

Weighted automata

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Abstract. Weighted automata are classical finite automata in which the transitions carry weights. These weights may model quantitative properties like the amount of resources needed for executing a transition or the probability or reliability of its successful execution. Using weighted automata, we may also count the number of successful paths labeled by a given word.

As an introduction into this field, we present selected classical and recent results concentrating on the expressive power of weighted automata.

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1 Introduction

27

28 Classical automata provide acceptance mechanisms for words. The starting point of
 29 weighted automata is to determine the number of ways a word can be accepted or the
 30 amount of resources used for this. The behavior of weighted automata thus associates
 31 a quantity or weight to every word. The goal of this chapter is to study the possible
 32 behaviors.

33 Historically, weighted automata were introduced in the seminal paper by Schützen-
 34 berger [85]. A close relationship to probabilistic automata was mutually influential in the
 35 beginning [77, 19, 95]. For the domain of weights and their computations, the algebraic
 36 structure of semirings proved to be very fruitful. This soon led to a rich mathematical
 37 theory including applications for purely language theoretic questions as well as practical
 38 applications in digital image compression and algorithms for natural language process-
 39 ing. Excellent treatments of this are provided by the books [38, 84, 95, 66, 11, 82] and
 40 the surveys in the recent handbook [30].

41 In this chapter, we describe the behavior of weighted automata by equivalent for-
 42 malisms. These include rational expressions and series, algebraic means like linear pre-
 43 sentations and semimodules, decomposition into simple behaviors, and quantitative log-
 44 ics. We also touch on decidability questions (including Colcombet’s new proof of a cel-
 45 ebrated result by Krob) and languages naturally associated to the behaviors of weighted
 46 automata.

47 We had to choose from the substantial amount of theory and applications of this topic
 48 and our choice is biased by our personal interests. We hope to wet the reader’s appetite
 49 for this exciting field and for consulting the abovementioned books.

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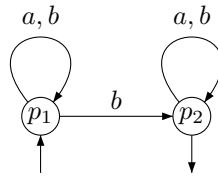


Figure 1. A nondeterministic finite automaton

2 Weighted automata and their behavior

52

53 We start with a simple automaton exemplifying different possible interpretations of its
 54 behavior. We identify a common feature that will permit us to consider them as instances
 55 of the unified concept of a weighted automaton. So let $\Sigma = \{a, b\}$ and $Q = \{p_1, p_2\}$ and
 56 consider the automaton from Figure 1.

57 **Example 2.1.** Classically, the language accepted describes the behavior of a finite au-
 58 tomaton. In our case, this is the language $\Sigma^* b \Sigma^*$.

59 Now set $\text{in}(p_1) = \text{out}(p_2) = \text{true}$, $\text{out}(p_1) = \text{in}(p_2) = \text{false}$, and $\text{wt}(p, c, q) = \text{true}$
 60 if (p, c, q) is a transition of the automaton and false otherwise. Then a word $a_1 a_2 \dots a_n$
 61 is accepted by the automaton if and only if

$$\bigvee_{q_0, q_1, \dots, q_n \in Q} \left(\text{in}(q_0) \wedge \bigwedge_{1 \leq i \leq n} \text{wt}(q_{i-1}, a_i, q_i) \wedge \text{out}(q_n) \right)$$

62 evaluates to true.

63 **Example 2.2.** For any word $w \in \Sigma^*$, let $f(w)$ denote the number of accepting paths
 64 labeled w . In our case, $f(w)$ equals the number of occurrences of the letter b .

65 Set $\text{in}(p_1) = \text{out}(p_2) = 1$, $\text{out}(p_1) = \text{in}(p_2) = 0$, and $\text{wt}(p, c, q) = 1$ if (p, c, q) is a
 66 transition of the automaton and 0 otherwise. Then $f(a_1 \dots a_n)$ equals

$$\sum_{q_0, q_1, \dots, q_n \in Q} \left(\text{in}(q_0) \cdot \prod_{1 \leq i \leq n} \text{wt}(q_{i-1}, a_i, q_i) \cdot \text{out}(q_n) \right). \quad (2.1)$$

67 Note that the above two examples would in fact work correspondingly for any finite
 68 automaton. The following two examples are specific for the particular automaton from
 69 Fig. 1.

70 **Example 2.3.** Define the functions in and out as in Example 2.2. But this time, set
 71 $\text{wt}(p, c, q) = 1$ if (p, c, q) is a transition of the automaton and $p = p_1$, $\text{wt}(p_2, c, p_2) = 2$
 72 for $c \in \Sigma$, and $\text{wt}(p, c, q) = 0$ otherwise. If we now evaluate the formula (2.1) for a word
 73 $w \in \Sigma^*$, we obtain the value of the word w if understood as a binary number where a
 74 stands for the digit 0 and b for the digit 1.

75 **Example 2.4.** Let the deficit of a word $v \in \Sigma^*$ be the number $|v|_b - |v|_a$ where $|v|_a$ is
 76 the number of occurrences of a in v and $|v|_b$ is defined analogously. We want to compute
 77 using the automaton from Fig. 1 the maximal deficit of a prefix of a word w . To this
 78 aim, set $\text{in}(p_1) = \text{out}(p_2) = 0$ and $\text{out}(p_1) = \text{in}(p_2) = -\infty$. Furthermore, we set
 79 $\text{wt}(p_1, b, p_i) = 1$ for $i = 1, 2$, $\text{wt}(p_1, a, p_1) = -1$, $\text{wt}(p_2, c, p_2) = 0$ for $c \in \Sigma$, and
 80 $\text{wt}(p, c, q) = -\infty$ in the remaining cases. Then the maximal deficit of a prefix of the
 81 word $w = a_1 a_2 \dots a_n \in \Sigma^* b \Sigma^*$ equals

$$\max_{q_0, q_1, \dots, q_n \in Q} \left(\text{in}(q_0) + \sum_{1 \leq i \leq n} \text{wt}(q_{i-1}, a_i, q_i) + \text{out}(q_n) \right).$$

82 The similarities between the above examples naturally lead to the definition of a
 83 weighted automaton.

84 **Definition 2.1.** Let S be a set and Σ an alphabet. A *weighted automaton over S and Σ* is
 85 a quadruple $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ where

- 86 • Q is a finite set of states,
- 87 • $\text{in}, \text{out}: Q \rightarrow S$ are weight functions for entering and leaving a state, resp., and
- 88 • $\text{wt}: Q \times \Sigma \times Q \rightarrow S$ is a transition weight function.

89 The rôle of S in the examples above is played by $\{\text{true}, \text{false}\}$, \mathbb{N} , and $\mathbb{Z} \cup \{-\infty\}$,
 90 resp., i.e., we reformulated all the examples as weighted automata over some appropriate
 91 set S .

92 Note also the similarity of the description of the behaviors in all the examples above.
 93 We now introduce semirings that formalize the similarities between the operations $\vee, +$,
 94 and \max on the one hand, and \wedge, \cdot , and $+$ on the other:

95 **Definition 2.2.** A *semiring* is a structure $(S, +_S, \cdot_S, 0_S, 1_S)$ such that

- 96 • $(S, +_S, 0_S)$ is a commutative monoid,
- 97 • $(S, \cdot_S, 1_S)$ is a monoid,
- 98 • multiplication distributes over addition from the left and from the right, and
- 99 • $0_S \cdot_S s = s \cdot_S 0_S = 0_S$ for all $s \in S$.

100 If no confusion can occur, we often write S for the semiring $(S, +_S, \cdot_S, 0_S, 1_S)$.

101 It is easy to check that the structures $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$, $(\mathbb{N}, +, \cdot, 0, 1)$, and $(\mathbb{Z} \cup$
 102 $\{-\infty\}, \max, +, -\infty, 0)$ are semirings (with $0 = \text{false}$ and $1 = \text{true}$, \mathbb{B} is the semiring
 103 underlying Example 2.1); many further examples are given in [29] and throughout this
 104 chapter. The theory of semirings is described in [49]. The notion of a semiring allows
 105 us to give a common definition of the behavior of weighted automata that subsumes all
 106 those from our examples and, with the language semiring $(\mathcal{P}(\Gamma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, we even
 107 capture the important notion of a transducer [9]; here $\mathcal{P}(\Gamma^*)$ denotes the powerset of Γ^* .

108 **Definition 2.3.** Let S be a semiring and \mathcal{A} a weighted automaton over S . A *path in \mathcal{A}* is
 109 an alternating sequence $P = q_0 a_1 q_1 \dots a_n q_n \in Q(\Sigma Q)^*$. Its *run weight* is the product

$$\text{rweight}(P) = \prod_{0 \leq i < n} \text{wt}(q_i, a_{i+1}, q_{i+1})$$

110 (for $n = 0$, this is defined to be 1); the *weight* of P is then defined by

$$\text{weight}(P) = \text{in}(q_0) \cdot \text{rweight}(P) \cdot \text{out}(q_n).$$

111 Furthermore, the *label* of P is the word $\text{label}(P) = a_1 a_2 \dots a_n$. Then the *behavior* of
112 the weighted automaton \mathcal{A} is the function $\|\mathcal{A}\|: \Sigma^* \rightarrow S$ with

$$\|\mathcal{A}\|(w) = \sum_{\substack{P \text{ path with} \\ \text{label}(P)=w}} \text{weight}(P). \quad (2.2)$$

113 Whereas classical automata determine whether a word is accepted or not, weighted
114 automata over the natural semiring \mathbb{N} allow us to *count* the number of successful paths
115 labeled by a word (cf. Example 2.2). Over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$,
116 weighted automata can be viewed as determining the maximal amount of resources needed
117 for the execution of a given sequence of actions. Thus, weighted automata determine
118 quantitative properties.

119 **Notational convention** We write $P: p \xrightarrow{w} q$ for “ P is a path in the weighted automa-
120 ton \mathcal{A} from p to q with label w ”. From now on, all weighted automata will be over some
121 semiring $(S, +, \cdot, 0, 1)$. We will call functions from Σ^* into S *series*. For such a series r ,
122 it is customary to write (r, w) for $r(w)$. The set of all series from Σ^* into S will be de-
123 noted by $S \langle\langle \Sigma^* \rangle\rangle$. If \mathcal{A} is a weighted automaton, then we get in particular $\|\mathcal{A}\| \in S \langle\langle \Sigma^* \rangle\rangle$
124 and in the above definition, we could have written $(\|\mathcal{A}\|, w)$ instead of $\|\mathcal{A}\|(w)$.

125 **Definition 2.4.** A series $r \in S \langle\langle \Sigma^* \rangle\rangle$ is *recognizable* if it is the behavior of some weighted
126 automaton. The set of all recognizable series is denoted by $S^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$.

127 For a series $r \in S \langle\langle \Sigma^* \rangle\rangle$, the *support* of r is the set $\text{supp}(r) = \{w \in \Sigma^* \mid (r, w) \neq 0\}$.
128 Also, for a language $L \subseteq \Sigma^*$, we write $\mathbb{1}_L$ for the series with $(\mathbb{1}_L, w) = 1_S$ if $w \in L$ and
129 $(\mathbb{1}_L, w) = 0_S$ otherwise; $\mathbb{1}_L$ is called the *characteristic series* of L . From Example 2.1,
130 it should be clear that a series r in $\mathbb{B} \langle\langle \Sigma^* \rangle\rangle$ is recognizable if and only if the language
131 $\text{supp}(r)$ is regular. Later, we will see that many properties of regular languages transfer
132 to recognizable series (sometimes with very similar proofs). But first, we want to point
133 out some differences.

134 **Example 2.5.** Let $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ and consider the series r with $(r, wa) =$
135 $\{aw\}$ for all words $w \in \Sigma^*$ and letters $a \in \Sigma$, and $(r, \varepsilon) = \emptyset$. Then $r \in S^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$,
136 but there is no deterministic transducer whose behavior equals r . Hence deterministic
137 weighted automata are in general weaker than general weighted automata, i.e., a funda-
138 mental property of finite automata (see Chapter 1) does not transfer to weighted automata.

139 **Example 2.6.** Let $S = (\mathbb{N}, +, \cdot, 0, 1)$ and $a \in \Sigma$. We consider the series r with $(r, aa) =$
140 2 and $(r, w) = 0$ for $w \neq aa$. Then there are 4 different (deterministic) weighted automata
141 with three states and behavior r (and none with only two states). Hence, another funda-
142 mental property of finite automata, namely the existence of unique minimal deterministic
143 automata, does not transfer.

144 Recall that the existence of a unique minimal deterministic automaton for a regular
 145 language can be used to decide whether two finite automata accept the same language.
 146 Above, we saw that this approach cannot be used for weighted automata over the semi-
 147 ring $(\mathbb{N}, +, \cdot, 0, 1)$, but other methods work in this case. However, there are no universal
 148 methods since the equivalence problem over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$
 149 is undecidable, see Section 8.

150 3 Linear presentations

Let S be a semiring and Q_1 and Q_2 sets. We will consider a function from $Q_1 \times Q_2$ into S
 as a matrix whose rows and columns are indexed by elements of Q_1 and Q_2 , respectively.
 Therefore, we will write $M_{p,q}$ for $M(p, q)$ where $M \in S^{Q_1 \times Q_2}$, $p \in Q_1$, and $q \in Q_2$.
 For finite sets Q_1, Q_2, Q_3 , this allows us to define the sum and the product of two matrices
 as usual:

$$(K + M)_{p,q} = K_{p,q} + M_{p,q} \quad (M \cdot N)_{p,r} = \sum_{q \in Q_2} M_{p,q} \cdot N_{q,r}$$

151 for $K, M \in S^{Q_1 \times Q_2}$, $N \in S^{Q_2 \times Q_3}$, $p \in Q_1$, $q \in Q_2$, and $r \in Q_3$. Since in semirings,
 152 multiplication distributes over addition from both sides, matrix multiplication is associa-
 153 tive. For a finite set Q , the *unit matrix* $E \in S^{Q \times Q}$ with $E_{p,q} = 1$ for $p = q$ and $E_{p,q} = 0$
 154 otherwise is the neutral element of the multiplication of matrices. Hence $(S^{Q \times Q}, \cdot, E)$ is
 155 a monoid. It is useful to note that with pointwise addition of matrices, $S^{Q \times Q}$ even forms
 156 a semiring.

157 **Lemma 3.1.** *Let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be a weighted automaton and define a mapping*
 158 $\mu: \Sigma^* \rightarrow S^{Q \times Q}$ *by*

$$\mu(w)_{p,q} = \sum_{P: p \xrightarrow{w} q} \text{rweight}(P). \quad (3.1)$$

159 *Then μ is a homomorphism from the free monoid Σ^* to the multiplicative monoid of*
 160 *matrices $(S^{Q \times Q}, \cdot, E)$.*

161 *Proof.* Let $P = p_0 a_1 p_1 \dots a_n p_n$ be a path with label uv and let $|u| = k$. Then $P_1 =$
 162 $p_0 a_1 \dots a_k p_k$ is a u -labeled path, $P_2 = p_k a_{k+1} \dots a_n p_n$ is a v -labeled path, and we
 163 have $\text{rweight}(P) = \text{rweight}(P_1) \cdot \text{rweight}(P_2)$. This simple observation, together with
 164 distributivity in the semiring S , allows us to prove the claim. \square

165 Now let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be a weighted automaton. Define $\lambda \in S^{\{1\} \times Q}$ and
 166 $\gamma \in S^{Q \times \{1\}}$ by $\lambda_{1,q} = \text{in}(q)$ and $\gamma_{q,1} = \text{out}(q)$. With the homomorphism μ from
 167 Lemma 3.1, we obtain for any word $w \in \Sigma^*$ (where we identify a $\{1\} \times \{1\}$ -matrix with
 168 its entry):

$$(|\mathcal{A}|, w) = \sum_{p,q \in Q} \lambda_{1,p} \cdot \mu(w)_{p,q} \cdot \gamma_{q,1} = \lambda \cdot \mu(w) \cdot \gamma. \quad (3.2)$$

169 Subsequently, we consider λ (as usual) as a row vector and γ as a column vector and we
 170 simply write $\lambda, \gamma \in S^Q$.

171 This motivates the following definition.

172 **Definition 3.1** (Schützenberger [85]). A *linear presentation* of dimension Q (where Q is
 173 some finite set) is a triple (λ, μ, γ) such that $\lambda, \gamma \in S^Q$ and $\mu: (\Sigma^*, \cdot, \varepsilon) \rightarrow (S^{Q \times Q}, \cdot, E)$
 174 is a monoid homomorphism. It defines the series $r = \|(\lambda, \mu, \gamma)\|$ with

$$(r, w) = \lambda \cdot \mu(w) \cdot \gamma \quad (3.3)$$

175 for all $w \in \Sigma^*$.

176 Above, we saw that any weighted automaton can be transformed into an equivalent
 177 linear presentation. Now we describe the converse transformation. So let (λ, μ, γ) be a
 178 linear presentation of dimension Q . For $a \in \Sigma$ and $p, q \in Q$, set $\text{wt}(p, a, q) = \mu(a)_{p,q}$,
 179 $\text{in}(q) = \lambda_q$, and $\text{out}(q) = \gamma_q$, and define $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$. Since the morphism μ is
 180 uniquely determined by its restriction to Σ , the linear representation associated with \mathcal{A} is
 181 precisely (λ, μ, γ) , so by Equation (3.2) we obtain $\|\mathcal{A}\| = \|(\lambda, \mu, \gamma)\|$. Hence we showed

182 **Theorem 3.2.** *Let S be a semiring, Σ an alphabet, and $r \in S \langle\langle \Sigma^* \rangle\rangle$. Then r is recogniz-*
 183 *able if and only if there exists a linear presentation (λ, μ, γ) with $r = \|(\lambda, \mu, \gamma)\|$.*

184 This theorem explains why some authors use linear presentations to define recogniz-
 185 able series or even weighted automata.

186 4 The Kleene-Schützenberger theorem

187 The goal of this section is to derive a generalization of Kleene's classical result on the co-
 188 incidence of rational and regular languages in the realm of series over semirings. There-
 189 fore, first we introduce operations in $S \langle\langle \Sigma^* \rangle\rangle$ that correspond to the language-theoretic
 190 operations union, intersection, concatenation, and Kleene iteration.

191 Let r_1 and r_2 be series. Pointwise addition is defined by

$$(r_1 + r_2, w) = (r_1, w) + (r_2, w).$$

192 Clearly, this operation is associative and has the constant series with value 0 as neutral
 193 element. Furthermore, it generalizes the union of languages since, in the Boolean semi-
 194 ring \mathbb{B} , we have $\text{supp}(r_1 + r_2) = \text{supp}(r_1) \cup \text{supp}(r_2)$ and $\mathbb{1}_{K \cup L} = \mathbb{1}_K + \mathbb{1}_L$.

195 Any family of languages has a union, so one is tempted to also define the sum of
 196 arbitrary sets of series. But this fails in general since it would require the sum of infinitely
 197 many elements of the semiring S (which, e.g. in $(\mathbb{N}, +, \cdot, 0, 1)$, does not exist). But certain
 198 families can be summed: a family $(r_i)_{i \in I}$ of series is *locally finite* if, for any word $w \in$
 199 Σ^* , there are only finitely many $i \in I$ with $(r_i, w) \neq 0$. For such families, we can define

$$\left(\sum_{i \in I} r_i, w \right) = \sum_{\substack{i \in I \text{ with} \\ (r_i, w) \neq 0}} (r_i, w).$$

200 Let $r_1, r_2 \in S \langle\langle \Sigma^* \rangle\rangle$. Pointwise multiplication is defined by

$$(r_1 \odot r_2, w) = (r_1, w) \cdot (r_2, w).$$

201 This operation is called *Hadamard product*, is clearly associative, has the constant series with value 1 as neutral element, and distributes over addition. If S is the Boolean semiring \mathbb{B} , then the Hadamard product corresponds to intersection:

$$\text{supp}(r_1 \odot r_2) = \text{supp}(r_1) \cap \text{supp}(r_2) \text{ and } \mathbb{1}_K \odot \mathbb{1}_L = \mathbb{1}_{K \cap L}$$

204 Other simple and natural operations are the *left* and *right scalar multiplication* that are defined by

$$(s \cdot r, w) = s \cdot (r, w) \text{ and } (r \cdot s, w) = (r, w) \cdot s$$

206 for $s \in S$ and $r \in S \langle\langle \Sigma^* \rangle\rangle$. These two scalar multiplications do not have natural counterparts in language theory.

208 The counterpart of singleton languages in the realm of series are monomials: a *monomial* is a series r with $|\text{supp}(r)| \leq 1$. With $w \in \Sigma^*$ and $s \in S$, we will write sw for the monomial r with $(r, w) = s$. Let r be an arbitrary series. Then the family of monomials $((r, w)w)_{w \in \Sigma^*}$ is locally finite and can therefore be summed. Then one obtains

$$r = \sum_{w \in \Sigma^*} (r, w)w = \sum_{w \in \text{supp}(r)} (r, w)w.$$

212 If the support of r is finite, then the second sum has only finitely many summands which is the reason to call r a *polynomial* in this case; the set of polynomials is denoted $S \langle \Sigma^* \rangle$, so $S \langle \Sigma^* \rangle \subseteq S \langle\langle \Sigma^* \rangle\rangle$. The similarity with polynomials makes it natural to define another product of the series r_1 and r_2 by

$$(r_1 \cdot r_2, w) = \sum_{w=uv} (r_1, u) \cdot (r_2, v).$$

216 Since the word w has only finitely many factorizations into u and v , the right-hand side has only finitely many summands and is therefore well-defined. This important product is called *Cauchy-product* of the series r_1 and r_2 . If r_1 and r_2 are polynomials, then $r_1 \cdot r_2$ is precisely the usual product of polynomials. For the Boolean semiring, we get

$$\text{supp}(r_1 \cdot r_2) = \text{supp}(r_1) \cdot \text{supp}(r_2) \text{ and } \mathbb{1}_K \cdot \mathbb{1}_L = \mathbb{1}_{K \cdot L},$$

220 i.e., the Cauchy-product is the counterpart of concatenation of languages. For any semiring S , the monomial 1ε is the neutral element of the Cauchy-product. It requires a short calculation to show that the Cauchy-product is associative and distributes over the addition of series. As a very useful consequence, $(S \langle\langle \Sigma^* \rangle\rangle, +, \cdot, 0, 1\varepsilon)$ is a semiring (note that the set of polynomials $S \langle \Sigma^* \rangle$ forms a subsemiring of this semiring). For the Boolean semiring \mathbb{B} , this semiring is isomorphic to $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, an isomorphism is given by $r \mapsto \text{supp}(r)$ with inverse $L \mapsto \mathbb{1}_L$.

227 In the theory of recognizable languages, the Kleene-iteration L^* of a language L is of central importance. It is defined as the union of all the powers L^n of L (for $n \geq 0$). To also define the iteration r^* of a series, one would therefore try to sum all finite powers r^n (defined by $r^0 = 1\varepsilon$ and $r^{n+1} = r^n \cdot r$). In general, the family $(r^n)_{n \geq 0}$ is not locally finite, so it cannot be summed. We therefore define the iteration r^* only for r proper: a

232 series r is *proper* if $(r, \varepsilon) = 0$. Then, for $n > |w|$, one has $(r^n, w) = 0$, so the family
 233 $(r^n)_{n \geq 0}$ is locally finite and we can set

$$r^* = \sum_{n \geq 0} r^n \text{ or equivalently } (r^*, w) = \sum_{0 \leq n \leq |w|} (r^n, w).$$

234 For the Boolean semiring and $L \subseteq \Sigma^+$, we get

$$\text{supp}(r^*) = (\text{supp}(r))^* \text{ and } (\mathbb{1}_L)^* = \mathbb{1}_{L^*}.$$

235 Recall that a language is rational if it can be constructed from the finite languages by
 236 union, concatenation, and Kleene-iteration. Here, we give the analogous definition for
 237 series:

238 **Definition 4.1.** A series from $S \langle\langle \Sigma^* \rangle\rangle$ is *rational* if it can be constructed from the mono-
 239 mials sa for $s \in S$ and $a \in \Sigma \cup \{\varepsilon\}$ by addition, Cauchy-product, and iteration (applied
 240 to proper series, only). The set of all rational series is denoted by $S^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$.

241 Observe that the class of rational series is closed under scalar multiplication since $s\varepsilon$
 242 is a monomial, $s \cdot r = s\varepsilon \cdot r$ and $r \cdot s = r \cdot s\varepsilon$ for $r \in S \langle\langle \Sigma^* \rangle\rangle$ and $s \in S$.

243 **Example 4.1.** Consider the Boolean semiring \mathbb{B} and $r \in \mathbb{B} \langle\langle \Sigma^* \rangle\rangle$. If r is a rational series,
 244 then the above formulas show that $\text{supp}(r)$ is a rational language since supp commutes
 245 with the rational operations $+$, \cdot , and $*$ for series and \cup , \cdot , and $*$ for languages. Now
 246 suppose that, conversely, $\text{supp}(r)$ is a rational language. To show that also r is a ratio-
 247 nal series, one needs that any rational language can be constructed in such a way that
 248 Kleene-iteration is only applied to languages in Σ^+ . Having ensured this, the remaining
 249 calculations are again straightforward. Thus, indeed, our notion of rational series is the
 250 counterpart of the notion of a rational language.

251 Hence, rational languages are precisely the supports of series in $\mathbb{B}^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$ and rec-
 252 ognizable languages are the supports of series in $\mathbb{B}^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$ (see above). Now Kleene's
 253 theorem from Chapter 1 implies $\mathbb{B}^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle = \mathbb{B}^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$. It is the aim of this section to
 254 prove this equality for arbitrary semirings. This is achieved by first showing that every
 255 rational series is recognizable. The other inclusion will be shown in Section 4.2.

256 4.1 Rational series are recognizable

257 For this implication, we prove that the set of recognizable series contains the monomials
 258 sa and $s\varepsilon$ and is closed under the necessary operations. To show this closure, we have
 259 two possibilities (a third one is sketched after the proof of Theorem 5.1): either the purely
 260 automata-theoretic approach that constructs weighted automata, or the more algebraic
 261 approach that handles linear presentations. We chose to give the automata constructions
 262 for monomials and addition, and the linear presentations for the Cauchy-product and the
 263 iteration. The reader might decide which approach she prefers and translate some of the
 264 constructions from one to the other.

265 There is a weighted automaton with just one state q and behavior the monomial $s\varepsilon$:
 266 just set $\text{in}(q) = s$, $\text{out}(q) = 1$ and $\text{wt}(q, a, q) = 0$ for all $a \in \Sigma$. For any $a \in \Sigma$, there

267 is a two-states weighted automaton with the monomial sa as behavior. If \mathcal{A}_1 and \mathcal{A}_2 are
 268 two weighted automata, then the behavior of their disjoint union equals $\|\mathcal{A}_1\| + \|\mathcal{A}_2\|$.

269 We next show that also the Cauchy-product of two recognizable series is recognizable:

270 **Lemma 4.1.** *If r_1 and r_2 are recognizable series, then so is $r_1 \cdot r_2$.*

Proof. By Theorem 3.2, the series r_i has a linear presentation $(\lambda^i, \mu^i, \gamma^i)$ of dimension Q^i with $Q^1 \cap Q^2 = \emptyset$. We define a row vector λ and a column vector γ of dimension $Q = Q^1 \cup Q^2$ as well as a matrix $\mu(w)$ for $w \in \Sigma^*$ of dimension $Q \times Q$:

$$\lambda = (\lambda^1 \quad 0) \quad \mu(w) = \begin{pmatrix} \mu^1(w) & \sum_{w=uv, v \neq \varepsilon} \mu^1(u) \gamma^1 \lambda^2 \mu^2(v) \\ 0 & \mu^2(w) \end{pmatrix} \quad \gamma = \begin{pmatrix} \gamma^1 \lambda^2 \gamma^2 \\ \gamma^2 \end{pmatrix}$$

The reader is invited to check that μ is actually a monoid homomorphism from $(\Sigma^*, \cdot, \varepsilon)$ into $(S^{Q \times Q}, \cdot, E)$, i.e., that (λ, μ, γ) is a linear presentation. One then gets

$$\begin{aligned} \lambda \cdot \mu(w) \cdot \gamma &= \lambda^1 \mu^1(w) \gamma^1 \lambda^2 \gamma^2 + \lambda^1 \sum_{\substack{w=uv \\ v \neq \varepsilon}} \mu^1(u) \gamma^1 \lambda^2 \mu^2(v) \gamma^2 \\ &= (r_1, w) \cdot (r_2, \varepsilon) + \sum_{\substack{w=uv \\ v \neq \varepsilon}} (r_1, u) (r_2, v) \\ &= (r_1 \cdot r_2, w). \end{aligned}$$

271 By Theorem 3.2, the series $\|(\lambda, \mu, \gamma)\| = r_1 \cdot r_2$ is recognizable. \square

272 **Lemma 4.2.** *Let r be a proper and recognizable series. Then r^* is recognizable.*

273 *Proof.* There exists a linear presentation (λ, μ, γ) of dimension Q with $r = \|(\lambda, \mu, \gamma)\|$.

274 Consider the homomorphism $\mu' : (\Sigma^*, \cdot, \varepsilon) \rightarrow (S^{Q \times Q}, \cdot, E)$ defined, for $a \in \Sigma$, by

$$\mu'(a) = \mu(a) + \gamma \lambda \mu(a).$$

Let $w = a_1 a_2 \dots a_n \in \Sigma^+$. Using distributivity of matrix multiplication or, alternatively, induction on n , it follows

$$\begin{aligned} \mu'(w) &= \prod_{1 \leq i \leq n} (\mu(a_i) + \gamma \lambda \mu(a_i)) \\ &= \sum_{\substack{w=w_1 \dots w_k \\ w_i \in \Sigma^+}} \left((\mu(w_1) + \gamma \lambda \mu(w_1)) \cdot \prod_{2 \leq j \leq k} \gamma \lambda \mu(w_j) \right). \end{aligned}$$

Note that $\lambda \gamma = \lambda \mu(\varepsilon) \gamma = (r, \varepsilon) = 0$. Hence we obtain

$$\begin{aligned} \lambda \mu'(w) \gamma &= \sum_{\substack{w=w_1 \dots w_k \\ w_i \in \Sigma^+}} \left(\lambda(\mu(w_1) + \gamma \lambda \mu(w_1)) \cdot \prod_{2 \leq j \leq k} \gamma \lambda \mu(w_j) \right) \gamma \\ &= \sum_{\substack{w=w_1 \dots w_k \\ w_i \in \Sigma^+}} \prod_{1 \leq j \leq k} \lambda \mu(w_j) \gamma \\ &= (r^*, w) \end{aligned}$$

as well as $\lambda \mu'(\varepsilon) \gamma = 0$. Hence $r^* = \|(\lambda, \mu', \gamma)\| + 1\varepsilon$ is recognizable. \square

Recall that the Hadamard-product generalizes the intersection of languages and that the intersection of regular languages is regular. The following result extends this latter fact to the weighted setting (since the Boolean semiring is commutative). We say that two subsets $S_1, S_2 \subseteq S$ commute, if $s_1 \cdot s_2 = s_2 \cdot s_1$ for all $s_1 \in S_1, s_2 \in S_2$.

Lemma 4.3. *Let S_1 and S_2 be two subsemirings of the semiring S such that S_1 and S_2 commute. If $r_1 \in S_1^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$ and $r_2 \in S_2^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$, then $r_1 \odot r_2 \in S^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle$.*

Proof. For $i = 1, 2$, let $\mathcal{A}_i = (Q_i, \text{in}_i, \text{wt}_i, \text{out}_i)$ be weighted automata over S_i with $\|\mathcal{A}_i\| = r_i$. We define the product automaton \mathcal{A} with states $Q_1 \times Q_2$ as follows:

$$\begin{aligned} \text{in}(p_1, p_2) &= \text{in}_1(p_1) \cdot \text{in}_2(p_2) \\ \text{wt}((p_1, p_2), a, (q_1, q_2)) &= \text{wt}_1(p_1, a, q_1) \cdot \text{wt}_2(p_2, a, q_2) \\ \text{out}(p_1, p_2) &= \text{out}_1(p_1) \cdot \text{out}_2(p_2) \end{aligned}$$

Then, $(\|\mathcal{A}\|, w) = (\|\mathcal{A}_1\| \odot \|\mathcal{A}_2\|, w)$ follows for all words w . For example, for a letter $a \in \Sigma$ we calculate as follows using the commutativity assumption and distributivity:

$$\begin{aligned} (\|\mathcal{A}\|, a) &= \sum_{(p_1, p_2), (q_1, q_2) \in Q} \left(\begin{array}{c} (\text{in}_1(p_1) \cdot \text{in}_2(p_2)) \cdot (\text{wt}_1(p_1, a, q_1) \cdot \text{wt}_2(p_2, a, q_2)) \\ \cdot (\text{out}_1(q_1) \cdot \text{out}_2(q_2)) \end{array} \right) \\ &= \sum_{(p_1, p_2), (q_1, q_2) \in Q} \left(\begin{array}{c} \text{in}_1(p_1) \cdot \text{wt}_1(p_1, a, q_1) \cdot \text{out}_1(q_1) \\ \cdot \text{in}_2(p_2) \cdot \text{wt}_2(p_2, a, q_2) \cdot \text{out}_2(q_2) \end{array} \right) \\ &= \left(\sum_{p_1, q_1 \in Q_1} \text{in}_1(p_1) \cdot \text{wt}_1(p_1, a, q_1) \cdot \text{out}_1(q_1) \right) \\ &\quad \cdot \left(\sum_{p_2, q_2 \in Q_2} \text{in}_2(p_2) \cdot \text{wt}_2(p_2, a, q_2) \cdot \text{out}_2(q_2) \right) \\ &= (\|\mathcal{A}_1\|, a) \cdot (\|\mathcal{A}_2\|, a) = (\|\mathcal{A}_1\| \odot \|\mathcal{A}_2\|, a) \end{aligned}$$

282

\square

283 We remark that the above lemma does not hold without the commutativity assumption:

284 **Example 4.2.** Let $\Sigma = \{a, b\}$, $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, and consider the recognizable
 285 series r given by $(r, w) = \{w\}$ for $w \in \Sigma^*$. Then $(r \odot r, w) = \{ww\}$ and pumping
 286 arguments show that $r \odot r$ is not recognizable.

287 As a consequence of Lemma 4.3, we obtain that “restrictions” of recognizable series
 288 to regular languages are again recognizable, more precisely:

289 **Corollary 4.4.** *Let $r \in S \langle\langle \Sigma^* \rangle\rangle$ be recognizable and let $L \subseteq \Sigma^*$ be a regular language.
 290 Then $r \odot \mathbb{1}_L$ is recognizable.*

291 *Proof.* Let \mathcal{A} be a deterministic automaton accepting L with set of states Q . Then weight
 292 by 1 those triples $(p, a, q) \in Q \times \Sigma \times Q$ that are transitions, the initial resp. final states
 293 with initial resp. final weight by 1, and all other triples resp. states with 0. This gives a
 294 weighted automaton with behavior $\mathbb{1}_L$. Since S commutes with its subsemiring generated
 295 by 1, Lemma 4.3 implies the result. \square

296 4.2 Recognizable series are rational

297 For this implication, we will transform a weighted automaton into a system of equations
 298 and then show that any solution of such a system is rational. The following lemma will
 299 be helpful and is also of independent interest (cf. [29, Section 5]).

300 **Lemma 4.5.** *Let $s, r, r' \in S \langle\langle \Sigma^* \rangle\rangle$ with r proper and $s = r \cdot s + r'$. Then $s = r^* r'$.*

Proof. Let $w \in \Sigma^*$. First observe that

$$\begin{aligned} s &= rs + r' \\ &= r(rs + r') + r' = r^2s + rr' + r' \\ &\vdots \\ &= r^{|w|+1}s + \sum_{0 \leq i \leq |w|} r^i r'. \end{aligned}$$

Since r is proper, we have $(r^i, u) = 0$ for all $u \in \Sigma^*$ and $i > |u|$. This implies

$$\begin{aligned} (r^* r', w) &= \sum_{w=uv} (r^*, u) \cdot (r', v) = \sum_{w=uv} \left(\sum_{0 \leq i \leq |w|} (r^i, u) \right) \cdot (r', v) = \sum_{0 \leq i \leq |w|} (r^i r', w) \\ &= (s, w). \end{aligned} \quad \square$$

Now let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be a weighted automaton. For $p \in Q$, define a new weighted automaton $\mathcal{A}_p = (Q, \text{in}_p, \text{wt}, \text{out})$ by $\text{in}_p(p') = 1$ for $p = p'$ and $\text{in}_p(p') = 0$ otherwise. Since all the entry weights of these weighted automata are 0 or 1, we have

$$\|\mathcal{A}\| = \sum_{(p,a,q) \in Q \times \Sigma \times Q} \text{in}(p) \text{wt}(p,a,q) a \cdot \|\mathcal{A}_q\| + \sum_{p \in Q} \text{in}(p) \text{out}(p) \varepsilon$$

and for all $p \in Q$

$$\|\mathcal{A}_p\| = \sum_{(p,a,q) \in Q \times \Sigma \times Q} \text{wt}(p,a,q)a \cdot \|\mathcal{A}_q\| + \text{out}(p)\varepsilon.$$

301 This transformation proves

302 **Lemma 4.6.** *Let r be a recognizable series. Then there are rational series $r_{ij}, r_i \in$
303 $S \langle\langle \Sigma^* \rangle\rangle$ with r_{ij} proper and a solution (s_1, \dots, s_n) with $s_1 = r$ of a system of equations*

$$\left(X_i = \sum_{1 \leq j \leq n} r_{ij} X_j + r_i \right)_{1 \leq i \leq n}. \quad (4.1)$$

304 A series s is *rational over the series* $\{s_1, \dots, s_n\}$ if it can be constructed from the
305 monomials and the series s_1, \dots, s_n by addition, Cauchy-product, and iteration (applied
306 to proper series, only).

307 We prove by induction on n that any solution of a system of the form (4.1) consists of
308 rational series. For $n = 1$, the system is a single equation of the form $X_1 = r_{11}X_1 + r_1$
309 with $r_{11}, r_1 \in S^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$ and r_{11} proper. Hence, by Lemma 4.5, the solution s_1 equals
310 $r_{11}^* r_1$ and is therefore rational. Now assume that any system with $n - 1$ unknowns has
311 only rational solutions and consider a solution (s_1, \dots, s_n) of (4.1). Then we have

$$s_n = r_{nn}s_n + \sum_{1 \leq j < n} r_{nj}s_j + r_n$$

312 and therefore by Lemma 4.5

$$s_n = r_{nn}^* \cdot \left(\sum_{1 \leq j < n} r_{nj}s_j + r_n \right).$$

313 In particular, s_n is rational over $\{s_1, s_2, \dots, s_{n-1}\}$ since r_{nj} and r_n are all rational. Since
314 (s_1, \dots, s_n) is a solution of the system (4.1), we obtain

$$s_i = \sum_{1 \leq j < n} (r_{ij} + r_{in}r_{nn}^*r_{nj})s_j + r_{in}r_{nn}^*r_n + r_i$$

315 for all $1 \leq i < n$. Since r_{ij} and r_{in} are proper and rational, so is $r_{ij} + r_{in}r_{nn}^*r_{nj}$. Hence
316 (s_1, \dots, s_{n-1}) is a solution of a system of equations of the form (4.1) with $n-1$ unknowns
317 implying by the induction hypothesis that the series s_1, \dots, s_{n-1} are all rational. Since
318 s_n is rational over s_1, \dots, s_{n-1} , it is therefore rational, too. This completes the inductive
319 proof of the following lemma.

320 **Lemma 4.7.** *Let $r_{ij}, r_i \in S^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$ with r_{ij} proper and let (s_1, \dots, s_n) be a solution of
321 the system of equations (4.1). Then all the series s_1, \dots, s_n are rational.*

322 From Lemmas 4.6 and 4.7, we obtain that any recognizable series is rational. Together
323 with Lemmas 4.1, 4.2, and the arguments from the beginning of Section 4.1, we obtain

324 **Theorem 4.8** (Schützenberger [85]). *Let S be a semiring, Σ an alphabet, and $r \in$
325 $S \langle\langle \Sigma^* \rangle\rangle$. Then r is recognizable if and only if it is rational, i.e., $S^{\text{rec}} \langle\langle \Sigma^* \rangle\rangle = S^{\text{rat}} \langle\langle \Sigma^* \rangle\rangle$.*

326 5 Semimodules

If, in the definition of a vector space, one replaces the underlying field by a semiring, one obtains a semimodule. More formally, let S be a semiring. An S -semimodule is a commutative monoid $(M, +, 0_M)$ together with a left scalar multiplication $S \times M \rightarrow M$ satisfying all the usual laws (with $s, s' \in S$ and $r, r' \in M$):

$$\begin{aligned} (s + s')r &= sr + s'r & (s \cdot s')r &= s(s'r) \\ s(r + r') &= sr + sr' & 1r &= r \\ 0r &= 0_M \end{aligned}$$

327 In our context, the most interesting example is the S -semimodule $S \langle\langle \Sigma^* \rangle\rangle$ of series
328 over Σ . The additive structure of the semimodule is pointwise addition and the left scalar
329 multiplication is as defined before.

330 A *subsemimodule* of the S -semimodule $(M, +, 0_M)$ is a set $N \subseteq M$ that is closed
331 under addition and left scalar multiplication. A set $X \subseteq M$ *generates* the subsemimod-
332 *ule* $N = \langle X \rangle$ if N is the least subsemimodule containing X . Equivalently, all elements of
333 N can be written as linear combinations of elements from X . The subsemimodule N is
334 *finitely generated* if it is generated by a finite set. A simple example of a subsemimodule
335 of $S \langle\langle \Sigma^* \rangle\rangle$ is the set of polynomials $S \langle \Sigma^* \rangle$, i.e. of series with finite support. But this
336 subsemimodule is not finitely generated. The set of constant series is a finitely generated
337 subsemimodule.

338 The following is specific for the semimodule of series. For $r \in S \langle\langle \Sigma^* \rangle\rangle$ and $u \in \Sigma^*$,
339 the series $u^{-1}r$ is defined by

$$(u^{-1}r, w) = (r, uw)$$

340 for all $w \in \Sigma^*$. A subsemimodule N of $S \langle\langle \Sigma^* \rangle\rangle$ is *stable* if $r \in N$ implies $u^{-1}r \in N$ for
341 all $u \in \Sigma^*$.

342 **Theorem 5.1** (Fliess [46] and Jacob [55]). *Let S be a semiring, Σ an alphabet, and
343 $r \in S \langle\langle \Sigma^* \rangle\rangle$. Then r is recognizable if and only if there exists a finitely generated and
344 stable subsemimodule N of $S \langle\langle \Sigma^* \rangle\rangle$ with $r \in N$.*

345 For the boolean semiring \mathbb{B} , any finitely generated subsemimodule of $\mathbb{B} \langle\langle \Sigma^* \rangle\rangle$ is finite.
346 Therefore the above equivalence extends the well-known result that a language is regular
347 if and only if it has finitely many left-quotients.

348 *Proof.* First, let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be a weighted automaton with $r = \|\mathcal{A}\|$. For
349 $q \in Q$, define $\text{in}_q: Q \rightarrow S$ by $\text{in}_q(q) = 1$ and $\text{in}_q(p) = 0$ for $p \neq q$, and let $\mathcal{A}_q =$
350 $(Q, \text{in}_q, \text{wt}, \text{out})$. Let N be the subsemimodule generated by $\{\|\mathcal{A}_q\| \mid q \in Q\}$. Since

351 $r = \|\mathcal{A}\| = \sum_{q \in Q} \text{in}(q) \|\mathcal{A}_q\|$, we get $r \in N$. Note that, for $a \in \Sigma$ and $p \in Q$, we have

$$a^{-1} \|\mathcal{A}_p\| = \sum_{q \in Q} \text{wt}(p, a, q) \|\mathcal{A}_q\|$$

352 which allows us to prove by simple calculations that N is stable.

Conversely, let N be finitely generated by $\{r_1, \dots, r_n\}$ and stable and let $r \in N$. For all $a \in \Sigma$ and $1 \leq i \leq n$, we have $a^{-1}r_i = \sum_{1 \leq j \leq n} s_{ij}r_j$ with suitable $s_{ij} \in S$. Then there exists a unique morphism $\mu: \Sigma^* \rightarrow S^{n \times n}$ with $\mu(a)_{ij} = s_{ij}$ for $a \in \Sigma$. By induction on the length of $w \in \Sigma^*$, we can show that $w^{-1}r_i = \sum_{1 \leq j \leq n} \mu(w)_{ij}r_j$. Hence

$$(r_i, w) = (w^{-1}r_i, \varepsilon) = \sum_{1 \leq j \leq n} \mu(w)_{ij}(r_j, \varepsilon).$$

353 Since $r \in N$, we have $r = \sum_{1 \leq i \leq n} \lambda_i r_i$ for some $\lambda_i \in S$. With $\gamma_j = (r_j, \varepsilon)$, we obtain

$$(r, w) = \sum_{1 \leq i, j \leq n} \lambda_i \cdot \mu(w)_{ij} \cdot \gamma_j = \lambda \cdot \mu(w) \cdot \gamma$$

354 showing that (λ, μ, γ) is a linear presentation of r . Hence r is recognizable by Theo-
355 rem 3.2. \square

356 Inductively, one can show that every rational series belongs to a finitely generated and
357 stable subsemimodule, cf. [11]. Together with the theorem above, this is an alternative
358 proof of the fact that every rational series is recognizable (cf. Theorem 4.8).

359 6 Nivat's theorem

360 Nivat's theorem [75] provides an insight into the concatenation of mappings and, as we
361 will see, recognizability of certain simple series. More precisely, it asserts that every
362 proper recognizable series $r \in S \langle\langle \Sigma^* \rangle\rangle$ can be decomposed into three particular rec-
363 ognizable series, namely an inverse monoid homomorphism $h^{-1}: \Sigma^* \rightarrow \mathcal{P}(\Gamma^*)$ with
364 $h: \Gamma^* \rightarrow \Sigma^*$, a recognizable "selection series" $\text{sel}: \Gamma^* \rightarrow \mathcal{P}(\Gamma^*)$ satisfying $(\text{sel}, v) \subseteq$
365 $\{v\}$, and a homomorphism $c: (\Gamma^*, \cdot, \varepsilon) \rightarrow (S, \cdot, 1)$. Conversely, assuming $h(a) \neq \varepsilon$ for
366 all $a \in \Gamma$, the composition of h^{-1} , sel , and c is recognizable.

367 A mapping $\text{sel}: \Gamma^* \rightarrow \mathcal{P}(\Gamma^*)$ is a *selection series* if $(\text{sel}, v) \subseteq \{v\}$ for all $v \in \Gamma^*$.
368 Let $\text{fin}(\Gamma^*)$ denote the set of all finite subsets of Γ^* . Then $(\text{fin}(\Gamma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ is a
369 (computable) subsemiring of $\mathcal{P}(\Gamma^*)$. For brevity, this subsemiring is denoted by $\text{fin}(\Gamma^*)$.

370 **Lemma 6.1.** (1) If $h: \Gamma^* \rightarrow \Sigma^*$ is a homomorphism with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, then
371 $h^{-1} \in \text{fin}(\Gamma^*) \langle\langle \Sigma^* \rangle\rangle$ with $(h^{-1}, w) = \{v \in \Gamma^* \mid h(v) = w\}$ is a recognizable
372 series.

373 (2) A selection series $\text{sel} \in \text{fin}(\Gamma^*) \langle\langle \Gamma^* \rangle\rangle$ is recognizable if and only if its support
374 $K = \{v \in \Gamma^* \mid v \in (\text{sel}, v)\}$ is regular.

375 (3) If $c: (\Gamma^*, \cdot, \varepsilon) \rightarrow (S, \cdot, 1)$ is a monoid homomorphism, then c is a recognizable
376 series in $S \langle\langle \Gamma^* \rangle\rangle$.

- 377 *Proof.* (1) Since $h(a) \neq \varepsilon$ for all letters $a \in \Gamma$, the set (h^{-1}, w) is indeed finite, i.e.,
 378 $h^{-1} \in \text{fin}(\Gamma^*) \langle\langle \Sigma^* \rangle\rangle$. Furthermore, this series equals $(\sum_{a \in \Gamma} \{a\} h(a))^*$ which is
 379 rational and therefore recognizable by Theorem 4.8. Alternatively, one observes
 380 that a weighted automaton with just one state suffices for this series where the a -
 381 transition gets weight $h^{-1}(a)$ for $a \in \Gamma$.
- 382 (2) We first prove the implication “ \Leftarrow ”. So let K be regular. Then, in an arbitrary finite
 383 automaton accepting K , weight any a -labeled transition with $\{a\}$ (for $a \in \Gamma$), and
 384 weight the initial and final states by $\{\varepsilon\}$. This gives a weighted automaton with
 385 behavior sel .
 386 The other direction follows from more general results on the support of recogniz-
 387 able series over positive semirings since $K = \text{supp}(\text{sel})$. A direct argument goes
 388 as follows: take a weighted automaton with behavior sel and delete all its weights
 389 (and all transitions with weight \emptyset). This results in a finite automaton that accepts
 390 the support of sel .
- 391 (3) This series is the behavior of a weighted automaton with just one state. \square

392 Next we show that morphisms and inverses of non-deleting morphisms preserve rec-
 393 ognizability which is also of independent interest.

394 **Lemma 6.2.** *Let $r \in S \langle\langle \Gamma^* \rangle\rangle$ be recognizable.*

- 395 (1) *If $h: \Sigma^* \rightarrow \Gamma^*$ is a homomorphism, then the series $r \circ h \in S \langle\langle \Sigma^* \rangle\rangle$ with $(r \circ h, w) =$
 396 $(r, h(w))$ is recognizable.*
- 397 (2) *If $h: \Gamma^* \rightarrow \Sigma^*$ is a homomorphism with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, then the series
 398 $r \circ h^{-1} \in S \langle\langle \Sigma^* \rangle\rangle$ with $(r \circ h^{-1}, w) = \sum_{v \in h^{-1}(w)} (r, v)$ is recognizable.*

399 Note that $h(a) \neq \varepsilon$ in the second statement implies $|h(v)| \geq |v|$. Hence, for any
 400 $w \in \Sigma^*$, there are only finitely many words v with $h(v) = w$. Hence the series is well-
 401 defined.

- 402 *Proof.* (1) If (λ, μ, γ) is a representation of r , then $\mu \circ h$ is a morphism and $(\lambda, \mu \circ h, \gamma)$
 403 represents $r \circ h$, as is easy to check.
- 404 (2) By Theorem 4.8, r is rational, and an inductive proof shows that $r \circ h^{-1}$ is rational,
 405 too. Hence it is recognizable by Theorem 4.8, again. \square

406 Next, if $c: \Gamma^* \rightarrow S$ is a mapping and $\text{sel}: \Gamma^* \rightarrow \text{fin}(\Gamma^*)$ is a selection series, then we
 407 define the series $c \circ \text{sel}: \Gamma^* \rightarrow S$ by

$$(c \circ \text{sel}, v) = \begin{cases} c(v) & \text{if } (\text{sel}, v) = \{v\} \\ 0 & \text{otherwise.} \end{cases}$$

408 **Theorem 6.3** (cf. Nivat [75]). *Let S be a semiring, Σ an alphabet, and $r \in S \langle\langle \Sigma^* \rangle\rangle$
 409 with $(r, \varepsilon) = 0$. Then r is recognizable if and only if there exist an alphabet Γ , a
 410 homomorphism $h: \Gamma^* \rightarrow \Sigma^*$ with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, a recognizable selection
 411 series $\text{sel} \in \text{fin}(\Gamma^*) \langle\langle \Gamma^* \rangle\rangle$, and a homomorphism $c: (\Gamma^*, \cdot, \varepsilon) \rightarrow (S, \cdot, 1)$ such that
 412 $r = c \circ \text{sel} \circ h^{-1}$.*

413 *Proof.* We first prove the implication “ \Leftarrow ”. Let $K = \text{supp}(\text{sel})$. By Lemma 6.1(2), K
 414 is regular. Note that $c \circ \text{sel} = c \odot \mathbb{1}_K$. Hence $c \circ \text{sel}$ is recognizable by Corollary 4.4.
 415 Therefore, $c \circ \text{sel} \circ h^{-1}$ is recognizable by Lemma 6.2(2).

Conversely, let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be a weighted automaton with $r = \|\mathcal{A}\|$. Set

$$\Gamma = (Q \uplus Q \times \{1\}) \times \Sigma \times (Q \uplus Q \times \{2\}),$$

$$h(p', a, q') = a, \text{ and}$$

$$c(p', a, q') = \begin{cases} \text{wt}(p', a, q') & \text{if } p', q' \in Q \\ \text{in}(p) \cdot \text{wt}(p, a, q') & \text{if } p' = (p, 1), q' \in Q \\ \text{wt}(p', a, q) \cdot \text{out}(q) & \text{if } p' \in Q, q' = (q, 2) \\ \text{in}(p) \cdot \text{wt}(p, a, q) \cdot \text{out}(q) & \text{if } p' = (p, 1), q' = (q, 2) \\ 0 & \text{otherwise} \end{cases}$$

416 for $(p', a, q') \in \Gamma$. Furthermore, let K be the set of words

$$((p_0, 1), a_1, p_1)(p_1, a_2, p_2) \dots (p_{n-1}, a_n, (p_n, 2))$$

417 with $p_i \in Q$ for all $0 \leq i \leq n$. Then K is regular and corresponds to the set of paths in \mathcal{A} .
 418 This allows us to prove $(r, w) = (\|\mathcal{A}\|, w) = \sum_{v \in h^{-1}(w) \cap K} c(v)$, i.e., $r = c \circ \text{sel}_K \circ h^{-1}$
 419 with $\text{sel}_K(v) = \{v\} \cap K$. But sel_K is recognizable by Lemma 6.1(2). \square

420 7 Weighted monadic second order logic

421 Fundamental results by Büchi, by Elgot and by Trakhtenbrot [18, 39, 92] state that a
 422 language is regular if and only if it is definable in monadic second order (MSO) logic.
 423 Here, we wish to extend this result to a quantitative setting and thereby obtain a further
 424 characterization of the recognizability of a series $r: \Sigma^* \rightarrow S$, using a weighted version
 425 of monadic second order logic. We follow [26, 28].

426 We will enrich MSO-logic by permitting all elements of S as atomic formulas. The
 427 semantics of a sentence from the weighted MSO-logic will be a series in $S \langle\langle \Sigma^* \rangle\rangle$. In
 428 general, this weighted MSO-logic is more expressive than weighted automata. But a
 429 suitable, syntactically defined restriction of the logic, which contains classical MSO-logic,
 430 has the same expressive power as weighted automata.

431 For the convenience of the reader we will recall basic background of classical MSO-
 432 logic, cf. [91, 57]. Let Σ be an alphabet. The syntax of formulas of $\text{MSO}(\Sigma)$, the monadic
 433 second order logic over Σ , is usually given by the grammar

$$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \vee \psi \mid \exists x.\varphi \mid \exists X.\varphi$$

434 where $a \in \Sigma$, x, y are first-order variables, and X is a set variable. We let $\text{Free}(\varphi)$ denote
 435 the set of all free variables of φ .

436 As usual, a word $w = a_1 \dots a_n \in \Sigma^*$ is represented by the relational structure
 437 $(\text{dom}(w), \leq, (R_a)_{a \in \Sigma})$ where $\text{dom}(w) = \{1, \dots, n\}$, \leq is the usual order on $\text{dom}(w)$
 438 and $R_a = \{i \in \text{dom}(w) \mid a_i = a\}$ for $a \in \Sigma$.

439 Let \mathcal{V} be a finite set of first-order or second-order variables. A (\mathcal{V}, w) -assignment

440 σ is a function mapping first-order variables in \mathcal{V} to elements of $\text{dom}(w)$ and second-
 441 order variables in \mathcal{V} to subsets of $\text{dom}(w)$. For a first-order variable x and $i \in \text{dom}(w)$,
 442 $\sigma[x \mapsto i]$ denotes the $(\mathcal{V} \cup \{x\}, w)$ -assignment which maps x to i and coincides with σ
 443 otherwise. Similarly, $\sigma[X \mapsto I]$ is defined for $I \subseteq \text{dom}(w)$. For $\varphi \in \text{MSO}(\Sigma)$ with
 444 $\text{Free}(\varphi) \subseteq \mathcal{V}$, the satisfaction relation $(w, \sigma) \models \varphi$ is defined as usual.

445 Subsequently, we will encode a pair (w, σ) as above as a word over the extended
 446 alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0, 1\}^{\mathcal{V}}$ (with $\Sigma_{\emptyset} = \Sigma$). We write a word $(a_1, \sigma_1) \dots (a_n, \sigma_n)$ over
 447 $\Sigma_{\mathcal{V}}$ as (w, σ) where $w = a_1 \dots a_n$ and $\sigma = \sigma_1 \dots \sigma_n$. We call (w, σ) *valid*, if it is empty
 448 or if for each first order variable $x \in \mathcal{V}$, there is a unique position i with $\sigma_i(x) = 1$. In
 449 this case, we identify σ with the (\mathcal{V}, w) -assignment that maps each first order variable x
 450 to the unique position i with $\sigma_i(x) = 1$ and each set variable X to the set of positions i
 451 with $\sigma_i(X) = 1$. Clearly the language

$$N_{\mathcal{V}} = \{(w, \sigma) \in \Sigma_{\mathcal{V}}^* \mid (w, \sigma) \text{ is valid}\}$$

452 is recognizable (here and later we write $\Sigma_{\mathcal{V}}^*$ for $(\Sigma_{\mathcal{V}})^*$). If $\text{Free}(\varphi) \subseteq \mathcal{V}$, we let

$$L_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}.$$

453 We simply write $\Sigma_{\varphi} = \Sigma_{\text{Free}(\varphi)}$, $N_{\varphi} = N_{\text{Free}(\varphi)}$, and $L(\varphi) = L_{\text{Free}(\varphi)}(\varphi)$.

454 By the Büchi-Elgot-Trakhtenbrot theorem [18, 39, 92], a language $L \subseteq \Sigma^*$ is regular
 455 if and only if it is definable by some MSO-sentence. In the proof of the implication \Rightarrow ,
 456 given an automaton, one constructs directly an MSO-sentence that defines the language
 457 of the automaton. For the other implication, one shows inductively the stronger fact that
 458 $L_{\mathcal{V}}(\varphi)$ is regular for each formula φ (where $\text{Free}(\varphi) \subseteq \mathcal{V}$). Our goal is to proceed
 459 similarly in the present weighted setting.

460 We start by defining the syntax of our weighted MSO-logic as in [26, 28] but we
 461 include arbitrary negation here.

Definition 7.1. The syntax of formulas of the *weighted MSO-logic* over S and Σ is given
 by the grammar

$$\begin{aligned} \varphi ::= & s \mid P_a(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \\ & \mid \exists x. \varphi \mid \forall x. \varphi \mid \exists X. \varphi \mid \forall X. \varphi \end{aligned}$$

462 where $s \in S$ and $a \in \Sigma$. We let $\text{MSO}(S, \Sigma)$ be the collection of all such weighted
 463 MSO-formulas φ .

464 Next we define the \mathcal{V} -semantics of formulas $\varphi \in \text{MSO}(S, \Sigma)$ as a series $\llbracket \varphi \rrbracket_{\mathcal{V}} : \Sigma_{\mathcal{V}}^* \rightarrow$
 465 S .

466 **Definition 7.2.** Let $\varphi \in \text{MSO}(S, \Sigma)$ and \mathcal{V} be a finite set of variables with $\text{Free}(\varphi) \subseteq \mathcal{V}$.
 467 The \mathcal{V} -semantics of φ is the series $\llbracket \varphi \rrbracket_{\mathcal{V}} \in S \langle\langle \Sigma_{\mathcal{V}}^* \rangle\rangle$ defined as follows. Let $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$.
 468 If (w, σ) is not valid, we put $\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma) = 0$. If (w, σ) with $w = a_1 \dots a_n$ is valid, we
 469 define $\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma) \in S$ inductively as in Table 1. Note that the product $\prod_{i \in \text{dom}(w)}$ is
 470 calculated following the natural order of the position in w . For the product $\prod_{X \subseteq \text{dom}(w)}$,
 471 we use the lexicographic order on the powerset of $\text{dom}(w)$.

472 For brevity, we write $\llbracket \varphi \rrbracket$ for $\llbracket \varphi \rrbracket_{\text{Free}(\varphi)}$. Note that if φ is a sentence, i.e. $\text{Free}(\varphi) = \emptyset$,
 473 then $\llbracket \varphi \rrbracket \in S \langle\langle \Sigma^* \rangle\rangle$.

Table 1. MSO(S, Σ) semantics

φ	$\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma)$	φ	$\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma)$
s	s	$\psi \vee \varrho$	$\llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma) + \llbracket \varrho \rrbracket_{\mathcal{V}}(w, \sigma)$
$P_a(x)$	$\begin{cases} 1 & \text{if } a_{\sigma(x)} = a \\ 0 & \text{otherwise} \end{cases}$	$\psi \wedge \varrho$	$\llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma) \cdot \llbracket \varrho \rrbracket_{\mathcal{V}}(w, \sigma)$
$x \leq y$	$\begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$	$\exists x. \psi$	$\sum_{i \in \text{dom}(w)} \llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma[x \mapsto i])$
$x \in X$	$\begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$	$\forall x. \psi$	$\prod_{i \in \text{dom}(w)} \llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma[x \mapsto i])$
$\neg \psi$	$\begin{cases} 1 & \text{if } \llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma) = 0 \\ 0 & \text{otherwise} \end{cases}$	$\exists X. \psi$	$\sum_{I \subseteq \text{dom}(w)} \llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma[X \mapsto I])$
		$\forall X. \psi$	$\prod_{I \subseteq \text{dom}(w)} \llbracket \psi \rrbracket_{\mathcal{V}}(w, \sigma[X \mapsto I])$

474 Similar definitions of the semantics occur in multivalued logic, cf. [51, 50]. In par-
475 ticular, a similar definition of the semantics of negated formulas is also used for Gödel
476 logics. We give several examples of possible interpretations of weighted formulas:

- 477 (1) Let S be an arbitrary bounded distributive lattice $(S, \vee, \wedge, 0, 1)$ with smallest el-
478 ement 0 and largest element 1. In this case, sums correspond to suprema, and
479 products to infima. For instance, we have $\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket$ for sentences φ, ψ .
480 Thus our logic may be interpreted as a multi-valued logic. In particular, if $S = \mathbb{B}$,
481 the 2-valued Boolean algebra, our semantics coincides with the usual semantics of
482 unweighted MSO-formulas, identifying characteristic series with their supports.
- 483 (2) The formula $\exists x. P_a(x)$ counts how often a occurs in the word. Here, *how often*
484 depends on the semiring: e.g., natural numbers, Boolean semiring, integers modulo
485 2, ...
- 486 (3) Let $S = (\mathbb{N}, +, \cdot, 0, 1)$ and assume φ does not contain constants $s \in \mathbb{N}$ and negation
487 is applied only to atomic formulas $P_a(x)$, $x \leq y$, or $x \in X$. Then $\llbracket \varphi \rrbracket(w, \sigma)$ gives
488 the number of ways a machine could present to show that $(w, \sigma) \models \varphi$. Indeed,
489 the machine could proceed inductively over the structure of φ . For the atomic
490 subformulas and their negations, the number should be 1 or 0 depending on whether
491 the formula holds or not. Now, if $\llbracket \varphi \rrbracket(w, \sigma) = m$ and $\llbracket \psi \rrbracket(w, \sigma) = n$, the number
492 for $\llbracket \varphi \vee \psi \rrbracket(w, \sigma)$ should be $m + n$ (since any reason for φ or ψ suffices), and for
493 $\llbracket \varphi \wedge \psi \rrbracket(w, \sigma)$ it should be $m \cdot n$ (since the machine could pair the reasons for φ
494 resp. ψ arbitrarily). Similarly, the machine could deal with existential and universal
495 quantifications.
- 496 (4) The semiring $S = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ is often used for settings with costs

497 or rewards as weights. For the semantics of formulas, a choice like in a disjunction
 498 or existential quantification is resolved by maximum. Conjunction is resolved by
 499 a sum of the costs, and $\forall x.\varphi$ can be interpreted by the sum of the costs from all
 500 positions x .

- 501 (5) Consider the reliability semiring $S = ([0, 1], \max, \cdot, 0, 1)$ and $\Sigma = \{a_1, \dots, a_n\}$.
 502 Assume that every letter a_i has a reliability $p_i \in [0, 1]$. Let $\varphi = \forall x. \bigvee_{i=1}^n (P_{a_i}(x) \wedge$
 503 $p_i)$. Then $(\llbracket \varphi \rrbracket, w)$ can be considered as the reliability of the word $w \in \Sigma^*$.
 504 (6) PCTL is a well-studied probabilistic extension of computational tree logic CTL
 505 that is applied in verification. As shown recently in [12], PCTL can be considered
 506 as a fragment of weighted MSO logic.

507 The following basic consistency property of the semantics definition can be shown by
 508 induction over the structure of the formula using also Lemma 6.2.

509 **Proposition 7.1.** *Let $\varphi \in \text{MSO}(S, \Sigma)$ and \mathcal{V} be a finite set of variables with $\text{Free}(\varphi) \subseteq$
 510 \mathcal{V} . Then*

$$\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma) = \llbracket \varphi \rrbracket(w, \sigma|_{\text{Free}(\varphi)})$$

511 *for each valid $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$. Also, the series $\llbracket \varphi \rrbracket$ is recognizable iff $\llbracket \varphi \rrbracket_{\mathcal{V}}$ is recognizable.*

512 Our goal is to compare the expressive power of suitable fragments of $\text{MSO}(S, \Sigma)$
 513 with weighted automata. Crucial for this will be closure properties of recognizable series
 514 under the constructs of our weighted logic. In general, neither negation, conjunction, or
 515 universal quantification preserves recognizability.

516 **Example 7.1.** Let $S = (\mathbb{Z}, +, \cdot, 0, 1)$ be the ring of integers and consider the sentence

$$\varphi = \exists x. P_a(x) \vee ((-1) \wedge \exists x. P_b(x)) .$$

517 Then $(\llbracket \varphi \rrbracket, w)$ is the difference of the numbers of occurrences of a and b in w . Note that
 518 $(\llbracket \neg \varphi \rrbracket, w) = 1$ if and only if these numbers are equal, so $\llbracket \neg \varphi \rrbracket = \mathbb{1}_L$ for a non-regular
 519 language L . Therefore $\llbracket \neg \varphi \rrbracket$ is not recognizable (see Theorem 9.2 below).

520 **Example 7.2.** Let $\Sigma = \{a, b\}$, $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, and $\varphi = \forall x. ((P_a(x) \wedge \{a\}) \vee$
 521 $(P_b(x) \wedge \{b\}))$. With r the series from Example 4.2, $\llbracket \varphi \rrbracket = r$ which is recognizable. On
 522 the other hand, $\llbracket \varphi \wedge \varphi \rrbracket = r \odot r$ is not recognizable.

523 **Example 7.3.** Let $S = (\mathbb{N}, +, \cdot, 0, 1)$. Then $(\llbracket \exists x.1 \rrbracket, w) = |w|$ and $(\llbracket \forall y. \exists x.1 \rrbracket, w) =$
 524 $|w|^{|w|}$ for each $w \in \Sigma^*$. So $\llbracket \exists x.1 \rrbracket$ is recognizable, but $\llbracket \forall y. \exists x.1 \rrbracket$ is not recogniz-
 525 able. Indeed, let $\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})$ be any weighted automaton over S . Let $M =$
 526 $\max\{\text{in}(p), \text{out}(p), \text{wt}(p, a, q) \mid p, q \in Q, a \in \Sigma\}$. Then $(\|\mathcal{A}\|, w) \leq |Q|^{|w|+1} \cdot M^{|w|+2}$
 527 for each $w \in \Sigma^*$, showing $\|\mathcal{A}\| \neq \llbracket \forall y. \exists x.1 \rrbracket$. Similarly, $(\llbracket \forall X.2 \rrbracket, w) = 2^{2^{|w|}}$ for each
 528 $w \in \Sigma^*$, and $\llbracket \forall X.2 \rrbracket$ is not recognizable due to its growth.

529 These examples lead us to consider fragments of $\text{MSO}(S, \Sigma)$. As in [12], we define
 530 the syntax of *Boolean formulas* of $\text{MSO}(S, \Sigma)$ by

$$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg \varphi \mid \varphi \wedge \varphi \mid \forall x. \varphi \mid \forall X. \varphi$$

531 where $a \in \Sigma$. Note that in comparison to the syntax of $\text{MSO}(\Sigma)$, we only replaced
 532 disjunction by conjunction and existential by universal quantifications. Now, clearly,
 533 $\llbracket \varphi \rrbracket_{\mathcal{V}}(w, \sigma) \in \{0, 1\}$ for each Boolean formula φ and $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$ if $\text{Free}(\varphi) \subseteq \mathcal{V}$.
 534 Expressing disjunction and existential quantifications by negation and conjunction resp.
 535 universal quantifications, for each $\varphi \in \text{MSO}(\Sigma)$ there is a Boolean formula ψ such
 536 that $\llbracket \psi \rrbracket = \mathbb{1}_{L(\varphi)}$, and conversely. Hence Boolean formulas capture the full power of
 537 $\text{MSO}(\Sigma)$.

538 Now the class of *almost unambiguous formulas* of $\text{MSO}(S, \Sigma)$ is the smallest class
 539 containing all constants $s \in S$ and all Boolean formulas which is closed under disjunction,
 540 conjunction, and negation.

541 It is useful to introduce the closely related notion of recognizable step functions: these
 542 are precisely the finite sums of series $s \mathbb{1}_L$ where $s \in S$ and $L \subseteq \Sigma^*$ is regular. By
 543 induction it follows that $\llbracket \varphi \rrbracket$ is a recognizable step function for any almost unambiguous
 544 formula $\varphi \in \text{MSO}(S, \Sigma)$. Conversely, if $r: \Sigma^* \rightarrow S$ is a recognizable step function,
 545 by the Büchi-Elgot-Trakhtenbrot theorem, we obtain an almost unambiguous sentence φ
 546 with $r = \llbracket \varphi \rrbracket$.

547 For $\varphi \in \text{MSO}(S, \Sigma)$, let $\text{const}(\varphi)$ be the set of all elements of S occurring in φ . We
 548 recall that two subsets $A, B \subseteq S$ commute, if $a \cdot b = b \cdot a$ for all $a \in A, b \in B$.

549 **Definition 7.3.** A formula $\varphi \in \text{MSO}(S, \Sigma)$ is *syntactically restricted*, if it satisfies the
 550 following conditions:

- 551 (1) for all subformulas $\psi \wedge \psi'$ of φ , the sets $\text{const}(\psi)$ and $\text{const}(\psi')$ commute or ψ or
 552 ψ' is almost unambiguous,
- 553 (2) whenever φ contains a subformula $\forall x.\psi$ or $\neg\psi$, then ψ is almost unambiguous,
- 554 (3) whenever φ contains a subformula $\forall X.\psi$, then ψ is Boolean.

555 We let $\text{srMSO}(S, \Sigma)$ denote the collection of all syntactically restricted formulas from
 556 $\text{MSO}(S, \Sigma)$.

557 Also, a formula $\varphi \in \text{MSO}(S, \Sigma)$ is called *existential*, if it has the form $\exists X_1. \dots \exists X_n.\psi$
 558 where ψ contains only first order quantifiers.

559 **Theorem 7.2** (Droste and Gastin [28]). *Let S be any semiring, Σ an alphabet, and*
 560 *$r: \Sigma^* \rightarrow S$ a series. The following are equivalent:*

- 561 (1) r is recognizable.
- 562 (2) $r = \llbracket \varphi \rrbracket$ for some syntactically restricted and existential sentence φ of $\text{MSO}(S, \Sigma)$.
- 563 (3) $r = \llbracket \varphi \rrbracket$ for some syntactically restricted sentence φ of $\text{MSO}(S, \Sigma)$.

564 *Proof (sketch).* (1) \rightarrow (2): We have $r = \|\mathcal{A}\|$ for some weighted automaton $\mathcal{A} =$
 565 $(Q, \text{in}, \text{wt}, \text{out})$. Then we can use the structure of \mathcal{A} to define a sentence φ as required
 566 such that $\|\mathcal{A}\| = \llbracket \varphi \rrbracket$.

567 (2) \rightarrow (3): Trivial.

568 (3) \rightarrow (1): By structural induction we show for each formula $\varphi \in \text{srMSO}(S, \Sigma)$ that
 569 $\llbracket \varphi \rrbracket = \|\mathcal{A}\|$ for some weighted automaton \mathcal{A} over Σ_{φ} and S_{φ} where $S_{\varphi} = \langle \text{const}(\varphi) \rangle$ is
 570 the subsemiring of S generated by the set $\text{const}(\varphi)$. For Boolean formulas, this is easy.
 571 For disjunction and existential quantification, we use closure properties of the class of rec-
 572 ognizable series. For conjunction, the assumption of Definition 7.3(1) and the particular

573 induction hypothesis allow us to employ the construction from Lemma 4.3. If $\varphi = \forall x.\psi$
 574 where ψ is almost unambiguous, we can use the description of $\llbracket\psi\rrbracket$ as a recognizable step
 575 function to construct a weighted automaton with the behavior $\llbracket\varphi\rrbracket$. \square

576 Note that the case $\varphi = \forall x.\psi$ requires a crucial new construction of weighted au-
 577 tomata which does not occur in the unweighted setting since, in general, we cannot reduce
 578 (weighted) universal quantification to existential quantification.

579 A semiring S is *locally finite* if each finitely generated subsemiring is finite. Examples
 580 include any bounded distributive lattice, thus in particular all Boolean algebras and the
 581 semiring $([0, 1], \max, \min, 0, 1)$. Another example is given by $([0, 1], \min, \oplus, 1, 0)$ with
 582 $x \oplus y = \min(1, x + y)$.

583 We call a formula $\varphi \in \text{MSO}(S, \Sigma)$ *weakly existential*, if whenever φ contains a sub-
 584 formula $\forall X.\psi$, then ψ is Boolean.

585 **Theorem 7.3** (Droste and Gastin [26, 28]). *Let S be locally finite and $r: \Sigma^* \rightarrow S$ a*
 586 *series. The following are equivalent:*

- 587 (1) r is recognizable.
- 588 (2) $r = \llbracket\varphi\rrbracket$ for some weakly existential sentence φ of $\text{MSO}(S, \Sigma)$.

589 *If moreover, S is commutative, these conditions are equivalent to the following one:*

- 590 (3) $r = \llbracket\varphi\rrbracket$ for some sentence φ of $\text{MSO}(S, \Sigma)$.

591 The proof uses the fact that if S is locally finite, then each recognizable series $r \in$
 592 $S \langle\langle \Sigma^* \rangle\rangle$ can be shown to be a recognizable step function.

593 Observe that Theorem 7.3 applies to all bounded distributive lattices and to all fi-
 594 nite semirings; in particular, with $S = \mathbb{B}$ it contains our starting point, the Büchi-Elgot-
 595 Trakhtenbrot theorem, as a very special case.

596 Given a syntactically restricted formula φ of $\text{MSO}(S, \Sigma)$, by the proofs of Theo-
 597 rem 7.2 we can *construct* a weighted automaton \mathcal{A} such that $\|\mathcal{A}\| = \llbracket\varphi\rrbracket$ (provided the
 598 operations of the semiring S are given in an effective way, i.e., S is *computable*). Since
 599 the equivalence problem for weighted automata over computable fields is decidable by
 600 Corollary 8.4 below, we obtain:

601 **Corollary 7.4.** *Let S be a computable field. Then the equivalence problem whether $\llbracket\varphi\rrbracket =$
 602 $\llbracket\psi\rrbracket$ for syntactically restricted sentences φ, ψ of $\text{MSO}(S, \Sigma)$ is decidable.*

603 In contrast, the equivalence problem for weighted automata is undecidable for the
 604 semirings $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ (Theorem 8.6).
 605 Since the proof of Theorem 7.2 is effective, for these semirings also the equivalence prob-
 606 lem for syntactically restricted sentences of $\text{MSO}(S, \Sigma)$ is undecidable.

607 8 Decidability of “ $r_1 = r_2$?”

608 In this section, we investigate when it is decidable whether two given recognizable series
 609 are equal. For this, we assume S to be a computable semiring, i.e., the underlying set of

610 S forms a decidable set and addition and multiplication can be performed effectively. In
 611 the first part, we fix one of the two series to be the constant series with value 0.

612 Let $P = (\lambda, \mu, \gamma)$ be a linear presentation of dimension Q of the series $r \in S \langle\langle \Sigma^* \rangle\rangle$.
 613 For $n \in \mathbb{N}$, let $U_n^P = \langle \{\lambda\mu(w) \mid w \in \Sigma^*, |w| \leq n\} \rangle$ and $U^P = \langle \{\lambda\mu(w) \mid w \in \Sigma^*\} \rangle$,
 614 so U_n^P and U^P are subsemimodules of $S^{\{1\} \times Q}$. Then $U_0^P \subseteq U_1^P \subseteq U_2^P \cdots \subseteq U^P =$
 615 $\bigcup_{n \in \mathbb{N}} U_n^P$, and each of the semimodules U_n^P is finitely generated.

616 **Lemma 8.1.** *The set of all pairs (P, n) such that P is a linear presentation and $U_n^P =$
 617 U_{n+1}^P is recursively enumerable (here, the homomorphism μ from the presentation P is
 618 given by its restriction to Σ).*

619 *Proof.* Note that $U_n^P = U_{n+1}^P$ if and only if every vector $\lambda\mu(w)$ with $|w| = n+1$ belongs
 620 to U_n^P if and only if for each $w \in \Sigma^*$ of length $n+1$,

$$\lambda\mu(w) = \sum_{\substack{v \in \Sigma^* \\ |v| \leq n}} s_v \lambda\mu(v)$$

621 for some $s_v \in S$. A non-deterministic Turing-machine can check the solvability of this
 622 equation by just guessing the coefficients s_v and checking the required equality. \square

623 **Corollary 8.2.** *Assume that, for any linear presentation P , U^P is a finitely generated
 624 semimodule. Then, from a linear presentation P of dimension Q , one can compute $n \in \mathbb{N}$
 625 with $U_n^P = U^P$ and finitely many vectors $x_1, \dots, x_m \in S^{\{1\} \times Q}$ with $\langle \{x_1, \dots, x_m\} \rangle =$
 626 U^P .*

627 *Proof.* Since U^P is finitely generated, there is some $n \in \mathbb{N}$ such that $U^P = U_n^P$ and
 628 therefore $U_n^P = U_{n+1}^P$. Hence, for some $n \in \mathbb{N}$, the pair (P, n) appears in the list from
 629 the previous lemma. Then $U^P = U_n^P = \langle \{\lambda\mu(v) \mid v \in \Sigma^*, |v| \leq n\} \rangle$. \square

630 Clearly, every finite semiring satisfies the condition of the corollary above, but not all
 631 semirings do.

632 **Example 8.1.** Let S be the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ and consider a presentation P with

$$\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix} \text{ and } \mu(w) = \begin{pmatrix} 1 & |w| \\ 0 & 1 \end{pmatrix}.$$

633 Then U_n^P is generated by all the vectors $(1 \ m)$ for $0 \leq m \leq n$ so that $(1 \ n+1) \in$
 634 $U_{n+1}^P \setminus U_n^P$; hence U^P is not finitely generated.

635 As a positive example, we have the following.

636 **Example 8.2.** If S is a skew-field (i.e., a semiring such that $(S, +, 0)$ and $(S \setminus \{0\}, \cdot, 1)$ are
 637 groups), then we can consider U_n^P as a vector space. Then the dimensions of the spaces
 638 $U_i^P \subseteq S^{\{1\} \times Q}$ are bounded by $|Q|$ and $\dim(U_i^P) \leq \dim(U_{i+1}^P)$ implying $U_{|Q|}^P = U^P$.
 639 Hence, for any skew-field S , in the corollary above we can set $n = |Q|$.

640 We only note that all Noetherian rings (that include all polynomial rings in several
 641 indeterminates over fields, by Hilbert's basis theorem) satisfy the assumption of Corol-
 642 lary 8.2.

643 **Theorem 8.3** (Schützenberger [85]). *Let S be a computable semiring such that, for any*
 644 *linear presentation P , U^P is a finitely generated semimodule. Then, for a linear presen-*
 645 *tation P , one can decide whether $\|P\| = 0$.*

646 *Proof.* We have to decide whether $y\gamma = 0$ for all vectors $y \in U^P$. By the previous
 647 lemma, we can compute a finite list x_1, \dots, x_m of vectors that generate U . So one only
 648 has to check whether $x_i\gamma = 0$ for $1 \leq i \leq m$. \square

649 **Example 8.3.** If S is a skew-field, a basis of U^P can be obtained in time $|\Sigma| \cdot |Q|^3$
 650 (where the operations in the skew-field S are assumed to require constant time). The
 651 algorithm actually computes a prefix-closed set of words $u_1, \dots, u_{\dim(U^P)}$ such that the
 652 vectors $\lambda\mu(u_i)$ form a basis of U^P (cf. [83]). This basis consists of at most $|Q|$ vectors
 653 (cf. Example 8.2), each of size $|Q|$. Hence $\|P\| = 0$ can be decided in time $|\Sigma||Q|^3$.

654 If S is a finite semiring, then $U^P = U_{|S|^{|Q|}}^P$. Hence the vectors $\lambda\mu(w)$ with $|w| \leq |S|^{|Q|}$
 655 form a generating set. To check whether $\lambda\mu(w)\gamma = 0$ for all such words w , time $|\Sigma|^{|S|^{|Q|}}$
 656 suffices. Within the same time bound, one can decide whether $\|P\| = 0$ holds.

657 **Corollary 8.4.** *Let S be a computable ring such that, for any linear presentation P ,*
 658 *U^P is a finitely generated semimodule. Then one can decide for two linear presentations*
 659 *P_1 and P_2 whether $\|P_1\| = \|P_2\|$.*

660 *Proof.* Since S is a ring, there is an element $-1 \in S$ with $x + (-1) \cdot x = 0$ for any $x \in S$.
 661 Replacing the initial vector λ from P_2 by $-\lambda$, one obtains a linear presentation for the
 662 series $(-1)\|P_2\|$. This yields a linear presentation P with $\|P\| = \|P_1\| + (-1)\|P_2\|$.
 663 Now $\|P_1\| = \|P_2\|$ if and only if $\|P\| = 0$ which is decidable by Theorem 8.3. \square

664 **Remark 8.5.** Let n_1 and n_2 be the dimensions of P_1 and P_2 , respectively. Then the linear
 665 presentation P from the proof above can be computed in time $n_1 \cdot n_2$ and has dimension
 666 $n_1 + n_2$. If S is a skew-field, then we can therefore decide whether $\|P_1\| = \|P_2\|$ in time
 667 $|\Sigma|(n_1 + n_2)^3$.

668 Let S be a finite semiring. Then from $s \in S$ and weighted automata for $\|P_1\|$ and
 669 for $\|P_2\|$, one can construct automata accepting $\{w \in \Sigma^* \mid (\|P_i\|, w) = s\}$ for $i = 1, 2$.
 670 This allows us to decide $\|P_1\| = \|P_2\|$ in doubly exponential time. If S is a finite ring,
 671 this result follows also from the proof of the corollary above and Example 8.3.

672 However, the following result is in sharp contrast to Corollary 8.4. For two series r and s
 673 with values in $\mathbb{N} \cup \{-\infty\}$, we write $r \leq s$ if $(r, w) \leq (s, w)$ for all words w .

674 **Theorem 8.6** (cf. Krob [63]). *There is a series $r_{\text{good}}: \Sigma^* \rightarrow \mathbb{N} \cup \{-\infty\}$ such that*
 675 *the sets of weighted automata \mathcal{A} over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ with*
 676 *$\|\mathcal{A}\| = r_{\text{good}}$ (with $r_{\text{good}} \leq \|\mathcal{A}\|$, resp.) are undecidable.*

677 We remark that analogous statements hold for the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$
 678 (where $r_{\text{good}} \geq \|\mathcal{A}\|$ is undecidable). As a consequence, the equivalence problem
 679 of weighted automata over these two semirings is undecidable (this undecidability was
 680 shown by Krob). The original proof by Krob is rather involved reducing Hilbert's 10th

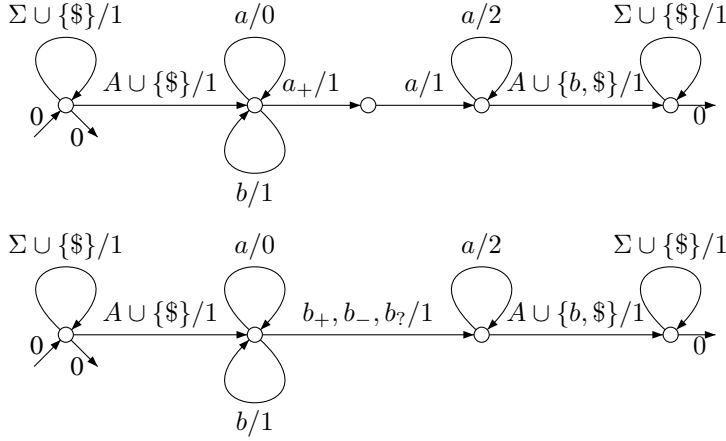


Figure 2. Automata for the proof of Theorem 8.6

681 problem to the equivalence problem. Colcombet presented a radical simplification at
 682 the Dagstuhl seminar “Advances and Applications of Automata on Words and Trees” in
 683 2010 starting from the undecidability of the question whether a 2-counter machine \mathcal{A}
 684 accepts the number 0 (this undecidable problem has also been used by Almagor, Boker
 685 and Kupferman in [3] to show the undecidabilities of the questions $\|\mathcal{A}\| = \|\mathcal{B}\|$ and
 686 $\|\mathcal{A}\| \leq \|\mathcal{B}\|$ for weighted automata over this semiring). The following is a slight exten-
 687 sion of Colcombet’s proof that he kindly allowed us to publish in this survey.

688 *Proof.* Let \mathcal{A} be a 2-counter machine, i.e., a nondeterministic finite automaton over the
 689 alphabet $A = \{a_+, a_-, a_?, b_+, b_-, b_?\}$. For a word $w \in (A \cup \{a, b, \$\})^*$, let $\pi_{\mathcal{A}}(w)$
 690 denote the projection onto A^* .

691 A counter trace is a word $w \in \$a^*(Aa^*b^*)^*\$$ such that $\pi_{\mathcal{A}}(w)$ is accepted by the
 692 finite automaton \mathcal{A} and, for any maximal factor of the form $a^m b^n c a^{m'} b^{n'}$ with $c \in A$,
 693 one of the following holds:

- $c = a_+, m' = m + 1$, and $n' = n$
- $c = a_-, m' + 1 = m$, and $n' = n$
- $c = a_?, m' = m = 0$, and $n' = n$
- $c = b_+, m' = m$, and $n' = n + 1$
- $c = b_-, m' = m$, and $n' + 1 = n$
- $c = b_?, m' = m$, and $n' = n = 0$

694 Then a number $m \in \mathbb{N}$ is accepted by the 2-counter machine \mathcal{A} if there exists a counter
 695 traces $w \in \$a^m(Aa^*b^*)^*\$$. By Minsky’s theorem, we can assume that the set of numbers
 696 m accepted by \mathcal{A} is undecidable. Let CT denote the set of all counter traces and let
 697 $\text{CT}_m = \text{CT} \cap \$a^m(Aa^*b^*)^*\$$ for $m \in \mathbb{N}$.

Note that no counter trace contains any factor from the following set:

$$\begin{aligned} & a_+(A \cup \{b, \$\}) \cup (A \cup \{\$\})b^*a_- \cup ab^*a_? \cup \{a_?a\} \\ & \cup b_+a^*(A \cup \{\$\}) \cup (A \cup \{\$, a\})b_- \cup b_?a^*b \cup \{bb_?\} \end{aligned}$$

699 Therefore, let $w \in \text{CT}_{\text{reg}}$ if $w \in \$a^*(Aa^*b^*)^*\$$ does not contain any such factor and
 700 if $\pi_{\mathcal{A}}(w)$ is accepted by the finite automaton \mathcal{A} (note that this set is regular). Furthermore,
 701 let $\text{CT}_{\text{reg}, m} = \text{CT}_{\text{reg}} \cap \$a^m(Aa^*b^*)^*\$$. We will now construct a recognizable series r

702 such that $(r, w) = |w|$ for $w \in \text{CT}$, $(r, w) > |w|$ for $w \in \text{CT}_{\text{reg}} \setminus \text{CT}$, and $(r, w) = -\infty$
 703 for $w \notin \text{CT}$.

704 Consider the first weighted automaton from Fig. 2 (where $\Sigma = A \cup \{a, b\}$). Its behav-
 705 ior maps a word $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$ to

$$\max\{|w|, |w| + \ell \mid \exists k \in \mathbb{N} : w \in \mathcal{A}(\Sigma^* A \cup \{\varepsilon\})a^k b^* a_+ a^{1+k+\ell}((A \cup \{b\})\Sigma^* \cup \{\varepsilon\})\mathcal{A}\}.$$

706 If we exchange the weights 0 and 2 at the two a -loops, the behavior for $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$
 707 yields

$$\max\{|w|, |w| + \ell \mid \exists k \in \mathbb{N} : w \in \mathcal{A}(\Sigma^* A \cup \{\varepsilon\})a^k b^* a_+ a^{1+k-\ell}((A \cup \{b\})\Sigma^* \cup \{\varepsilon\})\mathcal{A}\}.$$

708 By taking the union of these two weighted automata, we obtain a recognizable series r_{a_+}
 709 that maps $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$ to

$$\max\{|w|, |w| + \ell \mid \exists k \in \mathbb{N} : w \in \mathcal{A}(\Sigma^* A \cup \{\varepsilon\})a^k b^* a_+ a^{1+k \pm \ell}((A \cup \{b\})\Sigma^* \cup \{\varepsilon\})\mathcal{A}\}.$$

710 i.e., $(r_{a_+}, w) = |w|$ if and only if any maximal factor of w of the form $a^m b^* a_+ a^{1+n}$
 711 satisfies $m = n$ (and $(r_{a_+}, w) > |w|$ otherwise).

712 Similarly, one can construct a recognizable series r_{a_-} such that, for $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$,
 713 we have $(r_{a_-}, w) = |w|$ if and only if any maximal factor of w of the form $a^{1+m} b^* a_- a^n$
 714 satisfies $m = n$ (and $(r_{a_-}, w) > |w|$ otherwise).

715 Next consider the second weighted automaton from Fig. 2. Its behavior maps a word
 716 $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$ to

$$\max\{|w|, |w| + \ell \mid \exists k \in \mathbb{N} : w \in \mathcal{A}(\Sigma^* A \cup \{\varepsilon\})a^k b^* \{b_+, b_-, b_?\} a^{k+\ell}((A \cup \{b\})\Sigma^* \cup \{\varepsilon\})\mathcal{A}\}.$$

717 As above, we get a recognizable series r_a such that, for $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$, we have
 718 $(r_a, w) = |w|$ if and only if any maximal factor of w of the form $a^m b^* \{b_+, b_-, b_?\} a^n$
 719 satisfies $m = n$ (and $(r_a, w) > |w|$ otherwise).

720 Hence, there is a recognizable series r' such that, for a word $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$,
 721 we have $(r', w) = |w|$ if and only if any maximal factor of any of the following forms
 722 satisfies $m = n$:

$$\begin{array}{ccc} a^m b^* a_+ a^{1+n} & a^{1+m} b^* a_- a^n & a^m b^* \{b_+, b_-, b_?\} a^n \\ b^m b_+ a^* b^{1+n} & b^{1+m} b_- a^* b^n & b^m \{a_+, a_-, a_?\} a^* b^n \end{array}$$

723 For all other words $w \in \mathcal{A}^*(Aa^*b^*)^*\mathcal{A}$, we have $(r', w) > |w|$. From this series, we
 724 easily get the recognizable series $r = r' \odot \mathbb{1}_{\text{CT}_{\text{reg}}}$ satisfying

$$(r, w) \begin{cases} = |w| & \text{for } w \in \text{CT} \\ > |w| & \text{for } w \in \text{CT}_{\text{reg}} \setminus \text{CT} \\ = -\infty & \text{otherwise.} \end{cases}$$

725 Now define the recognizable series r_{good} and r_m for $m \in \mathbb{N}$ as follows:

$$(r_{\text{good}}, w) = \max(|w| + 1, (r, w)) \text{ and } (r_m, w) = \begin{cases} (r, w) & \text{for } w \in \text{CT}_{\text{reg}, m} \\ (r_{\text{good}}, w) & \text{otherwise.} \end{cases}$$

Then we have for $m \in \mathbb{N}$

$$\begin{aligned}
r_{\text{good}} = r_m &\iff (r_{\text{good}}, w) = (r_m, w) \text{ for all } w \in \text{CT}_{\text{reg}, m} \\
&\iff (r, w) > |w| \text{ for all } w \in \text{CT}_{\text{reg}, m} \\
&\iff \text{CT}_m = \emptyset \\
&\iff m \text{ is not accepted by the 2-counter machine } \mathcal{A}.
\end{aligned}$$

726 Since by our assumption on \mathcal{A} this last statement is undecidable, the first claim follows.

727 Note that $r_m \leq r_{\text{good}}$. Hence $r_m = r_{\text{good}}$ is equivalent to saying $r_{\text{good}} \leq r_m$.

728 Therefore the second claim holds as well. \square

729 9 Characteristic series and supports

730 The goal of this section is to investigate the regularity of the support of recognizable (char-
731 acteristic) series.

732 **Lemma 9.1.** *Let S be any semiring and $L \subseteq \Sigma^*$ a regular language. Then the charac-
733 teristic series $\mathbb{1}_L$ of L is recognizable.*

734 *Proof.* Take a deterministic finite automaton accepting L and weight the initial state, the
735 transitions, and the final states with 1 and all the non-initial states, the non-transitions,
736 and the non-final states with 0. Since every word has at most one successful path in the
737 deterministic finite automaton, the behavior of the weighted automaton constructed this
738 way is the characteristic series of L over S . \square

739 For all commutative semirings, also the converse of this lemma holds. This was first
740 shown for commutative rings where one actually has the following more general result:

741 **Theorem 9.2** (Schützenberger [85] and Sontag [89]). *Let S be a commutative ring, and
742 let $r \in S^{\text{rec}}\langle\langle \Sigma^* \rangle\rangle$ have finite image. Then $r^{-1}(s)$ is recognizable for any $s \in S$.*

743 It remains to consider commutative semirings that are not rings. Let S be a semiring.
744 A subset $I \subseteq S$ is called an *ideal*, if for all $a, b \in I$ and $s \in S$ we have $a + b, a \cdot s, s \cdot a \in$
745 I . Dually, a subset $F \subseteq S$ is called a *filter*, if for all $a, b \in F$ and $s \in S$ we have
746 $a \cdot b, s + a \in F$. Given a subset $A \subseteq S$, the smallest filter containing A is the set

$$F(A) = \{a_1 \cdots a_n + s \mid a_i \in A \text{ for } 1 \leq i \leq n, \text{ and } s \in S\}.$$

747 **Lemma 9.3** (Wang [94]). *Let S be a commutative semiring which is not a ring. Then
748 there is a semiring morphism onto \mathbb{B} .*

749 *Proof.* Consider the collection \mathcal{C} of all filters F of S with $0 \notin F$. Since S is not a ring,
750 we have $F(\{1\}) \in \mathcal{C}$. By Zorn's lemma, (\mathcal{C}, \subseteq) contains a maximal element M with
751 $F(\{1\}) \subseteq M$. We define $h: S \rightarrow \mathbb{B}$ by letting $h(s) = 1$ if $s \in M$, and $h(s) = 0$
752 otherwise. Clearly $h(0) = 0$ and $h(1) = 1$.

753 Now let $a, b \in S$. We claim that $h(a+b) = h(a) + h(b)$. By contradiction, we assume
 754 that $a, b \notin M$ but $a + b \in M$. Then $0 \in F(M \cup \{a\})$ and $0 \in F(M \cup \{b\})$. Since S is
 755 commutative, we have $0 = m \cdot a^n + s = m' \cdot b^{n'} + s'$ for some $m, m' \in M, n, n' \in \mathbb{N}$
 756 and $s, s' \in S$. This implies that $0 = m \cdot m' \cdot (a + b)^{n+n'} + s''$ for some $s'' \in S$. But now
 757 $a + b \in M$ implies $0 \in M$, a contradiction.

758 Finally, we claim that $h(a \cdot b) = h(a) \cdot h(b)$. If $a, b \in M$, then also $ab \in M$, showing
 759 our claim. Now assume $a \notin M$ but $ab \in M$. As above, we have $0 = m \cdot a^n + s$ for some
 760 $m \in M, n \in \mathbb{N}$, and $s \in S$. But then $0 = m \cdot a^n \cdot b^n + s \cdot b^n = m \cdot (ab)^n + sb^n \in M$ by
 761 $ab \in M$, a contradiction. \square

762 **Theorem 9.4** (Wang [94]). *Let S be a commutative semiring and $L \subseteq \Sigma^*$. Then L is*
 763 *regular iff $\mathbb{1}_L$ is recognizable.*

764 *Proof.* One implication is part of Lemma 9.1. Now assume that $\mathbb{1}_L$ is recognizable. If S
 765 is a ring, the result is immediate by Theorem 9.2. If S is not a ring, by Lemma 9.3 there
 766 is a semiring morphism h from S to \mathbb{B} . Let \mathcal{A} be a weighted automaton with $\|\mathcal{A}\| = \mathbb{1}_L$.
 767 In this automaton, replace all weights s by $h(s)$. The behavior of the resulting weighted
 768 automaton over the Boolean semiring \mathbb{B} is $\mathbb{1}_L \in \mathbb{B} \langle\langle \Sigma^* \rangle\rangle$. Hence L is regular. \square

769 Now we turn to supports of arbitrary recognizable series. Already for $S = \mathbb{Z}$, the
 770 ring of integers, such a language is not necessarily regular (cf. Example 7.1). One can
 771 characterize those semirings for which the support of any recognizable series is regular:

772 **Theorem 9.5** (Kirsten [59]). *For a semiring S , the following are equivalent:*

- 773 (1) *The support of every recognizable series over S is regular.*
 774 (2) *For any finitely generated semiring $S' \subseteq S$, there exists a finite semiring S_{fin} and*
 775 *a homomorphism $\eta: S' \rightarrow S_{\text{fin}}$ with $\eta^{-1}(0) = \{0\}$.*

776 It is not hard to see that positive (i.e., zero-sum- and zero-divisor-free) semirings like
 777 $(\mathbb{N}, +, \cdot, 0, 1)$ and locally finite semirings (like $(\mathbb{Z}/4\mathbb{Z})^\omega$ or bounded distributive lattices)
 778 satisfy condition (2) and therefore (1). By [60], also zero-sum-free commutative semi-
 779 rings like $\mathbb{N} \times \mathbb{N}$ satisfy condition (1) and therefore (2).

780 Given a semiring S , by Lemma 9.1, the class $\text{SR}(S)$ of all supports of recognizable
 781 series over S contains all regular languages. Closure properties of this class $\text{SR}(S)$ have
 782 been studied extensively, see e.g. [11]. A further result is the following.

783 **Theorem 9.6** (Restivo and Reutenauer [81]). *Let S be a field and $L \subseteq \Sigma^*$ a language*
 784 *such that L and its complement $\Sigma^* \setminus L$ both belong to $\text{SR}(S)$. Then L is regular.*

785 In contrast, we note the following result which was also observed by Kirsten:

786 **Theorem 9.7.** *There exists a semiring S such that $L \in \text{SR}(S)$ (and even $\mathbb{1}_L$ is recogniz-*
 787 *able) for any language L over any finite alphabet Σ .*

788 *Proof.* Let $\Gamma = \{a, b\}$ and $\Gamma_{\S} = \Gamma \cup \{\S\}$. Furthermore, let $\overline{\Gamma}_{\S} = \{\overline{\gamma} \mid \gamma \in \Gamma_{\S}\}$ be a
 789 disjoint copy of Γ_{\S} . The elements of the semiring S are the subsets of $\overline{\Gamma}_{\S}^* \Gamma_{\S}^*$ and the

790 addition of S is the union of these sets (with neutral element \emptyset). To define multiplication,
 791 let $L, M \in S$. Then $L \odot M$ consists of all words $uv \in \overline{\Gamma}_S^* \Gamma_S^*$ such that there exists a
 792 word $w \in \Gamma_S^*$ with $uw \in L$ and $\overline{w}^{\text{rev}}v \in M$. Alternatively, multiplication of L and M
 793 can be described as follows: concatenate any word from L with any word from M , delete
 794 any factors of the form $c\bar{c}$ for $c \in \Gamma_S$, and place the result into $L \odot M$ if and only if it
 795 belongs to $\overline{\Gamma}_S^* \Gamma_S^*$. For instance, we have

$$\begin{aligned} \{\overline{ab}\$\} \cdot \{\overline{\$a}, \overline{\$ba}, \overline{a}\} &= \{\overline{ab}\$\overline{\$a}, \overline{ab}\$\overline{\$ba}, \overline{ab}\$\overline{a}\} \text{ and} \\ \{\overline{ab}\$\} \odot \{\overline{\$a}, \overline{\$ba}, \overline{a}\} &= \{\overline{aba}, \overline{aa}\} \end{aligned}$$

796 since the above procedure, when applied to $\overline{ab}\$\overline{a}$ and \overline{a} , results in $\overline{ab}\$\overline{a} \notin \overline{\Gamma}_S^* \Gamma_S^*$. Then
 797 it is easily verified that $(S, \cup, \odot, \emptyset, \{\varepsilon\})$ is a semiring.

Now let $L \subseteq \Gamma^*$. Define the linear presentation $P = (\lambda, \mu, \gamma)$ of dimension 1 as follows:

$$\begin{aligned} \lambda_1 &= \{\$\} \odot L^{\text{rev}} \\ \mu(a)_{11} &= \{\overline{a}\} \text{ for } a \in \Gamma \\ \gamma_1 &= \{\overline{\$}\} \end{aligned}$$

798 For $v \in \Gamma^*$, one then obtains

$$(\|P\|, v) = \{\$\} \odot L^{\text{rev}} \odot \{\overline{v}\} \odot \{\overline{\$}\} = \begin{cases} \{\varepsilon\} & \text{if } v \in L \\ \emptyset & \text{otherwise.} \end{cases}$$

799 This proves that the characteristic series of L is recognizable for any $L \subseteq \Gamma^*$. To obtain
 800 this fact for any language $L \subseteq \Sigma^*$, let $h: \Sigma^* \rightarrow \Gamma^*$ be an injective homomorphism. Then

$$\mathbb{1}_L = \mathbb{1}_{h(L)} \circ h$$

801 which is recognizable by Lemma 6.2(1). □

802 An open problem is to characterize those (non-commutative) semirings S for which
 803 the support of every *characteristic* and recognizable series is regular.

804 10 Further results

805 Above, we could only touch on a few selected topics from the rich area of weighted
 806 automata. In this section, we wish to give pointers to many other research results and
 807 directions. For details as well as further topics, we refer the reader to the books [38,
 808 84, 66, 11, 82] and to the recent handbook [30] with extensive surveys including open
 809 problems.

810 **Recognizability** Some authors use linear presentations to define recognizable series [11].

811 The transition relation of weighted automata given in this chapter can alternatively be
 812 considered as a $Q \times Q$ -matrix whose entries are functions from Σ to S . A more general

813 approach is presented in [83, 82] where the entries are functions from Σ^* to S . Here, the
 814 free monoid Σ^* can even be replaced by an arbitrary monoid with a length function.

815 The surveys [40, 42, 43] contain an axiomatic treatment of iteration and weighted
 816 automata using the concept of Conway semirings (i.e., semirings equipped with a suitable
 817 $*$ -operation).

818 The abovementioned books contain many further properties of recognizable series
 819 including minimization, Fatou-properties, growth behavior, relationship to coding, and
 820 decidability and undecidability results.

821 The coincidence of aperiodic, starfree, and first-order definable languages [86, 73]
 822 has counterparts in the weighted setting [26, 27] for suitable semirings. An open prob-
 823 lem would be to investigate the relationship between dot-depth and quantifier-alternation
 824 (as in [90] for languages). Recently, the expressive power of weighted pebble automata
 825 and nested weighted automata was shown to equal that of a weighted transitive closure
 826 logic [13].

827 Recall that the distributivity of semirings permitted us to employ representations and
 828 algebraic proofs for many results. Using automata-theoretic constructions, one can obtain
 829 Kleene and Büchi type characterizations of recognizable series for strong bimonoids [35].
 830 These strong bimonoids can be viewed as semirings without distributivity assumption,
 831 also cf. [32].

832 **Weighted pushdown automata** A huge amount of research has dealt with weighted
 833 versions of pushdown automata and of context-free grammars. The books [84, 66] and
 834 the chapters [64, 78] survey the theory and also infer purely language-theoretic decid-
 835 ability results on unambiguous context-free languages. The list of equivalent formalisms
 836 (weighted pushdown automata, weighted context-free grammars, systems of algebraic
 837 equations) has recently been extended by a weighted logic [72].

838 **Quantitative automata** Motivated by practical questions on the behavior of technical
 839 systems, new kinds of behaviors of weighted automata have been investigated [20, 21].
 840 E.g., the run weight of a path could be the average of the weights of the transitions.
 841 Various decidability and undecidability results, closure properties, and properties of the
 842 expressive powers of these models have been established [20, 21, 32].

843 **Discrete structures** Weighted tree automata and transducers have been investigated,
 844 e.g., for program analysis and transformation [87] and for description logics [7]. Their
 845 investigation, e.g. [10, 15, 16, 65, 36], was also guided by results on weighted word
 846 automata and on tree transducers, for an extensive survey see [47].

847 Distributed behaviors can be modelled by Mazurkiewicz traces. The well-established
 848 theory of recognizable languages of traces [25] has a weighted counterpart including a
 849 weighted distributed automata model [45].

850 Automata models for other discrete structures like pictures [48], nested words [5],
 851 texts [37, 54], and timed words [4] have been studied extensively. Corresponding weighted
 852 automata models and their expressive power have been investigated in [44, 72, 71, 33, 79].

853 Weighted automata on infinite words were investigated for image processing [24] and
 854 used as devices to compute real functions [23]. A discounting parameter was employed

855 in [31, 34] in order to calculate the run weight of an infinite path. This led to Kleene-
856 Schützenberger and logical descriptions of the resulting behaviors. Alternatively, semi-
857 rings with infinitary sum and product operations allow us to define the behavior analog-
858 ously to the finitary case and to obtain corresponding results [41, 28]. Also the quan-
859 titative automata from above have been investigated for infinite words employing, e.g.,
860 accumulation points of averages to define the run weight of infinite paths [20, 21, 32].
861 Weighted Muller automata on ω -trees were studied in [7, 80, 70].

862 **Applications** Since the early 90s, weighted automata have been used for compressed
863 representations of images and movies which led to various algorithms for image transfor-
864 mation and processing, cf. [56, 1] for surveys.

865 Practical tools for multi-valued model checking have been developed based on weighted
866 automata over De Morgan algebras, cf. [22, 17, 67]. De Morgan algebras are particular
867 bounded distributive lattices and therefore locally finite semirings. Weighted automata
868 have also been crucially used to automatically prove termination of rewrite systems, cf.
869 [93] for an overview.

870 In network optimization problems, the max-plus-semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$
871 is often employed, see the corresponding chapter in this Handbook.

872 For quantitative evaluations, reachability questions, and scheduling optimization in
873 real-time systems, timed automata with cost functions form a vigorous current research
874 field [8, 6, 14].

875 In natural language processing, an interesting strand of applications is developing
876 where weighted tree automata play a central role, cf. [62, 69] for surveys. Toolkits for
877 handling weighted automata models are described in [61, 2]. A survey on algorithms for
878 weighted automata with references to many further applications is given in [74].

879 We close with three examples where weighted automata were employed to solve long-
880 standing open questions in language theory. First, the equivalence of deterministic multi-
881 tape automata was shown to be decidable in [52], cf. also [83]. Second, the equality
882 of an unambiguous context-free language and a regular language can be decided using
883 weighted pushdown automata [88], cf. also [76]. Third, the decidability and complexity
884 of determining the star-height of a regular language were determined using a variant of
885 weighted automata [53, 58].

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