Weighted automata

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2010 Mathematics Subject Classification: Primary: 68Q45 Secondary: 03B50, 03D05, 16Y60, 68Q70

Key words: weighted finite automata, semirings, rational formal power series

Abstract. Weighted automata are classical finite automata in which the transitions carry weights. These weights may model quantitative properties like the amount of resources needed for executing a transition or the probability or reliability of its successful execution. Using weighted automata, we may also count the number of successful paths labeled by a given word.

As an introduction into this field, we present selected classical and recent results concentrating on the expressive power of weighted automata.

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1 Introduction

Classical automata provide acceptance mechanisms for words. The starting point of weighted automata is to determine the number of ways a word can be accepted or the amount of resources used for this. The behavior of weighted automata thus associates a quantity or weight to every word. The goal of this chapter is to study the possible behaviors.

Historically, weighted automata were introduced in the seminal paper by Schützenberger [85]. A close relationship to probabilistic automata was mutually influential in the beginning [77, 19, 95]. For the domain of weights and their computations, the algebraic structure of semirings proved to be very fruitful. This soon led to a rich mathematical theory including applications for purely language theoretic questions as well as practical applications in digital image compression and algorithms for natural language processing. Excellent treatments of this are provided by the books [38, 84, 95, 66, 11, 82] and the surveys in the recent handbook [30].

In this chapter, we describe the behavior of weighted automata by equivalent formalisms. These include rational expressions and series, algebraic means like linear presentations and semimodules, decomposition into simple behaviors, and quantitative logics. We also touch on decidability questions (including Colcombet's new proof of a celebrated result by Krob) and languages naturally associated to the behaviors of weighted automata.

We had to choose from the substantial amount of theory and applications of this topic and our choice is biased by our personal interests. We hope to wet the reader's appetite for this exciting field and for consulting the abovementioned books.

Acknowledgement The authors would like to thank Werner Kuich for valuable suggestions regarding this chapter and Ingmar Meinecke for some improvements in Section 6.

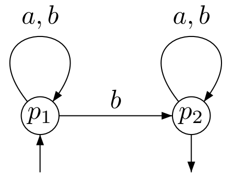

Figure 1. A nondeterministic finite automaton

2 Weighted automata and their behavior

We start with a simple automaton exemplifying different possible interpretations of its behavior. We identify a common feature that will permit us to consider them as instances of the unified concept of a weighted automaton. So let $\Sigma = \{a, b\}$ and $Q = \{p_1, p_2\}$ and consider the automaton from Figure 1.

Example 2.1. Classically, the language accepted describes the behavior of a finite automaton. In our case, this is the language $\Sigma^* b \Sigma^*$.

Now set $\mathrm{in}(p_1) = \mathrm{out}(p_2) = \mathrm{true}$, $\mathrm{out}(p_1) = \mathrm{in}(p_2) = \mathrm{false}$, and $\mathrm{wt}(p, c, q) = \mathrm{true}$ if (p, c, q) is a transition of the automaton and false otherwise. Then a word $a_1 a_2 \ldots a_n$ is accepted by the automaton if and only if

$$\bigvee_{q_0, q_1, \ldots, q_n \in Q} \left(\mathrm{in}(q_0) \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathrm{wt}(q_{i-1}, a_i, q_i) \wedge \mathrm{out}(q_n) \right)$$

evaluates to true.

Example 2.2. For any word $w \in \Sigma^*$, let $f(w)$ denote the number of accepting paths labeled w. In our case, $f(w)$ equals the number of occurrences of the letter b.

Set $\mathrm{in}(p_1) = \mathrm{out}(p_2) = 1$, $\mathrm{out}(p_1) = \mathrm{in}(p_2) = 0$, and $\mathrm{wt}(p, c, q) = 1$ if (p, c, q) is a transition of the automaton and 0 otherwise. Then $f(a_1 \ldots a_n)$ equals

$$\sum_{q_0, q_1, \ldots, q_n \in Q} \left(\mathrm{in}(q_0) \cdot \prod_{1 \leqslant i \leqslant n} \mathrm{wt}(q_{i-1}, a_i, q_i) \cdot \mathrm{out}(q_n) \right) . \tag{2.1}$$

Note that the above two examples would in fact work correspondingly for any finite automaton. The following two examples are specific for the particular automaton from Fig. 1.

Example 2.3. Define the functions in and out as in Example 2.2. But this time, set $\mathrm{wt}(p, c, q) = 1$ if (p, c, q) is a transition of the automaton and $p = p_1$, $\mathrm{wt}(p_2, c, p_2) = 2$ for $c \in \Sigma$, and $\mathrm{wt}(p, c, q) = 0$ otherwise. If we now evaluate the formula (2.1) for a word $w \in \Sigma^*$, we obtain the value of the word w if understood as a binary number where a stands for the digit 0 and b for the digit 1.

Example 2.4. Let the deficit of a word $v \in \Sigma^*$ be the number $|v|_b - |v|_a$ where $|v|_a$ is the number of occurrences of a in v and $|v|_b$ is defined analogously. We want to compute using the automaton from Fig. 1 the maximal deficit of a prefix of a word w. To this aim, set $\mathrm{in}(p_1) = \mathrm{out}(p_2) = 0$ and $\mathrm{out}(p_1) = \mathrm{in}(p_2) = -\infty$. Furthermore, we set $\mathrm{wt}(p_1, b, p_i) = 1$ for $i = 1, 2$, $\mathrm{wt}(p_1, a, p_1) = -1$, $\mathrm{wt}(p_2, c, p_2) = 0$ for $c \in \Sigma$, and $\mathrm{wt}(p, c, q) = -\infty$ in the remaining cases. Then the maximal deficit of a prefix of the word $w = a_1 a_2 \ldots a_n \in \Sigma^* b \Sigma^*$ equals

$$\max_{q_0, q_1, \ldots, q_n \in Q} \left(\mathrm{in}(q_0) + \sum_{1 \leqslant i \leqslant n} \mathrm{wt}(q_{i-1}, a_i, q_i) + \mathrm{out}(q_n) \right) .$$

The similarities between the above examples naturally lead to the definition of a weighted automaton.

Definition 2.1. Let S be a set and Σ an alphabet. A *weighted automaton over* S *and* Σ is a quadruple $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ where

- Q is a finite set of states,
- $\mathrm{in}, \mathrm{out}\colon Q \to S$ are weight functions for entering and leaving a state, resp., and
- $\mathrm{wt}\colon Q \times \Sigma \times Q \to S$ is a transition weight function.

The rôle of S in the examples above is played by $\{\mathrm{true}, \mathrm{false}\}$, $\mathbb{N}$, and $\mathbb{Z} \cup \{-\infty\}$, resp., i.e., we reformulated all the examples as weighted automata over some appropriate set S.

Note also the similarity of the description of the behaviors in all the examples above. We now introduce semirings that formalize the similarities between the operations $\vee$, $+$, and $\max$ on the one hand, and $\wedge$, $\cdot$, and $+$ on the other:

Definition 2.2. A *semiring* is a structure $(S, +_S, \cdot_S, 0_S, 1_S)$ such that

- $(S, +_S, 0_S)$ is a commutative monoid,
- $(S, \cdot_S, 1_S)$ is a monoid,
- multiplication distributes over addition from the left and from the right, and
- $0_S \cdot_S s = s \cdot_S 0_S = 0_S$ for all $s \in S$.

If no confusion can occur, we often write S for the semiring $(S, +_S, \cdot_S, 0_S, 1_S)$.

It is easy to check that the structures $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$, $(\mathbb{N}, +, \cdot, 0, 1)$, and $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$ are semirings (with $0 = \mathrm{false}$ and $1 = \mathrm{true}$, $\mathbb{B}$ is the semiring underlying Example 2.1); many further examples are given in [29] and throughout this chapter. The theory of semirings is described in [49]. The notion of a semiring allows us to give a common definition of the behavior of weighted automata that subsumes all those from our examples and, with the language semiring $(\mathcal{P}(\Gamma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, we even capture the important notion of a transducer [9]; here $\mathcal{P}(\Gamma^*)$ denotes the powerset of Γ^*.

Definition 2.3. Let S be a semiring and $\mathcal{A}$ a weighted automaton over S. A *path in* $\mathcal{A}$ is an alternating sequence $P = q_0 a_1 q_1 \ldots a_n q_n \in Q(\Sigma Q)^*$. Its *run weight* is the product

$$\mathrm{rweight}(P) = \prod_{0 \leqslant i < n} \mathrm{wt}(q_i, a_{i+1}, q_{i+1})$$

(for $n = 0$, this is defined to be 1); the *weight* of P is then defined by

$$\mathrm{weight}(P) = \mathrm{in}(q_0) \cdot \mathrm{rweight}(P) \cdot \mathrm{out}(q_n)\,.$$

Furthermore, the *label* of P is the word $\mathrm{label}(P) = a_1 a_2 \ldots a_n$. Then the *behavior* of the weighted automaton $\mathcal{A}$ is the function $||\mathcal{A}|| \colon \Sigma^* \to S$ with

$$||\mathcal{A}||(w) = \sum_{\substack{P \text{ path with} \\ \mathrm{label}(P)=w}} \mathrm{weight}(P)\,. \tag{2.2}$$

Whereas classical automata determine whether a word is accepted or not, weighted automata over the natural semiring $\mathbb{N}$ allow us to *count* the number of successful paths labeled by a word (cf. Example 2.2). Over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, weighted automata can be viewed as determining the maximal amount of resources needed for the execution of a given sequence of actions. Thus, weighted automata determine quantitative properties.

Notational convention We write $P \colon p \xrightarrow{w}_{\mathcal{A}} q$ for "P is a path in the weighted automaton $\mathcal{A}$ from p to q with label w". From now on, all weighted automata will be over some semiring $(S, +, \cdot, 0, 1)$. We will call functions from Σ^* into S *series*. For such a series r, it is customary to write (r, w) for $r(w)$. The set of all series from Σ^* into S will be denoted by $S\langle\!\langle \Sigma^* \rangle\!\rangle$. If $\mathcal{A}$ is a weighted automaton, then we get in particular $||\mathcal{A}|| \in S\langle\!\langle \Sigma^* \rangle\!\rangle$ and in the above definition, we could have written $(||\mathcal{A}||, w)$ instead of $||\mathcal{A}||(w)$.

Definition 2.4. A series $r \in S\langle\!\langle \Sigma^* \rangle\!\rangle$ is *recognizable* if it is the behavior of some weighted automaton. The set of all recognizable series is denoted by $S^{\mathrm{rec}}\langle\!\langle \Sigma^* \rangle\!\rangle$.

For a series $r \in S\langle\!\langle \Sigma^* \rangle\!\rangle$, the *support* of r is the set $\mathrm{supp}(r) = \{w \in \Sigma^* \mid (r, w) \neq 0\}$. Also, for a language $L \subseteq \Sigma^*$, we write $\mathbb{1}_L$ for the series with $(\mathbb{1}_L, w) = 1_S$ if $w \in L$ and $(\mathbb{1}_L, w) = 0_S$ otherwise; $\mathbb{1}_L$ is called the *characteristic series of* L. From Example 2.1, it should be clear that a series r in $\mathbb{B}\langle\!\langle \Sigma^* \rangle\!\rangle$ is recognizable if and only if the language $\mathrm{supp}(r)$ is regular. Later, we will see that many properties of regular languages transfer to recognizable series (sometimes with very similar proofs). But first, we want to point out some differences.

Example 2.5. Let $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ and consider the series r with $(r, wa) = \{aw\}$ for all words $w \in \Sigma^*$ and letters $a \in \Sigma$, and $(r, \varepsilon) = \emptyset$. Then $r \in S^{\mathrm{rec}}\langle\!\langle \Sigma^* \rangle\!\rangle$, but there is no deterministic transducer whose behavior equals r. Hence deterministic weighted automata are in general weaker than general weighted automata, i.e., a fundamental property of finite automata (see Chapter 1) does not transfer to weighted automata.

Example 2.6. Let $S = (\mathbb{N}, +, \cdot, 0, 1)$ and $a \in \Sigma$. We consider the series r with $(r, aa) = 2$ and $(r, w) = 0$ for $w \neq aa$. Then there are 4 different (deterministic) weighted automata with three states and behavior r (and none with only two states). Hence, another fundamental property of finite automata, namely the existence of unique minimal deterministic automata, does not transfer.

Recall that the existence of a unique minimal deterministic automaton for a regular language can be used to decide whether two finite automata accept the same language. Above, we saw that this approach cannot be used for weighted automata over the semiring $(\mathbb{N}, +, \cdot, 0, 1)$, but other methods work in this case. However, there are no universal methods since the equivalence problem over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ is undecidable, see Section 8.

3 Linear presentations

Let S be a semiring and Q_1 and Q_2 sets. We will consider a function from $Q_1 \times Q_2$ into S as a matrix whose rows and columns are indexed by elements of Q_1 and Q_2, respectively. Therefore, we will write $M_{p,q}$ for $M(p, q)$ where $M \in S^{Q_1 \times Q_2}$, $p \in Q_1$, and $q \in Q_2$. For finite sets Q_1, Q_2, Q_3, this allows us to define the sum and the product of two matrices as usual:

$$(K + M)_{p,q} = K_{p,q} + M_{p,q} \qquad\qquad (M \cdot N)_{p,r} = \sum_{q \in Q_2} M_{p,q} \cdot N_{q,r}$$

for $K, M \in S^{Q_1 \times Q_2}$, $N \in S^{Q_2 \times Q_3}$, $p \in Q_1$, $q \in Q_2$, and $r \in Q_3$. Since in semirings, multiplication distributes over addition from both sides, matrix multiplication is associative. For a finite set Q, the *unit matrix* $E \in S^{Q \times Q}$ with $E_{p,q} = 1$ for $p = q$ and $E_{p,q} = 0$ otherwise is the neutral element of the multiplication of matrices. Hence $(S^{Q \times Q}, \cdot, E)$ is a monoid. It is useful to note that with pointwise addition of matrices, $S^{Q \times Q}$ even forms a semiring.

Lemma 3.1. *Let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be a weighted automaton and define a mapping $\mu \colon \Sigma^* \to S^{Q \times Q}$ by*

$$\mu(w)_{p,q} = \sum_{P\colon p \xrightarrow{w}_{\mathcal{A}} q} \mathrm{rweight}(P)\,. \tag{3.1}$$

Then μ is a homomorphism from the free monoid Σ^* to the multiplicative monoid of matrices $(S^{Q \times Q}, \cdot, E)$.

Proof. Let $P = p_0 a_1 p_1 \ldots a_n p_n$ be a path with label uv and let $|u| = k$. Then $P_1 = p_0 a_1 \ldots a_k p_k$ is a u-labeled path, $P_2 = p_k a_{k+1} \ldots a_n p_n$ is a v-labeled path, and we have $\mathrm{rweight}(P) = \mathrm{rweight}(P_1) \cdot \mathrm{rweight}(P_2)$. This simple observation, together with distributivity in the semiring S, allows us to prove the claim. □

Now let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be a weighted automaton. Define $\lambda \in S^{\{1\} \times Q}$ and $\gamma \in S^{Q \times \{1\}}$ by $\lambda_{1,q} = \mathrm{in}(q)$ and $\gamma_{q,1} = \mathrm{out}(q)$. With the homomorphism μ from Lemma 3.1, we obtain for any word $w \in \Sigma^*$ (where we identify a $\{1\} \times \{1\}$-matrix with its entry):

$$(||\mathcal{A}||, w) = \sum_{p,q \in Q} \lambda_{1,p} \cdot \mu(w)_{p,q} \cdot \gamma_{q,1} = \lambda \cdot \mu(w) \cdot \gamma\,. \tag{3.2}$$

Subsequently, we consider λ (as usual) as a row vector and γ as a colum vector and we simply write $\lambda, \gamma \in S^Q$.

This motives the following definition.

Definition 3.1 (Schützenberger [85]). A *linear presentation* of dimension Q (where Q is some finite set) is a triple (λ, μ, γ) such that $\lambda, \gamma \in S^Q$ and $\mu\colon (\Sigma^*, \cdot, \varepsilon) \to (S^{Q\times Q}, \cdot, E)$ is a monoid homomorphism. It defines the series $r = ||(\lambda, \mu, \gamma)||$ with

$$(r, w) = \lambda \cdot \mu(w) \cdot \gamma \tag{3.3}$$

for all $w \in \Sigma^*$.

Above, we saw that any weighted automaton can be transformed into an equivalent linear presentation. Now we describe the converse transformation. So let (λ, μ, γ) be a linear presentation of dimension Q. For $a \in \Sigma$ and $p, q \in Q$, set $\mathrm{wt}(p, a, q) = \mu(a)_{p,q}$, $\mathrm{in}(q) = \lambda_q$, and $\mathrm{out}(q) = \gamma_q$, and define $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$. Since the morphism μ is uniquely determined by its restriction to Σ, the linear representation associated with $\mathcal{A}$ is precisely (λ, μ, γ), so by Equation (3.2) we obtain $||\mathcal{A}|| = ||(\lambda, \mu, \gamma)||$. Hence we showed

Theorem 3.2. *Let S be a semiring, Σ an alphabet, and $r \in S\langle\langle\Sigma^*\rangle\rangle$. Then r is recognizable if and only if there exists a linear presentation (λ, μ, γ) with $r = ||(\lambda, \mu, \gamma)||$.*

This theorem explains why some authors use linear presentations to define recognizable series or even weighted automata.

4 The Kleene-Schützenberger theorem

The goal of this section is to derive a generalization of Kleene's classical result on the coincidence of rational and regular languages in the realm of series over semirings. Therefore, first we introduce operations in $S\langle\langle\Sigma^*\rangle\rangle$ that correspond to the language-theoretic operations union, intersection, concatenation, and Kleene iteration.

Let r_1 and r_2 be series. Pointwise addition is defined by

$$(r_1 + r_2, w) = (r_1, w) + (r_2, w)\,.$$

Clearly, this operation is associative and has the constant series with value 0 as neutral element. Furthermore, it generalizes the union of languages since, in the Boolean semiring $\mathbb{B}$, we have $\mathrm{supp}(r_1 + r_2) = \mathrm{supp}(r_1) \cup \mathrm{supp}(r_2)$ and $\mathbb{1}_{K\cup L} = \mathbb{1}_K + \mathbb{1}_L$.

Any family of languages has a union, so one is tempted to also define the sum of arbitrary sets of series. But this fails in general since it would require the sum of infinitely many elements of the semiring S (which, e.g. in $(\mathbb{N}, +, \cdot, 0, 1)$, does not exist). But certain families can be summed: a family $(r_i)_{i\in I}$ of series is *locally finite* if, for any word $w \in \Sigma^*$, there are only finitely many $i \in I$ with $(r_i, w) \neq 0$. For such families, we can define

$$\left(\sum_{i\in I} r_i, w\right) = \sum_{\substack{i\in I \text{ with}\\ (r_i,w)\neq 0}} (r_i, w)\,.$$

Let $r_1, r_2 \in S\langle\langle\Sigma^*\rangle\rangle$. Pointwise multiplication is defined by

$$(r_1 \odot r_2, w) = (r_1, w) \cdot (r_2, w)\,.$$

This operation is called *Hadamard product*, is clearly associative, has the constant series with value 1 as neutral element, and distributes over addition. If S is the Boolean semiring $\mathbb{B}$, then the Hadamard product corresponds to intersection:

$$\mathrm{supp}(r_1 \odot r_2) = \mathrm{supp}(r_1) \cap \mathrm{supp}(r_2) \text{ and } \mathbb{1}_K \odot \mathbb{1}_L = \mathbb{1}_{K\cap L}$$

Other simple and natural operations are the *left* and *right scalar multiplication* that are defined by

$$(s \cdot r, w) = s \cdot (r, w) \text{ and } (r \cdot s, w) = (r, w) \cdot s$$

for $s \in S$ and $r \in S\langle\langle\Sigma^*\rangle\rangle$. These two scalar multiplications do not have natural counterparts in language theory.

The counterpart of singleton languages in the realm of series are monomials: a *monomial* is a series r with $|\mathrm{supp}(r)| \leqslant 1$. With $w \in \Sigma^*$ and $s \in S$, we will write sw for the monomial r with $(r, w) = s$. Let r be an arbitrary series. Then the family of monomials $((r, w)w)_{w\in\Sigma^*}$ is locally finite and can therefore be summed. Then one obtains

$$r = \sum_{w\in\Sigma^*} (r, w)w = \sum_{w\in\mathrm{supp}(r)} (r, w)w\,.$$

If the support of r is finite, then the second sum has only finitely many summands which is the reason to call r a *polynomial* in this case; the set of polynomials is denoted $S\langle\Sigma^*\rangle$, so $S\langle\Sigma^*\rangle \subseteq S\langle\langle\Sigma^*\rangle\rangle$. The similarity with polynomials makes it natural to define another product of the series r_1 and r_2 by

$$(r_1 \cdot r_2, w) = \sum_{w=uv} (r_1, u) \cdot (r_2, v)\,.$$

Since the word w has only finitely many factorizations into u and v, the right-hand side has only finitely many summands and is therefore well-defined. This important product is called *Cauchy-product* of the series r_1 and r_2. If r_1 and r_2 are polynomials, then $r_1 \cdot r_2$ is precisely the usual product of polynomials. For the Boolean semiring, we get

$$\mathrm{supp}(r_1 \cdot r_2) = \mathrm{supp}(r_1) \cdot \mathrm{supp}(r_2) \text{ and } \mathbb{1}_K \cdot \mathbb{1}_L = \mathbb{1}_{K\cdot L}\,,$$

i.e., the Cauchy-product is the counterpart of concatenation of languages. For any semiring S, the monomial 1ε is the neutral element of the Cauchy-product. It requires a short calculation to show that the Cauchy-product is associative and distributes over the addition of series. As a very useful consequence, $(S\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, 1\varepsilon)$ is a semiring (note that the set of polynomials $S\langle\Sigma^*\rangle$ forms a subsemiring of this semiring). For the Boolean semiring $\mathbb{B}$, this semiring is isomorphic to $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, an isomorphism is given by $r \mapsto \mathrm{supp}(r)$ with inverse $L \mapsto \mathbb{1}_L$.

In the theory of recognizable languages, the Kleene-iteration L^* of a language L is of central importance. It is defined as the union of all the powers L^n of L (for $n \geqslant 0$). To also define the iteration r^* of a series, one would therefore try to sum all finite powers r^n (defined by $r^0 = 1\varepsilon$ and $r^{n+1} = r^n \cdot r$). In general, the family $(r^n)_{n\geqslant 0}$ is not locally finite, so it cannot be summed. We therefore define the iteration r^* only for r proper: a series r is *proper* if $(r,\varepsilon) = 0$. Then, for $n > |w|$, one has $(r^n, w) = 0$, so the family $(r^n)_{n\geqslant 0}$ is locally finite and we can set

$$r^* = \sum_{n\geqslant 0} r^n \text{ or equivalently } (r^*, w) = \sum_{0\leqslant n\leqslant |w|} (r^n, w)\,.$$

For the Boolean semiring and $L \subseteq \Sigma^+$, we get

$$\mathrm{supp}(r^*) = (\mathrm{supp}(r))^* \text{ and } (\mathbb{1}_L)^* = \mathbb{1}_{L^*}\ .$$

Recall that a language is rational if it can be constructed from the finite languages by union, concatenation, and Kleene-iteration. Here, we give the analogous definition for series:

Definition 4.1. A series from $S\langle\langle\Sigma^*\rangle\rangle$ is *rational* if it can be constructed from the monomials sa for $s \in S$ and $a \in \Sigma \cup \{\varepsilon\}$ by addition, Cauchy-product, and iteration (applied to proper series, only). The set of all rational series is denoted by $S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

Observe that the class of rational series is closed under scalar multiplication since $s\varepsilon$ is a monomial, $s \cdot r = s\varepsilon \cdot r$ and $r \cdot s = r \cdot s\varepsilon$ for $r \in S\langle\langle\Sigma^*\rangle\rangle$ and $s \in S$.

Example 4.1. Consider the Boolean semiring $\mathbb{B}$ and $r \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$. If r is a rational series, then the above formulas show that $\mathrm{supp}(r)$ is a rational language since supp commutes with the rational operations $+$, $\cdot$, and * for series and $\cup$, $\cdot$, and * for languages. Now suppose that, conversely, $\mathrm{supp}(r)$ is a rational lanuage. To show that also r is a rational series, one needs that any rational language can be constructed in such a way that Kleene-iteration is only applied to languages in Σ^+. Having ensured this, the remaining calculations are again straightforward. Thus, indeed, our notion of rational series is the counterpart of the notion of a rational language.

Hence, rational languages are precisely the supports of series in $\mathbb{B}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ and recognizable languages are the supports of series in $\mathbb{B}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ (see above). Now Kleene's theorem from Chapter 1 implies $\mathbb{B}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle = \mathbb{B}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$. It is the aim of this section to prove this equality for arbitrary semirings. This is achieved by first showing that every rational series is recognizable. The other inclusion will be shown in Section 4.2.

4.1 Rational series are recognizable

For this implication, we prove that the set of recognizable series contains the monomials sa and $s\varepsilon$ and is closed under the necessary operations. To show this closure, we have two possibilities (a third one is sketched after the proof of Theorem 5.1): either the purely automata-theoretic approach that constructs weighted automata, or the more algebraic approach that handles linear presentations. We chose to give the automata constructions for monomials and addition, and the linear presentations for the Cauchy-product and the iteration. The reader might decide which approach she prefers and translate some of the constructions from one to the other.

There is a weighted automaton with just one state q and behavior the monomial $s\varepsilon$: just set $\mathrm{in}(q) = s$, $\mathrm{out}(q) = 1$ and $\mathrm{wt}(q, a, q) = 0$ for all $a \in \Sigma$. For any $a \in \Sigma$, there is a two-states weighted automaton with the monomial sa as behavior. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two weighted automata, then the behavior of their disjoint union equals $||\mathcal{A}_1|| + ||\mathcal{A}_2||$.

We next show that also the Cauchy-product of two recognizable series is recognizable:

Lemma 4.1. *If r_1 and r_2 are recognizable series, then so is $r_1 \cdot r_2$.*

Proof. By Theorem 3.2, the series r_i has a linear presentation $(\lambda^i, \mu^i, \gamma^i)$ of dimension Q^i with $Q^1 \cap Q^2 = \emptyset$. We define a row vector λ and a column vector γ of dimension $Q = Q^1 \cup Q^2$ as well as a matrix $\mu(w)$ for $w \in \Sigma^*$ of dimension $Q \times Q$:

$$\lambda = \begin{pmatrix} \lambda^1 & 0 \end{pmatrix} \quad \mu(w) = \begin{pmatrix} \mu^1(w) & \sum\limits_{w=uv, v\neq\varepsilon} \mu^1(u)\gamma^1\lambda^2\mu^2(v) \\ 0 & \mu^2(w) \end{pmatrix} \quad \gamma = \begin{pmatrix} \gamma^1\lambda^2\gamma^2 \\ \gamma^2 \end{pmatrix}$$

The reader is invited to check that μ is actually a monoid homomorphism from $(\Sigma^*, \cdot, \varepsilon)$ into $(S^{Q\times Q}, \cdot, E)$, i.e., that (λ, μ, γ) is a linear presentation. One then gets

$$\begin{aligned} \lambda \cdot \mu(w) \cdot \gamma &= \lambda^1\ \mu^1(w)\ \gamma^1\lambda^2\gamma^2 + \lambda^1 \sum_{\substack{w=uv\\ v\neq\varepsilon}} \mu^1(u)\gamma^1\lambda^2\mu^2(v)\ \gamma^2 \\ &= (r_1, w) \cdot (r_2, \varepsilon) + \sum_{\substack{w=uv\\ v\neq\varepsilon}} (r_1, u)(r_2, v) \\ &= (r_1 \cdot r_2, w)\,. \end{aligned}$$

By Theorem 3.2, the series $||(\lambda, \mu, \gamma)|| = r_1 \cdot r_2$ is recognizable. □

Lemma 4.2. *Let r be a proper and recognizable series. Then r^* is recognizable.*

Proof. There exists a linear presentation (λ, μ, γ) of dimension Q with $r = ||(\lambda, \mu, \gamma)||$. Consider the homomorphism $\mu' \colon (\Sigma^*, \cdot, \varepsilon) \to (S^{Q\times Q}, \cdot, E)$ defined, for $a \in \Sigma$, by

$$\mu'(a) = \mu(a) + \gamma\,\lambda\,\mu(a)\,.$$

Let $w = a_1a_2 \dots a_n \in \Sigma^+$. Using distributivity of matrix multiplication or, alternatively, induction on n, it follows

$$\begin{aligned} \mu'(w) &= \prod_{1\leqslant i\leqslant n} (\mu(a_i) + \gamma\,\lambda\,\mu(a_i)) \\ &= \sum_{\substack{w=w_1\dots w_k\\ w_i\in\Sigma^+}} \left((\mu(w_1) + \gamma\,\lambda\,\mu(w_1)) \cdot \prod_{2\leqslant j\leqslant k} \gamma\,\lambda\,\mu(w_j) \right)\,. \end{aligned}$$

Note that $\lambda\,\gamma = \lambda\,\mu(\varepsilon)\,\gamma = (r, \varepsilon) = 0$. Hence we obtain

$$\begin{aligned}
\lambda\,\mu'(w)\,\gamma &= \sum_{\substack{w=w_1\dots w_k\\ w_i\in\Sigma^+}} \left(\lambda\,(\mu(w_1) + \gamma\,\lambda\,\mu(w_1)) \cdot \prod_{2\leqslant j\leqslant k} \gamma\,\lambda\,\mu(w_j)\right)\gamma \\
&= \sum_{\substack{w=w_1\dots w_k\\ w_i\in\Sigma^+}} \prod_{1\leqslant j\leqslant k} \lambda\,\mu(w_i)\,\gamma \\
&= (r^*, w)
\end{aligned}$$

as well as $\lambda\,\mu'(\varepsilon)\,\gamma = 0$. Hence $r^* = ||(\lambda, \mu', \gamma)|| + 1\varepsilon$ is recognizable. □

Recall that the Hadamard-product generalizes the intersection of languages and that the intersection of regular languages is regular. The following result extends this latter fact to the weighted setting (since the Boolean semiring is commutative). We say that two subsets $S_1, S_2 \subseteq S$ *commute*, if $s_1 \cdot s_2 = s_2 \cdot s_1$ for all $s_1 \in S_1$, $s_2 \in S_2$.

Lemma 4.3. *Let S_1 and S_2 be two subsemirings of the semiring S such that S_1 and S_2 commute. If $r_1 \in S_1^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ and $r_2 \in S_2^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$, then $r_1 \odot r_2 \in S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$.*

Proof. For $i = 1, 2$, let $\mathcal{A}_i = (Q_i, \mathrm{in}_i, \mathrm{wt}_i, \mathrm{out}_i)$ be weighted automata over S_i with $||\mathcal{A}_i|| = r_i$. We define the product automaton $\mathcal{A}$ with states $Q_1 \times Q_2$ as follows:

$$\begin{aligned}
\mathrm{in}(p_1, p_2) &= \mathrm{in}_1(p_1) \cdot \mathrm{in}_2(p_2) \\
\mathrm{wt}((p_1, p_2), a, (q_1, q_2)) &= \mathrm{wt}_1(p_1, a, q_1) \cdot \mathrm{wt}_2(p_2, a, q_2) \\
\mathrm{out}(p_1, p_2) &= \mathrm{out}_1(p_1) \cdot \mathrm{out}_2(p_2)
\end{aligned}$$

Then, $(||\mathcal{A}||, w) = (||\mathcal{A}_1|| \odot ||\mathcal{A}_2||, w)$ follows for all words w. For example, for a letter $a \in \Sigma$ we calculate as follows using the commutativity assumption and distributivity:

$$\begin{aligned}
(||\mathcal{A}||, a) &= \sum_{(p_1,p_2),(q_1,q_2)\in Q} \begin{pmatrix} (\mathrm{in}_1(p_1) \cdot \mathrm{in}_2(p_2)) \cdot (\mathrm{wt}_1(p_1, a, q_1) \cdot \mathrm{wt}_2(p_2, a, q_2)) \\ \cdot\, (\mathrm{out}_1(q_1) \cdot \mathrm{out}_2(q_2)) \end{pmatrix} \\
&= \sum_{(p_1,p_2),(q_1,q_2)\in Q} \begin{pmatrix} \mathrm{in}_1(p_1) \cdot \mathrm{wt}_1(p_1, a, q_1) \cdot \mathrm{out}_1(q_1) \\ \cdot\, \mathrm{in}_2(p_2) \cdot \mathrm{wt}_2(p_2, a, q_2) \cdot \mathrm{out}_2(q_2) \end{pmatrix} \\
&= \left(\sum_{p_1,q_1\in Q_1} \mathrm{in}_1(p_1) \cdot \mathrm{wt}_1(p_1, a, q_1) \cdot \mathrm{out}_1(q_1)\right) \\
&\qquad \cdot \left(\sum_{p_2,q_2\in Q_2} \mathrm{in}_2(p_2) \cdot \mathrm{wt}_2(p_2, a, q_2) \cdot \mathrm{out}_2(q_2)\right) \\
&= (||\mathcal{A}_1||, a) \cdot (||\mathcal{A}_2||, a) = (||\mathcal{A}_1|| \odot ||\mathcal{A}_2||, a)
\end{aligned}$$

□

We remark that the above lemma does not hold without the commutativity assumption:

Example 4.2. Let $\Sigma = \{a, b\}$, $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, and consider the recognizable series r given by $(r, w) = \{w\}$ for $w \in \Sigma^*$. Then $(r \odot r, w) = \{ww\}$ and pumping arguments show that $r \odot r$ is not recognizable.

As a consequence of Lemma 4.3, we obtain that "restrictions" of recognizable series to regular languages are again recognizable, more precisely:

Corollary 4.4. *Let $r \in S\langle\langle\Sigma^*\rangle\rangle$ be recognizable and let $L \subseteq \Sigma^*$ be a regular language. Then $r \odot \mathbb{1}_L$ is recognizable.*

Proof. Let $\mathcal{A}$ be a deterministic automaton accepting L with set of states Q. Then weight by 1 those triples $(p, a, q) \in Q \times \Sigma \times Q$ that are transitions, the initial resp. final states with initial resp. final weight by 1, and all other triples resp. states with 0. This gives a weighted automaton with behavior $\mathbb{1}_L$. Since S commutes with its subsemiring generated by 1, Lemma 4.3 implies the result. □

4.2 Recognizable series are rational

For this implication, we will transform a weighted automaton into a system of equations and then show that any solution of such a system is rational. The following lemma will be helpful and is also of independent interest (cf. [29, Section 5]).

Lemma 4.5. *Let $s, r, r' \in S\langle\langle\Sigma^*\rangle\rangle$ with r proper and $s = r \cdot s + r'$. Then $s = r^* r'$.*

Proof. Let $w \in \Sigma^*$. First observe that

$$\begin{aligned} s &= rs + r' \\ &= r(rs + r') + r' = r^2 s + rr' + r' \\ &\;\vdots \\ &= r^{|w|+1} s + \sum_{0 \leqslant i \leqslant |w|} r^i r' \,. \end{aligned}$$

Since r is proper, we have $(r^i, u) = 0$ for all $u \in \Sigma^*$ and $i > |u|$. This implies

$$\begin{aligned} (r^* r', w) &= \sum_{w=uv} (r^*, u) \cdot (r', v) = \sum_{w=uv} \left(\sum_{0 \leqslant i \leqslant |w|} (r^i, u) \right) \cdot (r', v) = \sum_{0 \leqslant i \leqslant |w|} (r^i r', w) \\ &= (s, w) \,. \end{aligned}$$

□

Now let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be a weighted automaton. For $p \in Q$, define a new weighted automaton $\mathcal{A}_p = (Q, \mathrm{in}_p, \mathrm{wt}, \mathrm{out})$ by $\mathrm{in}_p(p') = 1$ for $p = p'$ and $\mathrm{in}_p(p') = 0$ otherwise. Since all the entry weights of these weighted automata are 0 or 1, we have

$$||\mathcal{A}|| = \sum_{(p,a,q) \in Q \times \Sigma \times Q} \mathrm{in}(p)\mathrm{wt}(p, a, q) a \cdot ||\mathcal{A}_q|| + \sum_{p \in Q} \mathrm{in}(p)\mathrm{out}(p)\varepsilon$$

and for all $p \in Q$

$$||\mathcal{A}_p|| = \sum_{(p,a,q)\in Q\times\Sigma\times Q} \mathrm{wt}(p,a,q)a \cdot ||\mathcal{A}_q|| + \mathrm{out}(p)\varepsilon\,.$$

This transformation proves

Lemma 4.6. *Let r be a recognizable series. Then there are rational series $r_{ij}, r_i \in S\langle\langle\Sigma^*\rangle\rangle$ with r_{ij} proper and a solution $(s_1,\dots,s_n)$ with $s_1 = r$ of a system of equations*

$$\left(X_i = \sum_{1\leqslant j\leqslant n} r_{ij}X_j + r_i\right)_{1\leqslant i\leqslant n}\,. \tag{4.1}$$

A series s is *rational over the series* $\{s_1,\dots,s_n\}$ if it can be constructed from the monomials and the series $s_1,\dots,s_n$ by addition, Cauchy-product, and iteration (applied to proper series, only).

We prove by induction on n that any solution of a system of the form (4.1) consists of rational series. For $n = 1$, the system is a single equation of the form $X_1 = r_{11}X_1 + r_1$ with $r_{11}, r_1 \in S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ and r_{11} proper. Hence, by Lemma 4.5, the solution s_1 equals $r_{11}^* r_1$ and is therefore rational. Now assume that any system with $n-1$ unknowns has only rational solutions and consider a solution $(s_1,\dots,s_n)$ of (4.1). Then we have

$$s_n = r_{nn}s_n + \sum_{1\leqslant j<n} r_{nj}s_j + r_n$$

and therefore by Lemma 4.5

$$s_n = r_{nn}^* \cdot \left(\sum_{1\leqslant j<n} r_{nj}s_j + r_n\right)\,.$$

In particular, s_n is rational over $\{s_1, s_2,\dots,s_{n-1}\}$ since r_{nj} and r_n are all rational. Since $(s_1,\dots,s_n)$ is a solution of the system (4.1), we obtain

$$s_i = \sum_{1\leqslant j<n} (r_{ij} + r_{in}r_{nn}^*r_{nj})s_j + r_{in}r_{nn}^*r_n + r_i$$

for all $1 \leqslant i < n$. Since r_{ij} and r_{in} are proper and rational, so is $r_{ij} + r_{in}r_{nn}^*r_{nj}$. Hence $(s_1,\dots,s_{n-1})$ is a solution of a system of equations of the form (4.1) with $n-1$ unknowns implying by the induction hypothesis that the series $s_1,\dots,s_{n-1}$ are all rational. Since s_n is rational over $s_1,\dots,s_{n-1}$, it is therefore rational, too. This completes the inductive proof of the following lemma.

Lemma 4.7. *Let $r_{ij}, r_i \in S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ with r_{ij} proper and let $(s_1,\dots,s_n)$ be a solution of the system of equations (4.1). Then all the series $s_1,\dots,s_n$ are rational.*

From Lemmas 4.6 and 4.7, we obtain that any recognizable series is rational. Together with Lemmas 4.1, 4.2, and the arguments from the beginning of Section 4.1, we obtain

Theorem 4.8 (Schützenberger [85]). *Let S be a semiring, Σ an alphabet, and $r \in S\langle\langle\Sigma^*\rangle\rangle$. Then r is recognizable if and only if it is rational, i.e., $S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle = S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.*

5 Semimodules

If, in the definition of a vector space, one replaces the underlying field by a semiring, one obtains a semimodule. More formally, let S be a semiring. An S-*semimodule* is a commutative monoid $(M, +, 0_M)$ together with a left scalar multiplication $S \times M \to M$ satisfying all the usual laws (with $s, s' \in S$ and $r, r' \in M$):

$$(s + s')r = sr + s'r \qquad (s \cdot s')r = s(s'r)$$
$$s(r + r') = sr + sr' \qquad 1r = r$$
$$0r = 0_M$$

In our context, the most interesting example is the S-semimodule $S\langle\langle\Sigma^*\rangle\rangle$ of series over Σ. The additive structure of the semimodule is pointwise addition and the left scalar multiplication is as defined before.

A *subsemimodule* of the S-semimodule $(M, +, 0_M)$ is a set $N \subseteq M$ that is closed under addition and left scalar multiplication. A set $X \subseteq M$ *generates* the subsemimodule $N = \langle X\rangle$ if N is the least subsemimodule containing X. Equivalently, all elements of N can be written as linear combinations of elements from X. The subsemimodule N is *finitely generated* if it is generated by a finite set. A simple example of a subsemimodule of $S\langle\langle\Sigma^*\rangle\rangle$ is the set of polynomials $S\langle\Sigma^*\rangle$, i.e. of series with finite support. But this subsemimodule is not finitely generated. The set of constant series is a finitely generated subsemimodule.

The following is specific for the semimodule of series. For $r \in S\langle\langle\Sigma^*\rangle\rangle$ and $u \in \Sigma^*$, the series $u^{-1}r$ is defined by

$$(u^{-1}r, w) = (r, uw)$$

for all $w \in \Sigma^*$. A subsemimodule N of $S\langle\langle\Sigma^*\rangle\rangle$ is *stable* if $r \in N$ implies $u^{-1}r \in N$ for all $u \in \Sigma^*$.

Theorem 5.1 (Fliess [46] and Jacob [55]). *Let S be a semiring, Σ an alphabet, and $r \in S\langle\langle\Sigma^*\rangle\rangle$. Then r is recognizable if and only if there exists a finitely generated and stable subsemimodule N of $S\langle\langle\Sigma^*\rangle\rangle$ with $r \in N$.*

For the boolean semiring $\mathbb{B}$, any finitely generated subsemimodule of $\mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ is finite. Therefore the above equivalence extends the well-known result that a language is regular if and only if it has finitely many left-quotients.

Proof. First, let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be a weighted automaton with $r = ||\mathcal{A}||$. For $q \in Q$, define $\mathrm{in}_q : Q \to S$ by $\mathrm{in}_q(q) = 1$ and $\mathrm{in}_q(p) = 0$ for $p \neq q$, and let $\mathcal{A}_q = (Q, \mathrm{in}_q, \mathrm{wt}, \mathrm{out})$. Let N be the subsemimodule generated by $\{||\mathcal{A}_q|| \mid q \in Q\}$. Since

$r = ||\mathcal{A}|| = \sum_{q \in Q} \mathrm{in}(q)||\mathcal{A}_q||$, we get $r \in N$. Note that, for $a \in \Sigma$ and $p \in Q$, we have

$$a^{-1}||\mathcal{A}_p|| = \sum_{q \in Q} \mathrm{wt}(p,a,q)||\mathcal{A}_q||$$

which allows us to prove by simple calculations that N is stable.

Conversely, let N be finitely generated by $\{r_1, \dots, r_n\}$ and stable and let $r \in N$. For all $a \in \Sigma$ and $1 \leqslant i \leqslant n$, we have $a^{-1}r_i = \sum_{1 \leqslant j \leqslant n} s_{ij} r_j$ with suitable $s_{ij} \in S$. Then there exists a unique morphism $\mu\colon \Sigma^* \to S^{n \times n}$ with $\mu(a)_{ij} = s_{ij}$ for $a \in \Sigma$. By induction on the length of $w \in \Sigma^*$, we can show that $w^{-1}r_i = \sum_{1 \leqslant j \leqslant n} \mu(w)_{ij} r_j$. Hence

$$(r_i, w) = (w^{-1}r_i, \varepsilon) = \sum_{1 \leqslant j \leqslant n} \mu(w)_{ij}(r_j, \varepsilon)\,.$$

Since $r \in N$, we have $r = \sum_{1 \leqslant i \leqslant n} \lambda_i r_i$ for some $\lambda_i \in S$. With $\gamma_j = (r_j, \varepsilon)$, we obtain

$$(r, w) = \sum_{1 \leqslant i,j \leqslant n} \lambda_i \cdot \mu(w)_{ij} \cdot \gamma_j = \lambda \cdot \mu(w) \cdot \gamma$$

showing that (λ, μ, γ) is a linear presentation of r. Hence r is recognizable by Theorem 3.2. □

Inductively, one can show that every rational series belongs to a finitely generated and stable subsemimodule, cf. [11]. Together with the theorem above, this is an alternative proof of the fact that every rational series is recognizable (cf. Theorem 4.8).

6 Nivat's theorem

Nivat's theorem [75] provides an insight into the concatenation of mappings and, as we will see, recognizability of certain simple series. More precisely, it asserts that every proper recognizable series $r \in S\langle\langle \Sigma^* \rangle\rangle$ can be decomposed into three particular recognizable series, namely an inverse monoid homomorphism $h^{-1}\colon \Sigma^* \to \mathcal{P}(\Gamma^*)$ with $h\colon \Gamma^* \to \Sigma^*$, a recognizable "selection series" $\mathrm{sel}\colon \Gamma^* \to \mathcal{P}(\Gamma^*)$ satisfying $(\mathrm{sel}, v) \subseteq \{v\}$, and a homomorphism $c\colon (\Gamma^*, \cdot, \varepsilon) \to (S, \cdot, 1)$. Conversely, assuming $h(a) \neq \varepsilon$ for all $a \in \Gamma$, the composition of h^{-1}, sel, and c is recognizable.

A mapping $\mathrm{sel}\colon \Gamma^* \to \mathcal{P}(\Gamma^*)$ is a *selection series* if $(\mathrm{sel}, v) \subseteq \{v\}$ for all $v \in \Gamma^*$. Let $\mathrm{fin}(\Gamma^*)$ denote the set of all finite subsets of Γ^*. Then $(\mathrm{fin}(\Gamma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ is a (computable) subsemiring of $\mathcal{P}(\Gamma^*)$. For brevity, this subsemiring is denoted by $\mathrm{fin}(\Gamma^*)$.

Lemma 6.1. *(1) If $h\colon \Gamma^* \to \Sigma^*$ is a homomorphism with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, then $h^{-1} \in \mathrm{fin}(\Gamma^*)\langle\langle \Sigma^* \rangle\rangle$ with $(h^{-1}, w) = \{v \in \Gamma^* \mid h(v) = w\}$ is a recognizable series.*

(2) A selection series $\mathrm{sel} \in \mathrm{fin}(\Gamma^*)\langle\langle \Gamma^* \rangle\rangle$ is recognizable if and only if its support $K = \{v \in \Gamma^* \mid v \in (\mathrm{sel}, v)\}$ is regular.

(3) If $c\colon (\Gamma^*, \cdot, \varepsilon) \to (S, \cdot, 1)$ is a monoid homomorphism, then c is a recognizable series in $S\langle\langle \Gamma^* \rangle\rangle$.

Proof. (1) Since $h(a) \neq \varepsilon$ for all letters $a \in \Gamma$, the set (h^{-1}, w) is indeed finite, i.e., $h^{-1} \in \mathrm{fin}(\Gamma^*)\langle\langle\Sigma^*\rangle\rangle$. Furthermore, this series equals $(\sum_{a\in\Gamma}\{a\}h(a))^*$ which is rational and therefore recognizable by Theorem 4.8. Alternatively, one observes that a weighted automaton with just one state suffices for this series where the a-transition gets weight $h^{-1}(a)$ for $a \in \Gamma$.

(2) We first prove the implication "$\Leftarrow$". So let K be regular. Then, in an arbitrary finite automaton accepting K, weight any a-labeled transition with $\{a\}$ (for $a \in \Gamma$), and weight the initial and final states by $\{\varepsilon\}$. This gives a weighted automaton with behavior sel.

The other direction follows from more general results on the support of recognizable series over positive semirings since $K = \mathrm{supp}(\mathrm{sel})$. A direct argument goes as follows: take a weighted automaton with behavior sel and delete all its weights (and all transitions with weight $\emptyset$). This results in a finite automaton that accepts the support of sel.

(3) This series is the behavior of a weighted automaton with just one state. □

Next we show that morphisms and inverses of non-deleting morphisms preserve recognizability which is also of independent interest.

Lemma 6.2. *Let $r \in S\langle\langle\Gamma^*\rangle\rangle$ be recognizable.*

(1) If $h\colon \Sigma^* \to \Gamma^*$ is a homomorphism, then the series $r\circ h \in S\langle\langle\Sigma^*\rangle\rangle$ with $(r\circ h, w) = (r, h(w))$ is recognizable.

(2) If $h\colon \Gamma^* \to \Sigma^*$ is a homomorphism with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, then the series $r \circ h^{-1} \in S\langle\langle\Sigma^*\rangle\rangle$ with $(r \circ h^{-1}, w) = \sum_{v\in h^{-1}(w)}(r, v)$ is recognizable.

Note that $h(a) \neq \varepsilon$ in the second statement implies $|h(v)| \geqslant |v|$. Hence, for any $w \in \Sigma^*$, there are only finitely many words v with $h(v) = w$. Hence the series is well-defined.

Proof. (1) If (λ, μ, γ) is a representation of r, then $\mu\circ h$ is a morphism and $(\lambda, \mu\circ h, \gamma)$ represents $r \circ h$, as is easy to check.

(2) By Theorem 4.8, r is rational, and an inductive proof shows that $r \circ h^{-1}$ is rational, too. Hence it is recognizable by Theorem 4.8, again. □

Next, if $c\colon \Gamma^* \to S$ is a mapping and $\mathrm{sel}\colon \Gamma^* \to \mathrm{fin}(\Gamma^*)$ is a selection series, then we define the series $c \circ \mathrm{sel} : \Gamma^* \to S$ by

$$(c \circ \mathrm{sel}, v) = \begin{cases} c(v) & \text{if } (\mathrm{sel}, v) = \{v\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.3 (cf. Nivat [75]). *Let S be a semiring, Σ an alphabet, and $r \in S\langle\langle\Sigma^*\rangle\rangle$ with $(r, \varepsilon) = 0$. Then r is recognizable if and only if there exist an alphabet Γ, a homomorphism $h\colon \Gamma^* \to \Sigma^*$ with $h(a) \neq \varepsilon$ for all $a \in \Gamma$, a recognizable selection series $\mathrm{sel} \in \mathrm{fin}(\Gamma^*)\langle\langle\Gamma^*\rangle\rangle$, and a homomorphism $c\colon (\Gamma^*, \cdot, \varepsilon) \to (S, \cdot, 1)$ such that $r = c \circ \mathrm{sel} \circ h^{-1}$.*

Proof. We first prove the implication "$\Leftarrow$". Let $K = \mathrm{supp}(\mathrm{sel})$. By Lemma 6.1(2), K is regular. Note that $c \circ \mathrm{sel} = c \odot \mathbb{1}_K$. Hence $c \circ \mathrm{sel}$ is recognizable by Corollary 4.4. Therefore, $c \circ \mathrm{sel} \circ h^{-1}$ is recognizable by Lemma 6.2(2).

Conversely, let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be a weighted automaton with $r = ||\mathcal{A}||$. Set

$$\Gamma = (Q \uplus Q \times \{1\}) \times \Sigma \times (Q \uplus Q \times \{2\}),$$

$$h(p', a, q') = a, \text{ and}$$

$$c(p', a, q') = \begin{cases} \mathrm{wt}(p', a, q') & \text{if } p', q' \in Q \\ \mathrm{in}(p) \cdot \mathrm{wt}(p, a, q') & \text{if } p' = (p, 1), q' \in Q \\ \mathrm{wt}(p', a, q) \cdot \mathrm{out}(q) & \text{if } p' \in Q, q' = (q, 2) \\ \mathrm{in}(p) \cdot \mathrm{wt}(p, a, q) \cdot \mathrm{out}(q) & \text{if } p' = (p, 1), q' = (q, 2) \\ 0 & \text{otherwise} \end{cases}$$

for $(p', a, q') \in \Gamma$. Furthermore, let K be the set of words

$$((p_0, 1), a_1, p_1)(p_1, a_2, p_2) \ldots (p_{n-1}, a_n, (p_n, 2))$$

with $p_i \in Q$ for all $0 \leqslant i \leqslant n$. Then K is regular and corresponds to the set of paths in $\mathcal{A}$. This allows us to prove $(r, w) = (||\mathcal{A}||, w) = \sum_{v \in h^{-1}(w) \cap K} c(v)$, i.e., $r = c \circ \mathrm{sel}_K \circ h^{-1}$ with $\mathrm{sel}_K(v) = \{v\} \cap K$. But sel_K is recognizable by Lemma 6.1(2). $\square$

7 Weighted monadic second order logic

Fundamental results by Büchi, by Elgot and by Trakhtenbrot [18, 39, 92] state that a language is regular if and only if it is definable in monadic second order (MSO) logic. Here, we wish to extend this result to a quantitative setting and thereby obtain a further characterization of the recognizability of a series $r\colon \Sigma^* \to S$, using a weighted version of monadic second order logic. We follow [26, 28].

We will enrich MSO-logic by permitting all elements of S as atomic formulas. The semantics of a sentence from the weighted MSO-logic will be a series in $S\langle\langle\Sigma^*\rangle\rangle$. In general, this weighted MSO-logic is more expressive than weighted automata. But a suitable, syntactically defined restriction of the logic, which contains classical MSO-logic, has the same expressive power as weighted automata.

For the convenience of the reader we will recall basic background of classical MSO-logic, cf. [91, 57]. Let Σ be an alphabet. The syntax of formulas of $\mathrm{MSO}(\Sigma)$, the monadic second order logic over Σ, is usually given by the grammar

$$\varphi ::= \mathrm{P}_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where $a \in \Sigma$, x, y are first-order variables, and X is a set variable. We let $\mathrm{Free}(\varphi)$ denote the set of all free variables of φ.

As usual, a word $w = a_1 \ldots a_n \in \Sigma^*$ is represented by the relational structure $(\mathrm{dom}(w), \leqslant, (R_a)_{a \in \Sigma})$ where $\mathrm{dom}(w) = \{1, \ldots, n\}$, $\leqslant$ is the usual order on $\mathrm{dom}(w)$ and $R_a = \{i \in \mathrm{dom}(w) \mid a_i = a\}$ for $a \in \Sigma$.

Let $\mathcal{V}$ be a finite set of first-order or second-order variables. A $(\mathcal{V}, w)$-assignment

σ is a function mapping first-order variables in $\mathcal{V}$ to elements of $\mathrm{dom}(w)$ and second-order variables in $\mathcal{V}$ to subsets of $\mathrm{dom}(w)$. For a first-order variable x and $i \in \mathrm{dom}(w)$, $\sigma[x \mapsto i]$ denotes the $(\mathcal{V} \cup \{x\}, w)$-assignment which maps x to i and coincides with σ otherwise. Similarly, $\sigma[X \mapsto I]$ is defined for $I \subseteq \mathrm{dom}(w)$. For $\varphi \in \mathrm{MSO}(\Sigma)$ with $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$, the satisfaction relation $(w, \sigma) \models \varphi$ is defined as usual.

Subsequently, we will encode a pair (w, σ) as above as a word over the extended alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0, 1\}^{\mathcal{V}}$ (with $\Sigma_\emptyset = \Sigma$). We write a word $(a_1, \sigma_1) \dots (a_n, \sigma_n)$ over $\Sigma_{\mathcal{V}}$ as (w, σ) where $w = a_1 \dots a_n$ and $\sigma = \sigma_1 \dots \sigma_n$. We call (w, σ) *valid*, if it is empty or if for each first order variable $x \in \mathcal{V}$, there is a unique position i with $\sigma_i(x) = 1$. In this case, we identify σ with the $(\mathcal{V}, w)$-assignment that maps each first order variable x to the unique position i with $\sigma_i(x) = 1$ and each set variable X to the set of positions i with $\sigma_i(X) = 1$. Clearly the language

$$\mathrm{N}_{\mathcal{V}} = \{(w, \sigma) \in \Sigma_{\mathcal{V}}^* \mid (w, \sigma) \text{ is valid}\}$$

is recognizable (here and later we write $\Sigma_{\mathcal{V}}^*$ for $(\Sigma_{\mathcal{V}})^*$). If $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$, we let

$$L_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in \mathrm{N}_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}.$$

We simply write $\Sigma_\varphi = \Sigma_{\mathrm{Free}(\varphi)}$, $\mathrm{N}_\varphi = \mathrm{N}_{\mathrm{Free}(\varphi)}$, and $L(\varphi) = L_{\mathrm{Free}(\varphi)}(\varphi)$.

By the Büchi-Elgot-Trakhtenbrot theorem [18, 39, 92], a language $L \subseteq \Sigma^*$ is regular if and only if it is definable by some MSO-sentence. In the proof of the implication $\Rightarrow$, given an automaton, one constructs directly an MSO-sentence that defines the language of the automaton. For the other implication, one shows inductively the stronger fact that $L_{\mathcal{V}}(\varphi)$ is regular for each formula φ (where $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$). Our goal is to proceed similarly in the present weighted setting.

We start by defining the syntax of our weighted MSO-logic as in [26, 28] but we include arbitrary negation here.

Definition 7.1. The syntax of formulas of the *weighted MSO-logic* over S and Σ is given by the grammar

$$\begin{aligned}\varphi ::= s \mid \mathrm{P}_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \\ \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists X.\varphi \mid \forall X.\varphi\end{aligned}$$

where $s \in S$ and $a \in \Sigma$. We let $\mathrm{MSO}(S, \Sigma)$ be the collection of all such weighted MSO-formulas φ.

Next we define the $\mathcal{V}$-semantics of formulas $\varphi \in \mathrm{MSO}(S, \Sigma)$ as a series $[\![\varphi]\!]_{\mathcal{V}} \colon \Sigma_{\mathcal{V}}^* \to S$.

Definition 7.2. Let $\varphi \in \mathrm{MSO}(S, \Sigma)$ and $\mathcal{V}$ be a finite set of variables with $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$. The $\mathcal{V}$*-semantics* of φ is the series $[\![\varphi]\!]_{\mathcal{V}} \in S\langle\langle \Sigma_{\mathcal{V}}^* \rangle\rangle$ defined as follows. Let $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$. If (w, σ) is not valid, we put $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = 0$. If (w, σ) with $w = a_1 \dots a_n$ is valid, we define $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) \in S$ inductively as in Table 1. Note that the product $\prod_{i \in \mathrm{dom}(w)}$ is calculated following the natural order of the position in w. For the product $\prod_{X \subseteq \mathrm{dom}(w)}$, we use the lexicographic order on the powerset of $\mathrm{dom}(w)$.

For brevity, we write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$. Note that if φ is a sentence, i.e. $\mathrm{Free}(\varphi) = \emptyset$, then $[\![\varphi]\!] \in S\langle\langle \Sigma^* \rangle\rangle$.

Table 1. $\mathrm{MSO}(S,\Sigma)$ semantics

φ	$[\![\varphi]\!]_{\mathcal{V}}(w,\sigma)$	φ	$[\![\varphi]\!]_{\mathcal{V}}(w,\sigma)$
s	s	$\psi \vee \varrho$	$[\![\psi]\!]_{\mathcal{V}}(w,\sigma) + [\![\varrho]\!]_{\mathcal{V}}(w,\sigma)$
$\mathrm{P}_a(x)$	$\begin{cases} 1 & \text{if } a_{\sigma(x)} = a \\ 0 & \text{otherwise} \end{cases}$	$\psi \wedge \varrho$	$[\![\psi]\!]_{\mathcal{V}}(w,\sigma) \cdot [\![\varrho]\!]_{\mathcal{V}}(w,\sigma)$
$x \leqslant y$	$\begin{cases} 1 & \text{if } \sigma(x) \leqslant \sigma(y) \\ 0 & \text{otherwise} \end{cases}$	$\exists x.\psi$	$\sum_{i \in \mathrm{dom}(w)} [\![\psi]\!]_{\mathcal{V}}(w,\sigma[x \mapsto i])$
$x \in X$	$\begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$	$\forall x.\psi$	$\prod_{i \in \mathrm{dom}(w)} [\![\psi]\!]_{\mathcal{V}}(w,\sigma[x \mapsto i])$
$\neg\psi$	$\begin{cases} 1 & \text{if } [\![\psi]\!]_{\mathcal{V}}(w,\sigma) = 0 \\ 0 & \text{otherwise} \end{cases}$	$\exists X.\psi$	$\sum_{I \subseteq \mathrm{dom}(w)} [\![\psi]\!]_{\mathcal{V}}(w,\sigma[X \mapsto I])$
		$\forall X.\psi$	$\prod_{I \subseteq \mathrm{dom}(w)} [\![\psi]\!]_{\mathcal{V}}(w,\sigma[X \mapsto I])$

Similar definitions of the semantics occur in multivalued logic, cf. [51, 50]. In particular, a similar definition of the semantics of negated formulas is also used for Gödel logics. We give several examples of possible interpretations of weighted formulas:

(1) Let S be an arbitrary bounded distributive lattice $(S, \vee, \wedge, 0, 1)$ with smallest element 0 and largest element 1. In this case, sums correspond to suprema, and products to infima. For instance, we have $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$ for sentences φ, ψ. Thus our logic may be interpreted as a multi-valued logic. In particular, if $S = \mathbb{B}$, the 2-valued Boolean algebra, our semantics coincides with the usual semantics of unweighted MSO-formulas, identifying characteristic series with their supports.

(2) The formula $\exists x.\, \mathrm{P}_a(x)$ counts how often a occurs in the word. Here, *how often* depends on the semiring: e.g., natural numbers, Boolean semiring, integers modulo 2, $\ldots$.

(3) Let $S = (\mathbb{N}, +, \cdot, 0, 1)$ and assume φ does not contain constants $s \in \mathbb{N}$ and negation is applied only to atomic formulas $\mathrm{P}_a(x)$, $x \leqslant y$, or $x \in X$. Then $[\![\varphi]\!](w,\sigma)$ gives the number of ways a machine could present to show that $(w,\sigma) \models \varphi$. Indeed, the machine could proceed inductively over the structure of φ. For the atomic subformulas and their negations, the number should be 1 or 0 depending on whether the formula holds or not. Now, if $[\![\varphi]\!](w,\sigma) = m$ and $[\![\psi]\!](w,\sigma) = n$, the number for $[\![\varphi \vee \psi]\!](w,\sigma)$ should be $m + n$ (since any reason for φ or ψ suffices), and for $[\![\varphi \wedge \psi]\!](w,\sigma)$ it should be $m \cdot n$ (since the machine could pair the reasons for φ resp. ψ arbitrarily). Similarly, the machine could deal with existential and universal quantifications.

(4) The semiring $S = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ is often used for settings with costs

or rewards as weights. For the semantics of formulas, a choice like in a disjunction or existential quantification is resolved by maximum. Conjunction is resolved by a sum of the costs, and $\forall x.\varphi$ can be interpreted by the sum of the costs from all positions x.

(5) Consider the reliability semiring $S = ([0,1], \max, \cdot, 0, 1)$ and $\Sigma = \{a_1, \ldots, a_n\}$. Assume that every letter a_i has a reliability $p_i \in [0,1]$. Let $\varphi = \forall x. \bigvee_{i=1}^{n}(\mathrm{P}_{a_i}(x) \wedge p_i)$. Then $(\llbracket\varphi\rrbracket, w)$ can be considered as the reliability of the word $w \in \Sigma^*$.

(6) PCTL is a well-studied probabilistic extension of computational tree logic CTL that is applied in verification. As shown recently in [12], PCTL can be considered as a fragment of weighted MSO logic.

The following basic consistency property of the semantics definition can be shown by induction over the structure of the formula using also Lemma 6.2.

Proposition 7.1. *Let $\varphi \in \mathrm{MSO}(S,\Sigma)$ and $\mathcal{V}$ be a finite set of variables with* $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$. *Then*

$$\llbracket\varphi\rrbracket_{\mathcal{V}}(w,\sigma) = \llbracket\varphi\rrbracket(w, \sigma|_{\mathrm{Free}(\varphi)})$$

for each valid $(w,\sigma) \in \Sigma^*_{\mathcal{V}}$. Also, the series $\llbracket\varphi\rrbracket$ is recognizable iff $\llbracket\varphi\rrbracket_{\mathcal{V}}$ is recognizable.

Our goal is to compare the expressive power of suitable fragments of $\mathrm{MSO}(S,\Sigma)$ with weighted automata. Crucial for this will be closure properties of recognizable series under the constructs of our weighted logic. In general, neither negation, conjunction, or universal quantification preserves recognizability.

Example 7.1. Let $S = (\mathbb{Z}, +, \cdot, 0, 1)$ be the ring of integers and consider the sentence

$$\varphi = \exists x. P_a(x) \vee ((-1) \wedge \exists x. P_b(x)) \ .$$

Then $(\llbracket\varphi\rrbracket, w)$ is the difference of the numbers of occurrences of a and b in w. Note that $(\llbracket\neg\varphi\rrbracket, w) = 1$ if and only if these numbers are equal, so $\llbracket\neg\varphi\rrbracket = \mathbb{1}_L$ for a non-regular language L. Therefore $\llbracket\neg\varphi\rrbracket$ is not recognizable (see Theorem 9.2 below).

Example 7.2. Let $\Sigma = \{a, b\}$, $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, and $\varphi = \forall x.\big((P_a(x) \wedge \{a\}) \vee (P_b(x) \wedge \{b\})\big)$. With r the series from Example 4.2, $\llbracket\varphi\rrbracket = r$ which is recognizable. On the other hand, $\llbracket\varphi \wedge \varphi\rrbracket = r \odot r$ is not recognizable.

Example 7.3. Let $S = (\mathbb{N}, +, \cdot, 0, 1)$. Then $(\llbracket\exists x.1\rrbracket, w) = |w|$ and $(\llbracket\forall y.\exists x.1\rrbracket, w) = |w|^{|w|}$ for each $w \in \Sigma^*$. So $\llbracket\exists x.1\rrbracket$ is recognizable, but $\llbracket\forall y.\exists x.1\rrbracket$ is not recognizable. Indeed, let $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$ be any weighted automaton over S. Let $M = \max\{\mathrm{in}(p), \mathrm{out}(p), \mathrm{wt}(p,a,q) \mid p, q \in Q,\ a \in \Sigma\}$. Then $(||\mathcal{A}||, w) \leqslant |Q|^{|w|+1} \cdot M^{|w|+2}$ for each $w \in \Sigma^*$, showing $||\mathcal{A}|| \neq \llbracket\forall y.\exists x.1\rrbracket$. Similarly, $(\llbracket\forall X.2\rrbracket, w) = 2^{2^{|w|}}$ for each $w \in \Sigma^*$, and $\llbracket\forall X.2\rrbracket$ is not recognizable due to its growth.

These examples lead us to consider fragments of $\mathrm{MSO}(S,\Sigma)$. As in [12], we define the syntax of *Boolean formulas* of $\mathrm{MSO}(S,\Sigma)$ by

$$\varphi ::= \mathrm{P}_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x.\varphi \mid \forall X.\varphi$$

where $a \in \Sigma$. Note that in comparison to the syntax of $\mathrm{MSO}(\Sigma)$, we only replaced disjunction by conjunction and existential by universal quantifications. Now, clearly, $[\![\varphi]\!]_{\mathcal{V}}(w,\sigma) \in \{0,1\}$ for each Boolean formula φ and $(w,\sigma) \in \Sigma_{\mathcal{V}}^*$ if $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$. Expressing disjunction and existential quantifications by negation and conjunction resp. universal quantifications, for each $\varphi \in \mathrm{MSO}(\Sigma)$ there is a Boolean formula ψ such that $[\![\psi]\!] = \mathbb{1}_{L(\varphi)}$, and conversely. Hence Boolean formulas capture the full power of $\mathrm{MSO}(\Sigma)$.

Now the class of *almost unambiguous formulas* of $\mathrm{MSO}(S,\Sigma)$ is the smallest class containing all constants $s \in S$ and all Boolean formulas which is closed under disjunction, conjunction, and negation.

It is useful to introduce the closely related notion of recognizable step functions: these are precisely the finite sums of series $s\,\mathbb{1}_L$ where $s \in S$ and $L \subseteq \Sigma^*$ is regular. By induction it follows that $[\![\varphi]\!]$ is a recognizable step function for any almost unambiguous formula $\varphi \in \mathrm{MSO}(S,\Sigma)$. Conversely, if $r\colon \Sigma^* \to S$ is a recognizable step function, by the Büchi-Elgot-Trakhtenbrot theorem, we obtain an almost unambiguous sentence φ with $r = [\![\varphi]\!]$.

For $\varphi \in \mathrm{MSO}(S,\Sigma)$, let $\mathrm{const}(\varphi)$ be the set of all elements of S occurring in φ. We recall that two subsets $A, B \subseteq S$ commute, if $a \cdot b = b \cdot a$ for all $a \in A$, $b \in B$.

Definition 7.3. A formula $\varphi \in \mathrm{MSO}(S,\Sigma)$ is *syntactically restricted*, if it satisfies the following conditions:

(1) for all subformulas $\psi \wedge \psi'$ of φ, the sets $\mathrm{const}(\psi)$ and $\mathrm{const}(\psi')$ commute or ψ or ψ' is almost unambiguous,
(2) whenever φ contains a subformula $\forall x.\psi$ or $\neg\psi$, then ψ is almost unambiguous,
(3) whenever φ contains a subformula $\forall X.\psi$, then ψ is Boolean.

We let $\mathrm{srMSO}(S,\Sigma)$ denote the collection of all syntactically restricted formulas from $\mathrm{MSO}(S,\Sigma)$.

Also, a formula $\varphi \in \mathrm{MSO}(S,\Sigma)$ is called *existential*, if it has the form $\exists X_1. \dots \exists X_n.\psi$ where ψ contains only first order quantifiers.

Theorem 7.2 (Droste and Gastin [28]). *Let S be any semiring, Σ an alphabet, and $r\colon \Sigma^* \to S$ a series. The following are equivalent:*

(1) r is recognizable.
(2) $r = [\![\varphi]\!]$ for some syntactically restricted and existential sentence φ of $\mathrm{MSO}(S,\Sigma)$.
(3) $r = [\![\varphi]\!]$ for some syntactically restricted sentence φ of $\mathrm{MSO}(S,\Sigma)$.

Proof (sketch). (1) $\to$ (2): We have $r = ||\mathcal{A}||$ for some weighted automaton $\mathcal{A} = (Q, \mathrm{in}, \mathrm{wt}, \mathrm{out})$. Then we can use the structure of $\mathcal{A}$ to define a sentence φ as required such that $||\mathcal{A}|| = [\![\varphi]\!]$.

(2) $\to$ (3): Trivial.

(3) $\to$ (1): By structural induction we show for each formula $\varphi \in \mathrm{srMSO}(S,\Sigma)$ that $[\![\varphi]\!] = ||\mathcal{A}||$ for some weighted automaton $\mathcal{A}$ over Σ_φ and S_φ where $S_\varphi = \langle \mathrm{const}(\varphi) \rangle$ is the subsemiring of S generated by the set $\mathrm{const}(\varphi)$. For Boolean formulas, this is easy. For disjunction and existential quantification, we use closure properties of the class of recognizable series. For conjunction, the assumption of Definition 7.3(1) and the particular

induction hypothesis allow us to employ the construction from Lemma 4.3. If $\varphi = \forall x.\psi$ where ψ is almost unambiguous, we can use the description of $[\![\psi]\!]$ as a recognizable step function to construct a weighted automaton with the behavior $[\![\varphi]\!]$. $\square$

Note that the case $\varphi = \forall x.\psi$ requires a crucial new construction of weighted automata which does not occur in the unweighted setting since, in general, we cannot reduce (weighted) universal quantification to existential quantification.

A semiring S is *locally finite* if each finitely generated subsemiring is finite. Examples include any bounded distributive lattice, thus in particular all Boolean algebras and the semiring $([0,1], \max, \min, 0, 1)$. Another example is given by $([0,1], \min, \oplus, 1, 0)$ with $x \oplus y = \min(1, x + y)$.

We call a formula $\varphi \in \mathrm{MSO}(S, \Sigma)$ *weakly existential*, if whenever φ contains a subformula $\forall X.\psi$, then ψ is Boolean.

Theorem 7.3 (Droste and Gastin [26, 28]). *Let S be locally finite and $r \colon \Sigma^* \to S$ a series. The following are equivalent:*

(1) r is recognizable.

(2) $r = [\![\varphi]\!]$ for some weakly existential sentence φ of $\mathrm{MSO}(S, \Sigma)$.

If moreover, S is commutative, these conditions are equivalent to the following one:

(3) $r = [\![\varphi]\!]$ for some sentence φ of $\mathrm{MSO}(S, \Sigma)$.

The proof uses the fact that if S is locally finite, then each recognizable series $r \in S\langle\!\langle \Sigma^* \rangle\!\rangle$ can be shown to be a recognizable step function.

Observe that Theorem 7.3 applies to all bounded distributive lattices and to all finite semirings; in particular, with $S = \mathbb{B}$ it contains our starting point, the Büchi-Elgot-Trakhtenbrot theorem, as a very special case.

Given a syntactically restricted formula φ of $\mathrm{MSO}(S, \Sigma)$, by the proofs of Theorem 7.2 we can *construct* a weighted automaton $\mathcal{A}$ such that $||\mathcal{A}|| = [\![\varphi]\!]$ (provided the operations of the semiring S are given in an effective way, i.e., S is *computable*). Since the equivalence problem for weighted automata over computable fields is decidable by Corollary 8.4 below, we obtain:

Corollary 7.4. *Let S be a computable field. Then the equivalence problem whether $[\![\varphi]\!] = [\![\psi]\!]$ for syntactically restricted sentences φ, ψ of $\mathrm{MSO}(S, \Sigma)$ is decidable.*

In contrast, the equivalence problem for weighted automata is undecidable for the semirings $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ (Theorem 8.6). Since the proof of Theorem 7.2 is effective, for these semirings also the equivalence problem for syntactically restricted sentences of $\mathrm{MSO}(S, \Sigma)$ is undecidable.

8 Decidability of "$r_1 = r_2$?"

In this section, we investigate when it is decidable whether two given recognizable series are equal. For this, we assume S to be a computable semiring, i.e., the underlying set of

S forms a decidable set and addition and multiplication can be performed effectively. In the first part, we fix one of the two series to be the constant series with value 0.

Let $P = (\lambda, \mu, \gamma)$ be a linear presentation of dimension Q of the series $r \in S\langle\langle\Sigma^*\rangle\rangle$. For $n \in \mathbb{N}$, let $U_n^P = \langle\{\lambda\mu(w) \mid w \in \Sigma^*, |w| \leqslant n\}\rangle$ and $U^P = \langle\{\lambda\mu(w) \mid w \in \Sigma^*\}\rangle$, so U_n^P and U^P are subsemimodules of $S^{\{1\}\times Q}$. Then $U_0^P \subseteq U_1^P \subseteq U_2^P \cdots \subseteq U^P = \bigcup_{n\in\mathbb{N}} U_n^P$, and each of the semimodules U_n^P is finitely generated.

Lemma 8.1. *The set of all pairs (P, n) such that P is a linear presentation and $U_n^P = U_{n+1}^P$ is recursively enumerable (here, the homomorphism μ from the presentation P is given by its restriction to Σ).*

Proof. Note that $U_n^P = U_{n+1}^P$ if and only if every vector $\lambda\mu(w)$ with $|w| = n+1$ belongs to U_n^P if and only if for each $w \in \Sigma^*$ of length $n+1$,

$$\lambda\mu(w) = \sum_{\substack{v\in\Sigma^*\\|v|\leqslant n}} s_v \lambda\mu(v)$$

for some $s_v \in S$. A non-deterministic Turing-machine can check the solvability of this equation by just guessing the coefficients s_v and checking the required equality. □

Corollary 8.2. *Assume that, for any linear presentation P, U^P is a finitely generated semimodule. Then, from a linear presentation P of dimension Q, one can compute $n \in \mathbb{N}$ with $U_n^P = U^P$ and finitely many vectors $x_1, \dots, x_m \in S^{\{1\}\times Q}$ with $\langle\{x_1, \dots, x_m\}\rangle = U^P$.*

Proof. Since U^P is finitely generated, there is some $n \in \mathbb{N}$ such that $U^P = U_n^P$ and therefore $U_n^P = U_{n+1}^P$. Hence, for some $n \in \mathbb{N}$, the pair (P, n) appears in the list from the previous lemma. Then $U^P = U_n^P = \langle\{\lambda\mu(v) \mid v \in \Sigma^*, |v| \leqslant n\}\rangle$. □

Clearly, every finite semiring satisfies the condition of the corollary above, but not all semirings do.

Example 8.1. Let S be the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ and consider a presentation P with

$$\lambda = \begin{pmatrix}1 & 0\end{pmatrix} \text{ and } \mu(w) = \begin{pmatrix}1 & |w|\\ 0 & 1\end{pmatrix}.$$

Then U_n^P is generated by all the vectors $\begin{pmatrix}1 & m\end{pmatrix}$ for $0 \leqslant m \leqslant n$ so that $\begin{pmatrix}1 & n+1\end{pmatrix} \in U_{n+1}^P \setminus U_n^P$; hence U^P is not finitely generated.

As a positive example, we have the following.

Example 8.2. If S is a skew-field (i.e., a semiring such that $(S, +, 0)$ and $(S\setminus\{0\}, \cdot, 1)$ are groups), then we can consider U_n^P as a vector space. Then the dimensions of the spaces $U_i^P \subseteq S^{\{1\}\times Q}$ are bounded by $|Q|$ and $\dim(U_i^P) \leqslant \dim(U_{i+1}^P)$ implying $U_{|Q|}^P = U^P$. Hence, for any skew-field S, in the corollary above we can set $n = |Q|$.

We only note that all Noetherian rings (that include all polynomial rings in several indeterminates over fields, by Hilbert's basis theorem) satisfy the assumption of Corollary 8.2.

Theorem 8.3 (Schützenberger [85]). *Let S be a computable semiring such that, for any linear presentation P, U^P is a finitely generated semimodule. Then, for a linear presentation P, one can decide whether $||P|| = 0$.*

Proof. We have to decide whether $y\gamma = 0$ for all vectors $y \in U^P$. By the previous lemma, we can compute a finite list $x_1, \ldots, x_m$ of vectors that generate U. So one only has to check whether $x_i\gamma = 0$ for $1 \leqslant i \leqslant m$. □

Example 8.3. If S is a skew-field, a basis of U^P can be obtained in time $|\Sigma| \cdot |Q|^3$ (where the operations in the skew-field S are assumed to require constant time). The algorithm actually computes a prefix-closed set of words $u_1, \ldots, u_{\dim(U^P)}$ such that the vectors $\lambda\mu(u_i)$ form a basis of U^P (cf. [83]). This basis consists of at most $|Q|$ vectors (cf. Example 8.2), each of size $|Q|$. Hence $||P|| = 0$ can be decided in time $|\Sigma||Q|^3$.

If S is a finite semiring, then $U^P = U^P_{|S^Q|}$. Hence the vectors $\lambda\mu(w)$ with $|w| \leqslant |S|^{|Q|}$ form a generating set. To check whether $\lambda\mu(w)\gamma = 0$ for all such words w, time $|\Sigma|^{|S|^{|Q|}}$ suffices. Within the same time bound, one can decide whether $||P|| = 0$ holds.

Corollary 8.4. *Let S be a computable ring such that, for any linear presentation P, U^P is a finitely generated semimodule. Then one can decide for two linear presentations P_1 and P_2 whether $||P_1|| = ||P_2||$.*

Proof. Since S is a ring, there is an element $-1 \in S$ with $x + (-1) \cdot x = 0$ for any $x \in S$. Replacing the initial vector λ from P_2 by $-\lambda$, one obtains a linear presentation for the series $(-1)||P_2||$. This yields a linear presentation P with $||P|| = ||P_1|| + (-1)||P_2||$. Now $||P_1|| = ||P_2||$ if and only if $||P|| = 0$ which is decidable by Theorem 8.3. □

Remark 8.5. Let n_1 and n_2 be the dimensions of P_1 and P_2, respectively. Then the linear presentation P from the proof above can be computed in time $n_1 \cdot n_2$ and has dimension $n_1 + n_2$. If S is a skew-field, then we can therefore decide whether $||P_1|| = ||P_2||$ in time $|\Sigma|(n_1 + n_2)^3$.

Let S be a finite semiring. Then from $s \in S$ and weighted automata for $||P_1||$ and for $||P_2||$, one can construct automata accepting $\{w \in \Sigma^* \mid (||P_i||, w) = s\}$ for $i = 1, 2$. This allows us to decide $||P_1|| = ||P_2||$ in doubly exponential time. If S is a finite ring, this result follows also from the proof of the corollary above and Example 8.3.

However, the following result is in sharp contrast to Corollary 8.4. For two series r and s with values in $\mathbb{N} \cup \{-\infty\}$, we write $r \leqslant s$ if $(r, w) \leqslant (s, w)$ for all words w.

Theorem 8.6 (cf. Krob [63]). *There is a series $r_{\text{good}} \colon \Sigma^* \to \mathbb{N} \cup \{-\infty\}$ such that the sets of weighted automata $\mathcal{A}$ over the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ with $||\mathcal{A}|| = r_{\text{good}}$ (with $r_{\text{good}} \leqslant ||\mathcal{A}||$, resp.) are undecidable.*

We remark that analogous statements hold for the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ (where $r_{\text{good}} \geqslant ||\mathcal{A}||$ is undecidable). As a consequence, the equivalence problem of weighted automata over these two semirings is undecidable (this undecidability was shown by Krob). The original proof by Krob is rather involved reducing Hilbert's 10^{th}

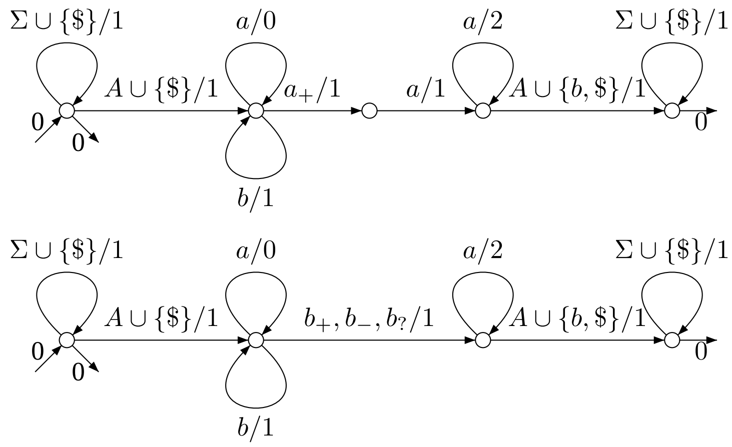

Figure 2. Automata for the proof of Theorem 8.6

problem to the equivalence problem. Colcombet presented a radical simplification at the Dagstuhl seminar "Advances and Applications of Automata on Words and Trees" in 2010 starting from the undecidability of the question whether a 2-counter machine $\mathcal{A}$ accepts the number 0 (this undecidable problem has also been used by Almagor, Boker and Kupferman in [3] to show the undecidabilities of the questions $||\mathcal{A}|| = ||\mathcal{B}||$ and $||\mathcal{A}|| \leqslant ||\mathcal{B}||$ for weighted automata over this semiring). The following is a slight extension of Colcombet's proof that he kindly allowed us to publish in this survey.

Proof. Let $\mathcal{A}$ be a 2-counter machine, i.e., a nondeterministic finite automaton over the alphabet $A = \{a_+, a_-, a_?, b_+, b_-, b_?\}$. For a word $w \in (A \cup \{a, b, \$\})^*$, let $\pi_A(w)$ denote the projection onto A^*.

A counter trace is a word $w \in \$\, a^*(Aa^*b^*)^*\,\$$ such that $\pi_A(w)$ is accepted by the finite automaton $\mathcal{A}$ and, for any maximal factor of the form $a^m b^n c a^{m'} b^{n'}$ with $c \in A$, one of the following holds:

- $c = a_+$, $m' = m + 1$, and $n' = n$
- $c = a_-$, $m' + 1 = m$, and $n' = n$
- $c = a_?$, $m' = m = 0$, and $n' = n$
- $c = b_+$, $m' = m$, and $n' = n + 1$
- $c = b_-$, $m' = m$, and $n' + 1 = n$
- $c = b_?$, $m' = m$, and $n' = n = 0$

Then a number $m \in \mathbb{N}$ is accepted by the 2-counter machine $\mathcal{A}$ if there exists a counter traces $w \in \$a^m(Aa^*b^*)^*\$$. By Minsky's theorem, we can assume that the set of numbers m accepted by $\mathcal{A}$ is undecidable. Let CT denote the set of all counter traces and let $\mathrm{CT}_m = \mathrm{CT} \cap \$a^m(Aa^*b^*)^*\$$ for $m \in \mathbb{N}$.

Note that no counter trace contains any factor from the following set:

$$a_+(A \cup \{b, \$\}) \cup (A \cup \{\$\})b^*a_- \cup ab^*a_? \cup \{a_?a\}$$
$$\cup\, b_+a^*(A \cup \{\$\}) \cup (A \cup \{\$, a\})b_- \cup b_?a^*b \cup \{bb_?\}$$

Therefore, let $w \in \mathrm{CT}_{\mathrm{reg}}$ if $w \in \$a^*(Aa^*b^*)^*\$$ does not contain any such factor and if $\pi_A(w)$ is accepted by the finite automaton $\mathcal{A}$ (note that this set is regular). Furthermore, let $\mathrm{CT}_{\mathrm{reg},m} = \mathrm{CT}_{\mathrm{reg}} \cap \$a^m(Aa^*b^*)^*\$$. We will now construct a recognizable series r

such that $(r,w)=|w|$ for $w\in\mathrm{CT}$, $(r,w)>|w|$ for $w\in\mathrm{CT}_{\mathrm{reg}}\setminus\mathrm{CT}$, and $(r,w)=-\infty$ for $w\notin\mathrm{CT}$.

Consider the first weighted automaton from Fig. 2 (where $\Sigma=A\cup\{a,b\}$). Its behavior maps a word $w\in\$a^*(Aa^*b^*)^*\$$ to

$$\max\{|w|,|w|+\ell\mid\exists k\in\mathbb{N}:w\in\$(\Sigma^*A\cup\{\varepsilon\})a^kb^*a_+a^{1+k+\ell}((A\cup\{b\})\Sigma^*\cup\{\varepsilon\})\$\}\,.$$

If we exchange the weights 0 and 2 at the two a-loops, the behavior for $w\in\$a^*(Aa^*b^*)^*\$$ yields

$$\max\{|w|,|w|+\ell\mid\exists k\in\mathbb{N}:w\in\$(\Sigma^*A\cup\{\varepsilon\})a^kb^*a_+a^{1+k-\ell}((A\cup\{b\})\Sigma^*\cup\{\varepsilon\})\$\}\,.$$

By taking the union of these two weighted automata, we obtain a recognizable series r_{a_+} that maps $w\in\$a^*(Aa^*b^*)^*\$$ to

$$\max\{|w|,|w|+\ell\mid\exists k\in\mathbb{N}:w\in\$(\Sigma^*A\cup\{\varepsilon\})a^kb^*a_+a^{1+k\pm\ell}((A\cup\{b\})\Sigma^*\cup\{\varepsilon\})\$\}\,.$$

i.e., $(r_{a_+},w)=|w|$ if and only if any maximal factor of w of the form $a^mb^*a_+a^{1+n}$ satisfies $m=n$ (and $(r_{a_+},w)>|w|$ otherwise).

Similarly, one can construct a recognizable series r_{a_-} such that, for $w\in\$a^*(Aa^*b^*)^*\$$, we have $(r_{a_-},w)=|w|$ if and only if any maximal factor of w of the form $a^{1+m}b^*a_-a^n$ satisfies $m=n$ (and $(r_{a_-},w)>|w|$ otherwise).

Next consider the second weighted automaton from Fig. 2. Its behavior maps a word $w\in\$a^*(Aa^*b^*)^*\$$ to

$$\max\{|w|,|w|+\ell\mid\ \exists k\in\mathbb{N}:w\in\$(\Sigma^*A\cup\{\varepsilon\})a^kb^*\{b_+,b_-,b_?\}\\ a^{k+\ell}((A\cup\{b\})\Sigma^*\cup\{\varepsilon\})\$\}\,.$$

As above, we get a recognizable series r_a such that, for $w\in\$a^*(Aa^*b^*)^*\$$, we have $(r_a,w)=|w|$ if and only if any maximal factor of w of the form $a^mb^*\{b_+,b_-,b_?\}a^n$ satisfies $m=n$ (and $(r_a,w)>|w|$ otherwise).

Hence, there is a recognizable series r' such that, for a word $w\in\$a^*(Aa^*b^*)^*\$$, we have $(r',w)=|w|$ if and only if any maximal factor of any of the following forms satisfies $m=n$:

$$\begin{array}{ccc} a^mb^*a_+a^{1+n} & a^{1+m}b^*a_-a^n & a^mb^*\{b_+,b_-,b_?\}a^n\\ b^mb_+a^*b^{1+n} & b^{1+m}b_-a^*b^n & b^m\{a_+,a_-,a_?\}a^*b^n \end{array}$$

For all other words $w\in\$a^*(Aa^*b^*)^*\$$, we have $(r',w)>|w|$. From this series, we easily get the recognizable series $r=r'\odot\mathbb{1}_{\mathrm{CT}_{\mathrm{reg}}}$ satisfying

$$(r,w)\begin{cases}=|w| & \text{for } w\in\mathrm{CT}\\ >|w| & \text{for } w\in\mathrm{CT}_{\mathrm{reg}}\setminus\mathrm{CT}\\ =-\infty & \text{otherwise.}\end{cases}$$

Now define the recognizable series r_{good} and r_m for $m\in\mathbb{N}$ as follows:

$$(r_{\mathrm{good}},w)=\max(|w|+1,(r,w))\text{ and }(r_m,w)=\begin{cases}(r,w) & \text{for } w\in\mathrm{CT}_{\mathrm{reg},m}\\ (r_{\mathrm{good}},w) & \text{otherwise.}\end{cases}$$

Then we have for $m \in \mathbb{N}$

$$\begin{aligned} r_{\text{good}} = r_m &\iff (r_{\text{good}}, w) = (r_m, w) \text{ for all } w \in \mathrm{CT}_{\text{reg},m} \\ &\iff (r, w) > |w| \text{ for all } w \in \mathrm{CT}_{\text{reg},m} \\ &\iff \mathrm{CT}_m = \emptyset \\ &\iff m \text{ is not accepted by the 2-counter machine } \mathcal{A}\,. \end{aligned}$$

Since by our assumption on $\mathcal{A}$ this last statement is undecidable, the first claim follows.

Note that $r_m \leqslant r_{\text{good}}$. Hence $r_m = r_{\text{good}}$ is equivalent to saying $r_{\text{good}} \leqslant r_m$. Therefore the second claim holds as well. □

9 Characteristic series and supports

The goal of this section to investigate the regularity of the support of recognizable (characteristic) series.

Lemma 9.1. *Let S be any semiring and $L \subseteq \Sigma^*$ a regular language. Then the characteristic series $\mathbb{1}_L$ of L is recognizable.*

Proof. Take a deterministic finite automaton accepting L and weight the initial state, the transitions, and the final states with 1 and all the non-initial states, the non-transitions, and the non-final states with 0. Since every word has at most one successful path in the deterministic finite automaton, the behavior of the weighted automaton constructed this way is the characteristic series of L over S. □

For all commutative semirings, also the converse of this lemma holds. This was first shown for commutative rings where one actually has the following more general result:

Theorem 9.2 (Schützenberger [85] and Sontag [89]). *Let S be a commutative ring, and let $r \in S^{\text{rec}}\langle\langle\Sigma^*\rangle\rangle$ have finite image. Then $r^{-1}(s)$ is recognizable for any $s \in S$.*

It remains to consider commutative semirings that are not rings. Let S be a semiring. A subset $I \subseteq S$ is called an *ideal*, if for all $a, b \in I$ and $s \in S$ we have $a + b,\ a \cdot s,\ s \cdot a \in I$. Dually, a subset $F \subseteq S$ is called a *filter*, if for all $a, b \in F$ and $s \in S$ we have $a \cdot b,\ s + a \in F$. Given a subset $A \subseteq S$, the smallest filter containing A is the set

$$\mathrm{F}(A) = \{a_1 \cdots a_n + s \mid a_i \in A \text{ for } 1 \leqslant i \leqslant n, \text{ and } s \in S\}\,.$$

Lemma 9.3 (Wang [94]). *Let S be a commutative semiring which is not a ring. Then there is a semiring morphism onto $\mathbb{B}$.*

Proof. Consider the collection $\mathcal{C}$ of all filters F of S with $0 \notin F$. Since S is not a ring, we have $\mathrm{F}(\{1\}) \in \mathcal{C}$. By Zorn's lemma, $(\mathcal{C}, \subseteq)$ contains a maximal element M with $\mathrm{F}(\{1\}) \subseteq M$. We define $h\colon S \to \mathbb{B}$ by letting $h(s) = 1$ if $s \in M$, and $h(s) = 0$ otherwise. Clearly $h(0) = 0$ and $h(1) = 1$.

Now let $a, b \in S$. We claim that $h(a+b) = h(a)+h(b)$. By contradiction, we assume that $a, b \notin M$ but $a + b \in M$. Then $0 \in \mathrm{F}(M \cup \{a\})$ and $0 \in \mathrm{F}(M \cup \{b\})$. Since S is commutative, we have $0 = m \cdot a^n + s = m' \cdot b^{n'} + s'$ for some $m, m' \in M$, $n, n' \in \mathbb{N}$ and $s, s' \in S$. This implies that $0 = m \cdot m' \cdot (a + b)^{n+n'} + s''$ for some $s'' \in S$. But now $a + b \in M$ implies $0 \in M$, a contradiction.

Finally, we claim that $h(a \cdot b) = h(a) \cdot h(b)$. If $a, b \in M$, then also $ab \in M$, showing our claim. Now assume $a \notin M$ but $ab \in M$. As above, we have $0 = m \cdot a^n + s$ for some $m \in M$, $n \in \mathbb{N}$, and $s \in S$. But then $0 = m \cdot a^n \cdot b^n + s \cdot b^n = m \cdot (ab)^n + sb^n \in M$ by $ab \in M$, a contradiction. □

Theorem 9.4 (Wang [94]). *Let S be a commutative semiring and $L \subseteq \Sigma^*$. Then L is regular iff $\mathbb{1}_L$ is recognizable.*

Proof. One implication is part of Lemma 9.1. Now assume that $\mathbb{1}_L$ is recognizable. If S is a ring, the result is immediate by Theorem 9.2. If S is not a ring, by Lemma 9.3 there is a semiring morphism h from S to $\mathbb{B}$. Let $\mathcal{A}$ be a weighted automaton with $||\mathcal{A}|| = \mathbb{1}_L$. In this automaton, replace all weights s by $h(s)$. The behavior of the resulting weighted automaton over the Boolean semiring $\mathbb{B}$ is $\mathbb{1}_L \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$. Hence L is regular. □

Now we turn to supports of arbitrary recognizable series. Already for $S = \mathbb{Z}$, the ring of integers, such a language is not necessarily regular (cf. Example 7.1). One can characterize those semirings for which the support of any recognizable series is regular:

Theorem 9.5 (Kirsten [59]). *For a semiring S, the following are equivalent:*

(1) The support of every recognizable series over S is regular.
(2) For any finitely generated semiring $S' \subseteq S$, there exists a finite semiring S_{fin} and a homomorphism $\eta\colon S' \to S_{\mathrm{fin}}$ with $\eta^{-1}(0) = \{0\}$.

It is not hard to see that positive (i.e., zero-sum- and zero-divisor-free) semirings like $(\mathbb{N}, +, \cdot, 0, 1)$ and locally finite semirings (like $(\mathbb{Z}/4\mathbb{Z})^\omega$ or bounded distributive lattices) satisfy condition (2) and therefore (1). By [60], also zero-sum-free commutative semirings like $\mathbb{N} \times \mathbb{N}$ satisfy condition (1) and therefore (2).

Given a semiring S, by Lemma 9.1, the class $\mathrm{SR}(S)$ of all supports of recognizable series over S contains all regular languages. Closure properties of this class $\mathrm{SR}(S)$ have been studied extensively, see e.g. [11]. A further result is the following.

Theorem 9.6 (Restivo and Reutenauer [81]). *Let S be a field and $L \subseteq \Sigma^*$ a language such that L and its complement $\Sigma^* \setminus L$ both belong to $\mathrm{SR}(S)$. Then L is regular.*

In contrast, we note the following result which was also observed by Kirsten:

Theorem 9.7. *There exists a semiring S such that $L \in \mathrm{SR}(S)$ (and even $\mathbb{1}_L$ is recognizable) for any language L over any finite alphabet Σ.*

Proof. Let $\Gamma = \{a, b\}$ and $\Gamma_\$ = \Gamma \cup \{\$\}$. Furthermore, let $\overline{\Gamma_\$} = \{\overline{\gamma} \mid \gamma \in \Gamma_\$\}$ be a disjoint copy of $\Gamma_\$$. The elements of the semiring S are the subsets of $\overline{\Gamma_\$}^*\Gamma_\$^*$ and the addition of S is the union of these sets (with neutral element $\emptyset$). To define multiplication, let $L, M \in S$. Then $L \odot M$ consists of all words $uv \in \overline{\Gamma_\$}^{*}\Gamma_\$^{*}$ such that there exists a word $w \in \Gamma_\$^{*}$ with $uw \in L$ and $\overline{w}^{\mathrm{rev}}v \in M$. Alternatively, multiplication of L and M can be described as follows: concatenate any word from L with any word from M, delete any factors of the form $c\overline{c}$ for $c \in \Gamma_\$$, and place the result into $L \odot M$ if and only if it belongs to $\overline{\Gamma_\$}^{*}\Gamma_\$^{*}$. For instance, we have

$$\begin{aligned} \{\overline{a}b\$\} \cdot \{\overline{\$}a, \overline{\$}\overline{b}a, \overline{a}\} &= \{\overline{a}b\$\overline{\$}a, \quad \overline{a}b\$\overline{\$}\overline{b}a, \quad \overline{a}b\$\overline{a}\} \text{ and} \\ \{\overline{a}b\$\} \odot \{\overline{\$}a, \overline{\$}\overline{b}a, \overline{a}\} &= \{\overline{a}ba, \qquad \overline{a}a\} \end{aligned}$$

since the above procedure, when applied to $\overline{a}b\$$ and $\overline{a}$, results in $\overline{a}b\$\overline{a} \notin \overline{\Gamma_\$}^{*}\Gamma_\$^{*}$. Then it is easily verified that $(S, \cup, \odot, \emptyset, \{\varepsilon\})$ is a semiring.

Now let $L \subseteq \Gamma^*$. Define the linear presentation $P = (\lambda, \mu, \gamma)$ of dimension 1 as follows:

$$\begin{aligned} \lambda_1 &= \{\$\} \odot L^{\mathrm{rev}} \\ \mu(a)_{11} &= \{\overline{a}\} \text{ for } a \in \Gamma \\ \gamma_1 &= \{\overline{\$}\} \end{aligned}$$

For $v \in \Gamma^*$, one then obtains

$$(||P||, v) = \{\$\} \odot L^{rev} \odot \{\overline{v}\} \odot \{\overline{\$}\} = \begin{cases} \{\varepsilon\} & \text{if } v \in L \\ \emptyset & \text{otherwise.} \end{cases}$$

This proves that the characteristic series of L is recognizable for any $L \subseteq \Gamma^*$. To obtain this fact for any language $L \subseteq \Sigma^*$, let $h\colon \Sigma^* \to \Gamma^*$ be an injective homomorphism. Then

$$\mathbb{1}_L = \mathbb{1}_{h(L)} \circ h$$

which is recognizable by Lemma 6.2(1). □

An open problem is to characterize those (non-commutative) semirings S for which the support of every *characteristic* and recognizable series is regular.

10 Further results

Above, we could only touch on a few selected topics from the rich area of weighted automata. In this section, we wish to give pointers to many other research results and directions. For details as well as further topics, we refer the reader to the books [38, 84, 66, 11, 82] and to the recent handbook [30] with extensive surveys including open problems.

Recognizability Some authors use linear presentations to define recognizable series [11].

The transition relation of weighted automata given in this chapter can alternatively be considered as a $Q \times Q$-matrix whose entries are functions from Σ to S. A more general approach is presented in [83, 82] where the entries are functions from Σ^* to S. Here, the free monoid Σ^* can even be replaced by an arbitrary monoid with a length function.

The surveys [40, 42, 43] contain an axiomatic treatment of iteration and weighted automata using the concept of Conway semirings (i.e., semirings equipped with a suitable *-operation).

The abovementioned books contain many further properties of recognizable series including minimization, Fatou-properties, growth behavior, relationship to coding, and decidability and undecidability results.

The coincidence of aperiodic, starfree, and first-order definable languages [86, 73] has counterparts in the weighted setting [26, 27] for suitable semirings. An open problem would be to investigate the relationship between dot-depth and quantifier-alternation (as in [90] for languages). Recently, the expressive power of weighted pebble automata and nested weighted automata was show to equal that of a weighted transitive closure logic [13].

Recall that the distributivity of semirings permitted us to employ representations and algebraic proofs for many results. Using automata-theoretic constructions, one can obtain Kleene and Büchi type characterizations of recognizable series for strong bimonoids [35]. These strong bimonoids can be viewed as semirings without distributivity assumption, also cf. [32].

Weighted pushdown automata A huge amount of research has dealt with weighted versions of pushdown automata and of context-free grammars. The books [84, 66] and the chapters [64, 78] survey the theory and also infer purely language-theoretic decidability results on unambiguous context-free languages. The list of equivalent formalisms (weighted pushdown automata, weighted context-free grammars, systems of algebraic equations) has recently been extended by a weighted logic [72].

Quantitative automata Motivated by practical questions on the behavior of technical systems, new kinds of behaviors of weighted automata have been investigated [20, 21]. E.g., the run weight of a path could be the average of the weights of the transitions. Various decidability and undecidability results, closure properties, and properties of the expressive powers of these models have been established [20, 21, 32].

Discrete structures Weighted tree automata and transducers have been investigated, e.g., for program analysis and transformation [87] and for description logics [7]. Their investigation, e.g. [10, 15, 16, 65, 36], was also guided by results on weighted word automata and on tree transducers, for an extensive survey see [47].

Distributed behaviors can be modelled by Mazurkiewicz traces. The well-established theory of recognizable languages of traces [25] has a weighted counterpart including a weighted distributed automata model [45].

Automata models for other discrete structures like pictures [48], nested words [5], texts [37, 54], and timed words [4] have been studied extensively. Corresponding weighted automata models and their expressive power have been investigated in [44, 72, 71, 33, 79].

Weighted automata on infinite words were investigated for image processing [24] and used as devices to compute real functions [23]. A discounting parameter was employed in [31, 34] in order to calculate the run weight of an infinite path. This led to Kleene-Schützenberger and logical descriptions of the resulting behaviors. Alternatively, semirings with infinitary sum and product operations allow us to define the behavior analogously to the finitary case and to obtain corresponding results [41, 28]. Also the quantitative automata from above have been investigated for infinite words employing, e.g., accumulation points of averages to define the run weight of infinite paths [20, 21, 32]. Weighted Muller automata on ω-trees were studied in [7, 80, 70].

Applications Since the early 90s, weighted automata have been used for compressed representations of images and movies which led to various algorithms for image transformation and processing, cf. [56, 1] for surveys.

Practical tools for multi-valued model checking have been developed based on weighted automata over De Morgan algebras, cf. [22, 17, 67]. De Morgan algebras are particular bounded distributive lattices and therefore locally finite semirings. Weighted automata have also been crucially used to automatically prove termination of rewrite systems, cf. [93] for an overview.

In network optimization problems, the max-plus-semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ is often employed, see the corresponding chapter in this Handbook.

For quantitative evaluations, reachability questions, and scheduling optimization in real-time systems, timed automata with cost functions form a vigorous current research field [8, 6, 14].

In natural language processing, an interesting strand of applications is developing where weighted tree automata play a central role, cf. [62, 69] for surveys. Toolkits for handling weighted automata models are described in [61, 2]. A survey on algorithms for weighted automata with references to many further applications is given in [74].

We close with three examples where weighted automata were employed to solve long-standing open questions in language theory. First, the equivalence of deterministic multi-tape automata was shown to be decidable in [52], cf. also [83]. Second, the equality of an unambiguous context-free language and a regular language can be decided using weighted pushdown automata [88], cf. also [76]. Third, the decidability and complexity of determining the star-height of a regular language were determined using a variant of weighted automata [53, 58].

References

[1] J. Albert and J. Kari. Digital image compression. In Droste et al. [30], chapter 11.

[2] C. Allauzen, M. Riley, J. Schalkwyk, W. Skut, and M. Mohri. Openfst: a general and efficient weighted finite-state transducer library. In *CIAA'07*, Lecture Notes in Computer Science vol. 4783, pages 11–23. Springer, 2007.

[3] S. Almagor, U. Boker, and O. Kupferman. What's decidable about weighted automata? In *ATVA 2011*, volume 6996 of *Lecture Notes in Computer Science*. Springer, 2011.

[4] R. Alur and D. Dill. A theory of timed automata. *Theoretical Computer Science*, 126:183–235, 1994.

[5] R. Alur and P. Madhusudan. Adding nesting structure to words. In *DLT'06*, Lecture Notes in Computer Science vol. 4036, pages 1–13. Springer, 2006.

[6] R. Alur, S. L. Torre, and G. Pappas. Optimal paths in weighted timed automata. *Theoretical Computer Science*, 318:297–322, 2004.

[7] F. Baader and R. Peñaloza. Automata-based axiom pinpointing. *Journal of Automated Reasoning*, 45(2):91–129, 2010.

[8] G. Behrmann, A. Fehnker, T. Hune, K. Larsen, P. Pettersson, J. Romijn, and F. Vaandrager. Minimum-cost reachability for priced timed automata. In *HSCC'01*, Lecture Notes in Computer Science vol. 2034, pages 147–161. Springer, 2001.

[9] J. Berstel. *Transductions and Context-Free Languages*. Teubner Studienbücher, Stuttgart, 1979.

[10] J. Berstel and C. Reutenauer. Recognizable formal power series on trees. *Theoretical Computer Science*, 18:115–148, 1982.

[11] J. Berstel and C. Reutenauer. *Rational Series and Their Languages*. Springer, 1988. An augmented and updated version appeared as *Noncommutative Rational Series With Applications*, Cambridge University Press, 2010.

[12] B. Bollig and P. Gastin. Weighted versus probabilistic logics. In V. Diekert and D. Nowotka, editors, *Developments in Language Theory*, volume 5583 of *Lecture Notes in Computer Science*, pages 18–38. Springer Berlin / Heidelberg, 2009.

[13] B. Bollig, P. Gastin, B. Monmege, and M. Zeitoun. Pebble weighted automata and transitive closure logics. In *ICALP'10*, Lecture Notes in Computer Science vol. 6199, pages 587–598. Springer, 2010.

[14] P. Bouyer, U. Fahrenberg, K. Larsen, N. Markey, and J. Srba. Infinite runs in weighted timed automata with energy constraints. In *FORMATS'08*, Lecture Notes in Computer Science vol. 5215, pages 33–47. Springer, 2008.

[15] S. Bozapalidis. Effective construction of the syntactic algebra of a recognizable series on trees. *Acta Informatica*, 28:351–363, 1991.

[16] S. Bozapalidis. Representable tree series. *Fundamenta Informaticae*, 21:367–389, 1994.

[17] G. Bruns and P. Godefroid. Model checking with multi-valued logics. In *ICALP'04*, Lecture Notes in Computer Science vol. 3142, pages 281–293. Springer, 2004.

[18] J. R. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.

[19] R. G. Bukharaev. *Theorie der stochastischen Automaten*. Teubner, 1995.

[20] K. Chatterjee, L. Doyen, and T. Henzinger. Quantitative languages. In *CSL'08*, Lecture Notes in Computer Science vol. 5213, pages 385–400. Springer, 2008.

[21] K. Chatterjee, L. Doyen, and T. Henzinger. Expressiveness and closure properties for quantitative languages. In *LICS'09*, pages 199–208. IEEE Comp. Soc. Press, 2009.

[22] M. Chechik, B. Devereux, and A. Gurfinkel. Model-checking infinite state-space systems with fine-grained abstractions using SPIN. In *SPIN'01*, Lecture Notes in Computer Science vol. 2057, pages 16–36. Springer, 2001.

[23] K. Čulik II and J. Karhumäki. Finite automata computing real functions. *SIAM J. of Computing*, pages 789–814, 1994.

[24] K. Čulik II and J. Kari. Image compression using weighted finite automata. *Computer and Graphics*, 17(3):305–313, 1993.

[25] V. Diekert and G. Rozenberg, editors. *The Book of Traces.* World Scientific Publ. Co., 1995.

[26] M. Droste and P. Gastin. Weighted automata and weighted logics. *Theoretical Computer Science*, 380:69–86, 2007.

[27] M. Droste and P. Gastin. On aperiodic and star-free formal power series in partially commuting variables. *Theory Comput. Syst.*, 42(4):608–631, 2008.

[28] M. Droste and P. Gastin. Weighted automata and weighted logics. In Droste et al. [30], chapter 5.

[29] M. Droste and W. Kuich. Semirings and formal power series. In Droste et al. [30], chapter 1.

[30] M. Droste, W. Kuich, and H. Vogler, editors. *Handbook of Weighted Automata*. EATCS Monographs in Theoretical Computer Science. Springer, 2009.

[31] M. Droste and D. Kuske. Skew and infinitary formal power series. *Theoretical Computer Science*, 366:199–227, 2006.

[32] M. Droste and I. Meinecke. Describing average- and longtime-behavior by weighted MSO logics. In *MFCS'10*, Lecture Notes in Computer Science vol. 6281, pages 537–548. Springer, 2010.

[33] M. Droste and K. Quaas. A Kleene-Schützenberger theorem for weighted timed automata. *Theoretical Computer Science*, 412:1140–1153, 2011.

[34] M. Droste and G. Rahonis. Weighted automata and weighted logics with discounting. *Theoretical Computer Science*, 410:3481–3494, 2009.

[35] M. Droste and H. Vogler. Kleene and Büchi theorems for weighted automata and multi-valued logics over arbitrary bounded lattices. In *DLT'10*, Lecture Notes in Computer Science vol. 6224, pages 160-172. Springer, 2010.

[36] M. Droste and H. Vogler. Weighted logics for unranked tree automata. *Theory of Computing Systems*, 48:23–47, 2011.

[37] A. Ehrenfeucht and G. Rozenberg. T-structures, T-functions, and texts. *Theoretical Computer Science*, 116:227–290, 1993.

[38] S. Eilenberg. *Automata, Languages, and Machines*, volume A. Academic Press, 1974.

[39] C. C. Elgot. Decision problems of finite automata design and related arithmetics. *Trans. Amer. Math. Soc.*, 98:21–51, 1961.

[40] Z. Ésik. Fixed point theories. In Droste et al. [30], chapter 2.

[41] Z. Ésik and W. Kuich. A semiring-semimodule generalization of ω-regular languages I+II. *Journal of Automata, Languages and Combinatorics*, 10:203–242 and 243–264, 2005.

[42] Z. Ésik and W. Kuich. Finite automata. In Droste et al. [30], chapter 3.

[43] Z. Ésik and W. Kuich. A unifying Kleene theorem for weighted finite automata. In C. Calude, G. Rozenberg, and A. Salomaa, editors, *Rainbow of Computer Science*, pages 76–89. Springer, 2011.

[44] I. Fichtner. Weighted picture automata and weighted logics. *Theory of Computing Systems*, 48(1):48–78, 2011.

[45] I. Fichtner, D. Kuske, and I. Meinecke. Traces, series-parallel posets, and pictures: A weighted study. In Droste et al. [30], chapter 10.

[46] M. Fliess. Matrices de Hankel. *Journal de Mathématiques Pures et Appliquées*, 53:197–222, 1974. Erratum in: *Journal de Mathématiques Pures et Appliquées*, 54:481, 1976.

[47] Z. Fülöp and H. Vogler. Weighted tree automata and tree transducers. In Droste et al. [30], chapter 9.

[48] D. Giammarresi, A. Restivo, S. Seibert, and W. Thomas. Monadic second-order logic over rectangular pictures and recognizability by tiling systems. *Inform. and Comput.*, 125:32–45, 1996.

[49] J. Golan. *Semirings and their Applications*. Kluwer Academic Publishers, 1999.

[50] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, 2001.

[51] P. Hájek. *Metamathematics of Fuzzy Logic*. Springer, 1998.

[52] T. Harju and J. Karhumäki. The equivalence problem of multitape finite automata. *Theoretical Computer Science*, 78:347–255, 1991.

[53] K. Hashigushi. Improved limitedness theorem on finite automata with distance functions. *Theoretical Computer Science*, 72:27–38, 1990.

[54] H. J. Hoogeboom and P. ten Pas. MSO definable text languages. In *MFCS'94*, Lecture Notes in Computer Science vol. 841, pages 413–422. Springer, 1994.

[55] G. Jacob. Représentations et substitutions matricielles dans la théorie algébraique des transductions. Thèse Sci. Math. Univ. Paris VII, 1975.

[56] J. Kari. Image processing using finite automata. In Z. Ésik, C. M. Vide, and C. Mitrana, editors, *Recent Advances in Formal Languages and Applications*, Studies in Computational Intelligience, pages 171–208. Springer, 2006.

[57] B. Khoussainov and A. Nerode. *Automata Theory and Its Applications*. Birkhäuser Boston, 2001.

[58] D. Kirsten. Distance desert automata and the star height problem. *RAIRO*, 39(3):455–509, 2005.

[59] D. Kirsten. An algebraic characterization of semirings for which the support of every recognizable series is recognizable. In *MFCS'09*, Lecture Notes in Comp. Science vol. 5734, pages 489–500. Springer, 2009.

[60] D. Kirsten. The support of a recognizable series over a zero-sumfree, commutative semiring is recognizable. In *DLT'09*, Lecture Notes in Computer Science vol. 5583, pages 326–333. Springer, 2009.

[61] K. Knight and J. May. Tiburon: A weighted tree automata toolkit. In *CIAA'06*, Lecture Notes in Computer Science vol. 4094, pages 102–113. Springer, 2006.

[62] K. Knight and J. May. Applications of weighted automata in natural language processing. In Droste et al. [30], chapter 14.

[63] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *International Journal of Algebra and Computation*, 4(3):405–425, 1994.

[64] W. Kuich. Semirings and formal power series: their relevance to formal languages and automata. In *Handbook of Formal Languages, vol. 1: Word, Language, Grammar*, pages 609–677. Springer, 1997.

[65] W. Kuich. Tree transducers and formal tree series. *Acta Cybernetica*, 14:135–149, 1999.

[66] W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 5 of *EATCS Monographs in Theoretical Computer Science*. Springer, 1986.

[67] O. Kupferman and Y. Lustig. Lattice automata. In *VMCAI'07*, Lecture Notes in Computer Science vol. 4349, pages 199–213. Springer, 2007.

[68] D. Kuske. Schützenberger's theorem on formal power series follows from Kleene's theorem. *Theoretical Computer Science*, 401:243–248, 2008.

[69] A. Maletti. Survey: Weighted extended top-down tree transducers — part II: Application in machine translation. *Fundam. Inform.*, 112(2-3):239-261, 2011.

[70] E. Mandrali and G. Rahonis. Recognizable tree series with discounting. *Acta Cybernetica*, 19:411–439, 2009.

[71] C. Mathissen. Definable transductions and weighted logics for texts. *Theoretical Computer Science*, 411(3):631–659, 2010.

[72] C. Mathissen. Weighted logics for nested words and algebraic formal power series. *Logical Methods in Computer Science*, 6(1:5):1–34, 2010.

[73] R. McNaughton and S. A. Papert. *Counter-Free Automata*. M.I.T. research monograph no. 65. The MIT Press, 1971.

[74] M. Mohri. Weighted automata algorithms. In Droste et al. [30], chapter 6.

[75] M. Nivat. Transductions des langages de Chomsky. *Ann. de l'Inst. Fourier*, 18:339–456, 1968.

[76] A. Panholzer. Gröbner bases and the definining polynomial of a context-free grammar generating function. *Journal of Automata, Languages and Combinatorics*, 10(1):79–97, 2005.

[77] A. Paz. *Introduction to Probabilistic Automata*. Academic Press, 1971.

[78] I. Petre and A. Salomaa. Algebraic systems and pushdown automata. In Droste et al. [30], chapter 7.

[79] K. Quaas. MSO logics for weighted timed automata. *Formal Methods in System Design*, 38(3):193–222, 2011.

[80] G. Rahonis. Weighted Muller tree automata and weighted logics. *Journal of Automata, Languages and Combinatorics*, 12(4):455–483, 2007.

[81] A. Restivo and C. Reutenauer. On cancellation properties of languages which are supports of rational power series. *J. Comput. Syst. Sci.*, 29(2):153–159, 1984.

[82] J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2009.

[83] J. Sakarovitch. Rational and recognisable power series. In Droste et al. [30], chapter 4.

[84] A. Salomaa and M. Soittola. *Automata-Theoretic Aspects of Formal Power Series*. Texts and Monographs in Computer Science. Springer, 1978.

[85] M. P. Schützenberger. On the definition of a family of automata. *Inf. Control*, 4:245–270, 1961.

[86] M. P. Schützenberger. On finite monoids having only trivial subgroups. *Inf. and Control*, 8:190–194, 1965.

[87] H. Seidl. Finite tree automata with cost functions. *Theoret. Comput. Sci.*, 126(1):113–142, 1994.

[88] A. Semenov. Algorithmitšeskie problemy dlja stepennykh rjadov i kontekst-nosvobodnykh grammatik. *Dokl. Adad. Nauk SSSR*, 212:50–52, 1973.

[89] E. D. Sontag. On some questions of rationality and decidability. *Journal of Computer and System Sciences*, 11(3):375–381, 1975.

[90] W. Thomas. Classifying regular events in symbolic logic. *Journal of Computer and System Sciences*, 25:360–376, 1982.

[91] W. Thomas. Languages, automata, and logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, pages 389–455. Springer Verlag, 1997.

[92] B. Trakhtenbrot. Finite automata and logic of monadic predicates (in Russian). *Doklady Akademii Nauk SSSR*, 140:326–329, 1961.

[93] J. Waldmann. Automatic termination. In R. Treinen, editor, *RTA*, volume 5595 of *Lecture Notes in Computer Science*, pages 1–16. Springer, 2009.

[94] H. Wang. On characters of semirings. *Houston J. Math*, 23(3):391–405, 1997.

[95] W. Wechler. *The Concept of Fuzzyness in Automata and Language Theory*. Akademie Verlag, Berlin, 1978.