. We shall denote the family of all such series over A and K by $K^{M-rec}\langle\langle A^\omega\rangle\rangle$.

Proposition 3 *The class $K^{M-rec}\langle\langle A^\omega\rangle\rangle$ is closed under sum and scalar products from the left. If K is commutative, $K^{M-rec}\langle\langle A^\omega\rangle\rangle$ is also closed under scalar products from the right. Furthermore, if K is a totally commutative complete semiring, then $K^{M-rec}\langle\langle A^\omega\rangle\rangle$ is also closed under taking Hadamard products.*

Proof. The closure properties under sum and scalar product are established by standard constructions on weighted automata: for the sum we can take the disjoint union of two automata, and for the scalar product we can multiply the initial weight with the given scalar. Therefore, it remains to treat the Hadamard product case.

Let $\mathcal{A} = (Q, in, wt, \mathcal{F})$ and $\mathcal{A}' = (Q', in', wt', \mathcal{F}')$ be two WMA. We construct the WMA $\widetilde{\mathcal{A}} = (\widetilde{Q}, \widetilde{in}, \widetilde{wt}, \widetilde{\mathcal{F}})$ in the following way. Its state set is $\widetilde{Q} = Q \times Q'$, and the initial distribution is given by

$$\widetilde{in}(q, q') = in(q)in'(q')$$

for all $(q, q') \in \widetilde{Q}$.

Its weight transition mapping is specified by

$$\widetilde{wt}((q, q'), a, (p, p')) = wt(q, a, p)wt'(q', a, p')$$

for all $(q, q'), (p, p') \in \widetilde{Q}$, $a \in A$.

Finally, the family $\widetilde{\mathcal{F}}$ is constructed as follows:

$$\widetilde{\mathcal{F}} = \{\widetilde{F} \mid pr(\widetilde{F}) \in \mathcal{F},\ pr'(\widetilde{F}) \in \mathcal{F}'\}$$

where $pr : \widetilde{Q} \to Q$, $pr' : \widetilde{Q} \to Q'$ are the projections of $\widetilde{Q}$ on Q and Q', respectively.

Let now $w = a_0 a_1 \ldots \in A^\omega$ and $\widetilde{P}_w$ be an infinite path of $\widetilde{\mathcal{A}}$ over w. Then there are two paths P_w and P'_w of $\mathcal{A}$ and $\mathcal{A}'$ respectively, obtained by projections of $\widetilde{P}_w$ in the obvious way. Let us assume that $\widetilde{P}_w$ is successful, i.e., there exists $\widetilde{F} \in \widetilde{\mathcal{F}}$ so that $In^{\widetilde{Q}}(\widetilde{P}_w) = \widetilde{F}$. This means that $F = pr(\widetilde{F}) \in \mathcal{F}$, $F' = pr'(\widetilde{F}) \in \mathcal{F}'$ and $In^Q(P_w) = F$ and $In^{Q'}(P'_w) = F'$. Thus P_w, P'_w are successful paths. Keeping the same notations, let us conversely assume P_w and P'_w to be successful paths of $\mathcal{A}$ and $\mathcal{A}'$ over w, respectively. Thus $In^Q(P_w) = F$ and $In^{Q'}(P'_w) = F'$ for some $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$. Now P_w and P'_w compose a path $\widetilde{P}_w$ of $\widetilde{\mathcal{A}}$ over w, and by construction of $\widetilde{\mathcal{F}}$, we have that $In^{\widetilde{Q}}(\widetilde{P}_w) \in \widetilde{\mathcal{F}}$, i.e., $\widetilde{P}_w$ is successful.

Thus, for each successful path $\widetilde{P}_w$ of $\widetilde{\mathcal{A}}$ over $w \in A^\omega$, there are two uniquely determined successful paths P_w of $\mathcal{A}$ and P'_w of $\mathcal{A}'$ over w, and vice-versa. Taking also into account the extended commutativity law (8) of K, we see that $weight(\widetilde{P}_w) = weight(P_w)weight(P'_w)$. Furthermore, we obtain

$$\begin{aligned}
\left(\left\|\widetilde{\mathcal{A}}\right\|, w\right) &= \sum_{\widetilde{P}_w} weight(\widetilde{P}_w) \\
&= \sum_{P_w, P'_w} weight(P_w)weight(P'_w) \\
&= \left(\sum_{P_w} weight(P_w)\right)\left(\sum_{P'_w} weight(P'_w)\right) \\
&= (\|\mathcal{A}\|, w)(\|\mathcal{A}'\|, w)
\end{aligned}$$

where the sums are taken over all successful paths $\widetilde{P}_w, P_w$ and P'_w of $\widetilde{\mathcal{A}}, \mathcal{A}$ and $\mathcal{A}'$, respectively. Thus

$$\left\|\widetilde{\mathcal{A}}\right\| = \|\mathcal{A}\| \odot \|\mathcal{A}'\|$$

as required. ■

Proposition 4 *Let A, B be two alphabets and $h : A^\omega \to B^\omega$ be a non-deleting homomorphism. Then $h : K \langle\langle A^\omega \rangle\rangle \to K \langle\langle B^\omega \rangle\rangle$ preserves Muller recognizability.*

Proof. Consider a WMA $\mathcal{A} = (Q, in, wt, \mathcal{F})$ over A and K. For each $q, q' \in Q$ and each $a \in A$ with $h(a) = b_1 \dots b_n$, ($b_i \in B$, $1 \leq i \leq n$), we introduce the new states $p_1, \dots, p_{n-1}$ not belonging to Q, and let P be the set of all new states obtained in this way. Let $\mathcal{A}' = (Q', in', wt', \mathcal{F}')$ be the WMA over B and K whose components are given by

- $Q' = Q \cup P$,
- $in'(q) = \begin{cases} in(q) & \text{if } q \in Q \\ 0, & \text{otherwise} \end{cases}$ for $q \in Q'$.

Keeping the above notations we set $wt'(q, b_1, p_1) = wt(q, a, q')$ and $wt'(p_1, b_2, p_2) = \dots = wt'(p_{n-1}, b_n, q') = 1$ if $n \geq 2$, and $wt'(q, b_1, q') = \sum_{\substack{a \in A \\ h(a) = b_1}} wt(q, a, q')$ if $n = 1$, and $wt'(t) = 0$ for all transitions t not of this form. Finally let $\mathcal{F}' = \{F \cup P' \mid F \in \mathcal{F} \text{ and } P' \subseteq P\}$.

Consider now a word $u \in A^\omega$ and a path P_u of $\mathcal{A}$ over u. We construct a path P'_w of $\mathcal{A}'$ over $w = h(u)$ by replacing each transition (q, a, q') of P_u by the sequence $(q, b_1, p_1)(p_1, b_2, p_2) \dots (p_{n-1}, b_n, q')$ as above if $n \geq 2$, and by the transition (q, b_1, q') if $n = 1$. Clearly, P_u is successful iff P'_w is successful. Conversely, given a word $w \in B^\omega$ and a path P'_w of $\mathcal{A}'$ over w, by the distributivity law (7) we have $weight(P'_w) = \sum_{h(P_u) = P'_w} weight(P_u)$. Thus

$$\begin{aligned}
(\|\mathcal{A}'\|, w) &= \sum_{P'_w} weight(P'_w) \\
&= \sum_{\substack{P_u \\ u \in h^{-1}(w)}} weight(P_u) \\
&= \sum_{u \in h^{-1}(w)} (\|\mathcal{A}\|), u) \\
&= (h(\|\mathcal{A}\|), w)
\end{aligned}$$

i.e.,

$$\|\mathcal{A}'\| = h(\|\mathcal{A}\|).$$

■

Proposition 5 *Let $h : A^\omega \to B^\omega$ be a strict alphabetic homomorphism, i.e., $h(A) \subseteq B$. Then for any $T \in K^{M-rec} \langle\langle B^\omega \rangle\rangle$, the series $h^{-1}(T)$ is also Muller recognizable.*

Proof. Assume that $\mathcal{A} = (Q, in, wt, \mathcal{F})$ is a WMA over B and K, such that $T = \|\mathcal{A}\|$. We consider the WMA $\mathcal{A}' = (Q, in, wt', \mathcal{F})$ over A and K with weight mapping $wt' : Q \times A \times Q \to K$ given by

$$wt'(q, a, q') = wt(q, h(a), q')$$

for each $q, q' \in Q$ and $a \in A$.
Then it is not difficult to formally check that $\|\mathcal{A}'\| = h^{-1}(T)$. ■

Proposition 6 *The characteristic series $1_L : A^\omega \to K$ of any ω-recognizable language $L \subseteq A^\omega$ is Muller recognizable.*

Proof. Let $\mathcal{A} = (Q, q_0, \delta, \mathcal{F})$ be a deterministic Muller automaton accepting L. We consider the WMA $\mathcal{A}' = (Q, in, wt, \mathcal{F})$ over A and K, with

- $in(q) = \begin{cases} 1 & \text{if } q = q_0 \\ 0 & \text{otherwise} \end{cases}$,

- $wt(q, a, q') = \begin{cases} 1 & \text{if } \delta(q, a) = q' \\ 0 & \text{otherwise} \end{cases}$

  for all $q, q' \in Q$ and $a \in A$.

It is evident that for any $w \in A^\omega$ it holds that

$$(\|\mathcal{A}'\|, w) = 1 \quad \text{iff} \quad w \in \mathcal{L}_\omega(\mathcal{A})$$

and thus the series $1_L = \|\mathcal{A}'\|$ is Muller recognizable. ■

We will call a power series $S : A^\omega \to K$ a *Muller recognizable step function* if $S = \sum_{1 \leq j \leq n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq A^\omega$ ($1 \leq j \leq n$ and $n \in \mathbb{N}$) are ω-recognizable languages. Then by Propositions 3 and 6, S is Muller recognizable.

3 Weighted monadic second order logic

Weighted monadic second order logic (MSO logic for short) was introduced by Droste and Gastin in [9] in order to obtain a logical characterization of recognizable formal power series over finite words.

In this section, we recall from [9] the basic definitions for the reader's convenience, but now we interpret the semantics of weighted MSO-formulas as formal power series over infinite words.

Let A be a finite alphabet and $\mathcal{V}$ a finite set of first and second order variables. An infinite word $w \in A^\omega$ is represented by the relational structure $(\omega, \leq, (R_a)_{a \in A})$ where $R_a = \{i \mid w(i) = a\}$ for $a \in A$. A $(w, \mathcal{V})$-*assignment* σ is a mapping associating first order variables from $\mathcal{V}$ to elements of ω, and second order variables from $\mathcal{V}$ to subsets of ω. If x is a first order variable and $i \in \omega$, then $\sigma[x \to i]$ denotes the $(w, \mathcal{V} \cup \{x\})$-assignment which associates i to x

and acts as σ on $\mathcal{V}\setminus\{x\}$. For a second order variable X and $I \subseteq \omega$, the notation $\sigma[X \to I]$ has a similar meaning.

In order to encode pairs (w, σ) for all $w \in A^\omega$ and any $(w, \mathcal{V})$-assignment σ, we use an extended alphabet $A_\mathcal{V} = A \times \{0,1\}^\mathcal{V}$. Each pair (w, σ) is a word in $A_\mathcal{V}^\omega$ where w is the projection over A and σ is the projection over $\{0,1\}^\mathcal{V}$. Then σ is a valid $(w, \mathcal{V})$-assignment if for each first order variable $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, we identify σ with the $(w, \mathcal{V})$-assignment so that for each first order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for each second order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions labelled with 1 along the X-row.

It is not difficult to see that the set

$$N_\mathcal{V} = \{(w, \sigma) \in A_\mathcal{V}^\omega \mid \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment}\}$$

is ω-recognizable.

Let φ be an MSO-formula [37, 38]. We shall write A_φ for $A_{Free(\varphi)}$ and $N_\varphi = N_{Free(\varphi)}$. The fundamental Büchi theorem [6] states that for $Free(\varphi) \subseteq \mathcal{V}$ the language

$$\mathcal{L}_\mathcal{V}(\varphi) = \{(w, \sigma) \in N_\mathcal{V} \mid (w, \sigma) \models \varphi\}$$

defined by φ over $A_\mathcal{V}$ is ω-recognizable. We simply write $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$.

Conversely, each ω-recognizable language $L \subseteq A^\omega$ is definable by an MSO-sentence φ, i.e., $L = \mathcal{L}(\varphi)$.

Now we turn to weighted logics.

Definition 7 *The syntax of formulas of the weighted MSO logic is given by*

$$\begin{aligned}\varphi := k \mid P_a(x) \mid \neg P_a(x) \mid x \leq y \mid \neg(x \leq y) \mid x \in X \mid \neg(x \in X) \\ \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x \centerdot \varphi \mid \exists X \centerdot \varphi \mid \forall x \centerdot \varphi \mid \forall X \centerdot \varphi\end{aligned}$$

where $k \in K$, $a \in A$. We shall denote by $MSO(K, A)$ the set of all such weighted MSO-formulas φ.

Next we represent the semantics of the formulas in $MSO(K, A)$ as formal power series over the extended alphabet $A_\mathcal{V}$ and the semiring K. *We assume K to be a totally commutative complete semiring.*

Definition 8 *Let $\varphi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The semantics of φ is a formal power series $\|\varphi\|_\mathcal{V} \in K\langle\langle A_\mathcal{V}^\omega \rangle\rangle$. Consider an element $(w, \sigma) \in A_\mathcal{V}^\omega$. If σ is not a valid assignment, then we put $\|\varphi\|_\mathcal{V}(w, \sigma) = 0$. Otherwise, we inductively define $\|\varphi\|_\mathcal{V}(w, \sigma) \in K$ as follows:*

- $\|k\|_\mathcal{V}(w, \sigma) = k$

- $\|P_a(x)\|_\mathcal{V}(w, \sigma) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$

- $\|x \leq y\|_\mathcal{V}(w, \sigma) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$

- $\|x \in X\|_{\mathcal{V}}(w,\sigma) = \begin{cases} 1 & \textit{if } \sigma(x) \in \sigma(X) \\ 0 & \textit{otherwise} \end{cases}$

- $\|\neg\varphi\|_{\mathcal{V}}(w,\sigma) = \begin{cases} 1 & \textit{if } \|\varphi\|_{\mathcal{V}}(w,\sigma) = 0 \\ 0 & \textit{if } \|\varphi\|_{\mathcal{V}}(w,\sigma) = 1 \end{cases}$, *provided that* φ *is of the form* $P_a(x)$, $(x \leq y)$ *or* $(x \in X)$

- $\|\varphi \vee \psi\|_{\mathcal{V}}(w,\sigma) = \|\varphi\|_{\mathcal{V}}(w,\sigma) + \|\psi\|_{\mathcal{V}}(w,\sigma)$

- $\|\varphi \wedge \psi\|_{\mathcal{V}}(w,\sigma) = \|\varphi\|_{\mathcal{V}}(w,\sigma) \cdot \|\psi\|_{\mathcal{V}}(w,\sigma)$

- $\|\exists x \centerdot \varphi\|_{\mathcal{V}}(w,\sigma) = \sum_{i \in \omega} \|\varphi\|_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i])$

- $\|\exists X \centerdot \varphi\|_{\mathcal{V}}(w,\sigma) = \sum_{I \subseteq \omega} \|\varphi\|_{\mathcal{V}\cup\{X\}}(w,\sigma[X \to I])$

- $\|\forall x \centerdot \varphi\|_{\mathcal{V}}(w,\sigma) = \prod_{i \in \omega} \|\varphi\|_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i])$.

To define the semantics of $\forall X \centerdot \varphi$*, we assume that in* K *products over index sets of size continuum exist. Then we put:*

- $\|\forall X \centerdot \varphi\|_{\mathcal{V}}(w,\sigma) = \prod_{I \subseteq \omega} \|\varphi\|_{\mathcal{V}\cup\{X\}}(w,\sigma[X \to I])$.

We note that the additional assumption on products in K does not create a problem since later we exclude universal second order quantification from our constructions. Also as in [9], we have restricted negation to atomic formulas. The reason is that if K is not a Boolean algebra, then it is difficult to define the semantics of the negation of an arbitrary formula. Our restriction is not essential in comparison to classical MSO logics, since any MSO-formula φ is equivalent (both logically and in the sense of defining the same ω-language) to one in which negation is applied only to atomic formulas.

We simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in K\langle\langle A^\omega \rangle\rangle$.

Next, we present several examples of possible interpretations for weighted formulas, for details see [9].

(i) Consider the semiring $K = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ of extended natural numbers and assume that φ does not contain constants $k \in K$. Then we may interpret $\|\varphi\|(w,\sigma)$ as the number of proofs we have that (w,σ) satisfies formula φ.

(ii) The formula $\exists x \centerdot P_a(x)$ counts how often the letter a occurs in the word. Of course *how often* depends on the semiring: Boolean semiring, extended natural numbers, etc.

(iii) For any formula φ without constants over the fuzzy semiring F, we have that $\|\varphi\|(w,\sigma) \neq 0$ iff (w,σ) satisfies φ.

(iv) Let K be an arbitrary finite Boolean algebra $(B, \vee, \wedge, ^-, 0, 1)$. In this case, infinite sums correspond to suprema and infinite products to infima. For any formula φ, we can define the semantics of $\neg\varphi$, by $\|\neg\varphi\|(w,\sigma) := \overline{\|\varphi\|(w,\sigma)}$. Especially, for $K = \mathbf{B}$ the 2-valued Boolean algebra our semantics coincides with the usual semantics of classical MSO-formulas, identifying characteristic series with their supports.

The reader may observe that the above definition is valid for each formula $\varphi \in MSO(K, A)$ and each finite set $\mathcal{V}$ of variables containing $Free(\varphi)$. Next we state that the semantics $\|\varphi\|_{\mathcal{V}}$ depends only on $Free(\varphi)$. More precisely,

Proposition 9 *Let $\varphi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables such that $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$\|\varphi\|_{\mathcal{V}}(w,\sigma) = \|\varphi\|(w, \sigma|_{Free(\varphi)})$$

for each $(w,\sigma) \in A^{\omega}_{\mathcal{V}}$, where σ is a valid $(w, \mathcal{V})$-assignment. Furthermore, $\|\varphi\|$ is Muller recognizable iff $\|\varphi\|_{\mathcal{V}}$ is Muller recognizable.

Proof. We use the same techniques as in the proof of Proposition 3.3 in [9], taking into account Propositions 3, 4, 5 and 6. ■

Let now $Z \subseteq MSO(K, A)$. A series $S : A^{\omega} \to K$ is called *Z-definable* if there is a sentence $\varphi \in Z$ so that $S = \|\varphi\|$.

It has been proved in [9] that universal quantifiers do not preserve in general the recognizability property of power series over finite words. Thus the authors worked on a restricted framework of weighted MSO logics.

Definition 10 *(cf. [9]) A formula $\varphi \in MSO(K, A)$ will be called* restricted *if it contains no universal quantification of the form $\forall X . \psi$, and whenever φ contains a universal first order quantification $\forall x . \psi$, then $\|\psi\|$ is a Muller recognizable step function.*

The subclass of all restricted formulas of $MSO(K, A)$ will be denoted by $RMSO(K, A)$. Moreover, a formula $\varphi \in RMSO(K, A)$ is *restricted existential* if it is of the form $\exists X_1, \ldots, X_n . \psi$ with $\psi \in RMSO(K, A)$ and ψ contains no set quantification. All such restricted existential formulas will compose the class $REMSO(K, A)$. We let $K^{rmso}\langle\langle A^{\omega}\rangle\rangle$ (resp. $K^{remso}\langle\langle A^{\omega}\rangle\rangle$) comprise all series from $K\langle\langle A^{\omega}\rangle\rangle$ which are definable by some sentence in $RMSO(K, A)$ (resp. in $REMSO(K, A)$).

4 The main result

In this section we establish our main result:

Theorem 11 *Let A be an alphabet and K any totally commutative complete semiring. Then*

$$K^{M-rec}\langle\langle A^{\omega}\rangle\rangle = K^{rmso}\langle\langle A^{\omega}\rangle\rangle = K^{remso}\langle\langle A^{\omega}\rangle\rangle .$$

First we show by induction on the structure of RMSO-formulas that $K^{rmso}\langle\langle A^\omega\rangle\rangle \subseteq K^{M-rec}\langle\langle A^\omega\rangle\rangle$. The next Lemmas 12,13,14 are established in a similar way as the corresponding ones in [9] for the finitary case. Thus, we only indicate their proofs.

Lemma 12 *Let $\varphi \in MSO(K, A)$ be atomic. Then $\|\varphi\|$ is Muller recognizable.*

Proof. Assume that $\varphi = k \in K$. Then the one state WMA $\mathcal{A} = (\{q\}, in, wt, \{\{q\}\})$ with $in(q) = k$ and $wt(q, a, q) = 1$ for all $a \in A$, recognizes $\|\varphi\| = k1_{A^\omega}$. Now assume that φ is of the form $P_a(x)$ or $(x \leq y)$ or $(x \in X)$, or φ is the negation of one of these formulas. Then φ can be considered as a classical MSO-formula and thus it is recognizable by a deterministic Muller automaton $\mathcal{A} = (Q, q_0, \delta, \mathcal{F})$ over A_φ. Then, we transform $\mathcal{A}$ to the WMA $\mathcal{A}'$ by assigning the initial weight 1 to q_0, and the weight 1 to each transition of $\mathcal{A}$. Obviously, $\mathcal{A}'$ recognizes $\|\varphi\|$. ∎

Lemma 13 *Let $\varphi, \psi \in MSO(K, A)$ such that $\|\varphi\|$ and $\|\psi\|$ are Muller recognizable. Then $\|\varphi \vee \psi\|$ and $\|\varphi \wedge \psi\|$ are also Muller recognizable.*

Proof. It follows from Propositions 3 and 9. ∎

Lemma 14 *For $\varphi \in MSO(K, A)$ the series $\|\exists x\,.\,\varphi\|$ and $\|\exists X\,.\,\varphi\|$ are Muller recognizable provided that $\|\varphi\|$ is Muller recognizable.*

Proof. We establish the case $\exists x\,.\,\varphi$. The other one is verified in a similar way. Let $\mathcal{V} = Free(\exists x.\varphi)$. Then $x \notin \mathcal{V}$. We consider the non-deleting homomorphism

$$h : A^\omega_{\mathcal{V}\cup\{x\}} \to A^\omega_{\mathcal{V}}$$

which erases the x-row. Then, for any $(w, \sigma) \in A^\omega_{\mathcal{V}}$, we have that

$$\|\exists x\,.\,\varphi\|_{\mathcal{V}}(w,\sigma) = \sum_{i\in\omega} \|\varphi\|_{\mathcal{V}\cup\{x\}}(w, \sigma[x \to i]) = h\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}\right)(w,\sigma).$$

Since $Free(\varphi) \subseteq \mathcal{V} \cup \{x\}$, by Propositions 4 and 9, we deduce that $\|\exists x\,.\,\varphi\| \in K^{M-rec}\langle\langle A^\omega\rangle\rangle$. ∎

The following result deals with the universal first order quantifier. Its proof contains our main new construction.

Lemma 15 *Let $\varphi \in MSO(K, A)$ such that $\|\varphi\|$ is a Muller recognizable step function. Then the series $\|\forall x\,.\,\varphi\|$ is Muller recognizable.*

Proof. Let $\mathcal{W} = Free(\varphi)$ and $\mathcal{V} = Free(\forall x\,.\,\varphi) = \mathcal{W} \setminus \{x\}$. Let also $\|\varphi\| = \sum_{1\leq j\leq n} k_j 1_{L_j}$ with ω-recognizable languages $L_j \subseteq A^\omega_{\mathcal{W}}$ $(1 \leq j \leq n)$. Without any loss, we can assume that the family $(L_j)_{1\leq j\leq n}$ is a partition of $A^\omega_{\mathcal{W}}$.

We assume first that $x \in \mathcal{W}$, and we consider the alphabet $\widetilde{A} = A \times \{1, \ldots, n\}$. A word in $\widetilde{A}^\omega_{\mathcal{V}}$ can be written as a triple (w, v, σ) where $(w, \sigma) \in A^\omega_{\mathcal{V}}$, and v is a mapping from ω to $\{1,\dots,n\}$. Consider the language $\widetilde{L} \subseteq \widetilde{A}_{\mathcal{V}}^{\omega}$ to be the collection of all words $(w,v,\sigma) \in \widetilde{A}_{\mathcal{V}}^{\omega}$, so that for all $i \in \omega$ and $j \in \{1,\dots,n\}$

$$v(i) = j \quad \text{implies} \quad (w,\sigma[x \to i]) \in L_j.$$

We observe that for each $(w,\sigma) \in A_{\mathcal{V}}^{\omega}$ there is a unique v such that $(w,v,\sigma) \in \widetilde{L}$ due to the fact that $(L_j)_{1\le j\le n}$ is a partition of $A_{\mathcal{W}}^{\omega}$.

We shall prove that $\widetilde{L}$ is ω-recognizable. For each $j \in \{1,\dots,n\}$, we let $\widetilde{L}_j$ be the set of all $(w,v,\sigma) \in \widetilde{A}_{\mathcal{V}}^{\omega}$, so that for all $i \in \omega$ we have that $v(i) = j$ implies $(w,\sigma[x \to i]) \in L_j$. Obviously $\widetilde{L} = \bigcap\limits_{1\le j\le n} \widetilde{L}_j$, thus it suffices to prove that each language $\widetilde{L}_j$ is ω-recognizable. Let $j \in \{1,\dots,n\}$ be fixed. We present two arguments to show that $\widetilde{L}_j$ is ω-recognizable. The first argument is shorter, but the second argument has better complexity for doing direct constructions.

Argument 1. We employ Büchi's theorem.

Since L_j is ω-recognizable, there is an MSO-sentence φ_j over $A_{\mathcal{W}}$ such that $L_j = \mathcal{L}(\varphi_j)$. Now we construct from φ_j a sentence φ_j' over $\widetilde{A}_{\mathcal{W}}$ by replacing all occurrences of $P_{(a,s)}(y)$ in φ_j where $a \in A$ and $s \in \{0,1\}^{\mathcal{W}}$ by $\bigvee\limits_{1\le l\le n} P_{(a,l,s)}(y)$. For any $(w,v,\tau) \in \widetilde{A}_{\mathcal{W}}^{\omega}$ we conclude that $(w,v,\tau) \models \varphi_j'$ iff $(w,\tau) \models \varphi_j$. Next, we modify φ_j' to obtain a formula φ_j'' over $\widetilde{A}_{\mathcal{V}}$, as follows. Each occurrence of $P_{(a,l,s)}(y)$ in φ_j' is replaced by $P_{(a,l,s')}(y) \wedge (x = y)$ if $s(x) = 1$, and by $P_{(a,l,s')}(y) \wedge \neg(x = y)$ if $s(x) = 0$, where s' is the restriction of s to $\mathcal{V} = \mathcal{W} \setminus \{x\}$. Then φ_j'' has x as its only free variable. Moreover, for all $(w,v,\sigma)[x \to i] \in \left(\widetilde{A}_{\mathcal{V}}\right)^{\omega}_{\varphi_j''}$ where $i \in \omega$ we have

$$(w,v,\sigma)[x \to i] \models \varphi_j'' \quad \text{iff} \quad (w,v,\sigma[x \to i]) \models \varphi_j'.$$

Now put

$$\widetilde{\varphi_j} = \forall x \,\centerdot \left(\left(\bigvee_{(a,s')\in A_{\mathcal{V}}} P_{(a,j,s')}(x) \right) \to \varphi_j'' \right)$$

where $\to$ is the usual implication. Then $\widetilde{\varphi_j}$ is an MSO-sentence over $\widetilde{A}_{\mathcal{V}}$. We claim that $\mathcal{L}(\widetilde{\varphi_j}) = \widetilde{L}_j$.

Indeed, let $(w,v,\sigma) \in \widetilde{A}_{\mathcal{V}}^{\omega}$. Then $(w,v,\sigma) \models \widetilde{\varphi_j}$ iff for all $i \in \omega$, whenever $v(i) = j$ then

$$(w,v,\sigma)[x \to i] \models \varphi_j''.$$

As noted before, the latter holds

$$\begin{aligned} &\text{iff} \quad (w,v,\sigma[x \to i]) \models \varphi_j' \\ &\text{iff} \quad \quad (w,\sigma[x \to i]) \models \varphi_j \\ &\text{iff} \quad \quad (w,\sigma[x \to i]) \in L_j. \end{aligned}$$

This implies our claim, and hence $\widetilde{L}_j$ is ω-recognizable.

This argument for the corresponding result for finite words [9] is essentially due to Droste and Gastin (unpublished). For the corresponding result for finite pictures it appeared in Mäurer [29].

Argument 2. We give a direct automata-theoretic proof.

Let $\mathcal{A}_j = (Q, q_0, \delta, \mathcal{F})$ be a deterministic Muller automaton over $A_{\mathcal{W}}$ accepting L_j, and let $card(Q) = m$. We shall construct a deterministic Muller automaton $\widetilde{\mathcal{A}}_j = (\widetilde{Q}, \widetilde{q_0}, \widetilde{\delta}, \widetilde{\mathcal{F}})$ over $\widetilde{A}_{\mathcal{V}}$ recognizing $\widetilde{L}_j$. Let $\overline{q}$ be a new state not belonging to Q. We set $\widetilde{Q} = Q \times ((Q \cup \{\overline{q}\}) \times \{0,1\})^m$, $\widetilde{q_0} = (q_0, (\overline{q}, 0), \ldots, (\overline{q}, 0))$, and

$$\widetilde{\mathcal{F}} = \left\{ \widetilde{F} \mid pr_k(\widetilde{F}) \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}' \cup \mathcal{F}'', \quad \text{for each } 2 \leq k \leq m+1 \right\}$$

where

- $\mathcal{F}_i = \{F \times \{i\} \mid F \in \mathcal{F}\}$ for $i = 0, 1$,
- $\mathcal{F}' = \{R' \subseteq (Q \cup \{\overline{q}\}) \times \{0,1\} \mid \overline{q} \in pr_{Q \cup \{\overline{q}\}}(R')\}$,
- $\mathcal{F}'' = \{R'' \subseteq Q \times \{0,1\} \mid pr_{\{0,1\}}(R'') = \{0,1\}\}$,

and

- $pr_k : \widetilde{Q} \to (Q \cup \{\overline{q}\}) \times \{0,1\}$ $(2 \leq k \leq m+1)$ is the k-th projection of $\widetilde{Q}$ onto $(Q \cup \{\overline{q}\}) \times \{0,1\}$,
- $pr_{Q \cup \{\overline{q}\}} : (Q \cup \{\overline{q}\}) \times \{0,1\} \to Q \cup \{\overline{q}\}$ is the projection of $(Q \cup \{\overline{q}\}) \times \{0,1\}$ onto $Q \cup \{\overline{q}\}$, and
- $pr_{\{0,1\}} : (Q \cup \{\overline{q}\}) \times \{0,1\} \to \{0,1\}$ is the projection of $(Q \cup \{\overline{q}\}) \times \{0,1\}$ onto $\{0,1\}$.

The transition mapping $\widetilde{\delta}$ is determined as follows. For any $(q_1, (q_2, z_2), \ldots, (q_{m+1}, z_{m+1})), (q_1', (q_2', z_2'), \ldots, (q_{m+1}', z_{m+1}')) \in \widetilde{Q}$ and $(a, l, s) \in \widetilde{A}_{\mathcal{V}}$, define $\widetilde{\delta}((q_1, (q_2, z_2), \ldots, (q_{m+1}, z_{m+1})), (a, l, s)) = (q_1', (q_2', z_2'), \ldots, (q_{m+1}', z_{m+1}'))$ as follows. We put $\delta(q_1, (a, s[x \to 0])) = q_1'$, and for each $2 \leq k \leq m+1$ let

$$q_k'' = \begin{cases} \overline{q} & \text{if } q_k = \overline{q} \\ \delta(q_k, (a, s[x \to 0])) & \text{if } q_k \in Q \end{cases}.$$

Now for each $2 \leq k \leq m+1$ define:

- $q_k^+ = \overline{q}$ if there exists $2 \leq t < k$ such that $q_t'' = q_k''$, and
- $q_k^+ = q_k''$ otherwise.

Case 1: $l \neq j$. Put $(q_k', z_k') = (q_k^+, z_k)$ for any $2 \leq k \leq m+1$.

Case 2: $l = j$.

(i) If there exists $2 \leq g \leq m+1$ such that $q_g^+ = \delta(q_1, (a, s[x \to 1]))$ then put

- $(q_k', z_k') = (q_k^+, z_k)$ for all $2 \leq k \leq m+1$.

(ii) Otherwise, note that $q_g^+ \in (Q \setminus \{\delta(q_1, (a, s[x \to 1]))\}) \cup \{\overline{q}\}$ for each $2 \leq g \leq m+1$, and $card(Q) = m$. Hence we can choose the smallest $r \leq m+1$ such that $q_r^+ = \overline{q}$ and put

- $q_r' = \delta(q_1, (a, s[x \to 1]))$ and $z_k' \equiv (1 + z_k) \bmod 2$, and
- $(q_k', z_k') = (q_k^+, z_k)$ for any $k \neq r$.

Let us explain informally the operation of the automaton $\widetilde{\mathcal{A}}_j$. Given any word $(w, v, \sigma) \in \widetilde{A}_{\mathcal{V}}^{\omega}$, we want to show that $(w, v, \sigma) \in \widetilde{L}_j$ iff for each $i \in \omega$, $v(i) = j$ implies $(w, \sigma[x \to i]) \in L_j$. This means that our deterministic automaton $\widetilde{\mathcal{A}}_j$ should accept the word (w, v, σ) iff the original automaton $\mathcal{A}_j$ accepts all the words $(w, \sigma[x \to i])$ with $i \in \omega$ and $v(i) = j$. In turn, this means that the unique path $\widetilde{P}_{(w,v,\sigma)}$ of $\widetilde{\mathcal{A}}_j$ over (w, v, σ) is successful iff all the unique paths $P_{(w,\sigma[x \to i])}$ ($i \in \omega$ and $v(i) = j$) of $\mathcal{A}_j$ over $(w, \sigma[x \to i])$ are successful. Thus, $\widetilde{\mathcal{A}}_j$ should have the ability to check simultaneously all the paths $P_{(w,\sigma[x \to i])}$ ($i \in \omega$ and $v(i) = j$) of $\mathcal{A}_j$ and should verify whether they are successful or not. This process will be implemented by the last m components of the states of $\widetilde{\mathcal{A}}_j$. More precisely, let $(w, v, \sigma) = (a_i, l_i, s_i)_{i \geq 0}$, $(a_i, l_i, s_i) \in \widetilde{A}_{\mathcal{V}}$ for any $i \geq 0$, and $\widetilde{P}_{(w,v,\sigma)} = \left((q_1^i, (q_2^i, z_2^i), \ldots, (q_{m+1}^i, z_{m+1}^i)), (a_i, l_i, s_i), (q_1^{i+1}, (q_2^{i+1}, z_2^{i+1}), \ldots, (q_{m+1}^{i+1}, z_{m+1}^{i+1}))\right)_{i \geq 0}$.
For any $1 \leq k \leq m+1$, we denote by $pr_k\left(\widetilde{P}_{(w,v,\sigma)}\right)$ the k-th projection of $\widetilde{P}_{(w,v,\sigma)}$ defined by $pr_1\left(\widetilde{P}_{(w,v,\sigma)}\right) = \left(q_1^i, (a_i, l_i, s_i), q_1^{i+1}\right)_{i \geq 0}$ and $pr_k\left(\widetilde{P}_{(w,v,\sigma)}\right) = \left((q_k^i, z_k^i), (a_i, l_i, s_i), (q_k^{i+1}, z_k^{i+1})\right)_{i \geq 0}$ for $2 \leq k \leq m+1$.
Furthermore, assume that the sequence $0 \leq i_1 < i_2 < \ldots$ contains all the occurrences of j along (w, v, σ). Then the automaton $\widetilde{\mathcal{A}}_j$ simulates the path $P_{(w,\sigma[x \to \emptyset])}$ of $\mathcal{A}_j$ over $(w, \sigma[x \to \emptyset])$ along its first projection, where the notation $\sigma[x \to \emptyset]$ means that at any place $i \geq 0$ we have $s_i[x \to 0]$ (observe that this is not a valid assignment). The states of any other k-th projection ($2 \leq k \leq m+1$) of $\widetilde{P}_{(w,v,\sigma)}$ remain stable at $(\overline{q}, 0)$, as long as the automaton acts on the prefix $(a_i, l_i, s_i)_{0 \leq i < i_1}$. Then at i_1, $\widetilde{\mathcal{A}}_j$ sets $(q_2^{i_1+1}, z_2^{i_1+1}) = (\delta(q_1^{i_1}, (a_{i_1}, s_{i_1}[x \to 1])), 1)$. In the sequel, $\widetilde{\mathcal{A}}_j$ runs the suffix path $P_{(w,\sigma[x \to i_1])}|_{>i_1}$ along its second projection.

At the second occurrence of j, i.e., $l_{i_2} = j$ the automaton $\widetilde{\mathcal{A}}_j$ will start to run the suffix path $P_{(w,\sigma[x \to i_2])}|_{>i_2}$. More precisely, if $q_2^{i_2+1} = \delta(q_1^{i_2}, (a_{i_2}, s_{i_2}[x \to 1]))$ then it is clear that $P_{(w,\sigma[x \to i_1])}|_{>i_2} = P_{(w,\sigma[x \to i_2])}|_{>i_2}$. Thus, $\widetilde{\mathcal{A}}_j$ simulates the suffix path $P_{(w,\sigma[x \to i_2])}|_{>i_2}$ along the second projection of $\widetilde{P}_{(w,v,\sigma)}$. Otherwise, it sets $(q_3^{i_2+1}, z_3^{i_2+1}) = (\delta(q_1^{i_2}, (a_{i_2}, s_{i_2}[x \to 1])), 1)$. Now the following happens. Either $q_2^{i'} = q_3^{i'}$ for an index $i' > i_2 + 1$, or $q_2^i \neq q_3^i$ for any $i > i_2 + 1$. Since $(w, \sigma[x \to i_1])|_{>i'} = (w, \sigma[x \to i_2])|_{>i'}$ in the first case we get $P_{(w,\sigma[x \to i_1])}|_{>i'} = P_{(w,\sigma[x \to i_2])}|_{>i'}$. Thus, the automaton sets $q_3^{i'} = \overline{q}$ and runs the suffix path $P_{(w,\sigma[x \to i_2])}|_{>i'}$ along the second projection of $\widetilde{P}_{(w,v,\sigma)}$. In the latter case, $\widetilde{\mathcal{A}}_j$ simulates the suffix path $P_{(w,\sigma[x \to i_2])}|_{>i_2}$ along the third projection of $\widetilde{P}_{(w,v,\sigma)}$.

The same procedure is followed for each occurrence of j. It will follow for each $p \geq 1$ and for each path $P_{(w,\sigma[x\to i_p])}$ ($p \geq 1$) that there exists a $2 \leq k \leq m+1$ such that the suffix $P_{(w,\sigma[x\to i_p])}|_{>i_p} = pr_{Q\cup\{\overline{q}\}}\left(pr_k\left(\widetilde{P}_{(w,v,\sigma)}|_{>i_p}\right)\right)$. Moreover, we shall have $pr_{\{0,1\}}\left(pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}|_{>i_p}\right)\right)\right) = \{0\}$ or $\{1\}$. The contribution of the set $\{0,1\}$ in the states is to ensure that there is a time where a path of the original automaton is simulated along the same state components of the new automaton. Now the path $\widetilde{P}_{(w,v,\sigma)}$ is successful iff for each $2 \leq k \leq m+1$, $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}' \cup \mathcal{F}''$. On the other hand, we have that for each $p \geq 1$, $In^Q\left(P_{(w,\sigma[x\to i_p])}\right) \subseteq Q$. Thus, $\widetilde{P}_{(w,v,\sigma)}$ is successful iff $P_{(w,\sigma[x\to i_p])}$ (for each $p \geq 1$) is successful, i.e., iff for each $i \in \omega$, $v(i) = j$ implies $(w, \sigma[x \to i) \in L_j$.

A word $(w, v, \sigma) \in \widetilde{A}^{\omega}_{\mathcal{V}}$ with $v(i) \neq j$ for any $i \in \omega$, is accepted by $\widetilde{\mathcal{A}}_j$ with the successful path $\widetilde{P}_{(w,v,\sigma)}$ with $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) = \{(\overline{q}, 0)\}$ for any $2 \leq k \leq m+1$.

On the other hand, taking into account the definition of $\widetilde{\delta}$, for any word $(w, v, \sigma) \in \widetilde{A}^{\omega}_{\mathcal{V}}$ it cannot happen that for any $2 \leq k \leq m+1$, $pr_{Q\cup\{\overline{q}\}}\left(pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ $= R_k \cup \{\overline{q}\}$, for some $\emptyset \neq R_k \subseteq Q$, or $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}''$.

Next, we formally show that

$$(w, v, \sigma) \in \mathcal{L}_\omega\left(\widetilde{\mathcal{A}}_j\right) \quad \text{iff} \quad \text{whenever } v(i) = j \text{ then } (w, \sigma[x \to i]) \in \mathcal{L}_\omega\left(\mathcal{A}_j\right).$$

Let $(w, v, \sigma) = (a_i, l_i, s_i)_{i\geq 0} \in \mathcal{L}_\omega\left(\widetilde{\mathcal{A}}_j\right)$ and $\widetilde{P}_{(w,v,\sigma)} = ((q_1^i, (q_2^i, z_2^i), \ldots, (q_{m+1}^i, z_{m+1}^i)),$ $(a_i, l_i, s_i), (q_1^{i+1}, (q_2^{i+1}, z_2^{i+1}), \ldots, (q_{m+1}^{i+1}, z_{m+1}^{i+1})))_{i\geq 0}$ be the unique successful path of $\widetilde{\mathcal{A}}_j$ over (w, v, σ). Thus, for each $2 \leq k \leq m+1$, $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}' \cup \mathcal{F}''$. We distinguish the following cases:
(a) $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}' \cup \mathcal{F}''$ for each $2 \leq k \leq m+1$, and
(b) there exists $2 \leq k \leq m+1$ such that $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1$.

Case (a): $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}' \cup \mathcal{F}''$ for each $2 \leq k \leq m+1$.

(a.1) $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) = \{(\overline{q}, 0)\}$ for each $2 \leq k \leq m+1$. By definition of $\widetilde{\delta}$ this implies that for any $i \in \omega$, $v(i) \neq j$.

(a.2) $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \{\{(\overline{q}, 0)\}, \{(\overline{q}, 1)\}\}$ for each $2 \leq k \leq m+1$ and there is a k' such that $pr_{k'}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) = \{(\overline{q}, 1)\}$. By definition of $\widetilde{\delta}$ Case 2 (ii) this means that there is an index $i \geq 0$ and a $2 \leq g \leq m+1$ such that $q_g^i \in Q$. But then we should have $q_2^i \in Q$ which implies that for any $i' \geq i$, $q_2^{i'} \in Q$. Thus, $pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \subseteq Q \times \{1\}$ which is a contradiction.

(a.3) There exists $2 \leq k \leq m+1$ and a set $\emptyset \neq R \subseteq Q$ such that $pr_{Q\cup\{\overline{q}\}}\left(pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = R \cup \{\overline{q}\}$. Consider an index $i \geq 0$ such that

$q_k^i \in R$. Then by definition of $\widetilde{\delta}$, we obtain that $q_2^i \in Q$. In turn, this implies that for any $i' \geq i$, it holds $q_2^{i'} \in Q$. Thus, $pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \subseteq Q \times \{1\}$ which is a contradiction.

(a.4) There exists $2 \leq k \leq m+1$ and a set $\emptyset \neq R \subseteq Q$ such that $pr_{\{0,1\}}\left(pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = \{0,1\}$. Then, the same argument used in (a.3) contradicts this case.

Case (b): There exists $2 \leq k \leq m+1$ such that $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1$. Then, there is at least one index $i \in \omega$ such that $v(i) = j$. Let us assume that the non-empty finite or infinite sequence $0 \leq i_1 < i_2 < \ldots$ contains all the occurrences of j along (w, v, σ), i.e., $l_1 = l_2 = \ldots = j$. It is clear that

$$(w, \sigma[x \to i_p])|_{>i_p} = (w, \sigma[x \to i_1])|_{>i_p}$$

for any $p \geq 1$.

Next, we show by induction on p that for any $p \geq 1$, the path $P_{(w,\sigma[x\to i_p])}$ of $\mathcal{A}_j$ over $(w, \sigma[x \to i_p])$ is successful. We start with $p = 1$. Let $P_{(w,\sigma[x\to i_1])} = \left(t_i^{i_1}\right)_{i\geq 0}$ be defined by $t_i^{i_1} = (\widehat{q}^i, \alpha_i, \delta(\widehat{q}^i, \alpha_i))$ where $\widehat{q}^0 = q_0$ and

$$\alpha_i = \begin{cases} (a_i, s_i[x \to 0]) & \text{if } 0 \leq i < i_1 \text{ or } i > i_1 \\ (a_i, s_i[x \to 1]) & \text{if } i = i_1 \end{cases}.$$

By definition of $\widetilde{\delta}$, for any $i \geq 0$ we get

$$\widehat{q}^i = \begin{cases} q_1^i & \text{if } 0 \leq i \leq i_1 \\ q_2^i, & \text{if } i > i_1 \end{cases}.$$

Thus, $P_{(w,\sigma[x\to i_1])}|_{>i_1} = pr_{Q\cup\{\widetilde{q}\}}\left(pr_2\left(\widetilde{P}_{(w,v,\sigma)}|_{>i_1}\right)\right)$. On the other hand $In^Q\left(P_{(w,\sigma[x\to i_1])}|_{>i_1}\right) = In^Q\left(P_{(w,\sigma[x\to i_1])}\right) \subseteq Q$ which implies that $pr_{Q\cup\{\widetilde{q}\}}\left(pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) \subseteq Q$. Since $pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1$, (in fact it holds $pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_1$, by definition of $\widetilde{\delta}$) we have $In^Q\left(P_{(w,\sigma[x\to i_1])}\right) \in \mathcal{F}$, as wanted.

Now, we deal with the second occurrence of j, i.e., with the case $p = 2$ and $l_2 = j$. Let $P_{(w,\sigma[x\to i_2])} = \left(t_i^{i_2}\right)_{i\geq 0}$ with $t_i^{i_2} = \left(q^{*i}, \beta_i, \delta(q^{*i}, \beta_i)\right)$ where $q^{*0} = q_0$ and

$$\beta_i = \begin{cases} (a_i, s_i[x \to 0]) & \text{if } 0 \leq i < i_2 \text{ or } i > i_2 \\ (a_i, s_i[x \to 1]) & \text{if } i = i_2 \end{cases}.$$

Taking into account the definition of $\widetilde{\delta}$, we distinguish the next cases:

Subcase (A): $q_2^i \neq q^{*i}$ for any $i > i_2$. Then we have

$$q^{*i} = \begin{cases} q_1^i & \text{if } 0 \leq i \leq i_2 \\ q_3^i & \text{if } i > i_2 \end{cases}.$$

Thus $In^Q\left(P_{(w,\sigma[x\to i_2])}|_{>i_2}\right) = pr_{Q\cup\{\widetilde{q}\}}\left(pr_3\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ and $z_3^i = 1$ for each $i > i_2$. This implies that $pr_{Q\cup\{\widetilde{q}\}}\left(pr_3\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) \in \mathcal{F}$, i.e.,

$pr_3\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_1$. We conclude that $In^Q\left(P_{(w,\sigma[x\to i_2])}\right) \in \mathcal{F}$.

Subcase (B): $q_2^{i_2+1} = q^{*i_2+1}$. Then we have

$$q^{*i} = \begin{cases} q_1^i & \text{if } 0 \leq i \leq i_2 \\ q_2^i & \text{if } i > i_2 \end{cases}.$$

Thus, $In^Q\left(P_{(w,\sigma[x\to i_2])}|_{>i_2}\right) = pr_{Q\cup\{\widetilde{q}\}}\left(pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ and so $In^Q\left(P_{(w,\sigma[x\to i_2])}\right) \in \mathcal{F}$.

Subcase (C): $q_2^{i_2+1} \neq q^{*i_2+1}$ and there exists an index $f > i_2+1$ such that $q_2^f = q^{*f}$. In this case, we get

$$q^{*i} = \begin{cases} q_1^i & \text{if } 0 \leq i \leq i_2 \\ q_3^i & \text{if } i_2 < i < f \\ q_2^i & \text{if } i \geq f \end{cases}.$$

Again, we obtain $In^Q\left(P_{(w,\sigma[x\to i_2])}|_{>i_2}\right) = pr_{Q\cup\{\widetilde{q}\}}\left(pr_2\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ and thus $In^Q\left(P_{(w,\sigma[x\to i_2])}\right) \in \mathcal{F}$.

We conclude that the path $P_{(w,\sigma[x\to i_2])}$ is successful.

For the induction step, let $p > 1$ and let us assume that the path $P_{(w,\sigma[x\to i_{p'}])}$ is successful for any $1 \leq p' < p$. Let $P_{(w,\sigma[x\to i_p])} = \left(t_i^{i_p}\right)_{i\geq 0}$ be determined by $t_i^{i_p} = (q^{\#i}, \gamma_i, \delta(q^{\#i}, \gamma_i))$ where $q^{\#0} = q_0$ and

$$\gamma_i = \begin{cases} (a_i, s_i[x\to 0]) & \text{if } 0 \leq i < i_p \text{ or } i > i_p \\ (a_i, s_i[x\to 1]) & \text{if } i = i_p \end{cases}.$$

We distinguish the following cases:

Subcase (A′): $q_g^i \neq q^{\#i}$ for any $i > i_p$ and any $2 \leq g \leq m+1$ such that $q_g^{i_p} \neq \overline{q}$. Then, we shall have $q_r^i = q^{\#i}$ for each $i > i_p$, where $r \leq m+1$ is the smallest index such that $q_r^{i_p} = \overline{q}$. Moreover, we have $z_r^{i_p+1} \equiv (1 + z_r^{i_p}) \bmod 2$ and since $q_r^i = q^{\#i}$ for each $i > i_p$, by $\widetilde{\delta}$ we get $z_r^i = z_r^{i_p+1}$ for each $i > i_p$. Thus we have $pr_{Q\cup\{\widetilde{q}\}}\left(pr_r\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = In^Q\left(P_{(w,\sigma[x\to i_p])}\right) \subseteq Q$, $pr_{\{0,1\}}\left(pr_r\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = \left\{z_r^{i_p+1}\right\}$ (i.e., it is either $\{0\}$ or $\{1\}$), and since $\widetilde{P}_{(w,v,\sigma)}$ is successful, we obtain $pr_r\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1$, i.e., $In^Q\left(P_{(w,\sigma[x\to i_p])}\right) \in \mathcal{F}$, as required.

Subcase (B′): $q_g^{i_p+1} = q^{\#i_p+1}$ for some $2 \leq g \leq m+1$.

Then, by Case 2 (ii) of the construction of $\widetilde{\delta}$ it follows that after at most $g-1$ steps we can determine a $2 \leq g' \leq g$ such that $pr_{Q\cup\{\widetilde{q}\}}\left(pr_{g'}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = In^Q\left(P_{(w,\sigma[x\to i_p])}\right) \subseteq Q$, and $pr_{\{0,1\}}\left(pr_{g'}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ is either $\{0\}$ or $\{1\}$. Taking into account that $\widetilde{P}_{(w,v,\sigma)}$ is successful, we get $pr_{g'}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in \mathcal{F}_0 \cup \mathcal{F}_1$ and thus $In^Q\left(P_{(w,\sigma[x\to i_p])}\right) \in \mathcal{F}$.

Subcase (C′): $q_g^{i_p+1} \neq q^{\#i_p+1}$ for any $2 \leq g \leq m+1$ and there exists an index

$f > i_p + 1$ and a $2 \le g' \le m+1$ such that $q_{g'}^{f} = q^{\# f}$. Let f be the least index with the above property, and let $r \le m+1$ be the smallest index such that $q_r^{i_p} = \overline{q}$; this exists since $card(Q) = m$. By definition of $\widetilde{\delta}$, for each $0 \le i \le f$ we have

$$q^{\# i} = \begin{cases} q_1^i & \text{if } 0 \le i \le i_p \\ q_r^i & \text{if } i_p < i < f \\ q_{g'}^i & \text{if } i = f \end{cases} .$$

Then by Case 2 (ii) of the construction of $\widetilde{\delta}$ we obtain after at most $g'-1$ steps a $2 \le g'' \le g' \le m+1$ such that $q^{\# i} = q_{g''}^i$ for any $i > f$. Thus $In^Q\left(P_{(w,\sigma[x \to i_p])}\right) = pr_{Q \cup \{\overline{q}\}}\left(pr_{g''}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ and $pr_{\{0,1\}}\left(pr_{g''}\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ equals either $\{0\}$ or $\{1\}$. Since $\widetilde{P}_{(w,v,\sigma)}$ is successful, we have that $P_{(w,\sigma[x \to i_p])} \in \mathcal{F}_0 \cup \mathcal{F}_1$ i.e., $P_{(w,\sigma[x \to i_p])}$ is also successful.

We have shown that for each $i \in \omega$ with $v(i) = j$, $(w, \sigma[x \to i]) \in \mathcal{L}_\omega(\mathcal{A}_j)$.

Next, we show the converse implication. To this end, let $(w,v,\sigma) = (a_i, l_i, s_i)_{i \ge 0} \in \widetilde{A}_{\mathcal{V}}^{\omega}$ such that for each $i \in \omega$ with $v(i) = j$, $(w, \sigma[x \to i]) \in L_j$. Keeping the above notations we shall show that the path $\widetilde{P}_{(w,v,\sigma)}$ is successful.

First, assume that $l_i \ne j$ for each $i \in \omega$. Then for each $2 \le k \le m+1$, we have $pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) = \{(\overline{q}, 0)\}$. Thus, $In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right) \in \widetilde{\mathcal{F}}$ as wanted.

Now, let $0 \le i_1 < i_2 < \ldots$ be the sequence containing all the occurrences of j along (w,v,σ). We claim that for each $2 \le k \le m+1$

$$\begin{aligned} pr_k\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \in &\left\{In^Q\left(P_{(w,\sigma[x \to i_p])}\right) \times \{0\} \mid p \ge 1\right\} \cup \\ &\left\{In^Q\left(P_{(w,\sigma[x \to i_p])}\right) \times \{1\} \mid p \ge 1\right\} \cup \mathcal{F}' \cup \mathcal{F}''. \end{aligned}$$

Indeed, choose any index $2 \le f \le m+1$ such that $pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right) \notin \mathcal{F}' \cup \mathcal{F}''$. Then $pr_{Q \cup \{\overline{q}\}}\left(pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) \subseteq Q$ and $pr_{\{0,1\}}\left(pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right)$ is either $\{0\}$ or $\{1\}$. Let $pr_f\left(\widetilde{P}_{(w,v,\sigma)}\right) = \left((q_f^i, z_f^i), (a_i, v_i, s_i), \left(q_f^{i+1}, z_f^{i+1}\right)\right)_{i \ge 0}$. Since $pr_{Q \cup \{\overline{q}\}}\left(pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) \subseteq Q$ there exists an index i' such that $q_f^i \in Q$ for any $i \ge i'$. We choose i' to be the least index with this property. Then by definition of $\widetilde{\delta}$ we have $l_{i'-1} = j$. On the other hand, $pr_{\{0,1\}}\left(pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = \{0\}$ or $\{1\}$, implies that there is an index $i'' \ge i'$ such that for each $i \ge i''$, $pr_{\{0,1\}}\left(\left(pr_f\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)^i\right)$ equals either $\{0\}$ or $\{1\}$. According to $\widetilde{\delta}$, this means that $\widetilde{\mathcal{A}}_j$ does not apply to any f-component state of the suffix path $\widetilde{P}_{(w,v,\sigma)}|_{>i''}$ any operation described in Case 2 (ii) of the construction of $\widetilde{\delta}$. Then by Case 2 (i) of $\widetilde{\delta}$, we obtain that $pr_{Q \cup \{\overline{q}\}}\left(pr_f\left(\widetilde{P}_{(w,v,\sigma)}|_{>i_p}\right)\right) = P_{(w,\sigma[x \to i_p])}|_{>i_p}$, for some $i_p \ge i''$ ($p \ge 1$). Hence, we get $pr_{Q \cup \{\overline{q}\}}\left(pr_f\left(In^{\widetilde{Q}}\left(\widetilde{P}_{(w,v,\sigma)}\right)\right)\right) = In^Q\left(P_{(w,\sigma[x \to i_p])}\right)$ which implies our claim. Thus, $(w,v,\sigma) \in \mathcal{L}_\omega\left(\widetilde{\mathcal{A}}_j\right)$.

We conclude that $\mathcal{L}_\omega\left(\widetilde{\mathcal{A}}_j\right) = \widetilde{L}_j$ as required.

Now, we can effectively construct a deterministic Muller automaton say $\widetilde{\mathcal{A}} = (Q, q_0, \delta, \mathcal{F})$ over $\widetilde{A}_{\mathcal{V}}$ accepting $\widetilde{L}$. Then we convert $\widetilde{\mathcal{A}}$ to a WMA $\mathcal{A} = (Q, in, wt, \mathcal{F})$ in the following manner:

- $in(q) = \begin{cases} 1 & \text{if } q = q_0 \\ 0 & \text{otherwise} \end{cases}$, and

- $wt(q,(a,j,s),q') = \begin{cases} k_j & \text{if } \delta(q,(a,j,s)) = q' \\ 0 & \text{otherwise} \end{cases}$,

for all $q, q' \in Q$, $a \in A$, $s \in \{0,1\}^{\mathcal{V}}$.

Since $\widetilde{\mathcal{A}}$ is deterministic, for each $(w,v,\sigma) \in \widetilde{L}$ there is a unique path $P_w = (t_i)_{i\geq 0}$ in $\widetilde{\mathcal{A}}$ such that $\|\mathcal{A}\|(w,v,\sigma) = weight(P_w) = \prod_{i\geq 0} wt(t_i)$ in $\mathcal{A}$, whereas $\|\mathcal{A}\|(w,v,\sigma) = 0$ for each $(w,v,\sigma) \in \widetilde{A}_{\mathcal{V}}^{\omega} \setminus \widetilde{L}$.

For each $i \geq 0$ note that if $v(i) = j$, then $wt(t_i) = k_j$ by construction of $\mathcal{A}$, and also $(w, \sigma[x \to i]) \in L_j$, so $\|\varphi\|_{\mathcal{V}\cup\{x\}}(w,\sigma[x\to i]) = k_j$.

We consider now the strict alphabetic homomorphism

$$h : \widetilde{A}_{\mathcal{V}}^{\omega} \to A_{\mathcal{V}}^{\omega}$$

defined by $h(a,k,s) = (a,s)$ for each $(a,k,s) \in \widetilde{A}_{\mathcal{V}}$.

Then for any $(w,\sigma) \in A_{\mathcal{V}}^{\omega}$ and the unique v such that $(w,v,\sigma) \in \widetilde{L}$, we have

$$\begin{aligned} h(\|\mathcal{A}\|)(w,\sigma) &= \|\mathcal{A}\|(w,v,\sigma) \\ &= \prod_{i\geq 0} wt(t_i) \\ &= \prod_{i\in\omega} \|\varphi\|_{\mathcal{V}\cup\{x\}}(w,\sigma[x\to i]) \\ &= \|\forall x \,.\, \varphi\|(w,\sigma). \end{aligned}$$

We conclude that $\|\forall x \,.\, \varphi\| = h(\|\mathcal{A}\|)$ which by Proposition 4 is Muller recognizable.

To complete our proof it remains to treat the case $x \notin \mathcal{W}$. Then $\mathcal{V} = \mathcal{W}$. Consider the formula $\varphi' = \varphi \wedge (x \leq x)$. Then $\|\varphi'\|$ is Muller recognizable by Lemma 13, and moreover, $\|\varphi\|_{\mathcal{V}\cup\{x\}} = \|\varphi'\|_{\mathcal{V}\cup\{x\}}$. We thus obtain, $\|\forall x \,.\, \varphi\| = \|\forall x \,.\, \varphi'\|$ which has already been proved to be Muller recognizable. ■

Now we can establish one inclusion of our main result:

Proposition 16 $K^{rmso}\langle\langle A^{\omega}\rangle\rangle \subseteq K^{M-rec}\langle\langle A^{\omega}\rangle\rangle$.

Proof. This is immediate by Lemmas 12,13,14 and 15. ■

In the sequel, we shall establish the inclusion $K^{M-rec}\langle\langle A^{\omega}\rangle\rangle \subseteq K^{remso}\langle\langle A^{\omega}\rangle\rangle$. The notion of unambiguous weighted MSO-formulas and some notation from [9] will be useful for our constructions. Note that here these notions are interpreted over infinite words.

Definition 17 *The class of* unambiguous *formulas in $MSO(K, A)$ is defined inductively as follows:*

- *All atomic formulas of the form $P_a(x)$, $(x \leq y)$ or $(x \in X)$ and their negations are unambiguous.*
- *If φ, ψ are unambiguous, then $\varphi \wedge \psi$, $\forall x \,.\, \varphi$ and $\forall X \,.\, \varphi$ are also unambiguous.*
- *If φ, ψ are unambiguous and $Supp(\|\varphi\|) \cap Supp(\|\psi\|) = \varnothing$, then $\varphi \vee \psi$ is unambiguous.*
- *Let φ be unambiguous and $\mathcal{V} = Free(\varphi)$. If for any $(w, \sigma) \in A^{\omega}_{\mathcal{V}}$ there is at most one element $i \in \omega$ such that $\|\varphi\|_{\mathcal{V} \cup \{x\}} (w, \sigma[x \rightarrow i]) \neq 0$, then $\exists x \,.\, \varphi$ is unambiguous.*
- *Let φ be unambiguous and $\mathcal{V} = Free(\varphi)$. If for any $(w, \sigma) \in A^{\omega}_{\mathcal{V}}$ there is at most one subset $I \subseteq \omega$, such that $\|\varphi\|_{\mathcal{V} \cup \{X\}} (w, \sigma[X \rightarrow I]) \neq 0$, then $\exists X \,.\, \varphi$ is unambiguous.*

Then the following three results can be derived just as in [9]. Proposition 20 may also be of independent interest.

Proposition 18 *Let $\varphi \in MSO(K, A)$ be unambiguous. We may also regard φ as a classical MSO-formula defining the language $\mathcal{L}(\varphi) \subseteq A^{\omega}_{\varphi}$. Then $\|\varphi\| = 1_{\mathcal{L}(\varphi)}$ is a Muller recognizable step function.*

Lemma 19 *For each classical MSO-formula φ not containing set quantifications (but possibly including atomic formulas of the form $(x \in X)$), we can effectively construct two unambiguous $MSO(K, A)$-formulas φ^+ and φ^- such that $\|\varphi^+\| = 1_{\mathcal{L}(\varphi)}$ and $\|\varphi^-\| = 1_{\mathcal{L}(\neg\varphi)}$.*

Proposition 20 *For any classical MSO-sentence φ, we can effectively construct an unambiguous $MSO(K, A)$-sentence ψ defining the same language, i.e., $\|\psi\| = 1_{\mathcal{L}(\varphi)}$.*

If φ is an unambiguous formula and ψ any $MSO(K, A)$-formula, we put

$$(\varphi \rightarrow \psi) := \varphi^- \vee (\varphi^+ \wedge \psi).$$

In particular, if $\varphi = (x \in K)$ and $\psi = k$ where $k \in K$, then for any $(w, \sigma) \in A^{\omega}_{\mathcal{V}}$ with σ a valid $(w, \mathcal{V})$-assignment, we have

$$\|(x \in X) \rightarrow k\|_{\mathcal{V}} (w, \sigma) = \begin{cases} k & \text{if } \sigma(x) \in \sigma(X) \\ 1 & \text{otherwise} \end{cases}$$

hence $\|(x \in X) \rightarrow k\|_{\mathcal{V}}$ is a Muller recognizable step function, and we get

$$\|\forall x \,.\, ((x \in X)) \rightarrow k)\|_{\mathcal{V}} (w, \sigma) = k^{|\sigma(X)|}.$$

We also let

$$\min(y) := \forall x \,.\, y \leq x$$

and

$$(y = x+1) := ((x \leq y) \wedge \neg(y \leq x) \wedge \forall z \,.\, (z \leq x \vee y \leq z))^{+} .$$

Furthermore, for set variables $X_1, \ldots, X_m$ we put

$$\text{partition}(X_1, \ldots, X_m) := \forall x \,.\, \bigvee_{i=1,\ldots,m} \left((x \in X_i) \wedge \bigwedge_{j \neq i} \neg(x \in X_j) \right) .$$

Now we are ready to show

Proposition 21 $K^{M-rec}\langle\langle A^{\omega}\rangle\rangle \subseteq K^{remso}\langle\langle A^{\omega}\rangle\rangle$.

Proof. Let $\mathcal{A} = (Q, in, wt, \mathcal{F})$ be a WMA over A and K. We set $T = Q \times A \times Q$. For each triple $(p, a, q) \in T$, we consider a set variable $X_{p,a,q}$ and let $\mathcal{V} = \{X_{p,a,q} \mid (p, a, q) \in T\}$. We choose an enumeration $X_1, \ldots, X_m$ of $\mathcal{V}$, where $m = |Q|^2 |A|$. Now we define the unambiguous formula

$$\psi(X_1, \ldots, X_m) := \text{partition}(X_1, \ldots, X_m)^{+} \wedge \bigwedge_{(p,a,q) \in T} \forall x \,.\, ((x \in X_{p,a,q}) \rightarrow P_a(x))^{+} \wedge$$

$$\forall x \,.\, \forall y \,.\, \left((y = x+1) \rightarrow \bigvee_{\substack{p,q,r \in Q \\ a,b \in A}} (x \in X_{p,a,q}) \wedge (y \in X_{q,b,r}) \right)^{+} .$$

Let $w = a_0 a_1 \ldots \in A^{\omega}$. We show that there is a bijection between the set of paths in $\mathcal{A}$ over w, and the set of $(w, \mathcal{V})$-assignments σ satisfying ψ, i.e., such that $\|\psi\|(w, \sigma) = 1$. To this end, we consider a path $P_w = (t_i)_{i \geq 0}$ of $\mathcal{A}$ over w, with $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. We define the $(w, \mathcal{V})$-assignment σ_{P_w} by $\sigma_{P_w}(X_{p,a,q}) = \{i \mid (q_i, a_i, q_{i+1}) = (p, a, q)\}$. It is obvious that $\|\psi\|(w, \sigma_{P_w}) = 1$. Conversely, let σ be a $(w, \mathcal{V})$-assignment such that $\|\psi\|(w, \sigma) = 1$. Then, for any $x \in \omega$ there are uniquely determined $p, q \in Q$ and $a \in A$, due to $\text{partition}(X_1, \ldots, X_m)$, so that $x \in \sigma(X_{p,a,q})$. Furthermore, for $y = x+1$ we shall also have $y \in X_{q,b,r}$ for unique $r \in Q$ and $b \in A$. In this way, we obtain a unique path P_w of $\mathcal{A}$ over w, such that $\sigma_{P_w} = \sigma$.

We consider now the formula

$$
\begin{aligned}
\varphi(X_1,\ldots,X_m) := \psi(X_1,\ldots,X_m) \wedge \left(\forall x \centerdot \bigwedge_{(p,a,q)\in T} ((x \in X_{p,a,q}) \to wt(p,a,q))\right) \wedge \\
\left(\exists y \centerdot \left(\min(y) \wedge \bigvee_{(p,a,q)\in T} ((y \in X_{p,a,q}) \wedge in(p))\right)\right) \wedge \\
\bigvee_{F\in\mathcal{F}} \left(\exists x \centerdot \forall y \centerdot (x \le y) \to \left(\begin{array}{c} \left(\bigvee_{\substack{(p,a,q)\in T\\ q\in F}} y \in X_{p,a,q}\right) \wedge \\ \bigwedge_{q\in F} \exists z \centerdot \left((y \le z) \wedge \bigvee_{\substack{p\in Q\\ a\in A}} z \in X_{p,a,q}\right)\end{array}\right)\right)^{+}.
\end{aligned}
$$

Intuitively, the semantics of the formula following $\bigvee_{F\in\mathcal{F}} \exists x \centerdot \forall y$, checks if a path P_w of $\mathcal{A}$ over a word $w \in A^\omega$, is successful or not, by taking a non-zero or the zero value, respectively.

Consider now an infinite word $w = a_0 a_1 \ldots \in A^\omega$, a path $P_w = (t_i)_{i\ge 0}$ of $\mathcal{A}$ over w, and let σ_{P_w} its associated assignment. For $t \in T$, we let $P_w(t) = \{i \mid t_i = t\}$. If P_w is not successful, by the semantics of the formula started with $\bigvee_{F\in\mathcal{F}} \exists x \centerdot \forall y$, we obtain $\|\varphi\|_{\mathcal{V}}(w, \sigma_{P_w}) = 0$. Otherwise, it holds that

$$\|\varphi\|_{\mathcal{V}}(w, \sigma_{P_w}) = in(q_0) \cdot \prod_{i\ge o} wt(q_i, a_i, q_{i+1}) = weight(P_w).$$

Let $\xi = \exists X_1 \ldots \exists X_m \centerdot \varphi(X_1,\ldots,X_m)$. Due to the above bijection, we get for $w \in A^\omega$

$$
\begin{aligned}
\|\xi\|(w) &= \sum_{\sigma\ (w,\mathcal{V})-assignment} \|\varphi\|_{\mathcal{V}}(w,\sigma) \\
&= \sum_{P_w} \|\varphi\|_{\mathcal{V}}(w, \sigma_{P_w}) \\
&= \sum_{P_w \text{ successful}} weight(P_w) \\
&= (\|\mathcal{A}\|, w).
\end{aligned}
$$

We conclude that

$$\|\mathcal{A}\| = \|\xi\| \in K^{remso}\langle\langle A^\omega\rangle\rangle.$$

■

Now the *proof of Theorem 11* is immediate by Propositions 16 and 21.

Corollary 22 (Büchi's Theorem) *An infinitary language is ω-recognizable iff it is definable by a MSO-sentence.*

5 Weighted Büchi automata

In this section, we consider weighted automata over infinite words with Büchi acceptance condition which were introduced by Ésik and Kuich [17, 18]. We show their equivalence to our model with Muller acceptance condition.

Let A be any alphabet and K be a totally complete semiring.

Definition 23 *([17, 18]) A* weighted Büchi automaton *(WBA for short) over A and K is a quadruple $\mathcal{A} = (Q, in, wt, F)$, where Q is the* finite state set*, $in : Q \to K$ is the* initial distribution*, $wt : Q \times A \times Q \to K$ is a* mapping assigning weights to the transitions of $\mathcal{A}$*, and $F \subseteq Q$ is the* final state set.

Given an infinite word $w = a_0 a_1 \ldots \in A^\omega$, a *path P_w of $\mathcal{A}$ over w* is an infinite sequence of transitions $P_w := (t_i)_{i \geq 0}$, so that $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. The *weight of P_w* is the value

$$weight(P_w) := in(q_0) \cdot \prod_{i \geq 0} wt(t_i).$$

The path P_w is called *successful* if at least one final state appears infinitely often, i.e.,

$$In^Q(P_w) \cap F \neq \emptyset.$$

The *behavior of $\mathcal{A}$* is the infinitary power series

$$\|\mathcal{A}\| : A^\omega \to K$$

with coefficients specified for $w \in A^\omega$ by

$$(\|\mathcal{A}\|, w) = \sum_{P_w} weight(P_w)$$

where the sum is taken over all successful paths P_w of $\mathcal{A}$ over w.

A series $S : A^\omega \to K$ is called *ω-recognizable* if there is a WBA $\mathcal{A}$ such that $S = \|\mathcal{A}\|$. The class of all ω-recognizable series over A and K is denoted by $K^{\omega-rec}\langle\langle A^\omega \rangle\rangle$.

Next, we show that the class of ω-recognizable series over A and K coincides with the class of Muller recognizable series over A and K. We shall need some preliminary matter.

A WMA $\mathcal{A} = (Q, in, wt, \mathcal{F})$ *has a unique initial state* if there is a distinguished state $q_0 \in Q$ such that for any $q \in Q$

$$in(q) = \begin{cases} 1 & \text{if } q = q_0 \\ 0 & \text{otherwise} \end{cases}$$

and $q_0 \notin F$ for each $F \in \mathcal{F}$.

Lemma 24 *For any WMA $\mathcal{A} = (Q, in, wt, \mathcal{F})$ we can effectively construct a WMA $\mathcal{A}' = (Q', in', wt', \mathcal{F}')$ with a unique initial state such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

Proof. Let q' be a new state not belonging to Q. We set $Q' = Q \cup \{q'\}$ and $\mathcal{F}' = \mathcal{F}$. The initial distribution in' and the weight mapping wt' are defined by:

- $in'(q) = 0 \quad$ for all $q \in Q$,
- $in'(q') = 1$,
- $wt'(p, a, q) = wt(p, a, q) \quad$ for all $p, q \in Q, a \in A$, and
- $wt'(q', a, q) = \sum_{p \in Q} in(p) \cdot wt(p, a, q) \quad$ for all $q \in Q, a \in A$.

To any other transition we assign value zero.

It is a routine matter to formally show that the WMA $\mathcal{A}$ and $\mathcal{A}'$ have the same behavior. ■

Theorem 25 *Let A be an alphabet and K any totally complete semiring. Then*

$$K^{\omega-rec}\left\langle\left\langle A^{\omega}\right\rangle\right\rangle = K^{M-rec}\left\langle\left\langle A^{\omega}\right\rangle\right\rangle.$$

Proof. Let $\mathcal{A} = (Q, in, wt, F)$ be a WBA over A and K. Then we construct the WMA $\mathcal{A}' = (Q, in, wt, \mathcal{F})$ by letting $\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$. Thus given an infinite word $w \in A^{\omega}$, any successful path P_w of $\mathcal{A}$ over w is also successful in $\mathcal{A}'$, and vice-versa. Thus $(\|\mathcal{A}'\|, w) = (\|\mathcal{A}\|, w)$, showing $\|\mathcal{A}'\| = \|\mathcal{A}\|$.
Conversely, let $\mathcal{A} = (Q, in, wt, \mathcal{F})$ be a WMA over A and K with unique initial state.
Case 1: $card(\mathcal{F}) = 1$, i.e., $\mathcal{F} = \{F\}$ for a set $F \subseteq Q$.
We consider the WBA $\widetilde{\mathcal{A}} = (\widetilde{Q}, \widetilde{in}, \widetilde{wt}, \widetilde{F})$ whose components are specified by the next clauses.

- $\widetilde{Q} = Q \cup (Q \times \mathcal{P}(Q))$,
- $\widetilde{in}(q) = in(q) \quad$ for all $q \in Q$, and

  $\widetilde{in}(q, R) = 0 \quad$ for all $q \in Q$ and $R \subseteq Q$,
- $\widetilde{F} = \{(q, F) \mid q \in F\}$.

The weight mapping of $\widetilde{\mathcal{A}}$ is determined by

- $\widetilde{wt}(p, a, q) = wt(p, a, q) \quad$ for all $p, q \in Q, a \in A$,
- $\widetilde{wt}(p, a, (q, \{q\})) = wt(p, a, q) \quad$ for all $p \in Q \setminus F, q \in F, a \in A$,
- $\widetilde{wt}((p, R), a, (q, R \cup \{q\})) = wt(p, a, q) \quad$ for all $p, q \in F, R \subsetneq F, a \in A$,
- $\widetilde{wt}((p, F), a, (q, \{q\})) = wt(p, a, q) \quad$ for all $p, q \in F, a \in A$.

To any other transition we assign value zero.

Let now $w \in A^{\omega}$ and $P_w = (t_i)_{i \geq 0}$ be a successful path of $\mathcal{A}$ over w, with $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. We can assume that q_0 is the initial state of $\mathcal{A}$, since otherwise $weight(P_w) = 0$. Thus $q_0 \notin F$. Then there is a unique minimal "start point" on P_w, say state q_j, so that $q_j \in Q \setminus F$ and each state after q_j along P_w belongs to F. Let us consider the path $\widetilde{P_w} = \left(\widetilde{t_i}\right)_{i \geq 0}$ of $\widetilde{\mathcal{A}}$ over w, in the following way:

- for all $0 \leq i < j$, $\widetilde{t_i} = t_i$,
- for $i = j$, $\widetilde{t_j} = (q_j, a_j, (q_{j+1}, \{q_{j+1}\}) \in Q \times A \times (Q \times \mathcal{P}(Q))$,
- for all $i > j$, either
  $\widetilde{t_i} = ((q_i, R_i), a_i, (q_{i+1}, R_i \cup \{q_{i+1}\}))$ if $R_i \subsetneq F$ and $\widetilde{t}_{i-1} = ((q_{i-1}, R_{i-1}), a_{i-1}, (q_i, R_i))$ or if $i - 1 = j$ and $R_i = \{q_i\} \subsetneq F$,
  or
  $\widetilde{t_i} = ((q_i, F), a_i, (q_{i+1}, \{q_{i+1}\}))$ if $\widetilde{t}_{i-1} = ((q_{i-1}, R_{i-1}), a_{i-1}, (q_i, F))$ or if $i - 1 = j$ and $F = \{q_i\}$

Since P_w is successful, it is clear that the path $\widetilde{P_w}$ of $\widetilde{\mathcal{A}}$ is also successful and $\widetilde{wt}(\widetilde{t_i}) = wt(t_i)$ for all $i \geq 0$. Thus $weight\left(\widetilde{P_w}\right) = weight(P_w)$. On the other hand, the reader should observe, that we can find infinitely many successful paths of $\widetilde{\mathcal{A}}$ over w with reference to P_w, just by changing from Q to $Q \times P(Q)$ at any state q_i with $i > j$, along P_w. But according to the mapping $\widetilde{wt}$ the weight of each such path equals 0. We conclude that for any successful path P_w of $\mathcal{A}$ over w with non-zero weight, there exists a unique successful path $\widetilde{P_w}$ of $\widetilde{\mathcal{A}}$ over w, with the same weight. The converse is established in a similar manner. We obtain that for any word $w \in A^{\omega}$,

$$\begin{aligned}
(\|\mathcal{A}\|, w) &= \sum_{P_w} weight(P_w) \\
&= \sum_{\widetilde{P_w}} weight\left(\widetilde{P_w}\right) \\
&= (\left\|\widetilde{\mathcal{A}}\right\|, w).
\end{aligned}$$

Case 2: $card(\mathcal{F}) > 1$, say $\mathcal{F} = \{F_1, \ldots, F_n\}$.

For each $j \in \{1, \ldots, n\}$, let $\mathcal{A}_j = (Q, in, wt, \{F_j\})$. Clearly, $\|\mathcal{A}\| = \sum_{j=1}^{n} \|\mathcal{A}_j\|$. By Case 1, for each j there exists a WBA $\widetilde{\mathcal{A}_j}$ with $\|\mathcal{A}_j\| = \left\|\widetilde{\mathcal{A}_j}\right\|$. A disjoint union of these automata produces a WBA $\widetilde{\mathcal{A}}$ with $\left\|\widetilde{\mathcal{A}}\right\| = \sum_{j=1}^{n} \left\|\widetilde{\mathcal{A}_j}\right\|$. Hence $\left\|\widetilde{\mathcal{A}}\right\| = \|\mathcal{A}\|$, and the proof is completed. ■

6 Conclusion

We introduced weighted Muller automata over totally complete semirings. We verified that the family of their behaviors coincides with the class of infinitary formal power series obtained as semantics of weighted restricted MSO-sentences, provided that the underlying semiring is totally commutative complete and also with the family of behaviors of weighted Büchi automata investigated by Ésik and Kuich [17, 18]. We do not know if this family coincides with the class of series specified by all weighted MSO-sentences. Also, the question arises whether Theorem 11, in particular the construction of a WMA $\mathcal{A}$ for a given MSO-formula φ can be made effective. The problem is the universal quantifier: Given a WMA for φ as described in the proof of Lemma 15, how do we obtain the values k_j and WMA for the languages L_j? In the case of finite words and given a field K, Droste and Gastin [9] could use results from the literature on formal power series to obtain a construction. Therefore, also in our situation we should consider specific semirings.

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