Weighted automata and weighted logics with discounting

Manfred Droste
Institute for Computer Science
Leipzig University
D-04009 Leipzig, Germany
droste@informatik.uni-leipzig.de

George Rahonis
Department of Mathematics
Aristotle University of Thessaloniki
54124 Thessaloniki, Greece
grahonis@math.auth.gr

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Abstract

We introduce a weighted logic with discounting and we establish Büchi's and Elgot's theorem for weighted automata over finite words and arbitrary commutative semirings. Then we investigate Büchi and Muller automata with discounting over the max-plus and the min-plus semiring. We show their expressive equivalence with weighted MSO-sentences with discounting. In this case our logic has a purely syntactic definition. For the finite case, we obtain a purely syntactically defined weighted logic if the underlying semiring is additively locally finite.

Keywords: Weighted automata, Weighted Büchi and Muller automata, Formal power series, Weighted MSO logic, Discounting.

1 Introduction

In automata theory, Büchi's and Elgot's fundamental theorems [?, ?] established the coincidence of regular languages of finite or infinite words with languages definable in monadic second-order logic. At the same time, Schützenberger [?] characterized the behaviors of finite automata enriched with weights for the transitions as rational formal power series. Both of these results have led to various extensions and also to practical applications, e.g. in verification of finite state programs [?, ?, ?], in digital image compression [?, ?, ?] and in speech-to-text processing [?, ?]. For surveys and monographs on weighted automata see [?, ?, ?, ?]. Recently, in [?] a logic with weights was developed for finite words and shown to be expressively equivalent to weighted automata.

It is the goal of this paper to provide a weighted logic for infinite words which is again expressively equivalent to weighted automata, thereby combining Büchi's and Schützenberger's approaches to achieve a quantitative model for non-terminating behavior. Whereas in the results of [?] for finite words the weights can be taken in an arbitrary semiring, it is clear that for weighted automata on infinite words questions of summability and convergence arise. Therefore we assume that the weights are taken in the real numbers, and we ensure convergence of infinite sums by discounting: in a path, later transitions get less weight. This method of discounting is classical in mathematical economics for systems with non-terminating behavior, also in Markov decision processes and game theory [?, ?]. Recently, for a theory of systems engineering, it was investigated in [?]. For weighted automata, it was introduced in [?], and the discounting behaviors of weighted Büchi automata were characterized as the ω-rational formal power series; this was further investigated in [?, ?].

As semirings, here we consider the max-plus and the min-plus semiring which are fundamental in max-plus algebra [?, ?] and algebraic optimization problems [?].

As our main contributions, we will:

(1) extend the weighted logic of [?] to weighted automata with discounting for finite words and arbitrary commutative semirings as investigated in [?, ?, ?]; our present form of discounting is slightly more general;

(2) provide for the max-plus and min-plus semirings of real numbers a weighted logic with discounting which is expressively equivalent to the weighted Büchi automata on infinite words of [?]; we will also show equivalence to weighted Muller automata;

(3) show that for a large class of semirings, a purely syntactically defined fragment of the weighted logics suffices to achieve the equivalences of (1) and (2).

In our approach, it was not clear how to define a discounted semantics of weighted formulas. Somewhat surprisingly, we can almost completely take over the *undiscounted* semantics as given in [?], changing only the semantics of the universal quantifier. For the general result of [?], the weighted formula employed require certain semantically described restrictions; clearly, a purely syntactic definition would be desirable. In (3), we present a new, purely syntactic definition of a class of weighted formulas and show that they are expressively equivalent to the weighted automata with discounting of (1) and (2). For these formulas, the equivalent automata can be constructed effectively. Our arguments combine the methods of [?, ?, ?], suitably adjusted to the discounted setting.

We note that a different approach of weighted automata acting on infinite words has been considered before in connection with digital image processing by Culik and Karhumäki [?]. Another approach requires the semirings to be *complete*, i.e., to have (built-in) infinitary sum and product operations. This was investigated deeply e.g. in [?, ?, ?]. Recently, in [?] we presented weighted Büchi and Muller automata and a weighted logics for complete semirings and showed their expressive equivalence. The present paper shows the robustness of the weighted logics approach also for infinite words in case of discounting. For weighted logics and automata on trees, pictures, traces and texts we refer the reader to [?, ?, ?, ?].

2 Weighted automata with discounting

Let A be a finite alphabet. The set of all finite (resp. infinite) words over A is denoted as usually by A^* (resp. A^ω). We let ε denote the empty word. The length of a finite word w is denoted by $|w|$. If w is finite (resp. infinite) for any $0 \leq i \leq |w|$ (resp. $i \geq 0$) we shall denote by $w_{\leq i}$ the finite prefix of w with length i. Obviously $w_{\leq 0} = \varepsilon$.

A *semiring* $(K, +, \cdot, 0, 1)$ (denoted simply also by K) is called *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in K$. The following structures constitute important examples of commutative semirings: the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of *natural numbers*; the *arctical semiring* or *max-plus semiring* $\mathbb{R}_{\max} = (\mathbb{R}_+ \cup \{-\infty\}, \max, +, -\infty, 0)$ where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ and $-\infty + x = -\infty$ for each $x \in \mathbb{R}_+$; the *tropical* or *min-plus semiring* $(\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$; each bounded distributive lattice with the operations supremum and infimum, in particular the *fuzzy semiring* $([0,1], \sup, \inf, 0, 1)$ and the *Boolean semiring* $\mathbf{B} = (\{0,1\}, \vee, \wedge, 0, 1)$.

The semiring K is called *additively locally finite* if each finitely generated submonoid of $(K, +, 0)$ is finite. Important examples of such semirings include: all idempotent semirings, in particular the arctical and the tropical semirings and all bounded distributive lattices; all fields of characteristic p, for any prime p; all products $K_1 \times \ldots \times K_n$ (with operations defined pointwise) of additively locally finite semirings K_i $(1 \leq i \leq n)$; the semiring of polynomials $(K[X], +, \cdot, 0, 1)$ over a variable X and an additively locally finite semiring

K.

A homomorphism $f : K \to K$ is an *endomorphism* of K. The set $End(K)$ of all endomorphisms of K is a monoid with operation the usual mapping composition $\circ$ and unit element the identity mapping id on K. If no confusion arises, we shall simply denote the operation $\cdot$ of K and the composition operation $\circ$ of $End(K)$ by concatenation.

Example 1 *Let $K = \mathbb{R}_{\max}$, the max-plus semiring. Choose any $p \in \mathbb{R}_+$ and put $p \cdot (-\infty) = -\infty$. Then the mapping $\overline{p} : \mathbb{R}_{\max} \to \mathbb{R}_{\max}$ given by $x \longmapsto p \cdot x$ is an endomorphism of $\mathbb{R}_{\max}$ which can be considered as a discounting of $\mathbb{R}_{\max}$. Conversely, every endomorphism of $\mathbb{R}_{\max}$ is of this form (cf. [?]). The same result can be proved for $\mathbb{R}_{\min}$ where $p \cdot \infty = \infty$.*

A Φ-*discounting over A and K* is a family $\Phi = (\Phi_a)_{a\in A}$ of endomorphisms of K, i.e. $\Phi_a \in End(K)$ for all $a \in A$. Then Φ induces a monoid morphism $\Phi : A^* \to End(K)$ determined by $\Phi(w) = \Phi_{a_0} \circ \Phi_{a_1} \circ \ldots \circ \Phi_{a_{n-1}}$ for any $w = a_0 a_1 \ldots a_{n-1} \in A^+$, ($a_i \in A$ for $0 \le i \le n-1$), and $\Phi(\varepsilon) = id$. We shall use the notation $\Phi_w = \Phi(w)$ for any $w \in A^*$. In particular, due to Example **??**, for $K = \mathbb{R}_{\max}$ (or $\mathbb{R}_{\min}$), a Φ'-discounting over A and $\mathbb{R}_{\max}$ will be of the form $\Phi' = (\overline{p_a})_{a\in A}$ where $0 \le p_a < 1$ for all $a \in A$. For any finite word $w = a_0 a_{1\ldots a_{n-1}} \in A^+$ ($a_i \in A$ for $0 \le i \le n-1$) we put $p_w = \prod_{a\in A} p_a^{|w|_a}$ where $|w|_a$ denotes the number of a's in w. Then $\Phi'_w(x) = p_w \cdot x$ for each $x \in \mathbb{R}_{\max}$. Note that if $m_{\Phi'} = \max\{p_a \mid a \in A\}$ then $0 \le m_{\Phi'} < 1$ and $p_w \le m_{\Phi'}^{|w|}$ for each $w \in A^*$.

A *finitary* (resp. *infinitary*) *formal power series* or *series* for short is a mapping $S : A^* \to K$ (resp. $S : A^\omega \to \mathbb{R}_{\max}$). The class of all finitary (resp. infinitary) series over A and K (resp. $\mathbb{R}_{\max}$) is denoted by $K\langle\langle A^*\rangle\rangle$ (resp. $\mathbb{R}_{\max}\langle\langle A^\omega\rangle\rangle$). We refer the reader to [**?**, **?**] for notions and results on finitary series, and to [**?**, **?**] for infinitary ones.

For the rest of the paper we fix a finite alphabet A, a semiring K *and a Φ-discounting (resp.* Φ'-discounting) *over A and K* (resp. $\mathbb{R}_{\max}$).

Definition 2 *A* weighted automaton over A and K *is a quadruple $\mathcal{A} = (Q, in, wt, out)$, where Q is the* finite state set, *$in : Q \to K$ is the* initial distribution*, $wt : Q \times A \times Q \to K$ is a* mapping assigning weights to the transitions *of the automaton, and $out : Q \to K$ is the* final distribution*.*

Now we define the Φ-behavior of $\mathcal{A}$ as follows. Given a word $w = a_0 a_1 \ldots a_{n-1} \in A^*$, a *path of $\mathcal{A}$ over w* is a finite sequence of transitions $P_w := (t_i)_{0\le i\le n-1}$ so that $t_i = (q_i, a_i, q_{i+1})$ for all $0 \le i \le n-1$. We define the *running weight $rwt(P_w)$ of P_w* by $rwt(P_w) = \prod_{0\le i\le n-1} \Phi_{w_{<i}}(wt(t_i))$. Then the Φ-*weight* (or simply *weight*) of P_w is the value $weight(P_w) := in(q_0) \cdot rwt(P_w) \cdot \Phi_w(out(q_{n+1}))$. The Φ-*behavior* (or simply *behavior*) of $\mathcal{A}$ is the formal power series $\|\mathcal{A}\| : A^* \to K$ whose coefficients are given by $(\|\mathcal{A}\|, w) = \sum_{P_w} weight(P_w)$ for any $w \in A^*$.

A series $S : A^* \to K$ is said to be Φ-*recognizable* if there is a weighted automaton $\mathcal{A}$ over A and K so that $S = \|\mathcal{A}\|$. We shall denote by $K^{\Phi-rec}\langle\langle A^*\rangle\rangle$ the class of all Φ-recognizable series over A and K. A power series $S : A^* \to K$ is called a *recognizable step function* if $S = \sum_{1\le j\le n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq A^*$ ($1 \le j \le n$ and $n \in \mathbb{N}$) are recognizable languages.

For intuition, note that if $K = \mathbb{R}_{\max}$ and $\Phi_a = \overline{p_a}$ for some $p_a \in (0,1)$ ($a \in A$), say, as in Example **??**, then in the computation of $rwt(P_w)$ later transitions get less weight, hence $\|\mathcal{A}\|$ models a discounted behavior of $\mathcal{A}$. If Φ is the *trivial* discounting, i.e. Φ_a is the identity on K for each $a \in A$, then the Φ-behavior coincides with the usual behavior of weighted automata.

By standard arguments (cf. [?]) we can show that: (a) the class of Φ-recognizable series (resp. recognizable step functions) is closed under sum and scalar products; furthermore, if K is commutative, then it is closed closed under Hadamard products; (b) given two finite alphabets A, B and a strict alphabetic homomorphism $h : A^* \to B^*$, i.e. such that $h(A) \subseteq B$, then $h : K\langle\langle A^*\rangle\rangle \to K\langle\langle B^*\rangle\rangle$ and $h^{-1} : K\langle\langle B^*\rangle\rangle \to K\langle\langle A^*\rangle\rangle$ preserve Φ-recognizability; (c) if $L \subseteq A^*$ is a recognizable language then its characteristic series $1_L \in K\langle\langle A^*\rangle\rangle$ is Φ-recogizable. The next result is new and its proof is non-trivial.

Proposition 3 *Let K be additively locally finite. Let A, B be two finite alphabets and $h : A^* \to B^*$ be a strict alphabetic homomorphism. If $S \in K\langle\langle A^*\rangle\rangle$ is a recognizable step function then $h(S) \in K\langle\langle B^*\rangle\rangle$ is also a recognizable step function.*

Next, we turn to weighted automata over infinite words. More precisely, we present two automata models acting on infinite words. Weighted Büchi automata with discounting were introduced and investigated in [?]. Here we define this model in a slightly more generalized form. On the other hand, weighted Muller automata were studied in [?] in connection to weighted MSO logics over infinite words. Our Muller automaton model here is equipped with a discounting Φ' so that convergence problems will not be encountered. The max-plus $\mathbb{R}_{\max}$ and the min-plus $\mathbb{R}_{\min}$ will be our underlying semirings. But now we intend to compute over infinite words, hence we will use sup and inf instead of max and min, respectively. The problem of summing up infinitely many factors will be faced by using a discounting parameter.

Definition 4 *(a) A* weighted Muller automaton *(WMA for short)* over A and $\mathbb{R}_{\max}$ *is a quadruple $\mathcal{A} = (Q, in, wt, \mathcal{F})$, where Q is the* finite state set, *$in : Q \to \mathbb{R}_{\max}$ is the* initial distribution, *$wt : Q \times A \times Q \to \mathbb{R}_{\max}$ is a* mapping assigning weights to the transitions *of the automaton, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the* family of final state sets.

(b) A WBA $\mathcal{A}$ is a weighted Büchi automaton *(WBA for short) if there is a* set $F \subseteq Q$ *such that $\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$.*

Given an infinite word $w = a_0 a_1 \ldots \in A^\omega$, a *path P_w of $\mathcal{A}$ over w* is an infinite sequence of transitions $P_w := (t_i)_{i \geq 0}$, so that $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. The *Φ'-weight of P_w* (or simply *weight*) is the value $weight(P_w) := in(q_0) \cdot \sum\limits_{i \geq 0} p_{w_{\leq i}} \cdot wt(t_i)$. Observe that this infinite sum converges; its value is bounded by $M \cdot \sum\limits_{i \geq 0} m^i_{\Phi'} = M \cdot 1/(1 - m_{\Phi'})$, where $M = max\{wt(t) \mid t \in Q \times A \times Q\}$. We denote by $In^Q(P_w)$ the set of states that appear infinitely many times in P_w, i.e., $In^Q(P_w) = \{q \in Q \mid \exists^\omega i : t_i = (q, a_i, q_{i+1})\}$. The path P_w is called *successful* if the set of states that appear infinitely often along P_w constitute a final state set, i.e., $In^Q(P_w) \in \mathcal{F}$. The *$\Phi'$-behavior* (or simply *behavior*) *of $\mathcal{A}$* is the infinitary power series $\|\mathcal{A}\| : A^\omega \to \mathbb{R}_{\max}$ with coefficients specified for $w \in A^\omega$ by $(\|\mathcal{A}\|, w) = \sup\limits_{P_w}(weight(P_w))$ where the supremum is taken over all successful paths P_w of $\mathcal{A}$ over w. Again, this supremum exists in $\mathbb{R}_{\max}$ since the values $weight(P_w)$ are uniformly bounded.

A series $S : A^\omega \to \mathbb{R}_{\max}$ is called *Φ'-Muller recognizable* (resp. *Φ'-Büchi recognizable* or *Φ'-ω-recognizable*) if there is a WMA (resp. WBA) $\mathcal{A}$, such that $S = \|\mathcal{A}\|$. The class of all Φ'-Muller recognizable (resp. Φ'-ω-recognizable series) over A and $\mathbb{R}_{\max}$ is denoted by $\mathbb{R}^{\Phi'-M-rec}_{\max}\langle\langle A^\omega\rangle\rangle$ (resp. $\mathbb{R}^{\Phi'-\omega-rec}_{\max}\langle\langle A^\omega\rangle\rangle$). We will call an infinitary series $S : A^\omega \to \mathbb{R}_{\max}$

Muller recognizable step function (or *ω-recognizable step function)* if $S = \max_{1 \le j \le n} (k_j + 1_{L_j})$ where $k_j \in \mathbb{R}_{\max}$ and $L_j \subseteq A^\omega$ $(1 \le j \le n$ and $n \in \mathbb{N})$ are ω-recognizable languages.

Droste and Kuske [**?**] consider WBA over $\mathbb{R}_{\max}$ where $p_a = p$ $(0 \le p < 1)$ for any $a \in A$.

Our first main result (in the next theorem) can be proved using similar arguments as in Theorem 25 in [**?**].

Theorem 5 $\mathbb{R}_{\max}^{\Phi'-\omega-rec} \langle\langle A^\omega \rangle\rangle = \mathbb{R}_{\max}^{\Phi'-M-rec} \langle\langle A^\omega \rangle\rangle$.

The next proposition refers to closure properties of Φ'-ω-recognizable series and ω-recognizable step functions.

Proposition 6 *(a) The class* $\mathbb{R}_{\max}^{\Phi'-\omega-rec} \langle\langle A^\omega \rangle\rangle$ *(resp. of ω-recognizable step functions) is closed under max, scalar sum and sum.*

(b) Let A, B *be two alphabets,* $h : A^\omega \to B^\omega$ *be a strict alphabetic homomorphism and* $S \in \mathbb{R}_{\max} \langle\langle A^\omega \rangle\rangle$ *be a* Φ'*-ω-recognizable series (resp. ω-recognizable step function). Then* $h(S) \in \mathbb{R}_{\max} \langle\langle B^\omega \rangle\rangle$ *is* Φ'*-ω-recognizable (resp. ω-recognizable step function). Furthermore,* $h^{-1} : \mathbb{R}_{\max} \langle\langle B^\omega \rangle\rangle \to \mathbb{R}_{\max} \langle\langle A^\omega \rangle\rangle$ *preserves* Φ'*-ω-recognizability.*

(c) Let $L \subseteq A^\omega$ *be an ω-recognizable language. Then its characteristic series* $1_L \in \mathbb{R}_{\max} \langle\langle A^\omega \rangle\rangle$ *is* Φ'*-ω-recognizable.*

3 Weighted MSO logic with discounting

In this section, we introduce our weighted monadic second order logic with discounting (weighted MSO logic with discounting, for short) and we interpret the semantics of MSO-formulas in this logic as formal power series. Let $\mathcal{V}$ be a finite set of first and second order variables. A word $w \in A^*$ (resp. $w \in A^\omega$) is represented by the relational structure $(dom(w), \le, (R_a)_{a \in A})$ where $dom(w) = \{0, \ldots, |w|-1\}$ (resp. $dom(w) = \omega = \{0, 1, 2, ...\}$), $\le$ is the natural order and $R_a = \{i \mid w(i) = a\}$ for $a \in A$. A $(w, \mathcal{V})$-*assignment* σ is a mapping associating first order variables from $\mathcal{V}$ to elements of $dom(w)$, and second order variables from $\mathcal{V}$ to subsets of $dom(w)$. If x is a first order variable and $i \in dom(w)$, then $\sigma[x \to i]$ denotes the $(w, \mathcal{V} \cup \{x\})$-assignment which associates i to x and acts as σ on $\mathcal{V} \setminus \{x\}$. For a second order variable X and $I \subseteq dom(w)$, the notation $\sigma[X \to I]$ has a similar meaning.

In order to encode pairs (w, σ) for all $w \in A^*$ (resp. $w \in A^\omega$) and any $(w, \mathcal{V})$-assignment σ, we use an extended alphabet $A_{\mathcal{V}} = A \times \{0,1\}^{\mathcal{V}}$. Each pair (w, σ) is a word in $A_{\mathcal{V}}^*$ (resp. in A^ω) where w is the projection over A and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Then σ is a valid $(w, \mathcal{V})$-assignment if for each first order variable $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, we identify σ with the $(w, \mathcal{V})$-assignment so that for each first order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for each second order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions labelled with 1 along the X-row.

It is well-known that the set $N_{\mathcal{V}} = \{(w, \sigma) \in A_{\mathcal{V}}^* \mid \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment}\}$ (resp. $N_{\mathcal{V}}^\omega = \{(w, \sigma) \in A_{\mathcal{V}}^\omega \mid \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment}\}$) is recognizable (resp. ω-recognizable).

Let φ be an MSO-formula [**?**, **?**, **?**]. Then Büchi's and Elgot's theorem [**?**, **?**] states that for $Free(\varphi) \subseteq \mathcal{V}$ the language $\mathcal{L}_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}$ defined by φ over $A_{\mathcal{V}}$ is recognizable. Conversely, each recognizable language $L \subseteq A^*$ is definable by an MSO-sentence φ, i.e., $L = \mathcal{L}(\varphi)$.

The fundamental Büchi's theorem [?] for infinite words proves that the language $\mathcal{L}^\omega_\mathcal{V}(\varphi) = \{(w,\sigma) \in N^\omega_\mathcal{V} \mid (w,\sigma) \models \varphi\}$ defined by φ over $A_\mathcal{V}$ is ω-recognizable. Conversely, each ω-recognizable language $L \subseteq A^\omega$ is definable by an MSO-sentence φ, i.e., $L = \mathcal{L}^\omega(\varphi)$.

We simply write $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$ (resp. $\mathcal{L}^\omega(\varphi) = \mathcal{L}^\omega_{Free(\varphi)}(\varphi)$).

Now we turn to weighted MSO logic with discounting.

Definition 7 *The syntax of formulas of the weighted MSO logic with Φ-discounting over K is given by*

$$\varphi := k \mid P_a(x) \mid \neg P_a(x) \mid Last(x) \mid \neg Last(x) \mid S(x,y) \mid \neg S(x,y) \\ \mid x \in X \mid \neg(x \in X) \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x \centerdot \varphi \mid \exists X \centerdot \varphi \mid \forall x \centerdot \varphi$$

where $k \in K$, $a \in A$. We shall denote by $MSO(K,A)$ the set of all such weighted MSO-formulas φ.

Definition 8 *Let $\varphi \in MSO(K,A)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The Φ-semantics of φ is a formal power series $\|\varphi\|_\mathcal{V} \in K\langle\langle A^*_\mathcal{V}\rangle\rangle$. Consider an element $(w,\sigma) \in A^*_\mathcal{V}$. If σ is not a valid assignment, then we put $\|\varphi\|_\mathcal{V}(w,\sigma) = 0$. Otherwise, we inductively define $(\|\varphi\|_\mathcal{V},(w,\sigma)) \in K$ as follows:*

- $(\|k\|_\mathcal{V},(w,\sigma)) = k$
- $(\|P_a(x)\|_\mathcal{V},(w,\sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$
- $(\|Last(x)\|_\mathcal{V},(w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x) = |w|-1 \\ 0 & \text{otherwise} \end{cases}$
- $(\|S(x,y)\|_\mathcal{V},(w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x)+1 = \sigma(y) \\ 0 & \text{otherwise} \end{cases}$
- $(\|x \in X\|_\mathcal{V},(w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$
- $(\|\neg\varphi\|_\mathcal{V},(w,\sigma)) = \begin{cases} 1 & \text{if } (\|\varphi\|_\mathcal{V},(w,\sigma)) = 0 \\ 0 & \text{if } (\|\varphi\|_\mathcal{V},(w,\sigma)) = 1 \end{cases}$, *provided that φ is of the form $P_a(x)$, $Last(x)$, $S(x,y)$ or $(x \in X)$*
- $(\|\varphi \vee \psi\|_\mathcal{V},(w,\sigma)) = (\|\varphi\|_\mathcal{V},(w,\sigma)) + (\|\psi\|_\mathcal{V},(w,\sigma))$
- $(\|\varphi \wedge \psi\|_\mathcal{V},(w,\sigma)) = (\|\varphi\|_\mathcal{V},(w,\sigma)) \cdot (\|\psi\|_\mathcal{V},(w,\sigma))$
- $(\|\exists x \centerdot \varphi\|_\mathcal{V},(w,\sigma)) = \sum_{i \in dom(w)} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}},(w,\sigma[x \to i])\right)$
- $(\|\exists X \centerdot \varphi\|_\mathcal{V},(w,\sigma)) = \sum_{I \subseteq dom(w)} \left(\|\varphi\|_{\mathcal{V}\cup\{X\}},(w,\sigma[X \to I])\right)$
- $(\|\forall x \centerdot \varphi\|_\mathcal{V},(w,\sigma)) = \prod_{i \in dom(w)} \Phi_{w \leq i}\left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}},(w,\sigma[x \to i])\right)\right)$

where the product is taken in the natural order.

We simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ is a sentence, i.e, it has no free variables, then $\|\varphi\| \in K\langle\langle A^*\rangle\rangle$. We note that if Φ is the trivial discounting, then the Φ-semantics coincides (apart from the slight changes in the syntax) with the semantics of weighted formulas as defined in [**?**]. The reader should find in [**?**, **?**] examples of possible interpretations of weighted MSO formulas. In the following, we give an example employing discounting.

Example 9 *Consider the alphabet* $A = \{a, b, c\}$, *the max-plus semiring* $\mathbb{R}_{\max}$ *and the discounting* $\Phi = \{\overline{p_a}, \overline{p_b}, \overline{p_c}\}$ *with* $p_a = p_b = 1$ *and* $p_c = 0$. *Let* $\varphi \in MSO(\mathbb{R}_{\max}, A)$ *given by* $\varphi = \forall x \,.\, (\neg P_a(x) \vee (P_a(x) \wedge \exists y \,.\, (S(x, y) \wedge P_b(y) \wedge 1)))$. *Then, for any word* $w \in A^*$ *the MSO-formula* φ *counts in* w *the occurrences of the subword ab before the first appearance of* c.

Now, we turn to weighted MSO logics over infinite words. Let Φ' be a discounting over $\mathbb{R}_{\max}$. The syntax of formulas of the weighted MSO logic with Φ'-discounting over $\mathbb{R}_{\max}$ is almost the same as in the finite case (cf. Definition **??**). The only difference is that we exclude $Last(x)$ and we add the atomic formula $x \leq y$ and its negation. We shall denote by $MSO(\mathbb{R}_{\max}, A)$ the set of all weighted MSO-formulas over $\mathbb{R}_{\max}$. Let $\varphi \in MSO(\mathbb{R}_{\max}, A)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The Φ'-*semantics* of φ is a formal power series $\|\varphi\|_{\mathcal{V}} \in \mathbb{R}_{\max}\langle\langle A^{\omega}_{\mathcal{V}}\rangle\rangle$. For any $(w, \sigma) \in A^{\omega}_{\mathcal{V}}$, if σ is not a valid assignment, then we put $(\|\varphi\|_{\mathcal{V}}, (w, \sigma)) = 0$. Otherwise, we define $(\|\varphi\|_{\mathcal{V}}, (w, \sigma))$ as in Definition **??**, where $K = \mathbb{R}_{\max}$ and the semiring operations are taking suprema and addition in the reals; also we put

- $$(\|x \leq y\|_{\mathcal{V}}, (w, \sigma)) = \begin{cases} 0 & \text{if } \sigma(x) \leq \sigma(y) \\ -\infty & \text{otherwise} \end{cases} .$$

Observe that the definitions of semantics are valid for each formula $\varphi \in MSO(K, A)$ (resp. $\varphi \in MSO(A, \mathbb{R}_{\max})$) and each finite set $\mathcal{V}$ of variables containing $Free(\varphi)$. In fact, the Φ-semantics (resp. Φ'-semantics) $\|\varphi\|_{\mathcal{V}}$ depends only on $Free(\varphi)$. More precisely,

Proposition 10 *Let* $\varphi \in MSO(K, A)$ *(resp.* $\varphi \in MSO(\mathbb{R}_{\max}, A)$*) and* $\mathcal{V}$ *be a finite set of variables such that* $Free(\varphi) \subseteq \mathcal{V}$. *Then* $(\|\varphi\|_{\mathcal{V}}, (w, \sigma)) = \left(\|\varphi\|, (w, \sigma|_{Free(\varphi)})\right)$ *for each* $(w, \sigma) \in A^*_{\mathcal{V}}$ *(resp.* $(w, \sigma) \in A^{\omega}_{\mathcal{V}}$*) where* σ *is a valid* $(w, \mathcal{V})$*-assignment. Furthermore,* $\|\varphi\|$ *is* Φ*-recognizable (resp. a recognizable step function,* Φ'*-*ω*-recognizable, an* ω*-recognizable step function) iff* $\|\varphi\|_{\mathcal{V}}$ *is* Φ*-recognizable (resp. a recognizable step function,* Φ'*-*ω*-recognizable, an* ω*-recognizable step function).*

Let now $Z \subseteq MSO(K, A)$ (resp. $Z \subseteq MSO(\mathbb{R}_{\max}, A)$). A series $S \in K\langle\langle A^*\rangle\rangle$ (resp. $S \in \mathbb{R}_{\max}\langle\langle A^{\omega}\rangle\rangle$) is called Φ-Z-*definable* (resp. Φ'-Z-*definable)* if there is a sentence $\varphi \in Z$ such that $S = \|\varphi\|$. The main results of this section refer to comparison of Φ-Z-definable (resp. Φ'-Z-definable) with Φ-recognizable (resp. Φ'-ω-recognizable) series for suitable fragments Z in the context of our weighted MSO logic with discounting. First, we show that $K^{\Phi-rec}\langle\langle A^*\rangle\rangle$ is not in general closed under universal quantifications.

Example 11 *Let* $K = (\mathbb{N}, +, \cdot, 0, 1)$. *It is easy to see that the series* $T = \|\exists x \,.\, 1\|$ *is recognizable. Let* $S = \|\forall y \,.\, \exists x \,.\, 1\|$. *Then* $(S, w) = |w|^{|w|}$. *But if* $\mathcal{A}$ *is a weighted automaton, there is a constant* $C \in \mathbb{N}$ *such that for all* $w \in A^*$ *we have* $(\|\mathcal{A}\|, w) \leq C^{|w|}$. *Hence* S *is not recognizable. Note that* T *takes on infinitely many values. In contrast, over the max-plus semiring* $K = \mathbb{R}_{\max}$, T *takes on only two values, and the series* S *would be recognizable.*

The previous example states that unrestricted universal quantification is too strong to preserve Φ-recognizability, and thus motivates the following definitions.

Definition 12 *(cf. [?]) (a) A formula $\varphi \in MSO(K, A)$ will be called* restricted *if whenever φ contains a universal first order quantification $\forall x \centerdot \psi$, then $\|\psi\|$ is a recognizable step function.*

(b) A formula $\varphi \in MSO(K, A)$ will be called almost existential *if whenever φ contains a universal first order quantification $\forall x \centerdot \psi$, then ψ does not contain any universal quantifier.*

We let $RMSO(K, A)$ comprise all restricted formulas of $MSO(K, A)$. Furthermore, let $REMSO(K, A)$ contain all *restricted existential* MSO-formulas φ, i.e., φ is of the form $\exists X_1, \dots, X_n \centerdot \psi$ with $\psi \in RMSO(K, A)$ containing no set quantification. We shall denote by $AEMSO(K, A)$ the set of all almost existential formulas of $MSO(K, A)$. We let $K^{\Phi-rmso}\langle\langle A^*\rangle\rangle$ (resp. $K^{\Phi-remso}\langle\langle A^*\rangle\rangle$, $K^{\Phi-aemso}\langle\langle A^*\rangle\rangle$) contain all series $S \in K\langle\langle A^*\rangle\rangle$ which are Φ-definable by some sentence in $RMSO(K, A)$ (resp. in $REMSO(K, A)$, $AEMSO(K, A)$). For the case $K = \mathbb{R}_{\max}$ the corresponding classes of infinitary series $\mathbb{R}_{\max}^{\Phi'-rmso}\langle\langle A^\omega\rangle\rangle$ (resp. $\mathbb{R}_{\max}^{\Phi'-remso}\langle\langle A^\omega\rangle\rangle$, $\mathbb{R}_{\max}^{\Phi'-aemso}\langle\langle A^\omega\rangle\rangle$) are defined analogously.

Next, we state our second main result.

Theorem 13 *Let A be a finite alphabet, K any commutative semiring and Φ any discounting over A and K. Then*

(a) $K^{\Phi-rec}\langle\langle A^*\rangle\rangle = K^{\Phi-rmso}\langle\langle A^*\rangle\rangle = K^{\Phi-remso}\langle\langle A^*\rangle\rangle$.

(b) If K is additively locally finite, then $K^{\Phi-rec}\langle\langle A^*\rangle\rangle = K^{\Phi-aemso}\langle\langle A^*\rangle\rangle$.

In our proof of the inclusion $K^{\Phi-rmso}\langle\langle A^*\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle A^*\rangle\rangle$ resp. $K^{\Phi-aemso}\langle\langle A^*\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle A^*\rangle\rangle$, we proceed by induction on the structure of a restricted or almost existential formula φ and we exploit closure properties of Φ-recognizable series. A crucial point is dealing with the universal quantifier; here we analyze a corresponding result of [?] (for restricted formula) resp. we employ Proposition **??** (for almost existential formula). For the converse inclusion $K^{\Phi-rec}\langle\langle A^*\rangle\rangle \subseteq K^{\Phi-aemso}\langle\langle A^*\rangle\rangle$ (and also for $K^{\Phi-rec}\langle\langle A^*\rangle\rangle \subseteq K^{\Phi-remso}\langle\langle A^*\rangle\rangle$), given a weighted Muller automaton $\mathcal{A}$ we give an explicit $AEMSO(K, A)$-formula φ with $\|\mathcal{A}\| = \|\varphi\|$.

Observe that Theorem **??**, part (a) generalizes the main result of [?] which we obtain by letting Φ be the trivial discounting.

The last theorem contains our third main result. For its proof we use similar arguments as for the finitary case.

Theorem 14 *Let A be a finite alphabet and Φ' any discounting over A and $\mathbb{R}_{\max}$. Then*

$$\mathbb{R}_{\max}^{\Phi'-\omega-rec}\langle\langle A^\omega\rangle\rangle = \mathbb{R}_{\max}^{\Phi'-rmso}\langle\langle A^\omega\rangle\rangle = \mathbb{R}_{\max}^{\Phi'-remso}\langle\langle A^\omega\rangle\rangle = \mathbb{R}_{\max}^{\Phi'-aemso}\langle\langle A^\omega\rangle\rangle .$$

Corollary 15 (Büchi's Theorem) *An infinitary language is ω-recognizable iff it is definable by a EMSO-sentence.*

Finally, we turn to constructibility and decision problems.

Corollary 16 *Let K be a computable, additively locally finite, commutative semiring, or let $K = \mathbb{R}_{\max}$ or $K = \mathbb{R}_{\min}$. Let Φ be a discounting over A and K. Given an $AEMSO(K, A)$-formula φ whose atomic entries from K are effectively given, we can effectively compute a weighted automaton, resp. a weighted Muller automaton, $\mathcal{A}$ such that $\|\varphi\| = \|\mathcal{A}\|$.*

Unfortunately, for such semirings K as in Corollary **??**, the equality $\|\varphi\| = \|\varphi'\|$ on A^*, i.e. finite words, for two $AEMSO(K, A)$-sentences φ, φ' is in general undecidable. Consider $K = \mathbb{R}_{\max}$, and suppose there was a decision procedure for this equality. By the Theorem **??**, part (a) we would obtain a decision procedure for weighted automata $\mathcal{A}$, $\mathcal{A}'$ of whether $\|\mathcal{A}\| = \|\mathcal{A}'\|$ (as series over A^*). But this is impossible by a result of Krob [**?**]. Here the interesting open problem arises whether, due to the discounting we might achieve better decidability results for the semirings $\mathbb{R}_{\max}$ or $\mathbb{R}_{\min}$ over infinite words.

4 Conclusion

We introduced a weighted logics with discounting over finite words, and we proved its expressive equivalence to discounted behaviors of weighted automata. We gave a logic with a purely syntactic definition whenever the underlying semiring is additively locally finite. Then we investigated Büchi and Muller automata with discounting over the max-plus and min-plus semiring and we characterized their behaviors as definable series in a discounting weighted logic over infinite words. This logic also possesses a syntactic definition. In this way, we obtained an extension of classical and recent results of the theory of formal languages and formal power series, and this provides an automata (and thus algorithmic) and logical theoretic way to describe the discounting concept which is widely used in game theory and mathematical economics.

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