Weighted automata and weighted logics*

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Abstract. Weighted automata are used to describe quantitative properties in various areas such as probabilistic systems, image compression, speech-to-text processing. The behaviour of such an automaton is a mapping, called a formal power series, assigning to each word a weight in some semiring. We generalize Büchi's and Elgot's fundamental theorems to this quantitative setting. We introduce a weighted version of MSO logic and prove that, for commutative semirings, the behaviours of weighted automata are precisely the formal power series definable with particular sentences of our weighted logic. We also consider weighted first-order logic and show that aperiodic series coincide with the first-order definable ones, if the semiring is locally finite, commutative and has some aperiodicity property.
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1 Introduction

In automata theory, Büchi's and Elgot's fundamental theorems [3, 12] established the coincidence of regular languages with languages definable in monadic second-order logic. At the same time, Schützenberger [29] investigated finite automata with weights and characterized their behaviours as rational formal power series. Both of these results have inspired a wealth of extensions and further research, cf. [32, 28, 20, 2] for surveys and monographs, and also led to recent practical applications, e.g. in verification of finite-state programs (model checking, [23, 1, 21]), in digital image compression [5, 15, 17, 16] and in speech-to-text processing [26, 27, 4].

It is the goal of this paper to introduce a logic with weights and to analyze conditions under which the behaviours of weighted finite automata are precisely the series definable in our weighted monadic second-order logic. Our motivation for this *weighted logic* is as follows. First, weighted automata and their behaviour can be viewed as a quantitative extension of classical automata. The latter decide whether a given word is accepted or not, whereas weighted automata also

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compute e.g. the ressources, time or cost used or the probability of its success when executing the word. We would like to have an extension of Büchi's and Elgot's theorems to this setting. Second, classical logic for automata describes whether a certain property (e.g. "there exist three consecutive a's") holds for a given word or not. One could be interested in knowing how often this property holds, i.e. again in extending the previous qualitative statement to a quantitative one. Next we describe the syntax of our weighted logics. Its definition incorporates weights taken as elements from a given abstract semiring K, just as done for weighted automata in order to model a variety of applications and situations. Also, our syntax should extend classical (unweighted) MSO logics. The semantics of a weighted logic formula φ should be a formal power series over an extended alphabet and with values in K. It is possible to assign a natural semantics to atomic formulas, to disjunction and conjunction, and to existential and universal quantifications, but a problem arises with negation. It would be natural to define the semantics of $\neg\varphi$ elementwise. But if K is not a Boolean algebra, K does not have a natural complement operation. Therefore we restrict negation to atomic formulas whose semantics will take as values only 0 and 1 in K; then the negation of atomic formulas also has a natural semantics. In comparison to classical MSO-logic, this is not an essential restriction, since the negation of a classical MSO-formula is equivalent (in the sense of defining the same language) to one in which negation is applied only to atomic formulas. This requires us to include universal quantifications into our syntax (which we do). In this sense, our weighted MSO-logics then contains the classical MSO-logics which we obtain by letting $K = \mathbb{B}$, the 2-element Boolean algebra. We define the semantics of sentences φ of our weighted MSO-logic by structural induction over φ. Thus, as usually, we also define the semantics of a formula φ with free variables, here as a formal power series over an extended alphabet. But even for the semiring of natural numbers or the tropical semiring it turns out that neither universal first-order nor universal second-order quantification of formulas preserve recognizability, i.e. representability of their semantics as behaviour of a weighted automaton. Therefore, for restricted MSO-logic we exclude universal second-order quantification, and we permit universal first-order quantification only for formulas whose semantics takes finitely many values in K. Moreover, if we allow existential set quantifications only to occur at the beginning of a fomula, we arrive at restricted existential MSO-logic.

Now we give a summary of our results. First we show for any commutative semiring K that the behaviours of weighted automata with values in K are precisely the series definable by sentences of our restricted MSO-logic, or, equivalently, of our restricted existential MSO-logic. Second, if the semiring K is locally finite, we obtain that the semantics of all sentences of our full weighted MSO-logic are representable by weighted automata. Locally finite semirings were investigated in [6, 7]; they form a large class of semirings including e.g. all finite semirings, the max-min-semiring employed for capacity problems of networks, and all Boolean algebras. Thus we obtain Büchi's and Elgot's theorems as a particular consequence. Moreover, if the semiring K is a field or locally finite and is given in some effective way, then the constructions in our proofs yield effective conversions of sentences of our weighted logics to weighted automata, and viceversa, and we obtain also decision procedures.

Finally, we investigate weighted first-order logic. As is well-known, the first-order definable languages are precisely the starfree languages which in turn coincide with the the aperiodic ones [30, 24]. Aperiodic and starfree formal power series were introduced and investigated in [6, 7]. Easy examples show that even if the semiring K is finite, series definable in our weighted first-order logic need not be aperiodic. However, we obtain that the aperiodic series coincide with the first-order definable ones, if the semiring is commutative and both addition and multiplication satisfy a certain aperiodicity property. Such semirings include again all Boolean algebras, but also quite different ones like the truncated max-plus semiring or the semiring which occurs in the MV-algebra used to define the semantics of Łukasiewicz multi-valued logic [13]. For this last semiring a restriction of Łukasiewicz logic coincides with our weighted MSO-logic [31].

An extended abstract of this paper appeared in [8].

2 MSO-logic and weighted automata

In this section, we summarize for the convenience of the reader our notation used for classical MSO-logic and basic background of weighted automata. We assume that the reader is familiar with the basics of monadic second-order logic and Büchi's theorem for languages of finite words, cf. [32, 18]. Let A be an alphabet. The syntax of formulas of MSO-logic over A is given by

$$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \varphi \vee \psi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where a ranges over A, x, y are first-order variables and X is a set variable. We let $\mathrm{Free}(\varphi)$ be the set of all free variables of φ.

Let $w = w(1)\ldots w(n) \in A^*$ with $w(i) \in A$. The length of w is $|w| = n$. The word w is usually represented by the structure $(\{1, \ldots, |w|\}, \leq, (R_a)_{a\in A})$ where $R_a = \{i \mid w(i) = a\}$ $(a \in A)$.

Let $\mathcal{V}$ be a finite set of first-order and second-order variables. A $(\mathcal{V}, w)$-assignment σ is a function mapping first-order variables in $\mathcal{V}$ to elements of $\{1, \ldots, |w|\}$ and second-order variables in $\mathcal{V}$ to subsets of $\{1, \ldots, |w|\}$. If x is a first-order variable and $i \in \{1, \ldots, |w|\}$ then $\sigma[x \to i]$ is the $(\mathcal{V} \cup \{x\}, w)$-assignment which assigns x to i and acts like σ on all other variables. Similarly, $\sigma[X \to I]$ is defined for $I \subseteq \{1, \ldots, |w|\}$. The definition that (w, σ) satisfies φ, denoted $(w, \sigma) \models \varphi$, is as usual assuming that the domain of σ contains $\mathrm{Free}(\varphi)$. Note that $(w, \sigma) \models \varphi$ only depends on the restriction $\sigma_{|\,\mathrm{Free}(\varphi)}$ of σ to $\mathrm{Free}(\varphi)$.

As usual, a pair (w, σ) where σ is a $(\mathcal{V}, w)$-assignment will be encoded using an extended alphabet $A_{\mathcal{V}} = A \times \{0, 1\}^{\mathcal{V}}$. More precisely, we will write a word over $A_{\mathcal{V}}$ as a pair (w, σ) where w is the projection over A and σ is the projection over $\{0, 1\}^{\mathcal{V}}$. Now, σ represents a *valid* assignment over $\mathcal{V}$ if for each first-order variable $x \in \mathcal{V}$, the x-row of σ contains exactly one 1. In this case, we identify σ with the $(\mathcal{V}, w)$-assignment such that for each first-order variable $x \in \mathcal{V}$, $\sigma(x)$

is the position of the 1 on the x-row, and for each second-order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions carrying a 1 on the X-row. Clearly, the language

$$N_{\mathcal{V}} = \{(w, \sigma) \in A_{\mathcal{V}}^* \mid \sigma \text{ is a valid } (\mathcal{V}, w)\text{-assignment}\}$$

is recognizable. We simply write $A_\varphi = A_{\mathrm{Free}(\varphi)}$ and $N_\varphi = N_{\mathrm{Free}(\varphi)}$. By Büchi's theorem, if $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$ then the language

$$\mathcal{L}_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}$$

defined by φ over $A_{\mathcal{V}}$ is recognizable. Again, we simply write $\mathcal{L}(\varphi)$ for $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi)$. Conversely, each recognizable language L in A^* is definable by an MSO-sentence φ, so $L = \mathcal{L}(\varphi)$.

Next, we turn to basic definitions and properties of semirings, formal power series and weighted automata. For background, we refer the reader to [2, 20, 28].

A *semiring* is a structure $(K, +, \cdot, 0, 1)$ where $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot x = x \cdot 0 = 0$ for each $x \in K$. If the multiplication is commutative, we say that K is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. Important examples include

- the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ with the usual addition and multiplication,
- the Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$,
- the tropical semiring $\mathrm{Trop} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ (also known as min-plus semiring), with min and + extended to $\mathbb{N} \cup \{\infty\}$ in the natural way,
- the arctical semiring $\mathrm{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$,
- the semiring $([0, 1], \max, \cdot, 0, 1)$ which can be used to compute probabilities,
- the semiring of languages $(\mathcal{P}(A^*), \cup, \cap, \emptyset, A^*)$.

If K is a semiring and $n \in \mathbb{N}$, then $K^{n \times n}$ comprises all $(n \times n)$-matrices over K. With usual matrix multiplication $(K^{n \times n}, \cdot)$ is a monoid.

A formal power series is a mapping $S : A^* \to K$. It is usual to write (S, w) for $S(w)$. The set $\mathrm{Supp}(S) := \{w \in A^* \mid (S, w) \neq 0\}$ is called the *support* of S, and $\mathrm{Im}(S) = \{(S, w) \mid w \in A^*\}$ is the *image* of S. The set of all formal power series over K and A is denoted by $K\langle\langle A^*\rangle\rangle$. Now let $S, T \in K\langle\langle A^*\rangle\rangle$. The *sum* $S + T$ and the *Hadamard product* $S \odot T$ are both defined pointwise:

$$(S + T, w) := (S, w) + (T, w) \text{ and } (S \odot T, w) := (S, w) \cdot (T, w) \quad (w \in A^*).$$

Then $(K\langle\langle A^*\rangle\rangle, +, \odot, 0, 1)$ where 0 and 1 denote the constant series with values 0 resp. 1, is again a semiring.

For $L \subseteq A^*$, we define the *characteristic series* $\mathbb{1}_L : A^* \to K$ by $(\mathbb{1}_L, w) = 1$ if $w \in L$, and $(\mathbb{1}_L, w) = 0$ otherwise. If $K = \mathbb{B}$, the correspondence $L \mapsto \mathbb{1}_L$ gives a useful and natural semiring isomorphism from $(\mathcal{P}(A^*), \cup, \cap, \emptyset, A^*)$ onto $(\mathbb{B}\langle\langle A^*\rangle\rangle, +, \odot, 0, 1)$.

Now we turn to weighted automata. We fix a semiring K and an alphabet A. A *weighted finite automaton* over K and A is a quadruple $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ where Q is a finite set of states, $\mu : A \to K^{Q \times Q}$ is the transition weight function and

$\lambda, \gamma : Q \to K$ are weight functions for entering and leaving a state, respectively. Here $\mu(a)$ is a $(Q \times Q)$-matrix whose (p, q)-entry $\mu(a)_{p,q} \in K$ indicates the weight (cost) of the transition $p \xrightarrow{a} q$. Then μ extends uniquely to a monoid homomorphism (also denoted by μ) from A^* into $(K^{Q \times Q}, \cdot)$.

The *weight* of a path $P : q_0 \xrightarrow{a_1} q_1 \longrightarrow \ldots \longrightarrow q_{n-1} \xrightarrow{a_n} q_n$ in $\mathcal{A}$ is the product $\mathrm{weight}(P) := \lambda(q_0) \cdot \mu(a_1)_{q_0,q_1} \cdots \mu(a_n)_{q_{n-1},q_n} \cdot \gamma(q_n)$. This path has label $a_1 \ldots a_n$. The *weight* of a word $w = a_1 \ldots a_n \in A^*$ in $\mathcal{A}$, denoted $(||\,\mathcal{A}\,||, w)$, is the sum of $\mathrm{weight}(P)$ over all paths P with label w. One can check that

$$(||\,\mathcal{A}\,||, w) = \sum_{i,j} \lambda(i) \cdot \mu(w)_{ij} \cdot \gamma(j) = \lambda \cdot \mu(w) \cdot \gamma$$

with usual matrix multiplication, considering λ as a row vector and γ as a column vector. If $w = \varepsilon$, we have $(||\,\mathcal{A}\,||, \varepsilon) = \lambda \cdot \gamma$. The formal power series $||\,\mathcal{A}\,|| : A^* \to K$ is called the *behavior* of $\mathcal{A}$. A formal power series $S \in K\langle\langle A^* \rangle\rangle$ is called *recognizable*, if there exists a weighted finite automaton $\mathcal{A}$ such that $S = ||\,\mathcal{A}\,||$. Then we also call $\mathcal{A}$ or (λ, μ, γ) a *representation* of S. We let $K^{\mathrm{rec}}\langle\langle A^* \rangle\rangle$ be the collection of all recognizable formal power series over K and A.

Lemma 2.1 ([11]).

(a) Let $S, T \in K\langle\langle A^* \rangle\rangle$ be recognizable. Then $S + T$ is recognizable. If K is commutative, then $S \odot T$ is also recognizable.
(b) For any recognizable language $L \subseteq A^*$, the series $\mathbb{1}_L$ is recognizable.

Now let $h : A^* \to B^*$ be a homomorphism. If $T \in K\langle\langle B^* \rangle\rangle$, then $h^{-1}(T) := T \circ h \in K\langle\langle A^* \rangle\rangle$. That is, $(h^{-1}(T), w) = (T, h(w))$ for each $w \in A^*$. We say that h is *non-erasing*, if $h(a) \neq \varepsilon$ for any $a \in A$, or, equivalently, $|w| \leq |h(w)|$ for all $w \in A^*$. In this case, for $S \in K\langle\langle A^* \rangle\rangle$, define $h(S) : B^* \to K$ by $(h(S), v) := \sum_{w \in h^{-1}(v)} (S, w)$ $(v \in B^*)$, noting that the sum is finite since h is non-erasing.

Lemma 2.2 ([11]). *Let $h : A^* \to B^*$ be a homomorphism.*

(a) $h^{-1} : K\langle\langle B^* \rangle\rangle \to K\langle\langle A^* \rangle\rangle$ preserves recognizability.
(b) If h is non-erasing, then $h : K\langle\langle A^* \rangle\rangle \to K\langle\langle B^* \rangle\rangle$ preserves recognizability.

We say $S : A^* \to K$ is a *recognizable step function*, if $S = \sum_{i=1}^{n} k_i \cdot \mathbb{1}_{L_i}$ for some $n \in \mathbb{N}, k_i \in K$ and recognizable languages $L_i \subseteq A^*$ $(i = 1, \ldots, n)$. As is well-known, any recognizable step function is a recognizable power series.

3 Weighted logics

In this section, we introduce our weighted logics and study its first properties. We fix a semiring K and an alphabet A. For each $a \in A$, P_a denotes a unary predicate symbol.

Definition 3.1. *The syntax of formulas of the* weighted MSO-logic *is given by*

$$\begin{aligned}\varphi ::= & \; k \mid P_a(x) \mid \neg P_a(x) \mid x \leq y \mid \neg(x \leq y) \mid x \in X \mid \neg(x \in X) \\ & \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $k \in K$ and $a \in A$. We denote by $\mathrm{MSO}(K, A)$ *the collection of all such weighted* MSO*-formulas φ.*

As noted in the introduction, we do not permit negation of general formulas due to difficulties defining then their semantics: The semantics of a weighted logic formula φ should be a formal power series over an extended alphabet and with values in K. It would be natural to define the semantics of $\neg\varphi$ element-wise. But if K is not a Boolean algebra, K does not have a natural complement operation.

Therefore we restrict negation to atomic formulas whose semantics will take as values only 0 and 1 in K; thus the negation of atomic formulas also has a natural semantics. In comparison to classical (unweighted) MSO-logic, this is not an essential restriction, since the negation of a classical MSO-formula is equivalent (in the sense of defining the same language) to one in which negation is applied only to atomic formulas. In this sense, our weighted MSO-logics contains the classical MSO-logics which we obtain by letting $K = \mathbb{B}$. Note that in this case, the constant k in the logic is either 0 (false) or 1 (true).

Now we turn to the definition of the semantics of formulas $\varphi \in \mathrm{MSO}(K, A)$. As usual, a variable is said to be *free* in φ if there is an occurence of it in φ not in the scope of a quantifier. A pair (w, σ) where $w \in A^*$ and σ is a $(\mathcal{V}, w)$-assignment is represented by a word over the extended alphabet $A_{\mathcal{V}}$ as explained in Section 2.

Definition 3.2. *Let $\varphi \in \mathrm{MSO}(K, A)$ and $\mathcal{V}$ be a finite set of variables containing* $\mathrm{Free}(\varphi)$*. The* $\mathcal{V}$-semantics *of φ is a formal power series $[\![\varphi]\!]_{\mathcal{V}} \in K\langle\langle A_{\mathcal{V}}^*\rangle\rangle$. Let $(w, \sigma) \in A_{\mathcal{V}}^*$. If σ is not a valid $(\mathcal{V}, w)$-assignment, then we put $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = 0$. Otherwise, we define $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) \in K$ inductively as follows:*

$$[\![k]\!]_{\mathcal{V}}(w, \sigma) = k$$

$$[\![P_a(x)]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$$

$$[\![x \leq y]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$$

$$[\![x \in X]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\neg\varphi]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } [\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = 0 \\ 0 & \text{if } [\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = 1 \end{cases} \quad \begin{array}{l}\text{if } \varphi \text{ is of the form } P_a(x), \\ (x \leq y) \text{ or } (x \in X).\end{array}$$

$$[\![\varphi \vee \psi]\!]_{\mathcal{V}}(w, \sigma) = [\![\varphi]\!]_{\mathcal{V}}(w, \sigma) + [\![\psi]\!]_{\mathcal{V}}(w, \sigma)$$

$$[\![\varphi \wedge \psi]\!]_{\mathcal{V}}(w, \sigma) = [\![\varphi]\!]_{\mathcal{V}}(w, \sigma) \cdot [\![\psi]\!]_{\mathcal{V}}(w, \sigma)$$

$$\llbracket \exists x.\varphi \rrbracket_{\mathcal{V}}(w,\sigma) = \sum_{1\leq i\leq |w|} \llbracket \varphi \rrbracket_{\mathcal{V}\cup\{x\}}(w,\sigma[x\to i])$$

$$\llbracket \exists X.\varphi \rrbracket_{\mathcal{V}}(w,\sigma) = \sum_{I\subseteq\{1,\ldots,|w|\}} \llbracket \varphi \rrbracket_{\mathcal{V}\cup\{X\}}(w,\sigma[X\to I])$$

$$\llbracket \forall x.\varphi \rrbracket_{\mathcal{V}}(w,\sigma) = \prod_{1\leq i\leq |w|} \llbracket \varphi \rrbracket_{\mathcal{V}\cup\{x\}}(w,\sigma[x\to i])$$

$$\llbracket \forall X.\varphi \rrbracket_{\mathcal{V}}(w,\sigma) = \prod_{I\subseteq\{1,\ldots,|w|\}} \llbracket \varphi \rrbracket_{\mathcal{V}\cup\{X\}}(w,\sigma[X\to I])$$

where we fix some order on the power set of $\{1,\ldots,|w|\}$ *so that the last product is defined even if* K *is not commutative. We simply write* $\llbracket\varphi\rrbracket$ *for* $\llbracket\varphi\rrbracket_{\mathrm{Free}(\varphi)}$.

Note that if φ is a sentence, i.e. has no free variables, then $\llbracket\varphi\rrbracket \in K\langle\langle A^*\rangle\rangle$. We give several examples of possible interpretations for weighted formulas:

I. Let $K = (\mathbb{N},+,\cdot,0,1)$ and assume φ does not contain constants $k \in \mathbb{N}$. We may interpret $\llbracket\varphi\rrbracket(w,\sigma)$ as the number of proofs we have that (w,σ) satisfies formula φ. Indeed, for atomic formulas the number of proofs is clearly 0 or 1, depending on whether φ holds for (w,σ) or not. Now if e.g. $\llbracket\varphi\rrbracket(w,\sigma) = m$ and $\llbracket\psi\rrbracket(w,\sigma) = n$, the number of proofs that (w,σ) satisfies $\varphi\vee\psi$ should be $m+n$ (since any proof suffices), and for $\varphi\wedge\psi$ it should be $m\cdot n$ (since we may pair the proofs of φ and ψ arbitrarily). Similarly, the semantics of the existential and universal quantifiers can be interpreted.

II. The formula $\exists x.P_a(x)$ counts how often a occurs in the word. Here *how often* depends on the semiring: e.g. Boolean semiring, natural numbers, integers modulo 3, ...

III. Consider the probability semiring $K = ([0,1],\max,\cdot,0,1)$ and the alphabet $A = \{a_1,\ldots,a_n\}$. Assume that each letter a_i has a reliability k_i. Then, the series assigning to a word its reliability can be given by the first-order formula $\forall x.\bigvee_{1\leq i\leq n}(P_{a_i}(x)\wedge k_i)$.

IV. Let K be an arbitrary Boolean algebra $(B,\vee,\wedge,\bar{\ },0,1)$. In this case, sums correspond to suprema, and products to infima. Here we can define the semantics of $\neg\varphi$ for an arbitrary formula φ by $\llbracket\neg\varphi\rrbracket(w,\sigma) := \overline{\llbracket\varphi\rrbracket(w,\sigma)}$, the complement of $\llbracket\varphi\rrbracket(w,\sigma)$ in B. Then clearly $\llbracket\varphi\wedge\psi\rrbracket = \llbracket\neg(\neg\varphi\vee\neg\psi)\rrbracket$, $\llbracket\forall x.\varphi\rrbracket = \llbracket\neg(\exists x.\neg\varphi)\rrbracket$ and $\llbracket\forall X.\varphi\rrbracket = \llbracket\neg(\exists X.\neg\varphi)\rrbracket$. This may be interpreted as a multi-valued logics. In particular, if $K = \mathbb{B}$, the 2-valued Boolean algebra, our semantics coincides with the usual semantics of unweighted MSO-formulas, identifying characteristic series with their supports.

Observe that if $\varphi \in \mathrm{MSO}(K,A)$, we have defined a semantics $\llbracket\varphi\rrbracket_{\mathcal{V}}$ for each finite set of variables $\mathcal{V}$ containing $\mathrm{Free}(\varphi)$. Now we show that these semantics' are consistent with each other.

Proposition 3.3. *Let* $\varphi \in \mathrm{MSO}(K,A)$ *and* $\mathcal{V}$ *a finite set of variables containing* $\mathrm{Free}(\varphi)$. *Then*

$$\llbracket\varphi\rrbracket_{\mathcal{V}}(w,\sigma) = \llbracket\varphi\rrbracket(w,\sigma_{|\,\mathrm{Free}(\varphi)})$$

for each $(w,\sigma) \in A_{\mathcal{V}}^*$ such that σ is a valid $(\mathcal{V},w)$-assignment. In particular, $[\![\varphi]\!]$ is recognizable iff $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable.

Proof. We show our first claim by induction on φ. It is clear if φ is an atomic proposition and follows directly by induction for disjunctions and conjunctions. The interesting cases are the quantifications. We give the proof for $\varphi = \exists x.\psi$. The other cases are similar. Since σ is a valid $(\mathcal{V},w)$-assignment, $\sigma[x \to i]$ is a valid $(\mathcal{V} \cup \{x\}, w)$-assignment for all $i \in \{1,\ldots,|w|\}$. Since $\mathrm{Free}(\psi) \subseteq \mathcal{V} \cup \{x\}$, we get by induction

$$[\![\psi]\!]_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i]) = [\![\psi]\!](w,\sigma[x \to i]_{|\,\mathrm{Free}(\psi)}).$$

Also, for all $i \in \{1,\ldots,|w|\}$, $\sigma_{|\,\mathrm{Free}(\varphi)}[x \to i]$ is a valid $(\mathrm{Free}(\varphi) \cup \{x\}, w)$-assignment. Since $\mathrm{Free}(\psi) \subseteq \mathrm{Free}(\varphi) \cup \{x\}$, we get by induction

$$[\![\psi]\!]_{\mathrm{Free}(\varphi)\cup\{x\}}(w,\sigma_{|\,\mathrm{Free}(\varphi)}[x \to i]) = [\![\psi]\!](w,\sigma[x \to i]_{|\,\mathrm{Free}(\psi)}).$$

Therefore,

$$[\![\psi]\!]_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i]) = [\![\psi]\!]_{\mathrm{Free}(\varphi)\cup\{x\}}(w,\sigma_{|\,\mathrm{Free}(\varphi)}[x \to i])$$

and we get $[\![\varphi]\!]_{\mathcal{V}}(w,\sigma) = [\![\varphi]\!](w,\sigma_{|\,\mathrm{Free}(\varphi)})$ by definition of the semantics of $\varphi = \exists x.\psi$.

For the final claim, consider the projection $\pi : A_{\mathcal{V}} \to A_\varphi$. For $(w,\sigma) \in A_{\mathcal{V}}^*$, we have $\pi(w,\sigma) = (w,\sigma_{|\,\mathrm{Free}(\varphi)})$. If $[\![\varphi]\!]$ is recognizable then $[\![\varphi]\!]_{\mathcal{V}} = \pi^{-1}([\![\varphi]\!]) \odot \mathbb{1}_{N_{\mathcal{V}}}$ is recognizable by Lemmata 2.1 and 2.2. Here, we do not need to assume K commutative since $\mathbb{1}_{N_{\mathcal{V}}}$ is the characteristic series of a recognizable language and the values 0 and 1 taken by characteristic series commute with everything.

Conversely, let F comprise the empty word and all $(w,\sigma) \in A_{\mathcal{V}}^+$ such that σ assigns to each variable x (resp. X) in $\mathcal{V} \setminus \mathrm{Free}(\varphi)$ position 1, i.e., $\sigma(x) = 1$ (resp. $\sigma(X) = \{1\}$). Then F is recognizable, and for each $(w,\sigma') \in A_\varphi^*$ there is a unique element $(w,\sigma) \in F$ such that $\pi(w,\sigma) = (w,\sigma')$. Thus $[\![\varphi]\!] = \pi([\![\varphi]\!]_{\mathcal{V}} \odot \mathbb{1}_F)$, as is easy to check. Hence, if $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable then so is $[\![\varphi]\!]$ by Lemmata 2.1 and 2.2. $\square$

Now let $Z \subseteq \mathrm{MSO}(K,A)$. A series $S : A^* \to K$ is called *Z-definable*, if there is a sentence $\varphi \in Z$ such that $S = [\![\varphi]\!]$. The main goal of this paper is the comparison of Z-definable with recognizable series, for suitable fragments Z of $\mathrm{MSO}(K,A)$. Crucial for this will be closure properties of recognizable series under the constructs of our weighted logic. However, first we will show that $K^{\mathrm{rec}}\langle\langle A^*\rangle\rangle$ is in general not closed under universal quantification.

Example 3.4. Let $K = (\mathbb{N},+,\cdot,0,1)$. Then $[\![\forall x.2]\!](w) = 2^{|w|}$ and $[\![\forall y \forall x.2]\!](w) = (2^{|w|})^{|w|} = 2^{|w|^2}$. Clearly, the series $[\![\forall x.2]\!]$ is recognizable by the weighted automaton (Q,λ,μ,γ) with $Q = \{1\}$, $\lambda_1 = \gamma_1 = 1$ and $\mu_{1,1}(a) = 2$ for all $a \in A$. However, $[\![\forall y \forall x.2]\!]$ is not recognizable. Suppose there was an automaton $\mathcal{A}' = (Q',\lambda',\mu',\gamma')$ with behavior $[\![\forall y \forall x.2]\!]$. Let $M = \max\{|\lambda'_p|, |\gamma'_p|, |\mu'(a)_{p,q}| \mid$

$p, q \in Q', a \in A\}$. Then, for any $w \in A^*$ and for each path P labelled by w we have weight$(P) \leq M^{|w|+2}$ and since there are $|Q|^{|w|+1}$ paths labelled w we obtain $(||\mathcal{A}'||, w) \leq |Q'|^{|w|+1} \cdot M^{|w|+2}$, a contradiction with $(||\mathcal{A}'||, w) = 2^{|w|^2}$.

A similar argument applies also for the tropical and the arctical semirings. Observe that in all these cases, $[\![\forall x.2]\!]$ has infinite image.

Example 3.5. Let $K = (\mathbb{N}, +, \cdot, 0, 1)$. Then $[\![\forall X.2]\!](w) = 2^{2^{|w|}}$ for any $w \in A^*$, and as above $[\![\forall X.2]\!]$ is not recognizable due to its growth. Again, this counterexample also works for the tropical and the arctical semirings.

The examples show that unrestricted universal quantification is too strong to preserve recognizability. This motivates the following definition.

Definition 3.6. *We will call a formula $\varphi \in \mathrm{MSO}(K, A)$* restricted*, if it contains no universal set quantification of the form $\forall X.\psi$, and whenever φ contains a universal first-order quantification $\forall x.\psi$, then $[\![\psi]\!]$ is a recognizable step function.*

Note that this is not a purely syntactic definition since one restriction is on the semantics $[\![\psi]\!]$ of formulas. We show later that this restriction is decidable under suitable hypotheses on the semiring.

We let $\mathrm{RMSO}(K, A)$ comprise all restricted formulas of $\mathrm{MSO}(K, A)$. Furthermore, let $\mathrm{REMSO}(K, A)$ contain all restricted *existential* MSO-formulas φ, i.e. φ is of the form $\exists X_1, \ldots, X_n.\psi$ with $\psi \in \mathrm{RMSO}(K, A)$ containing no set quantification.

We let $K^{\mathrm{rmso}}\langle\langle A^*\rangle\rangle$ (resp. $K^{\mathrm{remso}}\langle\langle A^*\rangle\rangle$) contain all series $S \in K\langle\langle A^*\rangle\rangle$ which are definable by some sentence in $\mathrm{RMSO}(K, A)$ (resp. in $\mathrm{REMSO}(K, A)$). The main result of this paper is the following theorem. It will be proved in sections 4 and 5.

Theorem 3.7. *Let K be a commutative semiring and A an alphabet. Then,*

$$K^{\mathrm{rec}}\langle\langle A^*\rangle\rangle = K^{\mathrm{rmso}}\langle\langle A^*\rangle\rangle = K^{\mathrm{remso}}\langle\langle A^*\rangle\rangle.$$

4 Definable series are recognizable

In all of this section, let K be a semiring and A an alphabet. We wish to show that if K is commutative, then all RMSO-definable series $[\![\varphi]\!]$ over K and A are recognizable. We proceed by induction over the structure of RMSO-formulas.

Lemma 4.1. *Let $\varphi \in \mathrm{MSO}(K, A)$ be atomic. Then $[\![\varphi]\!]$ is recognizable.*

Proof. We distinguish between two cases.
<u>Case 1.</u> $\varphi = k$ where $k \in K$

The one-state automaton $(Q, \lambda, \mu, \gamma)$ with $Q = \{1\}$, $\lambda_1 = 1$, $\mu_{1,1}(a) = 1$ for all $a \in A$ and $\gamma_1 = k$ recognizes $[\![\varphi]\!] = k \cdot \mathbb{1}_{A^*}$.
<u>Case 2.</u> φ is of the form $P_a(x)$ or $(x \leq y)$ or $(x \in X)$, or φ is the negation of one of these formulas.

Considering φ as a formula of classical MSO-logic, it is easy (and well-known) to find a deterministic automaton $\mathcal{A}$ over the extended alphabet A_φ recognizing the pairs (w, σ) satisfying φ. Now we transform $\mathcal{A}$ into the corresponding weighted automaton $\mathcal{A}'$ in which the transitions of $\mathcal{A}$ get weight 1, the triples which are not transitions of $\mathcal{A}$ get weight 0, the initial state of $\mathcal{A}$ gets initial weight 1 and the other states get initial weight 0, and similarly for the final weights. Then $\mathcal{A}'$ recognizes $[\![\varphi]\!]$. □

Now we turn to disjunction and conjunction.

Lemma 4.2. *Let $\varphi, \psi \in \mathrm{MSO}(K, A)$ such that $[\![\varphi]\!]$ and $[\![\psi]\!]$ are recognizable series. Then $[\![\varphi \vee \psi]\!]$ is recognizable. If K is commutative, then $[\![\varphi \wedge \psi]\!]$ is also recognizable.*

Proof. Let $\mathcal{V} = \mathrm{Free}(\varphi) \cup \mathrm{Free}(\psi)$. By definition, we have $[\![\varphi \vee \psi]\!] = [\![\varphi]\!]_\mathcal{V} + [\![\psi]\!]_\mathcal{V}$ and $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!]_\mathcal{V} \odot [\![\psi]\!]_\mathcal{V}$. Hence the result follows from Lemma 2.1 and Proposition 3.3. □

Lemma 4.3. *Let $\varphi \in \mathrm{MSO}(K, A)$ such that $[\![\varphi]\!]$ is recognizable. Then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are recognizable series.*

Proof. Let $\mathcal{V} = \mathrm{Free}(\exists X.\varphi)$ and note that $X \notin \mathcal{V}$. Consider the projection $\pi : A^*_{\mathcal{V}\cup\{X\}} \to A^*_\mathcal{V}$ which erases the X-row. Let $(w, \sigma) \in A^*_\mathcal{V}$. Note that σ is a valid $(\mathcal{V}, w)$-assignment iff $\sigma[X \to I]$ is a valid $(\mathcal{V} \cup \{X\}, w)$-assignment for all $I \subseteq \{1, \ldots, |w|\}$. Hence, we have (even if σ is not a valid $(\mathcal{V}, w)$-assignment)

$$[\![\exists X.\varphi]\!](w, \sigma) = \sum_{I \subseteq \{1,\ldots,|w|\}} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w, \sigma[X \to I]) = \pi([\![\varphi]\!]_{\mathcal{V}\cup\{X\}})(w, \sigma).$$

The last equality holds since $\pi(w, \sigma') = (w, \sigma)$ iff $\sigma' = \sigma[X \to I]$ for some $I \subseteq \{1, \ldots, |w|\}$. Now, $\mathrm{Free}(\varphi) \subseteq \mathcal{V} \cup \{X\}$ and $[\![\varphi]\!]_{\mathcal{V}\cup\{X\}}$ is recognizable by Proposition 3.3. We deduce from Lemma 2.2 that $[\![\exists X.\varphi]\!]$ is recognizable.

We turn now to the case $\exists x.\varphi$. As above, we let $\mathcal{V} = \mathrm{Free}(\exists x.\varphi)$ and $x \notin \mathcal{V}$. Consider the projection $\pi : A^*_{\mathcal{V}\cup\{x\}} \to A^*_\mathcal{V}$ which erases the x-row. Let $(w, \sigma) \in A^*_\mathcal{V}$. Note that σ is a valid $(\mathcal{V}, w)$-assignment iff $\sigma[x \to i]$ is a valid $(\mathcal{V} \cup \{x\}, w)$-assignment for all $i \in \{1, \ldots, |w|\}$. Hence, we have (even if σ is not a valid $(\mathcal{V}, w)$-assignment)

$$[\![\exists x.\varphi]\!]_\mathcal{V}(w, \sigma) = \sum_{i \in \{1,\ldots,|w|\}} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(w, \sigma[x \to i]) = \pi([\![\varphi]\!]_{\mathcal{V}\cup\{x\}})(w, \sigma).$$

Here, the last equality holds since σ' is a valid $(\mathcal{V} \cup \{x\}, w)$-assignment and $\pi(w, \sigma') = (w, \sigma)$ iff $\sigma' = \sigma[x \to i]$ for some $i \in \{1, \ldots, |w|\}$. We conclude as above. □

The most interesting case here arises from universal quantification.

Lemma 4.4. *Let K be commutative and $\varphi \in \mathrm{MSO}(K, A)$ such that $[\![\varphi]\!]$ is a recognizable step function. Then $[\![\forall x.\varphi]\!]$ is recognizable.*

Proof. Let $\mathcal{W} = \mathrm{Free}(\varphi)$ and $\mathcal{V} = \mathrm{Free}(\forall x.\varphi) = \mathcal{W} \setminus \{x\}$. We may write $[\![\varphi]\!] = \sum_{j=1,\dots,n} k_j \cdot \mathbb{1}_{L_j}$ with $n \in \mathbb{N}$, $k_j \in K$ and recognizable languages $L_j \subseteq A^*_{\mathcal{W}}$ ($j = 1,\dots,n$) such that the languages L_j ($j = 1,\dots,n$) form a partition of $A^*_{\mathcal{W}}$.

First, we assume that $x \in \mathcal{W}$. Let $\widetilde{A} = A \times \{1,\dots,n\}$. A word in $(\widetilde{A}_{\mathcal{V}})^*$ will be written (w,ν,σ) where $(w,\sigma) \in A^*_{\mathcal{V}}$ and $\nu \in \{1,\dots,n\}^{|w|}$ is interpreted as a mapping from $\{1,\dots,|w|\}$ to $\{1,\dots,n\}$. Let $\widetilde{L}$ be the set of $(w,\nu,\sigma) \in (\widetilde{A}_{\mathcal{V}})^*$ such that for all $i \in \{1,\dots,|w|\}$ and $j \in \{1,\dots,n\}$ we have

$$\nu(i) = j \quad \text{implies} \quad (w, \sigma[x \to i]) \in L_j.$$

Observe that for each $(w,\sigma) \in A^*_{\mathcal{V}}$ there is a unique ν such that $(w,\nu,\sigma) \in \widetilde{L}$ since the L_j form a partition of $A^*_{\mathcal{W}}$.

We claim that $\widetilde{L}$ is recognizable. Indeed, for $j \in \{1,\dots,n\}$, let $\widetilde{L_j}$ be the set of all $(w,\nu,\sigma) \in (\widetilde{A}_{\mathcal{V}})^*$ such that for all $i \in \{1,\dots,|w|\}$ we have $\nu(i) = j$ implies $(w,\sigma[x \to i]) \in L_j$. Note that $\widetilde{L} = \bigcap_{1 \le j \le n} \widetilde{L_j}$. Therefore, it suffices to show that each language $\widetilde{L_j}$ is recognizable. For this, fix $j \in \{1,\dots,n\}$.

Let $\mathcal{A}_j = (Q, q_0, \delta, F)$ be a deterministic automaton, with transition function $\delta : Q \times A_{\mathcal{W}} \to Q$, recognizing L_j. We wish to construct a deterministic automaton $\widetilde{\mathcal{A}_j} = (\widetilde{Q}, \widetilde{q_0}, \widetilde{\delta}, \widetilde{F})$ recognizing $\widetilde{L_j}$. Intuitively, $\widetilde{\mathcal{A}_j}$ works as follows. When reading a word $(w,\nu,\sigma) \in (\widetilde{A}_{\mathcal{V}})^*$ and detecting that $\nu(i) = j$, the automaton $\widetilde{\mathcal{A}_j}$ should check if $(w,\sigma[x \to i]) \in L_j$. For this, $\widetilde{\mathcal{A}_j}$ uses a copy of the automaton $\mathcal{A}_j$. Note that the x-row in the word $(w,\sigma[x \to i])$ contains a 1 exactly at entry i. Therefore we let $\widetilde{\mathcal{A}_j}$ contain a *master copy* of $\mathcal{A}_j$ which works only on words of the form $(w,\sigma[x \to \emptyset])$[3] to start a new copy of $\mathcal{A}_j$ in the suitable state when reading $\nu(i) = j$. The number of copies needed is clearly bounded by the size of Q.

More precisely, let $\widetilde{Q} = Q \times \mathscr{P}(Q)$ where $\mathscr{P}(Q)$ denotes the power set of Q, put $\widetilde{q_0} = (q_0, \emptyset)$, and $\widetilde{F} = Q \times \mathscr{P}(F)$. Define $\widetilde{\delta} : \widetilde{Q} \times \widetilde{A}_{\mathcal{V}} \to \widetilde{Q}$ by

$$\widetilde{\delta}\big((p,P),(a,k,s)\big) = \big(\delta\big(p,(a,s[x \to 0])\big), P'\big)$$

where $(p,P) \in \widetilde{Q}$, $(a,k,s) \in \widetilde{A}_{\mathcal{V}}$, $s[x \to 0]$ is the mapping $s \in \{0,1\}^{\mathcal{V}}$ extended to $\mathcal{V} \cup \{x\}$ by $x \mapsto 0$, and

$$P' = \begin{cases} \big\{\delta\big(q,(a,s[x \to 0])\big) \mid q \in P\big\} & \text{if } k \neq j \\ \big\{\delta\big(q,(a,s[x \to 0])\big) \mid q \in P\big\} \cup \big\{\delta\big(p,(a,s[x \to 1])\big)\big\} & \text{if } k = j \end{cases}$$

It remains to show that $\widetilde{\mathcal{A}_j}$ recognizes $\widetilde{L_j}$. By induction on the length of a word $(w,\nu,\sigma) \in (\widetilde{A}_{\mathcal{V}})^*$, one can prove that

$$\widetilde{\delta}\big(\widetilde{q_0},(w,\nu,\sigma)\big) = \big(\delta\big(q_0,(w,\sigma[x \to \emptyset])\big), P'\big)$$

[3] Abusing notations, even if x is a first-order variable, we write $\sigma[x \to \emptyset]$ to denote the assignment σ extended by an x-row which is uniformly 0. Note that $\sigma[x \to \emptyset]$ is not a valid assignment.

where $P' = \{\delta(q_0, (w, \sigma[x \to i])) \mid 1 \le i \le |w|, \nu(i) = j\}$.

Also, for any $1 \le i \le |w|$, we have

$$(w, \sigma[x \to i]) \in L_j \quad \text{iff} \quad \delta(q_0, (w, \sigma[x \to i])) \in F.$$

It follows that $(w, \nu, \sigma) \in \widetilde{L_j}$ iff whenever $\nu(i) = j$ then $\delta(q_0, (w, \sigma[x \to i])) \in F$, and this holds iff $P' \subseteq F$, i.e. (w, ν, σ) is accepted by $\widetilde{A}_j$. Hence $\widetilde{A}_j$ recognizes $\widetilde{L_j}$ which implies our claim.

Hence there is a deterministic automaton $\widetilde{\mathcal{A}}$ over the alphabet $\widetilde{A}_{\mathcal{V}}$, recognizing $\widetilde{L}$. Now we obtain a weighted automaton $\mathcal{A}$ with the same state set by adding weights to the transitions of $\widetilde{\mathcal{A}}$ as follows: If $(p, (a, j, s), q)$ is a transition in $\widetilde{\mathcal{A}}$ with $(a, j, s) \in \widetilde{A}_{\mathcal{V}}$, we let this transition in $\mathcal{A}$ have weight k_j, i.e. $\mu_{\mathcal{A}}(a, j, s)_{p,q} = k_j$. All triples which are not transitions in $\widetilde{\mathcal{A}}$ get weight 0. Also, the initial state of $\widetilde{\mathcal{A}}$ gets initial weight 1 in $\mathcal{A}$, all non-initial states of $\widetilde{\mathcal{A}}$ get initial weight 0, and similarly for the final states and final weights.

Clearly, since $\widetilde{\mathcal{A}}$ is deterministic and accepts $\widetilde{L}$, the weight of $(w, \nu, \sigma) \in \widetilde{L}$ in $\mathcal{A}$ is $\prod_{1 \le j \le n} k_j^{|\nu^{-1}(j)|}$, and the weight of $(w, \nu, \sigma) \in \widetilde{A}^* \setminus \widetilde{L}$ in $\mathcal{A}$ is 0. Now let $h : (\widetilde{A}_{\mathcal{V}})^* \to A_{\mathcal{V}}^*$ be the projection mapping (w, ν, σ) to (w, σ). Then for any $(w, \sigma) \in A_{\mathcal{V}}^*$ and the unique ν such that $(w, \nu, \sigma) \in \widetilde{L}$ we obtain

$$h(||\,\mathcal{A}\,||)(w, \sigma) = \sum_{\rho} ||\,\mathcal{A}\,||(w, \rho, \sigma) = ||\,\mathcal{A}\,||(w, \nu, \sigma) = \prod_{1 \le j \le n} k_j^{|\nu^{-1}(j)|}.$$

Now we have

$$[\![\forall x.\varphi]\!](w, \sigma) = \prod_{1 \le i \le |w|} [\![\varphi]\!](w, \sigma[x \to i]) = \prod_{1 \le j \le n} k_j^{|\nu^{-1}(j)|}$$

where the last equality holds due to the form of φ. Hence $[\![\forall x.\varphi]\!] = h(||\,\mathcal{A}\,||)$ which is recognizable by Lemma 2.2.

Now assume that $x \notin \mathcal{W}$, so that $\mathcal{V} = \mathcal{W}$. Let $\varphi' = \varphi \wedge (x \le x)$. So $[\![\varphi']\!]$ is recognizable by Lemma 4.2, and clearly $[\![\varphi]\!]_{\mathcal{V} \cup \{x\}} = [\![\varphi']\!]_{\mathcal{V} \cup \{x\}}$. Thus $[\![\forall x.\varphi]\!]_{\mathcal{V}} = [\![\forall x.\varphi']\!]_{\mathcal{V}}$ which is recognizable by what we showed above. $\square$

Now the following result is immediate by Lemmata 4.1, 4.2, 4.3 and 4.4.

Theorem 4.5. *Let K be a commutative semiring, A an alphabet and $\varphi \in$ RMSO(K, A). Then $[\![\varphi]\!] \in K^{\mathrm{rec}}\langle\langle A^* \rangle\rangle$ is recognizable.*

Next we turn to decidability questions. We will employ decidability results from the theory of formal power series and our previous constructions.

Proposition 4.6. *Let $\varphi \in$ MSO$(\mathbb{Q}, A)$ where $\mathbb{Q}$ is the field of rational numbers. It is decidable whether φ is restricted, and in this case one can effectively compute a weighted automaton $\mathcal{A}_\varphi$ for $[\![\varphi]\!]$.*

Proof. We may assume that φ contains no universal set quantification. We proceed by structural induction on φ. Note that Proposition 3.3 and Lemmata 2.1, 2.2, 4.1, 4.2 and 4.3 are effective, meaning that if weighted automata are given for the *arguments* then weighted automata can be effectively computed for the *results*. Therefore, the only difficult case in the induction is $\forall x.\varphi$. We have to show that if $K = \mathbb{Q}$ then Lemma 4.4 is also effective.

Let $\mathcal{V} = \mathrm{Free}(\varphi)$. We start with a weighted automaton for φ. We have to decide first whether $[\![\varphi]\!]$ is a recognizable step function. We can compute a reduced representation $(Q, \lambda, \mu, \gamma)$ for $[\![\varphi]\!]$ ([2]). By the argument in [2, p. 105, proof of Cor. VI.2.5], $\mathrm{Im}([\![\varphi]\!])$ is finite iff $\mu(A_{\mathcal{V}}^*)$ is finite, and by a result of Jacob [Ja78], cf. [2, Cor VI.2.6 and p.105], the latter property is decidable.

Actually, $[\![\varphi]\!]$ is a recognizable step function if and only if $\mathrm{Im}([\![\varphi]\!])$ is finite. The condition is clearly necessary. Conversely, let $\mathrm{Im}([\![\varphi]\!])$ and thus $\mu(A_{\mathcal{V}}^*)$ be finite. Then compute $\mu(A_{\mathcal{V}}^*)$ and $\mathrm{Im}([\![\varphi]\!]) = \{\lambda \cdot \Gamma \cdot \gamma \mid \Gamma \in \mu(A_{\mathcal{V}}^*)\}$. For each $k \in \mathrm{Im}([\![\varphi]\!])$, the language $[\![\varphi]\!]^{-1}(k) := \{w \in A_{\mathcal{V}}^* \mid ([\![\varphi]\!], w) = k\}$ is saturated by μ, since if $u \in [\![\varphi]\!]^{-1}(k), v \in A_{\mathcal{V}}^*$ and $\mu(u) = \mu(v)$, then $([\![\varphi]\!], v) = \lambda \cdot \mu(v) \cdot \gamma = \lambda \cdot \mu(u) \cdot \gamma = ([\![\varphi]\!], u) = k$, so $v \in [\![\varphi]\!]^{-1}(k)$. Hence $[\![\varphi]\!] = \sum_{k \in \mathrm{Im}([\![\varphi]\!])} k \cdot \mathbb{1}_{[\![\varphi]\!]^{-1}(k)}$ and each language $[\![\varphi]\!]^{-1}(k)$ is recognized by the morphism μ. Therefore, $[\![\varphi]\!]$ is a recognizable step function

Finally, we have to show that a weighted automaton for $[\![\forall x.\varphi]\!]$ can effectively be computed. Note that from μ, one can effectively compute deterministic automata for the languages $[\![\varphi]\!]^{-1}(k)$ and then a deterministic automaton $\widetilde{\mathcal{A}}$ for the language $\widetilde{L}$ introduced in the proof of Lemma 4.4. Therefore, following the proof of Lemma 4.4, we can effectively compute a weighted automaton for $[\![\forall x.\varphi]\!]$. □

Corollary 4.7. *Let $\varphi, \psi \in \mathrm{RMSO}(\mathbb{Q}, A)$. Then it is decidable whether $[\![\varphi]\!] = [\![\psi]\!]$. It is also decidable whether $[\![\varphi]\!]$ and $[\![\psi]\!]$ differ only for finitely many words.*

Proof. By Proposition 4.6, the series $[\![\varphi]\!], [\![\psi]\!]$ and hence also $[\![\varphi]\!] - [\![\psi]\!] = [\![\varphi]\!] + (-1) \cdot [\![\psi]\!]$ are effectively recognizable. By [2, Propositions VI.1.1, VI.1.2], it is decidable whether such a series equals 0, or whether its support is finite. □

Note that in the two previous results we may replace $\mathbb{Q}$ by any "computable" field (see [2]).

5 Recognizable series are definable

In all of this section let K be a semiring and A an alphabet. We wish to show that if K is commutative, then all recognizable series are REMSO-definable. For this, the concept of an unambiguous MSO-formula will be useful. For these formulas, the Boolean semantics will coincide with the weighted semantics.

Definition 5.1. *The class of* unambiguous *formulas in* $\mathrm{MSO}(K, A)$ *is defined inductively as follows:*

1. *All atomic formulas of the form $P_a(x)$, $x \leq y$ or $(x \in X)$, and their negations are unambiguous.*

2. If φ, ψ are unambiguous, then $\varphi \wedge \psi$, $\forall x.\varphi$ and $\forall X.\varphi$ are also unambiguous.
3. If φ, ψ are unambiguous and $\mathrm{Supp}([\![\varphi]\!]) \cap \mathrm{Supp}([\![\psi]\!]) = \emptyset$, then $\varphi \vee \psi$ is unambiguous.
4. Let φ be unambiguous and $\mathcal{V} = \mathrm{Free}(\varphi)$. If for any $(w, \sigma) \in A^*_{\mathcal{V}}$ there is at most one element $i \in \{1, \ldots, |w|\}$ such that $[\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(w, \sigma[x \to i]) \neq 0$, then $\exists x.\varphi$ is unambiguous.
5. Let φ be unambiguous and $\mathcal{V} = \mathrm{Free}(\varphi)$. If for any $(w, \sigma) \in A^*_{\mathcal{V}}$ there is at most one subset $I \subseteq \{1, \ldots, |w|\}$ such that $[\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w, \sigma[X \to I]) \neq 0$, then $\exists X.\varphi$ is unambiguous.

Note that, as for unambiguous rational expressions, this is not a purely syntactic definition since some restrictions are on the semantics of formulas. This is not so important since we will show that any MSO formula can be effectively transformed into an unambiguous one which is equivalent for the Boolean semantics.

Proposition 5.2. *Let $\varphi \in \mathrm{MSO}(K, A)$ be unambiguous. We may also regard φ as a classical* MSO*-formula defining the language $\mathcal{L}(\varphi) \subseteq A^*_\varphi$. Then, $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ is a recognizable step function.*

Proof. Let $(w, \sigma) \in A^*_\varphi$. If $(w, \sigma) \notin N_\varphi$ then $[\![\varphi]\!](w, \sigma) = 0$ and $(w, \sigma) \notin \mathcal{L}(\varphi)$. Assume now that $(w, \sigma) \in N_\varphi$. We show by structural induction on φ that $[\![\varphi]\!](w, \sigma)$ equals 1 if $(w, \sigma) \models \varphi$ and equals 0 otherwise. This is clear for the atomic formulas and their negations. It is also trivial by induction for conjunction and universal quantifications. Using the unambiguity of the formulas, we also get the result by induction for disjunction and existential quantifications. Therefore, $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ and since $\mathcal{L}(\varphi)$ is a recognizable language in A^*_φ we obtain that $[\![\varphi]\!]$ is a recognizable step function. $\square$

Next we show that, conversely, classical MSO-formulas can be transformed into unambiguous formulas.

Lemma 5.3. *For each classical* MSO*-formula φ not containing set quantifications (but possibly including atomic formulas of the form $(x \in X)$) we can effectively construct two unambiguous $\mathrm{MSO}(K, A)$-formulas φ^+ and φ^- such that $[\![\varphi^+]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ and $[\![\varphi^-]\!] = \mathbb{1}_{\mathcal{L}(\neg\varphi)}$, i.e., for any $(w, \sigma) \in N_\varphi$ we have*

$$[\![\varphi^+]\!](w, \sigma) = 1 \iff (w, \sigma) \models \varphi$$
$$[\![\varphi^-]\!](w, \sigma) = 1 \iff (w, \sigma) \not\models \varphi.$$

Proof. We may assume (using also conjunction and universal quantification in our syntax or as abbreviations) that in φ negations are applied only to atomic formulas. Now we proceed by induction, and we only give the respective formulas φ^+ and φ^-, leaving the easy proofs to the reader.

1. If φ is atomic or negation of an atomic formula, put $\varphi^+ = \varphi$ and $\varphi^- = \neg\varphi$ with the convention that $\neg\neg\psi = \psi$.
2. $(\varphi \vee \psi)^+ = \varphi^+ \vee (\varphi^- \wedge \psi^+)$ and $(\varphi \vee \psi)^- = \varphi^- \wedge \psi^-$

3. $(\varphi \wedge \psi)^- = \varphi^- \vee (\varphi^+ \wedge \psi^-)$ and $(\varphi \wedge \psi)^+ = \varphi^+ \wedge \psi^+$
4. $(\exists x.\varphi)^+ = \exists x.(\varphi^+(x) \wedge \forall y.((x \leq y) \vee (\neg(x \leq y) \wedge \varphi^-(y))))$ and $(\exists x.\varphi)^- = \forall x.\varphi^-$
5. $(\forall x.\varphi)^- = \exists x.(\varphi^-(x) \wedge \forall y.((x \leq y) \vee (\neg(x \leq y) \wedge \varphi^+(y))))$ and $(\forall x.\varphi)^+ = \forall x.\varphi^+$. □

Proposition 5.4. *For each classical* MSO*-sentence φ, we can effectively construct an unambiguous* $\mathrm{MSO}(K, A)$*-sentence ψ defining the same language, i.e.* $[\![\psi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$.

Proof. The complement $\overline{\mathcal{L}(\varphi)}$ of $\mathcal{L}(\varphi)$ can be defined in existential MSO-logic, hence $\mathcal{L}(\varphi)$ is in universal MSO-logic. That is, $\mathcal{L}(\varphi) = \mathcal{L}(\rho)$ for some MSO-formula ρ of the form $\rho = \forall X_1, \ldots, X_n.\zeta$ such that ζ contains no set quantifications. Using Lemma 5.3, put $\psi = \forall X_1, \ldots, X_n.\zeta^+$. □

Now we aim at showing that recognizable series are definable. First, for $k \in K$, we define

$$((x \in X) \to k) := \neg(x \in X) \vee ((x \in X) \wedge k).$$

Hence for any word w and valid assignment σ, we have

$$[\![((x \in X) \to k)]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} k & \text{if } \sigma(x) \in \sigma(X) \\ 1 & \text{otherwise} \end{cases},$$

so $[\![((x \in X) \to k)]\!]_{\mathcal{V}}$ is a recognizable step function, and we get

$$[\![\forall x.((x \in X) \to k)]\!]_{\mathcal{V}}(w, \sigma) = k^{|\sigma(X)|}.$$

We introduce a few abbreviations. We let $\min(y) := \forall x.y \leq x$, and $\max(z) := \forall x.x \leq z$, and $(y = x + 1) := (x \leq y) \wedge \neg(y \leq x) \wedge \forall z.(z \leq x \vee y \leq z)$. If $X_1, \ldots, X_m$ are set variables, put

$$\mathrm{partition}(X_1, \ldots, X_m) := \forall x. \bigvee_{i=1,\ldots,m} \left((x \in X_i) \wedge \bigwedge_{j \neq i} \neg(x \in X_j) \right).$$

Now we show:

Theorem 5.5. *Let K be commutative. Then $K^{\mathrm{rec}}\langle\langle A^* \rangle\rangle \subseteq K^{\mathrm{remso}}\langle\langle A^* \rangle\rangle$.*

Proof. Let $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ be a weighted automaton over A. For each triple $(p, a, q) \in Q \times A \times Q$ choose a set variable $X_{p,a,q}$, and let $\mathcal{V} = \{X_{p,a,q} \mid p, q \in Q, a \in A\}$. We choose an enumeration $\overline{X} = (X_1, \ldots, X_m)$ of $\mathcal{V}$ with $m = |Q|^2 \cdot |A|$. Define the unambiguous formula

$$\psi(\overline{X}) := \mathrm{partition}(\overline{X})^+ \wedge \bigwedge_{p,a,q} \forall x.((x \in X_{p,a,q}) \to P_a(x))^+$$
$$\wedge \forall x \forall y. \Big((y = x + 1) \to \bigvee_{p,q,r \in Q, a,b \in A} (x \in X_{p,a,q}) \wedge (y \in X_{q,b,r}) \Big)^+.$$

Let $w = a_1 \dots a_n \in A^+$. We show that there is a bijection between the set of paths in $\mathcal{A}$ over w and the set of $(\mathcal{V}, w)$-assignments σ satisfying ψ, i.e., such that $[\![\psi]\!](w, \sigma) = 1$. Let $\rho = (q_0 \xrightarrow{a_1} q_1 \longrightarrow \cdots \xrightarrow{a_n} q_n)$ be a path in $\mathcal{A}$ over w. Define the $(\mathcal{V}, w)$-assignment σ_ρ by $\sigma_\rho(X_{p,a,q}) = \{i \mid (q_{i-1}, a_i, q_i) = (p, a, q)\}$. Clearly, we have $[\![\psi]\!](w, \sigma_\rho) = 1$. Conversely, let σ be a $(\mathcal{V}, w)$-assignment such that $[\![\psi]\!](w, \sigma) = 1$. Due to partition $\overline{X}$, for any $x \in \{1, \dots, n\}$ there are uniquely determined $p, q \in Q$ and $a \in A$ such that $x \in \sigma(X_{p,a,q})$ and if $y = x + 1 \leq n$, then $y \in \sigma(X_{q,b,r})$ for some uniquely determined $b \in A$, $r \in Q$. Hence we obtain a unique path $\rho = (q_0 \xrightarrow{a_1} q_1 \longrightarrow \cdots \xrightarrow{a_n} q_n)$ for w such that $\sigma_\rho = \sigma$.

Consider now the formula

$$\begin{aligned}\varphi(\overline{X}) := \psi(\overline{X}) \wedge & \bigwedge_{p,a,q} \forall x.((x \in X_{p,a,q}) \to \mu(a)_{p,q}) \\ & \wedge \exists y.\Big(\min(y) \wedge \bigvee_{p,a,q} (y \in X_{p,a,q}) \wedge \lambda_p\Big) \\ & \wedge \exists z.\Big(\max(z) \wedge \bigvee_{p,a,q} (z \in X_{p,a,q}) \wedge \gamma_q\Big).\end{aligned}$$

Let $\rho = (q_0 \xrightarrow{a_1} q_1 \longrightarrow \cdots \xrightarrow{a_n} q_n)$ be a path in $\mathcal{A}$ over w and let σ_ρ be the associated $(\mathcal{V}, w)$-assignment. We obtain

$$\begin{aligned}[\![\varphi]\!](w, \sigma_\rho) &= \left(\prod_{p,a,q} \mu(a)_{p,q}^{|\sigma_\rho(X_{p,a,q})|}\right) \cdot \lambda_{q_o} \cdot \gamma_{q_n} \\ &= \lambda_{q_0} \cdot \mu(a_1)_{q_0,q_1} \cdots \mu(a_n)_{q_{n-1},q_n} \cdot \gamma_{q_n}\end{aligned}$$

which is the weight of ρ in $\mathcal{A}$. Let $\xi = \exists X_1 \cdots \exists X_m . \varphi(X_1, \dots, X_m)$. Using the bijection above, we get for $w \in A^+$

$$\begin{aligned}[\![\xi]\!](w) &= \sum_{\sigma\ (\mathcal{V},w)\text{-assignment}} [\![\varphi]\!](w, \sigma) = \sum_{\rho \text{ path in } \mathcal{A} \text{ for } w} [\![\varphi]\!](w, \sigma_\rho) \\ &= \sum_{\rho \text{ path in } \mathcal{A} \text{ for } w} \text{weight}(\rho) = (||\mathcal{A}||, w).\end{aligned}$$

Note that $[\![\xi]\!](\varepsilon) = 0$ due to the subformula starting with $\exists y$ in φ. Hence, it remains to deal with $w = \varepsilon$. We have $(||\mathcal{A}||, \varepsilon) = \lambda \cdot \gamma$. Let $\zeta = (\lambda \cdot \gamma) \wedge \forall x. \neg(x \leq x)$. For $w \in A^+$ we have $[\![\zeta]\!](w) = [\![\forall x. \neg(x \leq x)]\!](w) = 0$. Now, $[\![\forall x. \neg(x \leq x)]\!](\varepsilon) = 1$ since an empty product is 1 by convention, hence we get $[\![\zeta]\!](\varepsilon) = \lambda \cdot \gamma$. Finally, $||\mathcal{A}|| = [\![\zeta \vee \xi]\!] \in K^{\text{remso}}\langle\!\langle A^* \rangle\!\rangle$. $\square$

Now Theorem 3.7 is immediate by Theorems 4.5 and 5.5.

Observe that the proof of Theorem 5.5 is constructive, i.e. given a weighted automaton $\mathcal{A}$, we effectively obtain an REMSO(K, A)-sentence φ with $[\![\varphi]\!] = \|\mathcal{A}\|$. Using this, from the theory of formal power series (cf. [28, 20, 2]) we immediately obtain undecidability results for the semantics of weighted MSO-sentences. For instance, it is undecidable whether a given REMSO-sentence φ

over $\mathbb{Q}$, the field of rational numbers, and an alphabet A, satisfies $\mathrm{Supp}(\llbracket\varphi\rrbracket) = A^*$. Also, by a result of Krob [19], the equality of given recognizable series over the tropical semiring is undecidable. Hence, the equality of two given $\mathrm{REMSO}(\mathrm{Trop}, A)$-sentences is also undecidable.

6 Locally finite semirings

In section 3 we gave examples of semirings K showing that the results of Theorem 3.7 and 4.5 in general do not hold for arbitrary $\mathrm{MSO}(K, A)$-sentences. In contrast, here we wish to show that for a large class of semirings K, all $\mathrm{MSO}(K, A)$-formulas have a recognizable semantics.

A semiring K is called *locally finite*, if each finitely generated subsemiring of K is finite. A monoid is called locally finite, if each finitely generated submonoid is finite. It is easy to check that a semiring $(K, +, \cdot, 0, 1)$ is locally finite iff both monoids $(K, +, 0)$ and $(K, \cdot, 1)$ are locally finite.

For example, any Boolean algebra $(B, \vee, \wedge, 0, 1)$ is locally finite. The max-min semiring $\mathbb{R}_{\max,\min} = (\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ of positive reals, used in operations research for maximum capacity problems of networks, is locally finite. In fact, more generally, any distributive lattice $(L, \vee, \wedge, 0, 1)$ with smallest element 0 and largest element 1 is a locally finite semiring. Examples of infinite but locally finite fields are provided by the algebraic closures of the finite fields $\mathbf{Z}/p\mathbf{Z}$ for any prime p. If K is a locally finite semiring, the matrix monoids $K^{n\times n}$ are locally finite for all n, cf. [6, 7] for further basic properties.

Lemma 6.1 ([6, 7]). *Let K be a locally finite semiring. Then any recognizable series $S : A^* \to K$ is a recognizable step function.*

Proof. Choose a representation $(Q, \lambda, \mu, \gamma)$ of S. Let K' be the subsemiring of K generated by $\{\lambda_p, \mu(a)_{p,q}, \gamma_q : p, q \in Q, a \in A\}$. Then K' is finite, hence so is $\mu(A^*) \subseteq {K'}^{Q\times Q}$ and also $\mathrm{Im}(S) = \{\lambda \cdot \Gamma \cdot \gamma : \Gamma \in \mu(A^*)\}$. So $S = \sum_{k\in \mathrm{Im}(S)} k \cdot \mathbb{1}_{S^{-1}(k)}$, and as in the proof of Proposition 4.6, each language $S^{-1}(k)$ is recognizable. ☐

Lemma 6.2 ([2, Cor. III.2.4,2.5]). *If $T : A^* \to \mathbb{N}$ is a recognizable series over the semiring $\mathbb{N}$ with natural addition and multiplication, then for all $a, b \in \mathbb{N}$, the languages $T^{-1}(a)$ and $T^{-1}(a + b\,\mathbb{N})$ are recognizable.*

Proposition 6.3. *Let K be a locally finite commutative semiring, $h : A^* \to B^*$ a non-erasing homomorphism, and $S : A^* \to K$ a recognizable series. Then the series $\Pi_h(S) : B^* \to K$ given by $(\Pi_h(S), w) := \prod_{v\in h^{-1}(w)}(S, v)$ $(w \in B^*)$ is recognizable.*

Proof. By Lemma 6.1, S has the form $S = \sum_{j=1}^{n} k_j \cdot \mathbb{1}_{L_j}$ with $n \in \mathbb{N}, k_j \in K$ and recognizable languages $L_j \subseteq A^*$ $(j = 1, \ldots, n)$ which form a partition of A^*. For any $w \in B^*$, let $m_j(w) := |h^{-1}(w) \cap L_j|$. Then $(\Pi_h(S), w) = \prod_{j=1}^{n} k_j^{m_j(w)}$. For each $j \in \{1, \ldots, n\}$, the submonoid of $(K, \cdot, 1)$ generated by $\{k_j\}$ is finite.

Choose a minimal $a_j \in \mathbb{N}$ such that $k_j^{a_j} = k_j^{a_j+x}$ for some $x > 0$, and let b_j be the smallest such $x > 0$. Then $\langle k_j \rangle = \{1, k_j, k_j^2, \ldots, k_j^{a_j+b_j-1}\}$, and for each $w \in B^*$, $k_j^{m_j(w)} = k_j^{d_j(w)}$ for some uniquely determined $d_j(w) \in \mathbb{N}$ with $0 \le d_j(w) \le a_j + b_j - 1$ and $m_j(w) \in d_j(w) + b_j\,\mathbb{N}$. Note that if $0 \le d < a_j$, then $k_j^{m_j(w)} = k_j^d$ iff $m_j(w) = d$, and if $a_j \le d < a_j + b_j$, then $k_j^{m_j(w)} = k_j^d$ iff $m_j(w) \in d + b_j\,\mathbb{N}$. Thus

$$(\Pi_h(S), w) = \prod_{j=1}^{n} k_j^{d_j(w)} = \sum_{\substack{d_1,\ldots,d_n: \\ 0 \le d_j < a_j + b_j \ (j=1,\ldots,n)}} k_1^{d_1} \cdots k_n^{d_n} \cdot \mathbb{1}_{M^1_{d_1} \cap \cdots \cap M^n_{d_n}}(w)$$

with $M_d^j := \{w \in B^* : k_j^{m_j(w)} = k_j^d\}$ $(0 \le d < a_j + b_j, 1 \le j \le n)$. Note that for each $1 \le j \le n$, the series $\mathbb{1}_{L_j} : A^* \to \mathbb{N}$ is recognizable, hence also the series $T_j = h(\mathbb{1}_{L_j}) : B^* \to \mathbb{N}$ is recognizable and satisfies $(h(\mathbb{1}_{L_j}), w) = \sum_{v \in h^{-1}(w)} (\mathbb{1}_{L_j}, v) = m_j(w)$ $(w \in B^*)$. Hence

$$M_d^j = \{w \in B^* \mid m_j(w) = d\} = T_j^{-1}(d) \text{ if } 0 \le d < a_j, \text{ and}$$
$$M_d^j = \{w \in B^* \mid m_j(w) \in d + b_j\,\mathbb{N}\} = T_j^{-1}(d + b_j\,\mathbb{N}) \text{ if } a_j \le d < a_j + b_j.$$

In each case, M_d^j is recognizable by Lemma 6.2. Hence $\Pi_h(S)$ is recognizable. $\square$

As a consequence, we obtain:

Theorem 6.4. *Let K be a locally finite commutative semiring and A an alphabet. Then $K^{rec}\langle\langle A^*\rangle\rangle = K^{mso}\langle\langle A^*\rangle\rangle$.*

Proof. The inclusion $K^{rec}\langle\langle A^*\rangle\rangle \subseteq K^{mso}\langle\langle A^*\rangle\rangle$ is immediate by Theorem 5.5, but we obtain an independent proof by Lemma 6.1, Büchi's Theorem and Proposition 5.4. For the converse, we prove by structural induction that $[\![\varphi]\!]$ is recognizable for any $\mathrm{MSO}(K, A)$-formula φ. We may apply Lemma 4.1, 4.2 and 4.3, and for universal first-order quantification we use Lemmas 6.1 and 4.4. Now the induction step for universal second-order quantification $\forall X.\varphi$ is immediate by Proposition 6.3, using a standard projection from $A_{\mathcal{V} \cup \{X\}}$ onto $A_{\mathcal{V}}$ where $\mathcal{V} = \mathrm{Free}(\varphi)$. $\square$

Again, given an $\mathrm{MSO}(K, A)$-formula φ, following the above proof we can effectively construct a weighted automaton $\mathcal{A}$ over K and A_φ such that $\|\mathcal{A}\| = [\![\varphi]\!]$. As a consequence of this and of corresponding decidability results given in [7, Cor. 4.5] for recognizable series over locally finite semirings, we immediately obtain:

Corollary 6.5. *Let K be a locally finite commutative semiring which is effectively given and let A be an alphabet. It is decidable*

(a) whether two given $\mathrm{MSO}(K, A)$*-formulas φ and ψ satisfy $[\![\varphi]\!] = [\![\psi]\!]$;*
(b) whether a given $\mathrm{MSO}(K, A)$*-formula φ satisfies* $\mathrm{Supp}([\![\varphi]\!]) = A_\varphi^*$.

7 Weighted first-order logic

In this section, we investigate weighted first-order logic and the relationship to aperiodic series. Most of our results will require additional assumptions on the semiring K.

Definition 7.1. *Let K be a semiring and A be an alphabet. A formula $\varphi \in \mathrm{MSO}(K, A)$ is called a (weighted) first-order formula, if φ does not contain any set variable. We let $\mathrm{FO}(K, A)$ contain all first-order formulas and $\mathrm{RFO}(K, A)$ all restricted first-order formulas over K and A. The collections of series definable by these formulas are denoted $K^{\mathrm{fo}}\langle\langle A^*\rangle\rangle$ and $K^{\mathrm{rfo}}\langle\langle A^*\rangle\rangle$, respectively.*

As is well-known, the first-order definable languages are precisely the starfree languages which in turn coincide with the the aperiodic ones [30, 24]. Aperiodic and starfree formal power series were introduced and investigated in [6, 7]. Recall that a monoid M is said to be *aperiodic*, if there exists some $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in M$. We call a monoid M *weakly aperiodic*, if for each $x \in M$ there exists $m \geq 0$ such that $x^m = x^{m+1}$. Clearly, a finite monoid is aperiodic iff it is weakly aperiodic.

A language $L \subseteq A^*$ is called *aperiodic* if there exists a finite aperiodic monoid M and a homomorphism $\varphi : A^* \to M$ which saturates L, i.e. $L = \varphi^{-1}(\varphi(L))$. Equivalently, the language L is aperiodic iff L is recognizable and there exists some $m \geq 0$ such that $uv^m w \in L$ iff $uv^{m+1}w \in L$ for all $u, v, w \in A^*$. The smallest such m is called the *index* of L and denoted $\mathrm{index}(L)$.

A series $S : A^* \to K$ is called *aperiodic*, if there exists a representation $S = (Q, \lambda, \mu, \gamma)$ with $\mu(A^*)$ aperiodic. Observe that then there exists some $m \geq 0$ such that for all $w \in A^*$ we have $\mu(w^m) = \mu(w^{m+1})$ and hence $(S, w^m) = (S, w^{m+1})$. The collection of all aperiodic series over K and A will be denoted $K^{\mathrm{aper}}\langle\langle A^*\rangle\rangle$.

We summarize some properties of aperiodic series derived in [6, 7].

Lemma 7.2 ([7]). *Let K be a semiring and A an alphabet.*

(a) If $L \subseteq A^*$ is an aperiodic language, then the series $\mathbb{1}_L$ is aperiodic.
(b) If $S, T : A^* \to K$ are aperiodic, then $S + T$, $k \cdot S$ and $S \cdot k$ are aperiodic for any $k \in K$. Moreover, if K is commutative, then $S \odot T$ is also aperiodic.
(c) If $S : B^* \to K$ is aperiodic and $h : A^* \to B^*$ is a morphism then $h^{-1}(S) : A^* \to K$ is also aperiodic.

Lemma 7.3 ([6, 7]). *Let K be locally finite, let A be an alphabet and let $S : A^* \to K$ be aperiodic. Then $S = \sum_{j=1}^{n} k_j \cdot \mathbb{1}_{L_j}$ is a recognizable step function with aperiodic languages L_j $(j = 1, \ldots, n)$.*

Next, we extend Proposition 3.3 to first-order formula and aperiodic series.

Proposition 7.4. *Let $\varphi \in \mathrm{FO}(K, A)$ and $\mathcal{V}$ a finite set of first-order variables containing $\mathrm{Free}(\varphi)$. Then $[\![\varphi]\!]$ is aperiodic if and only if $[\![\varphi]\!]_{\mathcal{V}}$ is aperiodic.*

Proof. From left to right, the proof of Proposition 3.3 applies, using Lemma 7.2 and the fact that $N_{\mathcal{V}}$ is aperiodic, which is easy to check.

For the converse we need a new proof since aperiodic languages and series are not closed under morphic image. For $\sigma \in (\{0,1\}^{\mathrm{Free}(\varphi)})^+$ we let $\sigma_0, \sigma_1 \in (\{0,1\}^{\mathcal{V}})^+$ be such that their projection on $\mathrm{Free}(\varphi)$ is σ and their projections on the other variables are in 0^+ and 10^*, respectively.

Assume that $[\![\varphi]\!]_{\mathcal{V}}$ is aperiodic and let $(Q, \lambda, \mu, \gamma)$ be a representation for $[\![\varphi]\!]_{\mathcal{V}}$ with $\mu(A_{\mathcal{V}}^*)$ aperiodic. Let $Q' = Q \uplus Q$ be the disjoint union of two copies of Q. For $(w, \sigma) \in A_\varphi^+$, we define

$$\mu'(w,\sigma) = \begin{pmatrix} \mu(w,\sigma_0) & 0 \\ \mu(w,\sigma_1) & 0 \end{pmatrix}.$$

Using the fact that for $\sigma, \sigma' \in (\{0,1\}^{\mathrm{Free}(\varphi)})^+$, we have $(\sigma\sigma')_0 = \sigma_0\sigma'_0$ and $(\sigma\sigma')_1 = \sigma_1\sigma'_0$, it is easy to check that μ' is a morphism.

Let $m \geq 0$ be such that $\mu((w,\tau)^{m+1}) = \mu((w,\tau)^m)$ for all $(w,\tau) \in A_{\mathcal{V}}^+$. Then, for all $(w,\sigma) \in A_\varphi^+$, we have

$$\mu'((w,\sigma)^{m+2}) = \begin{pmatrix} \mu(w,\sigma_0)\mu((w,\sigma_0)^{m+1}) & 0 \\ \mu(w,\sigma_1)\mu((w,\sigma_0)^{m+1}) & 0 \end{pmatrix} = \mu'((w,\sigma)^{m+1})$$

and we have shown that μ' is aperiodic. Consider now

$$\lambda' = (0 \quad \lambda) \qquad \gamma' = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$$

so that for $(w,\sigma) \in A_\varphi^+$, we have

$$\lambda'\mu'(w,\sigma)\gamma' = \lambda\mu(w,\sigma_1)\gamma = [\![\varphi]\!]_{\mathcal{V}}(w,\sigma_1) = [\![\varphi]\!](w,\sigma)$$

where the last equality follows from Proposition 3.3. Therefore,

$$[\![\varphi]\!] = ||(Q', \lambda', \mu', \gamma')|| + [\![\varphi]\!](\varepsilon) \cdot \mathbb{1}_{\{\varepsilon\}}$$

is aperiodic. ☐

Now we turn to the relationship between aperiodic and FO-definable series. First, from Lemma 7.2 and Proposition 7.4 we obtain:

Corollary 7.5. *Let $\varphi, \psi \in \mathrm{FO}(K, A)$ such that $[\![\varphi]\!]$ and $[\![\psi]\!]$ are aperiodic series. Then $[\![\varphi \vee \psi]\!]$ is aperiodic. If K is commutative, then $[\![\varphi \wedge \psi]\!]$ is also aperiodic.*

Next, we show that even if K is finite and commutative, in general we do not have $K^{\mathrm{aper}}\langle\langle A^*\rangle\rangle = K^{\mathrm{fo}}\langle\langle A^*\rangle\rangle$.

Example 7.6. Let $K = \mathbb{Z}/2\,\mathbb{Z}$, the field with two elements, and $S = [\![\exists x.1]\!]$. Then $S(w) = |w| \mod 2$ for any $w \in A^*$. Hence S is not aperiodic since otherwise we would obtain some $m \geq 1$ such that $S(a^m) = S(a^{m+1})$ $(a \in A)$, a contradiction. Note that here the monoid $(K, \cdot)$ is idempotent, and $(K, +)$ is not aperiodic.

Example 7.7. Let K be the tropical semiring and $T = [\![\forall x.1]\!]$. Then $T(w) = |w|$ for all $w \in A^*$, so T is not aperiodic. Note that $(\mathbb{N} \cup \{\infty\}, \min)$ is idempotent, but $(\mathbb{N} \cup \{\infty\}, +)$ is not weakly aperiodic.

These examples indicate that in order to achieve the inclusion $K^{\mathrm{fo}}\langle\!\langle A^*\rangle\!\rangle \subseteq K^{\mathrm{aper}}\langle\!\langle A^*\rangle\!\rangle$, we need some aperiodicity assumption both for $(K, +)$ and $(K, \cdot)$.

Lemma 7.8. *Let K be locally finite, let A be an alphabet, and let $\varphi \in \mathrm{FO}(K, A)$ such that $[\![\varphi]\!]$ is aperiodic.*

(a) If $(K, +)$ is weakly aperiodic then $[\![\exists x.\varphi]\!]$ is aperiodic.
(b) If $(K, \cdot)$ is weakly aperiodic and commutative, then $[\![\forall x.\varphi]\!]$ is aperiodic.

Proof. We try to prove (a) and (b) simultaneously as far as possible, following the argument for Proposition 6.3. Let $\mathcal{V} = \mathrm{Free}(\exists x.\varphi) = \mathrm{Free}(\forall x.\varphi)$. If $x \notin \mathrm{Free}(\varphi)$ then let $\varphi' = \varphi \wedge (x \leq x)$. By Corollary 7.5, we deduce that $[\![\varphi']\!]$ is aperiodic and using Proposition 3.3 we get $[\![\varphi]\!]_{\mathcal{V}\cup\{x\}} = [\![\varphi']\!]_{\mathcal{V}\cup\{x\}}$. Hence, we obtain $[\![\exists x.\varphi]\!] = [\![\exists x.\varphi']\!]$ and $[\![\forall x.\varphi]\!] = [\![\forall x.\varphi']\!]$. Therefore, we may assume that $\mathcal{W} = \mathrm{Free}(\varphi) = \mathcal{V} \cup \{x\}$.

By Lemma 7.3, we may write $[\![\varphi]\!] = \sum_{j=1}^{n} k_j \cdot \mathbb{1}_{L_j}$ with $n \in \mathbb{N}$, $k_j \in K$ and aperiodic languages $L_j \subseteq A^*_{\mathcal{W}}$ $(j = 1, \ldots, n)$. Since aperiodic languages are closed under boolean operations, we may assume that the languages $(L_j)_{1\leq j\leq n}$ are pairwise disjoint. We may also assume that $k_j \neq 0$ for all $1 \leq j \leq n$. Note that this implies $L_j \subseteq N_{\mathcal{W}} = \{(w, \sigma) \in A^*_{\mathcal{W}} \mid \sigma \text{ is a valid } (\mathcal{W}, w)\text{-assignment}\}$ for $1 \leq j \leq n$.

For $(w, \sigma) \in A^*_{\mathcal{V}}$, let $m_j(w, \sigma) = |\{i \mid 1 \leq i \leq |w|, (w, \sigma[x \to i]) \in L_j\}| \in \mathbb{N}$. We obtain

$$[\![\exists x.\varphi]\!](w, \sigma) = \sum_{i=1}^{|w|} \sum_{j=1}^{n} k_j \cdot \mathbb{1}_{L_j}(w, \sigma[x \to i]) = \sum_{j=1}^{n} k_j \cdot m_j(w, \sigma).$$

Since $(K, +)$ is weakly aperiodic, for each $j \in \{1, \ldots, n\}$ we can choose a minimal $a_j \in \mathbb{N}$ such that $k_j \cdot a_j = k_j \cdot (a_j + 1)$. Note that $k_j \cdot m_j(w, \sigma) = k_j \cdot a_j$ iff $m_j(w, \sigma) \geq a_j$ and if $0 \leq d < a_j$, then $k_j \cdot m_j(w, \sigma) = k_j \cdot d$ iff $m_j(w, \sigma) = d$. Hence

$$[\![\exists x.\varphi]\!] = \sum_{\substack{d_1,\ldots,d_n:\\ 0\leq d_j\leq a_j}} \left(\sum_{j=1}^{n} k_j \cdot d_j\right) \mathbb{1}_{M^1_{d_1}\cap\ldots\cap M^n_{d_n}}$$

with $M^j_d := \{(w, \sigma) \in A^*_{\mathcal{V}} \mid k_j \cdot m_j(w, \sigma) = k_j \cdot d\}$ for $0 \leq d \leq a_j$, $1 \leq j \leq n$. By Lemma 7.2, it remains to show that these languages M^j_d are aperiodic. As noted above,

$$M^j_d = \begin{cases} \{(w, \sigma) \in A^*_{\mathcal{V}} \mid m_j(w, \sigma) = d\} & \text{if } 0 \leq d < a_j \\ \{(w, \sigma) \in A^*_{\mathcal{V}} \mid m_j(w, \sigma) \geq a_j\} & \text{otherwise.} \end{cases}$$

We first show that the languages M^j_d are recognizable. Let $\pi : A_{\mathcal{W}} \to A_{\mathcal{V}}$ be the canonical projection erasing the x-row. Consider the semiring $\mathbb{N}$ of natural numbers. Since $L_j \subseteq N_{\mathcal{W}}$, we have for each $(w,\sigma) \in A_V^*$

$$\pi(\mathbb{1}_{L_j})(w,\sigma) = \sum_{(w,\sigma')\in\pi_{-1}(w,\sigma)} \mathbb{1}_{L_j}(w,\sigma') = \sum_{1\le i\le |w|} \mathbb{1}_{L_j}(w,\sigma[x\to i]) = m_j(w,\sigma).$$

The series $\pi(\mathbb{1}_{L_j})$ is recognizable by Lemmas 2.1 and 2.2 and we have $M_d^j = (\pi(\mathbb{1}_{L_j}))^{-1}(d)$ for $0 \le d < j$. Therefore, by Lemma 6.2 we deduce that M_d^j is recognizable for $0 \le d < a_j$. Finally, the language $M_{a_j}^j = A_{\mathcal{V}}^* \setminus \bigcup_{0\le d<a_j} M_d^j$ is also recognizable.

Since the class of aperiodic languages is closed under complements and intersections, it suffices to prove that $M_{\ge d}^j = \{(w,\sigma) \in A_{\mathcal{V}}^* \mid m_j(w,\sigma) \ge d\}$ is aperiodic for each $0 \le d \le a_j$. Note that $M_{\ge 0}^j = A_{\mathcal{V}}^*$ is aperiodic.

Fix $1 \le j \le n$ and $0 < d \le a_j$. Choose $m \ge (d+1)\cdot(\ell+1) \ge 2$ where $\ell = \mathrm{index}(L_j)$ and let $u,v,w \in A_{\mathcal{V}}^*$. We show that $uv^{m+1}w \in M_{\ge d}^j$ implies $uv^m w \in M_{\ge d}^j$. The converse implication, which is slightly simpler, can be shown similarly. So assume that $uv^{m+1}w \in M_{\ge d}^j$. Then $(uv^{m+1}w)[x \to i] \in L_j$ for at least d positions i with $1 \le i \le |uv^{m+1}w|$. Now choose exactly d such positions i. By choice of m we can find a consecutive sequence of $\ell+1$ copies of v such that all of the d chosen positions i lie outside of this sequence. Since $\ell = \mathrm{index}(L_j)$, we can remove an occurrence of v in this sequence and we obtain $(uv^m w)[x \to i] \in L_j$ for at least d positions of i with $1 \le i \le |uv^m w|$ (some of these positions might now have been shifted by $|v|$ to the left). Therefore, $uv^m w \in M_{\ge d}^j$. This proves our claim, showing that $M_{\ge d}^j$ is aperiodic.

Next we turn to part (b). We compute

$$[\![\forall x.\varphi]\!](w,\sigma) = \prod_{i=1}^{|w|} \sum_{j=1}^{n} k_j \cdot \mathbb{1}_{L_j}(w,\sigma[x\to i]) = \prod_{j=1}^{n} k_j^{m_j(w,\sigma)}$$

using commutativity of K. Since $(K,\cdot)$ is weakly aperiodic, there exists for each $j \in \{1,\dots,n\}$ a minimal $a'_j \in \mathbb{N}$ such that $k_j^{a'_j} = k_j^{a'_j+1}$. Similarly as above, we obtain

$$[\![\forall x.\varphi]\!] = \sum_{\substack{d_1,\dots,d_n:\\ 0\le d_j\le a'_j}} \left(\prod_{j=1}^{n} k_j^{d_j}\right) \mathbb{1}_{M'^1_{d_1}\cap\ldots\cap M'^n_{d_n}}$$

with $M'^j_d := \{(w,\sigma) \in A_{\mathcal{V}}^* \mid k_j^{m_j(w,\sigma)} = k_j^d\}$ for $0 \le d \le a'_j, 1 \le j \le n$. Now

$$M'^j_d = \begin{cases} \{(w,\sigma) \in A_{\mathcal{V}}^* \mid m_j(w,\sigma) = d\} & \text{if } 0 \le d < a'_j \\ \{(w,\sigma) \in A_{\mathcal{V}}^* \mid m_j(w,\sigma) \ge a'_j\} & \text{otherwise.} \end{cases}$$

As shown above, these languages are aperiodic, so $[\![\forall x.\varphi]\!]$ is aperiodic. $\square$

We just note here that by [6], Lemma 7.3 and hence Lemma 7.8 also hold for all semirings having Burnside matrix monoids (cf. [6] for definition of this notion). However, this generalization will not be needed subsequently.

We call a semiring K *weakly bi-aperiodic*, if both $(K,+)$ and $(K,\cdot)$ are weakly aperiodic. If K is also commutative, then in particular K is locally finite. Clearly, any idempotent monoid is weakly aperiodic. Thus the weakly bi-aperiodic semirings include all semirings in which both addition and multiplication are idempotent, and this class of semirings properly contains (cf. [14]) the class of all distributive lattices $(L,\vee,\wedge,0,1)$ with smallest element 0 and greatest element 1. There are further examples:

Example 7.9. Let $0 < d \in \mathbb{R}$. We let $\mathbb{R}^d_{\max}$ be the real max$-$plus semiring truncated at d, i.e. $\mathbb{R}^d_{\max} = ([0,d] \cup \{-\infty\}, \max, +_d, -\infty, 0)$ with $x +_d y := x + y$ if $x + y \leq d$, and $x +_d y := d$ if $x + y \geq d$. This semiring is weakly bi-aperiodic, and $(\mathbb{R}^d_{\max}, +_d)$ is weakly aperiodic but not aperiodic.

Example 7.10. Let $K = ([0,1], \max, \otimes, 0, 1)$ where $x \otimes y = \max(0, x + y - 1)$ be the semiring occuring in the MV-algebra used to define the semantics of Łukasiewicz multi-valued logic [13]. This semiring is weakly bi-aperiodic but $\otimes$ is not aperiodic. For this semiring a restriction of Łukasiewicz logic coincides with our weighted MSO-logic [31].

Now we show:

Theorem 7.11. *Let K be a commutative weakly bi-aperiodic semiring, and A an alphabet. Then*

$$K^{\mathrm{aper}}\langle\langle A^*\rangle\rangle = K^{\mathrm{rfo}}\langle\langle A^*\rangle\rangle = K^{\mathrm{fo}}\langle\langle A^*\rangle\rangle.$$

Proof. Let $S : A^* \to K$ be aperiodic. By Lemma 7.3, we have $S = \sum_{j=1}^{n} k_j \mathbb{1}_{L_j}$ where the L_j are aperiodic languages. Using McNaughton-Papert's theorem [24] and Schützenberger's theorem [30], we find first-order formulas φ_j such that $L_j = \mathcal{L}(\varphi_j)$ for each j. Now, using Lemma 5.3, we have $\mathbb{1}_{L_j} = [\![\varphi_j^+]\!]$. It remains to define $\varphi = \bigvee_{1 \leq j \leq n} k_j \wedge \varphi_j^+$ in order to obtain $S = [\![\varphi]\!]$ as desired. Therefore, $K^{\mathrm{aper}}\langle\langle A^*\rangle\rangle \subseteq K^{\mathrm{rfo}}\langle\langle A^*\rangle\rangle$.

Conversely, we prove for any $\mathrm{FO}(K,A)$-formula φ by induction on the structure of φ that $[\![\varphi]\!]$ is aperiodic. This is clear for atomic formulas and their negations by Lemma 7.2(a). For disjunction and conjunction we use Corollary 7.5 and for existential and universal quantification apply Lemma 7.8. $\square$

8 Conclusion

We believe that the present paper opens a new research road. Recently our approach has been extended to trees [10], pictures [22], traces [25] and infinite words [9], generalizing corresponding equivalence results for classical unweighted automata and MSO logic for these structures. This shows the robustness of our approach. One could also try to define weighted temporal logics and study not only expressiveness but also decidability and complexity of natural problems such as quantitative model checking.

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