On full groups of measure preserving and ergodic transformations with uncountable cofinalities*

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Abstract

The group of all measure preserving permutations of the unit interval and the full group of an ergodic transformation of the unit interval are shown to have uncountable cofinality and the Bergman property. Here, a group G is said to have the Bergman property, if for any generating subset E of G, already some bounded power of $E \cup E^{-1} \cup \{1\}$ covers G. This property arose in a recent interesting paper of Bergman where it was derived for the infinite symmetric groups. We give a general sufficient criterion for groups G to have the Bergman property. We show that the criterion applies to a range of other groups, including sufficiently transitive groups of measure preserving, non-singular, or ergodic transformations of the reals; it also applies to large groups of homeomorphisms of the rationals, the irrationals, or the Cantor set.

1 Introduction

Groups of measurable or ergodic transformations of the reals and their algebraic properties have been investigated intensively. Fathi [Fat78] showed that the group of measure-preserving transformations of the unit interval is simple, and Eigen [Eig81] derived this in the group of non-singular transformations of the reals for the full subgroups of any ergodic transformation. Already Anderson [And58] proved that certain homeomorphism groups of Hausdorff spaces like Cantor's space, or the rationals, are simple.

It is the goal of this paper to investigate these groups and sufficiently large subgroups with respect to their cofinality and the Bergman property. An infinite group G is said to have *uncountable cofinality*, if it cannot be expressed as the union of a countable ascending chain of proper subgroups. Serre [Ser80] considered groups of uncountable cofinality in his study of groups acting on trees. Sabbagh [Sab75] showed that any ω_1-existentially closed group has uncountable cofinality, and Macpherson and Neumann [MN90] derived this for the full symmetric group $\mathrm{Sym}(\Omega)$ of all permutations of an infinite set Ω; for further various results see [Gou92], [HHLS93], [Tho96], [Tho97], [DG02]. In a recent interesting paper, Bergman [Ber06] showed that the infinite symmetric groups $\mathrm{Sym}(\Omega)$ satisfy a property which we call

*MSC Primary : 20F05, 28D15; Secondary : 37A15, 22F10, 20B22 the *Bergman property* of a group G: for any generating subset E of G with $E = E^{-1}$ and $1 \in E$ there is a natural number k such that $G = E^k$. Soon after [Ber06], the Bergman property was established also for the homeomorphism groups of Cantor's set, the rationals and the irrationals [DG05], the order-automorphism group of the reals [DH05], the general linear groups of infinite-dimensional vector spaces [Tol06], and several other classes of groups, see [Mil04], [Ros05], [dC06], [KR07].

Here, we will investigate the groups of measure-preserving transformations of the reals and of the unit interval, groups of non-singular transformations and full groups of ergodic transformations. We will show that all these groups have both uncountable cofinality and the Bergman property. In order to achieve a uniform approach, we first prove a general result which gives a new sufficient condition for permutation groups G of an infinite set Ω to have uncountable cofinality and to satisfy the Bergman property. In fact, as shown in [DH05], a group G satisfies both of these two properties if and only if G has *uncountable strong cofinality*, i.e., G cannot be expressed as the union of a countable ascending chain of proper subsets H_i of G such that for each $i \in \omega$ we have $H_i = H_i^{-1}$ and $H_i H_i \subseteq H_j$ for some $j \geq i$. A sufficient criterion for permutation groups G to have uncountable strong cofinality was given in [DG05], but it relies on information about the normal subgroup structure of the group G which is often intricate to obtain. The present criterion avoids this and thus is simpler to use. We show that this criterion applies, besides to the groups mentioned before, to a range of groups of measure-preserving transformations or homeomorphisms of the reals or its subsets (including the groups investigated in [DG05]); thus all these groups have uncountable cofinality and the Bergman property.

2 Strong cofinality and the Bergman property

Let G be a group that is not finitely generated. A subset $H \subseteq G$ is *symmetric* if $H = H^{-1}$. A chain of subsets of G is *exhaustive* if it has union G. The *cofinality* of G is the smallest cardinal κ such that there is an exhaustive ascending chain $(H_i)_{i\in\kappa}$ of proper subgroups of G. Similarly, the *strong cofinality* of G is the smallest cardinal κ such that there is an exhaustive ascending chain $(H_i)_{i\in\kappa}$ of proper and symmetric subsets of G such that for each $i \in \kappa$ there is some $j \in \kappa$ with $H_i H_i \subseteq H_j$. Finally, G has the *Bergman property* if for each symmetric generating set $E \subseteq G$ with $1 \in E$ there is some $k \geq 1$ with $G = E^k$. As has been shown in [DH05], the strong cofinality of G is uncountable if and only if G has the Bergman property and uncountable cofinality.

Let Ω be a set and G a group of permutations of Ω. If $g \in G$, let $\operatorname{supp} g = \{x \in \Omega \mid xg \neq x\}$, the support of g. For $\Sigma \subseteq \Omega$ we denote by $G(\Sigma)$ the group of all elements of G having support inside Σ, i.e., $G(\Sigma) = \{g \in G \mid \operatorname{supp} g \subseteq \Sigma\}$. By id_Σ we denote the identity map on Σ. Now let $\mathfrak{K} \subseteq 2^\Omega$ be a non-empty collection of subsets of Ω. Let $\Gamma \subseteq \Omega$ and I a countable index set. A collection $(\Gamma_i)_{i\in I} \subseteq \mathfrak{K}$ of pairwise disjoint subsets of Γ is a $\mathfrak{K}$-*subpartition of* Γ. We say that $(\Gamma_i)_{i\in I}$ satisfies *local patching* if for each sequence $(g_i)_{i\in I} \subseteq G$ with $\Gamma_i g_i = \Gamma_i$ we have that $\cup_{i\in I} g_{i\restriction\Gamma_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in I}\Gamma_i} \in G$. Next consider the following conditions on G and $\mathfrak{K}$.

(A) There is a $\mathfrak{K}$-subpartition of Ω that permits local patching.

(B) For each $\Delta \in \mathfrak{K}$ there are some $N \geq 1$, some $\Sigma \in \mathfrak{K}$ with $\Sigma \subseteq \Delta$ and some $g \in G(\Delta)$ such that each element of $G(\Sigma)$ is a product of at most N conjugates of g or g^{-1} with conjugating elements from $G(\Delta)$.

(C) For each $\Sigma \in \mathfrak{K}$ there are some $M \geq 1$ and some finite set $F \subseteq G$ such that $G = \big(G(\Sigma) \cup F\big)^M$.

In this section we will show:

Theorem 2.1. *If G and $\mathfrak{K}$ satisfy the conditions (A)-(C), then G has uncountable strong cofinality.*

Droste and Göbel [DG05] define similar conditions on G and $\mathfrak{K}$. We want to point out that (B) is a much weaker version of their concept of *bounded conjugacy.* Essentially, we are satisfied by finding *some* $g \in G(\Delta)$ such that each element of $G(\Sigma)$ is a product of a bounded number of conjugates of $g^{\pm 1}$, whereas bounded conjugacy requires the same to hold for *all* $g \in G(\Delta)$ that have "large" support (cf. [DG05]). On the other hand, condition (C) is quite strong. However, later on, the groups we are interested in turn out to admit easy proofs of (C).

Assume that $(G, \mathfrak{K})$ satisfies (A)-(C). We establish the proof of Theorem 2.1 through a series of auxiliary results. For this we assume we are given an arbitrary countable exhaustive ascending chain $(H_i)_{i\in\omega}$ of symmetric subsets of G such that for all $i \in \omega$ we have $H_i H_i \subseteq H_j$ for some $j \in \omega$. Generally, for $H \subseteq G$ we say that H is *captured (by the chain)* if $H \subseteq H_i$ for some $i \in \omega$, and a set $\Sigma \subseteq \Omega$ is *full* for H if for every $g \in G(\Sigma)$ there is some $h \in H$ agreeing with g on Σ.

Lemma 2.2. *There are $i \in \omega$ and $\Delta \in \mathfrak{K}$ such that Δ is full for H_i.*

Proof. By (A) there is a $\mathfrak{K}$-subpartition $(\Gamma_i)_{i\in\omega}$ of Ω that permits local patching. Now suppose that no Γ_i is full for H_i. Then for each $i \in \omega$ choose $g_i \in G(\Gamma_i)$ which is not induced on Γ_i by any element of H_i. The map $g = \cup_{i\in\omega} g_{i\restriction\Gamma_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in I}\Gamma_i}$ lies in G. Since the chain $(H_i)_{i\in\omega}$ is exhaustive, there is some $n \in \omega$ with $g \in H_n$, but then $g_{\restriction\Gamma_n} = g_{n\restriction\Gamma_n}$, a contradiction. Hence $\Delta = \Gamma_i$ is full for H_i for some $i \in \omega$. □

Lemma 2.3. *There is some $\Sigma \in \mathfrak{K}$ such that $G(\Sigma)$ is captured by the chain $(H_i)_{i\in\omega}$.*

Proof. Let $i \in \omega$ and $\Delta \in \mathfrak{K}$ as provided by Lemma 2.2. We apply (B) and obtain some $N \geq 1$, some $\Sigma \in \mathfrak{K}$ with $\Sigma \subseteq \Delta$ and some $g \in G(\Delta)$ with the properties mentioned there. There is some $j_1 \in \omega$ with $g \in H_{j_1}$. Choose j large enough such that $(H_i H_{j_1} H_i)^n \subseteq H_j$ for all $n = 1, \ldots, N$. Now let $p \in G(\Sigma)$ be arbitrary. With (B) we find some $1 \leq n \leq N$ and $q_1, \ldots, q_n \in G(\Delta)$ such that $p = (g^{\pm 1})^{q_1} \cdots (g^{\pm 1})^{q_n}$. Since Δ is full for H_i there are $h_k \in H_i$ such that $h_{k\restriction\Delta} = q_{k\restriction\Delta}$ for all $k = 1, \ldots, n$. Because of $p_{\restriction\Delta^c} = g_{\restriction\Delta^c} = \mathrm{id}_{\Delta^c}$ we have $p = (g^{\pm 1})^{h_1} \cdots (g^{\pm 1})^{h_n} \in H_j$ and we conclude $G(\Sigma) \subseteq H_j$. □

Proof of Theorem 2.1. With Lemma 2.3 we find some $\Sigma \in \mathfrak{K}$ such that $G(\Sigma)$ is captured by the chain $(H_i)_{i\in\omega}$. Now let M and F be given as in (C). Since F is finite, $G(\Sigma) \cup F$ and later $\big(G(\Sigma) \cup F\big)^M$ are captured, too. Hence G equals some chain member. We conclude that G has uncountable strong cofinality. $\square$

We want to close this section with sufficient conditions for (B) and (C). A $\mathfrak{K}$-subpartition $(\Delta_i)_{i\in\mathbb{Z}}$ of Ω permits *shifted patching* if each sequence $(g_i)_{i\in\mathbb{Z}} \in G$ with $\Delta_i g_i = \Delta_{i+1}$ satisfies $\cup_{i\in\mathbb{Z}} g_{i\restriction\Delta_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in\mathbb{Z}}\Delta_i} \in G$. Let $\sqsubseteq \subseteq \mathfrak{K}\times\mathfrak{K}$ be a binary relation. Intuitively, the sets in $\mathfrak{K}$ will be "large" subsets of Ω having also a large complement, and, for $\Sigma, \Gamma \in \mathfrak{K}$, $\Sigma \sqsubseteq \Gamma$ means that $\Sigma \subseteq \Gamma$ and $\Gamma \setminus \Sigma$ is also large. Consider the following conditions on G and $(\mathfrak{K}, \sqsubseteq)$.

(1) For each $\Delta \in \mathfrak{K}$ there is a $\mathfrak{K}$-subpartition $(\Delta_i)_{i\in\mathbb{Z}}$ of Δ that permits shifted patching and all the Δ_i lie in the same G-orbit.

(2) $\Delta \sqsubseteq \Gamma \implies \Delta \subseteq \Gamma$

(3) $\Sigma, \Gamma, \Delta \in \mathfrak{K}$ with $\Sigma \sqsubseteq \Gamma \subseteq \Delta \implies \Sigma \sqsubseteq \Delta$

(4) $\Sigma \in \mathfrak{K} \implies$ there is $\Delta \in \mathfrak{K}$ with $\Delta \sqsubseteq \Sigma$

(5) $\Omega = \Sigma \cup \Sigma'$ and $\Gamma, \Sigma, \Sigma' \in \mathfrak{K} \implies \Gamma \cap \Sigma \in \mathfrak{K}$ or $\Gamma \cap \Sigma' \in \mathfrak{K}$

(6) $\Delta \sqsubseteq \Sigma$ and $f \in G \implies \Delta f \sqsubseteq \Sigma f$

(7) $\Delta, \Delta', \Gamma \in \mathfrak{K}$ with $\Delta, \Delta' \sqsubseteq \Gamma \implies$ there is some $g \in G(\Gamma)$ with $\Delta g = \Delta'$

(8) $\Delta \in \mathfrak{K}$ and $g \in G$ fixes Δ setwise $\implies g_{\restriction\Delta} \cup \mathrm{id}_{\Omega\setminus\Delta} \in G$

(9) $\Sigma \in \mathfrak{K} \implies$ there are $\Sigma' \in \mathfrak{K}, g \in G$ with $\Omega = \Sigma \cup \Sigma'$, $\Sigma \setminus \Sigma' \sqsubseteq \Sigma$, $\Sigma' \setminus \Sigma \sqsubseteq \Sigma'$, $\Sigma \cap \Sigma' \sqsubseteq \Sigma$, $\Sigma \cap \Sigma' \sqsubseteq \Sigma'$ and $\Sigma g = \Sigma'$

In order to illustrate conditions (1)-(9), we consider the following example. Let Ω be an infinite set and $G = \mathrm{Sym}(\Omega)$. We say that $\Sigma \subseteq \Omega$ is a *moiety* of Ω if $|\Sigma| = |\Omega \setminus \Sigma|$. Then let $\mathfrak{K}$ be the set of all moieties of Ω, and for $\Sigma, \Gamma \in \mathfrak{K}$ we put $\Sigma \sqsubseteq \Gamma$ if $\Sigma \subseteq \Gamma$ and $\Gamma \setminus \Sigma \in \mathfrak{K}$. The requirement of condition (1) that each $\Delta \in \mathfrak{K}$ has a $\mathfrak{K}$-*sub*partition (instead of a partition in $\mathfrak{K}$) is due to the structure of the Cantor set and its homeomorphism group (see Section 4, Theorem 4.1). There, $\mathfrak{K}$ will comprise all non-empty proper clopen subsets. Since the Cantor set is compact, it is not possible to partition a set from $\mathfrak{K}$ into infinitely many other sets from $\mathfrak{K}$.

For later use we observe that conditions (4) and (6) imply that whenever $\Sigma \in \mathfrak{K}$ and $f \in G$ then $\Sigma f \in \mathfrak{K}$, since $\sqsubseteq \subseteq \mathfrak{K}\times\mathfrak{K}$. Now we show:

Lemma 2.4. *Assume (1). Then condition (A) follows, and condition (B) holds with $N = 2$.*

Proof. Let $\Delta \in \mathfrak{K}$. With (1) we find a subpartition $(\Delta_i)_{i\in\mathbb{Z}}$ of Δ that permits shifted patching and such that all Δ_i lie in the same G-orbit. Hence, we find $g_i \in G$ mapping Δ_i onto Δ_{i+1} for all $i \in \mathbb{Z}$. By shifted patching we obtain that $g = \cup_{i\in\mathbb{Z}} g_{i\restriction\Delta_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in\mathbb{Z}}\Delta_i} \in G(\Delta)$. First we show that $(\Delta_i)_{i\in\mathbb{Z}}$ satisfies local patching. Let $(f_i)_{i\in\mathbb{Z}} \subseteq G$ with $\Delta_i f_i = \Delta_i$ for each $i \in \mathbb{Z}$. By shifted patching, $h = \cup_{i\in\mathbb{Z}}(f_i g_i)_{\restriction\Delta_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in\mathbb{Z}}\Delta_i} \in G$ and so $\cup_{i\in\mathbb{Z}} f_{i\restriction\Delta_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\in\mathbb{Z}}\Delta_i} = hg^{-1} \in G$.

To check condition (B), let f be an arbitrary element of $G(\Delta_0)$ and note that $\operatorname{supp} g^{-i}fg^i \subseteq \Delta_0 g^i = \Delta_i$. By local patching we have $k = \cup_{i\geq 0}(g^{-i}fg^i)_{\restriction\Delta_i} \cup \mathrm{id}_{\Omega\setminus\dot\cup_{i\geq 0}\Delta_i} \in G(\Delta)$ and we claim that $f = kg^{-1}k^{-1}g$. Note that k and k^{-1} are the identity on Δ_i for all $i < 0$. For $x \in \cup_{i<0}\Delta_i$ we have

$$xkg^{-1}k^{-1}g = xg^{-1}k^{-1}g = xg^{-1}g = x = xf.$$

For $x \in \Delta_0$ we have $xk = xf \in \Delta_0$ and observe that $xkg^{-1} \in \Delta_{-1}$. But k^{-1} is the identity on Δ_{-1}, hence $xkg^{-1}k^{-1}g = xkg^{-1}g = xk = xf$. Finally, let $x \in \Delta_i$ for $i > 0$. We have

$$xkg^{-1}k^{-1}g = x(g^{-i}fg^i)g^{-1}(g^{-i+1}f^{-1}g^{i-1})g = x = xf.$$

Since $f, g, k \in G(\Delta)$, our claim and hence the result, with $\Sigma = \Delta_0$, follow. $\square$

Lemma 2.5. *Assume (2)-(9). Then, for each $\Sigma \in \mathfrak{K}$, $G = \big(G(\Sigma) \cup \{g, g^{-1}\}\big)^5$ for some $g \in G$. In particular, condition (C) holds.*

Proof. Given an arbitrary $\Sigma \in \mathfrak{K}$, let $\Sigma' \in \mathfrak{K}$ and $g \in G$ as in (9). Since $\Sigma g = \Sigma'$ we have that $G(\Sigma') = G(\Sigma)^g$. Hence, it suffices to show $G = G(\Sigma) \cdot G(\Sigma') \cdot G(\Sigma)$. For this we adapt a classical idea given in [DNT86]. Let $f \in G$ and $\Gamma = \Sigma \cap \Sigma'$. As noted before, we have $\Gamma f \in \mathfrak{K}$ and by (5) we may assume that $\Gamma f \cap \Sigma \in \mathfrak{K}$. Now use (4) to find some $\Delta' \in \mathfrak{K}$ with $\Delta' \sqsubseteq \Gamma f \cap \Sigma$ and let $\Delta = \Delta' f^{-1}$. By (6), we have $\Delta \sqsubseteq (\Gamma f \cap \Sigma) f^{-1} \subseteq \Gamma \subseteq \Sigma$. With (3) we conclude $\Delta \sqsubseteq \Sigma$ and, from $\Delta' \sqsubseteq \Gamma f \cap \Sigma$, also $\Delta' \sqsubseteq \Sigma$. By (7) we find some $h \in G(\Sigma)$ mapping $\Delta' = \Delta f$ onto Δ. Hence $h' = (fh)_{\restriction\Delta} \cup \mathrm{id}_{\Omega\setminus\Delta} \in G(\Delta)$ by (8). With (2) and $\Delta \sqsubseteq \Sigma$ we have $G(\Delta) \subseteq G(\Sigma)$. Now, with $h'' = h(h')^{-1} \in G(\Sigma)$, we have that $xfh'' = x$ for all $x \in \Delta$. Let $\Sigma_1 = \Sigma \setminus \Sigma'$ and observe $\Sigma_1 \sqsubseteq \Sigma$ by (9). With $\Delta \sqsubseteq \Sigma$ and (7) we find some $k \in G(\Sigma)$ with $\Sigma_1 = \Delta k$. Now every element of Σ_1 is fixed by $k^{-1}fh''k$, thus $k^{-1}fh''k \in G(\Sigma')$. Hence $f \in k \cdot G(\Sigma') \cdot (h''k)^{-1} \subseteq G(\Sigma) \cdot G(\Sigma') \cdot G(\Sigma)$, as needed. $\square$

Theorem 2.6. *Let G and $\mathfrak{K}$ satisfy conditions (1)-(9). Then G has uncountable strong cofinality.*

Proof. With Lemmas 2.4 and 2.5 we obtain conditions (A), (B) and (C). Now apply Theorem 2.1. $\square$

3 Measure preserving transformations of $[0, 1]$

Let Ω be the unit interval $[0, 1]$ or the real line $\mathbb{R}$, equipped with the σ-algebra $\mathfrak{L}$ of all Lebesgue-measurable sets. We denote the Lebesgue measure by λ. All sets under consideration will be measurable. A transformation $f : \Omega \to \Omega$ is *measurable* if $\Sigma \in \mathfrak{L}$ implies

$\Sigma f^{-1} \in \mathfrak{L}$. Let f be a bijective measurable transformation. Then f is *bi-measurable* if its inverse is measurable, too. Subsequently, all transformations occuring here will be assumed to be bi-measurable. We say that f is *measure preserving* if $\lambda(\Sigma) = \lambda(\Sigma f)$ for all $\Sigma \in \mathfrak{L}$, and f is *non-singular* if $\lambda(\Sigma) = 0 \iff \lambda(\Sigma f) = 0$ for all $\Sigma \in \mathfrak{L}$. A set $\Sigma \subseteq \Omega$ is *invariant* under f if $\Sigma f = \Sigma$. Finally, f is *ergodic* if each measurable invariant set Σ satisfies $\lambda(\Sigma) = 0$ or $\lambda(\Omega \setminus \Sigma) = 0$. When considering measure preserving, non-singular or ergodic transformations, as usual we identify sets and functions that differ by a null set only.

Let S be the group of all bi-measurable transformations of Ω and G a subgroup of S. We say that G is *equitransitive* if any two sets $\Sigma, \Delta \in \mathfrak{L}$ with $\lambda(\Sigma) = \lambda(\Delta)$ and $\lambda(\Omega\backslash\Sigma) = \lambda(\Omega\backslash\Delta)$ lie in the same orbit of G. Furthermore, G is *full* if for any countable index set I, any two partitions $\Omega = \dot{\bigcup}_{i\in I} \Sigma_i = \dot{\bigcup}_{i\in I} \Gamma_i$ into sets $\Sigma_i, \Gamma_i \in \mathfrak{L}$ and any sequence $(g_i)_{i\in I} \subseteq G$ with $\Sigma_i g_i = \Gamma_i$ we have that $\cup_{i\in I} g_{i\restriction\Sigma_i} \in G$. Clearly, S, the group of all measure preserving transformations, and the group of all non-singular bi-measurable transformations are each full. Since the intersection of any collection of full groups is full, it follows that there is a smallest full group containing G which will be denoted $[G]$. Clearly, if G consists of measure preserving resp. non-singular transformations only, then so does $[G]$. The *full group of a transformation* $f \in S$ is the smallest full group containing f.

For the remainder of this section we are interested in the following scenario:

(*) *Ω is the unit interval, G consists of measure preserving transformations, and G is full and equitransitive.*

At least two special cases of these groups have been studied before. A prominent example is the group of all measure preserving transformations of the unit interval, which is well-known to be equitransitive. Fathi [Fat78] showed that this group is perfect and simple. Eigen [Eig81] considered the full group of an ergodic measure preserving transformation of the unit interval and obtained similar results as Fathi. The full group of an ergodic measure preserving transformation is equitransitive (cf. [Hal56]).

Now assume (*) and let $n \geq 1$. A transformation $f \in G$ is *n-point periodic* if almost every $x \in \Omega$ has an orbit of length n under the action of f. We say that f has *finite orbits* if almost every $x \in \Omega$ is contained in a finite orbit of f.

Lemma 3.1. *Each element of G with finite orbits is a commutator in G.*

Proof. First, let $f \in G$ be n-point periodic, where $n \geq 1$. With our assumptions on the group G we can use exactly the same proof as Eigen in [Eig81, Lemma 1] to show that f is a commutator in G. If f is a transformation with finite orbits, then let

$$\Sigma_n = \{x \in \Omega \mid x \text{ has an orbit of } f \text{ of length } n\}.$$

Clearly, each Σ_n is invariant under f and $f_{\restriction\Sigma_n}$ is an n-point periodic transformation of Σ_n, hence $f_{\restriction\Sigma_n} \cup \mathrm{id}_{\Omega\backslash\Sigma_n} = g_n h_n g_n^{-1} h_n^{-1}$, a commutator of elements $g_n, h_n \in G$ with support inside Σ_n. Thus we can patch the g_n resp. h_n together along $(\Sigma_n)_{n\geq 0}$ to obtain g resp. $h \in G$ and then f is a commutator of g and h. □

Theorem 3.2. *Let G be a full and equitransitive group of measure preserving transformations of the unit interval.*

(a) Each element of G is a product of 5 commutators; in particular, G is perfect.

(b) G is simple and each element of G is a product of 10 involutions.

(c) Let $g \in G$ with $\lambda(\operatorname{supp} g) = 1$. Then every element of G is a product of at most 24 conjugates of g or g^{-1}.

Proof. Each element of G with finite orbits is a commutator by Lemma 3.1. This fact and the assumptions on G suffice to establish the results (a)-(c) precisely as in Fathi [Fat78] for the group of all measure preserving transformations of the unit interval. □

Lemma 3.3. *Let $\Sigma, \Gamma \in \mathfrak{L}$ with $\lambda(\Sigma) = \lambda(\Gamma)$. Then there is an involution $g \in G(\Sigma \cup \Gamma)$ exchanging both of them.*

Proof. Note that $\Sigma \setminus \Gamma$ and $\Gamma \setminus \Sigma$ have the same measure and are disjoint. Since G is equitransitive, there is some $h \in G$ with $(\Sigma \setminus \Gamma)h = \Gamma \setminus \Sigma$. Now let $g = h_{\restriction \Sigma\setminus\Gamma} \cup {h^{-1}}_{\restriction \Gamma\setminus\Sigma} \cup \mathrm{id}_{\Omega \setminus ((\Sigma\setminus\Gamma) \dot\cup (\Gamma\setminus\Sigma))}$. □

Now let $\mathfrak{K}$ be the collection of all measurable subsets of $[0,1]$ of positive measure.

Corollary 3.4. *Condition (B) holds.*

Proof. Let $\Sigma \in \mathfrak{K}$ and let $I \subseteq \Omega$ be an interval with $\lambda(\Sigma) = \lambda(I)$. Let ψ be a measure preserving transformation that maps Σ onto I, and let φ be the affine bijection that maps I onto $[0,1]$. For $f \in G(\Sigma)$ let f' denote the transformation $\varphi^{-1}\psi^{-1}f\psi\varphi$. It is easy to see that $G' = \{f' \mid f \in G(\Sigma)\}$ is a full and equitransitive group on $[0,1]$ consisting of measure preserving transformations. Furthermore, there is some $g' \in G'$ with $\lambda(\operatorname{supp} g') = 1$ (for example, split Σ into two sets of equal measure and let g be an involution exchanging them, then g' is the desired element). Hence, with Theorem 3.2(c), it follows that each element of G' is a product of at most 24 conjugates of g' or g'^{-1}. Since the above map $f \mapsto f'$ provides a group isomorphism between $G(\Sigma)$ and G', we conclude that each element of $G(\Sigma)$ is a product of at most 24 conjugates of g or g^{-1} with exponents in $G(\Sigma)$. □

Lemma 3.5. *Let $\Sigma \in \mathfrak{L}$ and let $\Sigma = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3$ be a partition in $\mathfrak{L}$ with $\lambda(\Sigma_1) = \lambda(\Sigma_2)$. Then $G(\Sigma) = \big(G(\Sigma_1 \dot\cup \Sigma_2) \cup G(\Sigma_2 \dot\cup \Sigma_3)\big)^4$.*

Proof. Choose any $f \in G(\Sigma)$. For $i, j \in \{1,2,3\}$ let $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j f^{-1}$, thus $\Sigma_{i,j}$ is the set of all elements of Σ_i mapped to Σ_j under f. These nine sets form a partition of Σ and we observe

$$\Sigma_3 = \Sigma_{1,3}f \dot\cup \Sigma_{2,3}f \dot\cup \Sigma_{3,3}f = \Sigma_{3,1} \dot\cup \Sigma_{3,2} \dot\cup \Sigma_{3,3}.$$

Since f preserves the measure, we obtain $\lambda(\Sigma_{1,3}) + \lambda(\Sigma_{2,3}) = \lambda(\Sigma_{3,1}) + \lambda(\Sigma_{3,2})$. Hence we find some λ-measurable $\Gamma \subseteq \Sigma_{3,1} \dot\cup \Sigma_{3,2}$ with $\lambda(\Sigma_{2,3}) = \lambda(\Gamma)$ and by Lemma 3.3 there is some

$f_1 \in G(\Sigma_{2,3} \dot\cup \Gamma) \subseteq G(\Sigma_2 \cup \Sigma_3)$ exchanging $\Sigma_{2,3}$ and Γ. Since Σ_1 and Σ_2 have equal measure, we can find $f_2 \in G(\Sigma_1 \dot\cup \Sigma_2)$ exchanging both of them. We obtain

$$\Sigma_3 f^{-1} f_1 f_2 = (\Sigma_{1,3} \dot\cup \Sigma_{2,3} \dot\cup \Sigma_{3,3}) f_1 f_2 = (\Sigma_{1,3} \dot\cup \Gamma \dot\cup \Sigma_{3,3}) f_2 \subseteq \Sigma_2 \dot\cup \Sigma_3.$$

By Lemma 3.3 there is some $f_3 \in G(\Sigma_2 \dot\cup \Sigma_3)$ with $\Sigma_3 f^{-1} f_1 f_2 f_3 = \Sigma_3$. Let $h = f^{-1} f_1 f_2 f_3$. We just showed that h leaves Σ_3 and hence also $\Sigma_1 \dot\cup \Sigma_2$ setwise invariant. Let $f_4 = h_{\restriction \Sigma_1 \dot\cup \Sigma_2} \cup \mathrm{id}_{\Omega \setminus (\Sigma_1 \dot\cup \Sigma_2)}$ and $f_5 = h_{\restriction \Sigma_3} \cup \mathrm{id}_{\Omega \setminus \Sigma_3}$. Then $f_4, f_5 \in G$ by fullness and $h = f_4 f_5$. Finally, the statement follows from $f = f_1 f_2 (f_3 f_5^{-1}) f_4^{-1}$ and $f_1, f_3, f_5 \in G(\Sigma_2 \dot\cup \Sigma_3)$, $f_2, f_4 \in G(\Sigma_1 \dot\cup \Sigma_2)$. □

Corollary 3.6. *Condition (C) holds.*

Proof. Let $\Sigma \in \mathfrak{K}$ be a set of positive measure. It contains some subset Γ_0 of measure $m = (2/3)^n$ for some $n \geq 1$. We split $\Gamma_0 = \Sigma_1 \dot\cup \Sigma_2$ into two subsets of equal measure $\frac{1}{2}m$. Since $m \leq \frac{2}{3}$ we have $1 - m \geq \frac{1}{2}m$. Hence the complement of Γ_0 contains a set Σ_3 of measure $\frac{1}{2}m$. Since $\Sigma_1 \dot\cup \Sigma_2$ has the same measure as $\Sigma_2 \dot\cup \Sigma_3$ there is an involution $f \in G(\Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3)$ mapping $\Sigma_1 \dot\cup \Sigma_2$ onto $\Sigma_2 \dot\cup \Sigma_3$. Hence $G(\Sigma_2 \dot\cup \Sigma_3) = f^{-1} G(\Sigma_1 \dot\cup \Sigma_2) f \subseteq (G(\Gamma_0) \cup \{f, f^{-1}\})^3$. Now let $\Gamma_1 = \Sigma_1 \dot\cup \Sigma_2 \dot\cup \Sigma_3 = \Gamma_0 \dot\cup \Sigma_3$. With Lemma 3.5 we have $G(\Gamma_1) = \big(G(\Gamma_0) \cup G(\Sigma_2 \dot\cup \Sigma_3)\big)^4 = \big(G(\Gamma_0) \cup \{f, f^{-1}\}\big)^8$. Note that the measure of Γ_1, compared with the measure of Γ_0, increased by a factor $\frac{3}{2}$, hence $\lambda(\Gamma_1) = (2/3)^{n-1}$. We repeat this construction inductively obtaining $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, then $\lambda(\Gamma_n) = 1$ and so $G = G(\Gamma_n)$. By construction we have $G(\Gamma_{i+1}) = \big(G(\Gamma_i) \cup \{f_i, f_i^{-1}\}\big)^8$ for $i = 0, \ldots, n-1$ and some $f_i \in G$. Hence $G = G(\Gamma_n) = (G(\Gamma_0) \cup F)^{8^n}$ for some finite set $F \subseteq G$. Since $G(\Gamma_0) \subseteq G(\Sigma)$, the result follows. □

Theorem 3.7. *Let Ω be the unit interval and let G be a group of measure preserving transformations of Ω that is full and equitransitive. Then G has uncountable strong cofinality.*

Proof. As noted before, let $\mathfrak{K}$ be the collection of all subsets of Ω of positive measure. Condition (A) for G and $\mathfrak{K}$ is immediate, condition (B) follows from Corollary 3.4 and condition (C) is provided by Corollary 3.6. Theorem 2.1 implies the result. □

By the remarks above, as an immediate consequence we obtain:

Corollary 3.8. *Let Ω be the unit interval, and let G be either the group of all measure preserving transformations of Ω, or the full group of an ergodic measure preserving transformation of Ω. Then G has uncountable strong cofinality.*

4 Further applications

In this section, we wish to apply Theorem 2.6 to various permutation groups.

Theorem 4.1. *Assume one of the following.*

(i) [Ber06] *Let Ω be an infinite set and $G = \mathrm{Sym}(\Omega)$.*

(ii) [DG05] *Let $\Omega = \mathbb{R}$ and let G be the group of all Borel-automorphisms of $\mathbb{R}$.*

(iii) [DG05] *Let Ω be the topological space of the rationals $\mathbb{Q}$, the irrationals $\mathbb{R} \setminus \mathbb{Q}$ or the Cantor set and let G be the group of all homeomorphisms of Ω.*

(iv) Let $\Omega = \mathbb{R}$ and let G be a full and equitransitive group of measure preserving transformations of $\mathbb{R}$, identified modulo null sets.

(v) Let $\Omega = \mathbb{R}$ and let G be a full group consisting of non-singular transformations, identified modulo null sets. Furthermore, assume that G is transitive on the collection of all measurable subsets Σ of $\mathbb{R}$ for which both Σ and $\mathbb{R} \setminus \Sigma$ have positive measure.

Then G has uncountable strong cofinality.

Proof. The statement for (i) is the classical result of Bergman [Ber06], and (ii) and (iii) have been shown in [DG05]. We note that (iv) and (v) can be derived from Proposition 8.3 in Miller [Mil04]. However, these proofs of (ii)-(v) relied on the normal subgroup structure of G. Here, we give an alternative simpler proof, using Theorem 2.6 for all these cases. Let $\mathfrak{K}$ comprise all subsets $\Sigma \subseteq \Omega$ satisfying, respectively,

(i) $|\Sigma| = |\Omega \setminus \Sigma| = |\Omega|$,

(ii) Σ and $\Omega \setminus \Sigma$ are uncountable Borel sets,

(iii) Σ and $\Omega \setminus \Sigma$ are non-empty and clopen.

(iv) Σ is measurable and $\lambda(\Sigma) = \lambda(\Omega \setminus \Sigma) = \infty$,

(v) Σ is measurable and $\lambda(\Sigma), \lambda(\Omega \setminus \Sigma) > 0$.

Now, for $\Sigma, \Gamma \subseteq \Omega$, let $\Sigma \sqsubseteq \Gamma$ if $\Sigma \subseteq \Gamma$ and $\Sigma, \Gamma \setminus \Sigma \in \mathfrak{K}$ and in this case we say that Σ is a *moiety* of Γ. First we remark that in each case the sets of $\mathfrak{K}$ lie in a single G-orbit. Indeed, this is clear for (i). Any two uncountable Borel sets are of cardinality continuum and isomorphic via a Borel-automorphism by a result of Kuratowski [Kur66]. For case (iii), any two sets in $\mathfrak{K}$ are homeomorphic. Clearly, G is transitive on $\mathfrak{K}$ for the cases (iv) and (v). Next we want to verify conditions (1)-(9). In each of the cases, except the Cantor set, it is elementary that each moiety can be split into countably many moieties. If Ω is the Cantor set, in order to ensure condition (1), we choose a $\mathfrak{K}$-subpartition consisting of sets whose diameters converge to 0; then patching along this $\mathfrak{K}$-subpartition results in a map that is continuous. From these observations condition (1) follows. Conditions (2)-(6) and (8) can be verified elementary. For condition (7) let $\Delta, \Delta', \Gamma \in \mathfrak{K}$ with $\Delta, \Delta' \sqsubseteq \Gamma$. Since $\mathfrak{K}$ is a G-orbit, we can find $h, h' \in G$ with $\Delta h = \Delta'$ and $(\Gamma \setminus \Delta)h' = \Gamma \setminus \Delta'$. Then put $g = h_{\restriction \Delta} \cup h'_{\restriction \Gamma \setminus \Delta} \cup \mathrm{id}_{\Omega \setminus \Gamma}$. Condition (9) can be seen as follows. Let $\Sigma \in \mathfrak{K}$ be a moiety of Ω and split $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ in $\mathfrak{K}$, which is possible in all cases. Since $\mathfrak{K}$ is closed under complement, $\Sigma_3 = \Omega \setminus \Sigma$ is also a moiety. Then $\Sigma' = \Sigma_2 \dot\cup \Sigma_3$ satisfies all requirements. Now Theorem 2.6 implies the result. $\square$

As special cases of Theorem 4.1(iv) and (v) we obtain:

Corollary 4.2. *Let $\Omega = \mathbb{R}$. Assume that G is*

- *the full group of a measure preserving ergodic transformation, or*
- *the full group of an ergodic transformation that preserves no measure equivalent to the Lebesgue measure.*

Then G has uncountable strong cofinality.

Proof. From [HIK74], Lemma 6 resp. the Corollary preceeding it, it follows that G satisfies the prerequisites of Theorem 4.1(v) resp. (iv). $\square$

Next we wish to show that Theorem 2.6 can be also applied to a range of further permutation groups on the reals.

Example 4.3. We give a construction that yields a group H with uncountable strong cofinality. In view of Theorem 4.1, this group is a proper subgroup of G in the cases (ii) and (iii).

Let Ω be $\mathbb{R}$ or $\mathbb{R} \setminus \mathbb{Q}$. Let $\mathfrak{I} \subseteq 2^{\Omega}$ be the collection of all half-open intervals with rational endpoints, i.e., $\mathfrak{I} = \{[a,b) \mid a,b \in \mathbb{Q}\}$. We call an element of $\mathrm{Sym}(\Omega)$ a *translation* if it is of the form $x \mapsto x + r$ for some rational r. Furthermore, we call $g \in \mathrm{Sym}(\Omega)$ a *piecewise translation* if $g = \cup_{i \in I} g_{i \restriction \Sigma_i}$, where I is a countable index set, $(\Sigma_i)_{i \in I}$ is a partition of Ω with $\Sigma_i \in \mathfrak{I}$ and $g_i \in \mathrm{Sym}(\Omega)$ is a translation for all $i \in I$. Let H denote the set of all piecewise translations. It is straightforward to check that H is a subgroup of $\mathrm{Sym}(\Omega)$.

Next, let $\mathfrak{K}$ be the collection of all sets $\Sigma \subseteq \Omega$ such that both Σ and $\Omega \setminus \Sigma$ have infinite measure and can be expressed as a countable partition in $\mathfrak{I}$. (If $\Omega = \mathbb{R} \setminus \mathbb{Q}$, these sets $\Sigma \in \mathfrak{K}$ are precisely the non-empty proper clopen subsets of $\mathbb{R} \setminus \mathbb{Q}$.) Clearly, $\mathfrak{K}$ is invariant under the action of H. We claim that H is transitive on $\mathfrak{K}$ and give a sketch of the proof. We say that two sequences $(\Sigma_i)_{i \in \omega}, (\Gamma_i)_{i \in \omega} \subseteq \mathfrak{I}$ are *similar* if $\lambda(\Sigma_i) = \lambda(\Gamma_i)$ for all $i \in \omega$. We show that any two $\Sigma, \Gamma \in \mathfrak{K}$ have *similar partitions*, that means, Σ and Γ have partitions in $\mathfrak{I}$ consisting of similar sequences. By definition of $\mathfrak{K}$ we find *some* partitions $\Sigma = \dot{\bigcup}_{i \in \omega} \Sigma_i$ and $\Gamma = \dot{\bigcup}_{i \in \omega} \Gamma_i$ in $\mathfrak{I}$. First, find the smallest $n \geq 0$ such that $\lambda(\Sigma_0) \leq \sum_{j=0}^{n} \lambda(\Gamma_j)$. Then choose $x \in \mathbb{Q}$ such that $\lambda(\Sigma_0) = \sum_{j=0}^{n-1} \lambda(\Gamma_j) + \lambda(\Gamma_n \cap (-\infty, x))$. Now we can split $\Sigma_0 = \Sigma_0' \,\dot{\cup}\, \Sigma_1' \,\dot{\cup} \ldots \dot{\cup}\, \Sigma_n'$ in $\mathfrak{I}$ such that $\lambda(\Sigma_j') = \lambda(\Gamma_j)$ for $j = 0, \ldots, n-1$ and $\lambda(\Sigma_n') = \lambda(\Gamma_n \cap (-\infty, x))$. In the sequence $(\Sigma_i)_{i \in \omega}$ replace Σ_0 by $\Sigma_0', \ldots, \Sigma_n'$ and in $(\Gamma_i)_{i \in \omega}$ replace Γ_n by $\Gamma_n \cap (-\infty, x), \Gamma_n \cap [x, \infty)$. These two new sequences have the property that the first $n+1$ members have pairwise equal measure. Now proceed similar with the respective tails of these new sequences starting with the $(n+2)$nd members. This way we inductively construct two similar partitions of Σ and Γ.

Now let $\Sigma, \Gamma \in \mathfrak{K}$. We choose similar partitions $\Sigma = \dot{\bigcup}_{i \in \omega} \Sigma_i$ and $\Gamma = \dot{\bigcup}_{i \in \omega} \Gamma_i$ and also similar partitions $\Omega \setminus \Sigma = \dot{\bigcup}_{i \in \omega} \Sigma_i'$ and $\Omega \setminus \Gamma = \dot{\bigcup}_{i \in \omega} \Gamma_i'$. For all $i \in \omega$ we find translations h_i resp. h_i' with $\Sigma_i h_i = \Gamma_i$ resp. $\Sigma_i' h_i' = \Gamma_i'$. Then $(\cup_{i \in \omega} h_{i \restriction \Sigma_i}) \cup (\cup_{i \in \omega} h'_{i \restriction \Sigma_i'})$ is a piecewise translation that maps Σ onto Γ.

Finally, as in the proof of Theorem 4.1, let $\Sigma \sqsubseteq \Gamma$ if $\Sigma \subseteq \Gamma$ and $\Sigma, \Gamma \setminus \Sigma \in \mathfrak{K}$. We verified that $\mathfrak{K}$ is a H-orbit and with this it is easy to check conditions (1)-(9). Now Theorem 2.6 implies that H has uncountable strong cofinality. With respect to the cases (ii) and (iii) of Theorem 4.1, H is a proper subgroup of G, since H does not contain the transformation $x \mapsto -x$.

Essentially, the same construction yields several other examples for the cases (ii) and (iii). Instead of patching translations, exclusively, one might patch affine transformations with non-zero rational slope, for example, to obtain further groups with uncountable strong cofinality.

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