Customized TRS Invariants for 2D Vector Fields via Moment Normalization

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Abstract

The behavior of vector fields under translation, rotation and scaling differs with respect to the underlying application. Moment invariants that are customized to the specific problem can be constructed by means of normalization.

In this paper, we calculate general TRS (translation, rotation, and scaling) moment invariants for two-dimensional vector fields. As an example, we show explicitly how to customize the result for the detection of flow field patterns.

Key words: moment invariant, normalization, vector field, flow field, pattern recognition, TRS

1 Introduction

Moment invariants are one of the fundamental techniques to describe and compare real-valued objects because they are robust and easy to use. They are a number of values representing a function that do not change under certain transformations. Their invariance property allows to compare objects in one single step instead of having to compare every possible transformed version of it.

Two-dimensional invariants with respect to translation, rotation, and scaling (TRS) were introduced to the pattern recognition community by Hu [1]. The

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use of complex moments [2,3] simplified the construction of rotation invariants because of the easy way to describe rotations by means of complex exponentials. In the last decade, Flusser et al. [4–6] structured the theory of complex moment invariants into a clear framework. That paved the way for a generalization to vector-valued data. In [7], a comprehensive treatment of their work can be found.

Recently, Schlemmer et al. [8,9] applied their results to flow fields. They constructed a basis of flow field moment invariants, developed an algorithm that calculates them efficiently, and successfully used it to detect features in real-world data.

In contrast to the use of an independent bases [4,9], there is a different approach for the construction of moment invariants, called normalization [1,10,7]. First, the function is brought into a standard position by setting certain moments to given values. Then, all the remaining moments are used as the discriminating invariants. The transformation of the first step can take various forms even in the case where it is only the combination of translation, rotation, and scaling.

We will show how invariants with respect to all of these forms can be constructed by means of normalization. As an example, we will calculate the set of moment invariants that are customized to the problem of finding patterns in flow fields.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $p, q \in \mathbb{N}$, the moments $m_{p,q}$ are defined by

$$m_{p,q} = \int_{\mathbb{R}^2} x^p y^q f(x, y) \,\mathrm{d}x \,\mathrm{d}y. \tag{1}$$

For the analysis of functions over the plane, we can make use of the isomorphism between the Euclidean and the complex plane [11–13], interpret them as functions

$$f'(x_1, x_2) = f(x_1 + ix_2) = f(z) : \mathbb{C} \to \mathbb{R}, \tag{2}$$

and use the complex moments $c_{p,q}$. For $f : \mathbb{C} \to \mathbb{R}$, they are defined by

$$c_{p,q} = \int_{\mathbb{C}} z^p \overline{z}^q f(z) \,\mathrm{d}z. \tag{3}$$

Analogously, two-dimensional vector fields

$$\boldsymbol{v}(\boldsymbol{x}) = v_1(x_1, x_2)\boldsymbol{e}_1 + v_2(x_1, x_2)\boldsymbol{e}_2 : \mathbb{R}^2 \to \mathbb{R}^2, \tag{4}$$

with $v_1, v_2 : \mathbb{R}^2 \to \mathbb{R}$, can be interpreted as complex functions

$$f(z) = f(x_1 + ix_2) = f'(x_1, x_2) = v_1(x_1, x_2) + iv_2(x_1, x_2) : \mathbb{C} \to \mathbb{C}. \tag{5}$$

For $f : \mathbb{C} \to \mathbb{C}$, the definition of complex moments (3) can easily be generalized. We will work with complex functions during the calculations and keep in mind that the results are also valid for vector fields.

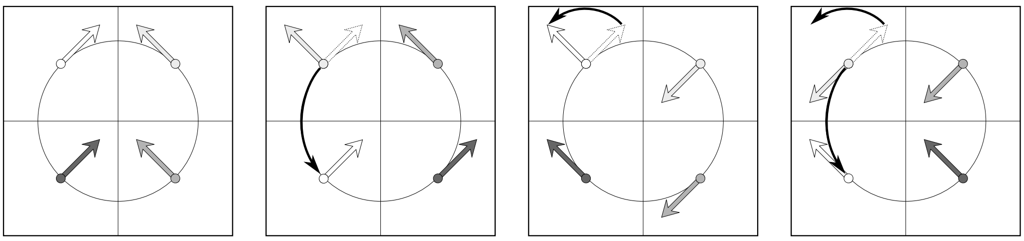
In order to customize the results to practical applications, we assume the functions to vanish outside an area $A \subseteq \mathbb{C}$ with characteristic function

$$\chi_A(z) = \begin{cases} 1, & \text{if } z \in A, \\ 0, & \text{else.} \end{cases} \tag{6}$$

Although the functions with infinite support are easier to deal with, they will not appear very often in real-world applications. For the sake of completeness, the case $A = \mathbb{C}$ is not excluded.

2 Translation, rotation, and scaling on vector fields

Vector fields can have very different properties under affine transformations. The specific behavior depends on the interpretation of the field. When working with vector fields, one has to distinguish at least three cases. In this paper, we show how moment invariants can be constructed that satisfy the different requirements.



Original vector field: $\boldsymbol{v}(\boldsymbol{x})$ — Inner rotation: $\boldsymbol{v}(\mathrm{R}_{-\alpha}(\boldsymbol{x}))$ — Outer rotation: $\mathrm{R}_\alpha(\boldsymbol{v}(\boldsymbol{x}))$ — Total rotation: $\mathrm{R}_\alpha(\boldsymbol{v}(\mathrm{R}_{-\alpha}(\boldsymbol{x})))$

Fig. 1. Effect of the rotation operator R_α applied to an example vector field in different ways.

In contrast to scalar fields, the term rotational misalignment is ambiguous for vector fields. A simple example rotated by $\frac{\pi}{2}$ can be found visualized in Figure 1. Let R_α be an operator, that describes a mathematically positive rotation by the angle α. Two vector fields $\boldsymbol{v}, \boldsymbol{v}' : \mathbb{R}^2 \to \mathbb{R}^2$ differ by an **inner rotation** if they suffice

$$\boldsymbol{v}'(\boldsymbol{x}) = \boldsymbol{v}(\mathrm{R}_{-\alpha}(\boldsymbol{x})). \tag{7}$$

It can be interpreted in the following way. The starting position of every vector is rotated by α. Then, the old vector is reattached at the new position, but it still points into the old direction. The inner rotation is suitable to describe the rotation of a 2D color image [1] or a complex-valued function over a plane. The color, or the complex value respectively, is represented as a vector and does not change when the underlying plane is turned.

Another kind of misalignment, we want to mention, is the **outer rotation**

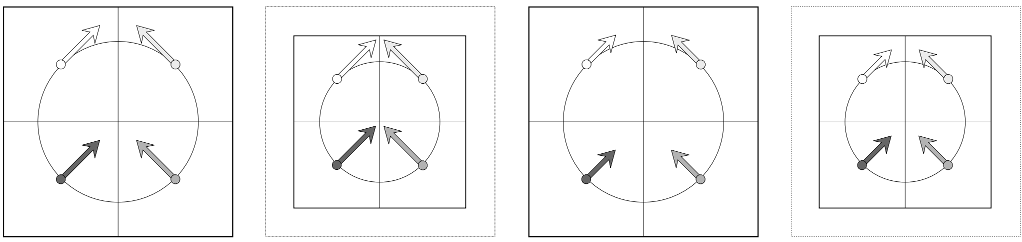
$$\boldsymbol{v}'(\boldsymbol{x}) = \mathrm{R}_{\alpha}(\boldsymbol{v}(\boldsymbol{x})). \tag{8}$$

Here, every vector on the vector field $\boldsymbol{v}'$ is the rotated copy of every vector in the vector field $\boldsymbol{v}$. The vectors are rotated independently from their positions. This kind of rotation appears, for example, in color images when the color space is rotated but the alignment of the picture is not changed, compare [14]. Another example is a phase shift in a complex-valued function describing the alternating current over a plane.

If the vector field is an isomorphic mapping $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{R}^2$, a third type is of interest, the **total rotation**

$$\boldsymbol{v}'(\boldsymbol{x}) = \mathrm{R}_{\alpha}(\boldsymbol{v}(\mathrm{R}_{-\alpha}(\boldsymbol{x}))). \tag{9}$$

Here, the positions and the vectors are stiffly connected during the rotation. It can be interpreted as a coordinate transform, such as when looking at the vector field from another point of view. Total rotations occur in physical vector fields, for example, in fluid mechanics and aerodynamics.



Original vector field: $\boldsymbol{v}(\boldsymbol{x})$ | Inner scaling: $\boldsymbol{v}(s^{-1}\boldsymbol{x})$ | Outer scaling: $s\boldsymbol{v}(\boldsymbol{x})$ | Total scaling: $s\boldsymbol{v}(s^{-1}\boldsymbol{x})$

Fig. 2. Effect of the scaling by $s \in \mathbb{R}$ applied to an example vector field in different ways.

[1] A proper color space is three-dimensional. Still, two-dimensional color spaces appear in some applications, for example, in the visualization of complex functions or in two-dimensional color maps.

A similar behavior occurs if we consider the scaling of vector fields $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{R}^2$ by the value $s \in \mathbb{R}$. The visualization for the scaling of the example from Figure 1 by the value $s = 0.75$ can be found in Figure 2. We can distinguish three cases. **Inner scaling**

$$\boldsymbol{v}'(\boldsymbol{x}) = \boldsymbol{v}(s^{-1}\boldsymbol{x}) \tag{10}$$

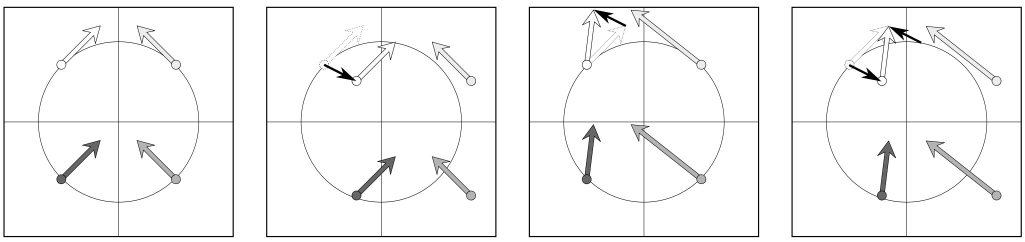
corresponds to the change in size of a color image, whereas the colors remain unchanged. On the other hand, **outer scaling**

$$\boldsymbol{v}'(\boldsymbol{x}) = s\boldsymbol{v}(\boldsymbol{x}) \tag{11}$$

is in accordance to the contrast of a color image. If we have isomorphic vector fields, analogous to the total rotation, the **total scaling**

$$\boldsymbol{v}'(\boldsymbol{x}) = s\boldsymbol{v}(s^{-1}\boldsymbol{x}) \tag{12}$$

can be interpreted as a change of coordinates for flow fields.



Original vector field: $\boldsymbol{v}(\boldsymbol{x})$ | Inner translation: $\boldsymbol{v}(\boldsymbol{x} - \boldsymbol{t})$ | Outer translation: $\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{t}$ | Total translation: $\boldsymbol{v}(\boldsymbol{x} - \boldsymbol{t}) + \boldsymbol{t}$

Fig. 3. Effect of the translation by $\boldsymbol{t} \in \mathbb{R}^2$ applied to an example vector field in different ways.

Analogous to the transforms above, one can also think of different kinds of translation. The **inner translation**

$$\boldsymbol{v}'(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x} - \boldsymbol{t}) \tag{13}$$

corresponds to moving the vector field to a new position, no matter whether the field is interpreted as a color image or as a flow field. The **outer translation**

$$\boldsymbol{v}'(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{t} \tag{14}$$

can be interpreted as a shift in the color space of a color image or the appearance of a background flow in a flow field. The **total translation**

$$\boldsymbol{v}'(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x} - \boldsymbol{t}) + \boldsymbol{t} \tag{15}$$

looks like a pure movement of the positions that is compensated such that the ends of the vectors do not leave the position they point to. We can currently not think of a useful interpretation of this transform that occurs in the real world.

3 Moment invariants of scalar functions

In this section, we state the classical method of the normalization of moments of real-valued functions. Even though the results are commonly known, we show how they can be achieved in order to pave the way to the following sections. In the classical case the transforms are always inner transforms. It is possible to show the results without the use of the characteristic function. They are valid for all integrable functions. But since we will definitely need the characteristic function to normalize with respect to outer translation, we want to introduce this structure here already to grant a smooth transfer to the more complicated transformations. We analyze the relation of the complex moments of a function over the area A

$$g(z) = f(z)\chi_A(z) : \mathbb{C} \to \mathbb{R} \tag{16}$$

to its inner transformed copy

$$g'(z) = f'(z)\chi_{A'}(z) \tag{17}$$

satisfying

$$\begin{aligned} f'(z) &= f\left(se^{i\alpha}z + t\right), \\ \chi_{A'}(z) &= \chi_A(se^{i\alpha}z + t) \end{aligned} \tag{18}$$

with the scaling factor $s \in \mathbb{R}^+$, translational difference $t \in \mathbb{C}$, and rotation angle $\alpha \in [-\pi, \pi]$. The moments of g' satisfy

$$
\begin{aligned}
c'_{p,q} =& \int_{\mathbb{C}} z^p \bar{z}^q g'(z)\,\mathrm{d}z \\
=& \int_{\mathbb{C}} z^p \bar{z}^q f\Big(se^{i\alpha}z + t\Big)\chi_A(se^{i\alpha}z + t)\,\mathrm{d}z \\
=& \int_{\mathbb{C}} (s^{-1}e^{-i\alpha}(z-t))^p(\overline{s^{-1}e^{-i\alpha}(z-t)})^q f(z)\chi_A(z)s^{-2}\,\mathrm{d}z \\
=& \int_{\mathbb{C}} s^{-p}e^{-i\alpha p}(z-t)^p s^{-q}e^{i\alpha q}(\overline{z-t})^q f(z)s^{-2}\,\mathrm{d}z \\
=& s^{-p-q-2}e^{i\alpha(q-p)}\int_{\mathbb{C}} (z-t)^p(\overline{z-t})^q f(z)\chi_A(z)\,\mathrm{d}z \\
=& s^{-p-q-2}e^{i\alpha(q-p)}\int_{\mathbb{C}} \sum_{k=0}^{p}\sum_{l=0}^{q}\begin{pmatrix}p\\k\end{pmatrix}\begin{pmatrix}q\\l\end{pmatrix}(-t)^{p-k}z^k(\overline{-t})^{q-l}\bar{z}^l f(z)\chi_A(z)\,\mathrm{d}z \\
=& s^{-p-q-2}e^{i\alpha(q-p)}\left(\sum_{k=0}^{p}\sum_{l=0}^{q}\begin{pmatrix}p\\k\end{pmatrix}\begin{pmatrix}q\\l\end{pmatrix}(-t)^{p-k}(\overline{-t})^{q-l}\int_A z^k\bar{z}^l f(z)\,\mathrm{d}z\right) \\
=& s^{-p-q-2}e^{i\alpha(q-p)}\left(\sum_{k=0}^{p}\sum_{l=0}^{q}\begin{pmatrix}p\\k\end{pmatrix}\begin{pmatrix}q\\l\end{pmatrix}(-t)^{p-k}(\overline{-t})^{q-l}c_{k,l}\right).
\end{aligned}
\tag{19}
$$

We want to create the function g^c that is the inner rotated, translated and scaled copy of g in standard position. That means, certain moments take given values. The choice of the preset moments is theoretically free. We only have to take care that the preset degrees of freedom match the degrees of freedom that represent the affine transform. We should further choose moments of small grades because of their robustness. The low order complex moments satisfy

$$
\begin{aligned}
c'_{0,0} =& s^{-2}c_{0,0}, \\
c'_{1,0} =& s^{-3}e^{-i\alpha}(-tc_{0,0} + c_{1,0}), \\
c'_{0,1} =& s^{-3}e^{i\alpha}(-\bar{t}c_{0,0} + c_{0,1}), \\
c'_{2,0} =& s^{-4}e^{-2i\alpha}(t^2c_{0,0} - 2tc_{1,0} + c_{2,0}).
\end{aligned}
\tag{20}
$$

From the first relation, we can see that a reasonable choice for a standard with respect to scale is

$$
c'_{0,0} = 1 \Leftrightarrow s = c_{0,0}^{\frac{1}{2}}, \tag{21}
$$

and from the second relation, we can find a standard with respect to translation from

$$
c'_{1,0} = 0 \Leftrightarrow t = \frac{c_{1,0}}{c_{0,0}}. \tag{22}
$$

Considering the equality

$$\overline{c_{p,q}} = c_{q,p} \tag{23}$$

for real-valued functions, that leaves us with

$$\begin{aligned}
c'_{0,0} =& 1, \\
c'_{1,0} =& 0, \\
c'_{0,1} =& c_{0,0}^{-\frac{3}{2}} e^{i\alpha}(-\frac{\overline{c_{1,0}}}{c_{0,0}} c_{0,0} + c_{0,1}) \\
=& 0, \\
c'_{2,0} =& c_{0,0}^{-2} e^{-2i\alpha}((\frac{c_{1,0}}{c_{0,0}})^2 c_{0,0} - 2\frac{c_{1,0}}{c_{0,0}} c_{1,0} + c_{2,0}) \\
=& e^{-2i\alpha}(\frac{c_{2,0}}{c_{0,0}^2} - \frac{c_{1,0}^2}{c_{0,0}^3}).
\end{aligned} \tag{24}$$

Since $c'_{0,1}$ vanishes together with $c'_{1,0}$, we can no longer use it for normalization or discrimination. In order to normalize with respect to rotation, we choose $c_{2,0}$ and move it to the positive real axis

$$c'_{2,0} \in \mathbb{R}^+ \Leftrightarrow \alpha = \frac{1}{2} \arg\Big(\frac{c_{2,0}}{c_{0,0}^2} - \frac{c_{1,0}^2}{c_{0,0}^3}\Big). \tag{25}$$

Theorem 1 *For an area A, its characteristic function $\chi_A(z)$, the function $g(z) = f(z)\chi_A(z) : \mathbb{C} \to \mathbb{R}$, and its complex moments $c_{0,0}, c_{1,0}, c_{2,0}$, we set*

$$\begin{aligned}
t^c =& \frac{c_{1,0}}{c_{0,0}}, \\
s^c =& \sqrt{c_{0,0}}, \\
\alpha^c =& \frac{1}{2} \arg\Big(\frac{c_{2,0}}{c_{0,0}^2} - \frac{c_{1,0}^2}{c_{0,0}^3}\Big).
\end{aligned} \tag{26}$$

Then, the normalized function

$$g^c(z) = f(s^c e^{i\alpha^c} z + t^c)\chi_A(s^c e^{i\alpha^c} z + t^c) \tag{27}$$

is invariant with respect to inner translation, rotation, and scaling.

PROOF. The assertion follows from straight calculation and can be found in Appendix A.

Summarizing, the normalized complex moments $c^c_{p,q}$ satisfy

$$c^c_{0,0} = 1, \quad c^c_{1,0} = 0, \quad c_{2,0} \in \mathbb{R}^+ \tag{28}$$

and can be calculated from the complex moments $c_{p,q}$ using the formula

$$c^c_{p,q} = s^{-p-q-2} e^{i\alpha(q-p)} \left(\sum_{k=0}^{p} \sum_{l=0}^{q} \binom{p}{k} \binom{q}{l} (-t)^{p-k} (\overline{-t})^{q-l} c_{k,l} \right). \tag{29}$$

All in all, the inner transform had four degrees of freedom: two for the translation, one for the scaling, and one for the rotation. The first two are one complex degree of freedom and were removed by setting $c_{1,0} = 0$. The next one was removed by setting $|c_{0,0}| = 1$. This is equivalent to setting $c_{0,0} = 1$, because $c_{0,0}$ is always real valued for real-valued functions. The last one was removed by setting $c_{2,0} \in \mathbb{R}^+$. Please note that, in order to achieve translation, rotation, and scaling invariants, these are the standard choices of moments but choosing other moments would have lead to similar results.

4 Moment invariants of complex functions

Now, we have the case that the affine transforms can not only be applied to the arguments but also to the values of the functions. That means that we have far more degrees of freedom. In order to normalize with respect to outer and inner transformations, we anlyze the relation of the moments of a function

$$g(z) = f(z)\chi_A(z) : \mathbb{C} \to \mathbb{C} \tag{30}$$

to the ones of its transformed copy

$$g'(z) = s_o e^{i\alpha_o} \Big(f(s_i e^{i\alpha_i} z + t_i) + t_o \Big) \chi_A(s_i e^{i\alpha_i} z + t_i) \tag{31}$$

with the inner and outer scaling factors $s_i, s_o \in \mathbb{R}^+$, translational differences $t_i, t_o \in \mathbb{C}$, rotation angles $\alpha_i, \alpha_o \in [-\pi, \pi]$, and the transformed area A' with the characteristic function $\chi_{A'}(z) = \chi_A(s_i e^{i\alpha_i} z + t_i)$. The moments of g' satisfy

$$
\begin{aligned}
c'_{p,q} &= \int_{\mathbb{C}} z^p \bar{z}^q g'(x,y)\,\mathrm{d}z \\
&= \int_{\mathbb{C}} z^p \bar{z}^q s_o e^{i\alpha_o}\Big(f(s_i e^{i\alpha_i} z + t_i) + t_o\Big)\chi_A(s_i e^{i\alpha_i} z + t_i)\,\mathrm{d}z \\
&= s_o e^{i\alpha_o} \int_{\mathbb{C}} (s_i^{-1} e^{-i\alpha_i}(z - t_i))^p (\overline{s_i^{-1} e^{-i\alpha_i}(z - t_i)})^q \Big(f(z) + t_o\Big)\chi_A(z) s_i^{-2}\,\mathrm{d}z \\
&= s_o e^{i\alpha_o} \int_{\mathbb{C}} s_i^{-p} e^{-i\alpha_i p}(z - t_i)^p s_i^{-q} e^{i\alpha_i q}(\overline{z - t_i})^q \Big(f(z) + t_o\Big)\chi_A(z) s_i^{-2}\,\mathrm{d}z \\
&= s_o e^{i\alpha_o} s_i^{-p-q-2} e^{i\alpha_i(q-p)} \int_{\mathbb{C}} (z - t_i)^p (\overline{z - t_i})^q \Big(f(z) + t_o\Big)\chi_A(z)\,\mathrm{d}z \\
&= s_o e^{i\alpha_o} s_i^{-p-q-2} e^{i\alpha_i(q-p)} \\
&\quad \int_{\mathbb{C}} \sum_{k=0}^{p}\sum_{l=0}^{q} \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} q \\ l \end{pmatrix} (-t_i)^{p-k} z^k (\overline{-t_i})^{q-l} \bar{z}^l \Big(f(z) + t_o\Big)\chi_A(z)\,\mathrm{d}z \\
&= s_o e^{i\alpha_o} s_i^{-p-q-2} e^{i\alpha_i(q-p)} \\
&\quad \Big(\sum_{k=0}^{p}\sum_{l=0}^{q} \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} q \\ l \end{pmatrix} (-t_i)^{p-k} (\overline{-t_i})^{q-l} (c_{k,l} + t_o \int_A z^k \bar{z}^l\,\mathrm{d}z)\Big)
\end{aligned} \tag{32}
$$

Now, we want to find the function g^c that is the normalized version of g. To do so, we have to choose four complex moments and use them to express the corresponding transformation parameters $s_i, s_o, t_i, t_o, \alpha_i, \alpha_o$. Again, we choose the moments of low order for their robustness. They satisfy

$$
\begin{aligned}
c'_{0,0} &= s_o e^{i\alpha_o} s_i^{-2}\Big(c_{0,0} + t_o \int_A \mathrm{d}z\Big), \\
c'_{1,0} &= s_o e^{i\alpha_o} s_i^{-3} e^{-i\alpha_i}\Big(- t_i(c_{0,0} + t_o \int_A \mathrm{d}z) + c_{1,0} + t_o \int_A z\,\mathrm{d}z\Big), \\
c'_{0,1} &= s_o e^{i\alpha_o} s_i^{-3} e^{i\alpha_i}\Big(- \overline{t_i}(c_{0,0} + t_o \int_A \mathrm{d}z) + c_{0,1} + t_o \int_A \bar{z}\,\mathrm{d}z\Big), \\
c'_{2,0} &= s_o e^{i\alpha_o} s_i^{-4} e^{-2i\alpha_i}\Big(t_i^2\Big(c_{0,0} + t_o \int_A \mathrm{d}z\Big) - 2t_i\Big(c_{1,0} + t_o \int_A z\,\mathrm{d}z\Big) \\
&\qquad + c_{2,0} + t_o \int_A z^2\,\mathrm{d}z\Big), \\
c'_{0,2} &= s_o e^{i\alpha_o} s_i^{-4} e^{2i\alpha_i}\Big(\overline{t_i}^2\Big(c_{0,0} + t_o \int_A \mathrm{d}z\Big) - 2\overline{t_i}\Big(c_{0,1} + t_o \int_A \bar{z}\,\mathrm{d}z\Big) \\
&\qquad + c_{0,2} + t_o \int_A \bar{z}^2\,\mathrm{d}z\Big).
\end{aligned} \tag{33}
$$

It seems most convenient to start expressing t_o by means of $c_{0,0}$. The standard position shall be the absence of background flow. So, we set

$$
c'_{0,0} = 0 \Leftrightarrow t_o = -\frac{c_{0,0}}{\int_A \mathrm{d}z}. \tag{34}
$$

That leaves us with

$$
\begin{aligned}
c'_{0,0} =& 0,\\
c'_{1,0} =& s_o e^{i\alpha_o} s_i^{-3} e^{-i\alpha_i}\Big(c_{1,0} + t_o \int_A z\,\mathrm{d}z\Big),\\
c'_{0,1} =& s_o e^{i\alpha_o} s_i^{-3} e^{i\alpha_i}\Big(c_{0,1} + t_o \int_A \bar{z}\,\mathrm{d}z\Big),\\
c'_{2,0} =& s_o e^{i\alpha_o} s_i^{-4} e^{-2i\alpha_i}\Big(-2t_i(c_{1,0} + t_o \int_A z\,\mathrm{d}z) + c_{2,0} + t_o \int_A z^2\,\mathrm{d}z\Big),\\
c'_{0,2} =& s_o e^{i\alpha_o} s_i^{-4} e^{2i\alpha_i}\Big(-2\overline{t_i}\Big(c_{0,1} + t_o \int_A \bar{z}\,\mathrm{d}z\Big) + c_{0,2} + t_o \int_A \bar{z}^2\,\mathrm{d}z\Big).
\end{aligned}
\tag{35}
$$

Since the first order moments do not depend on t_i after setting $c'_{0,0} = 0$, we choose $c'_{2,0}$ to be set to zero and express $s_o e^{i\alpha_o}$ by means of $c_{1,0}$. We choose the standard position with respect to outer rotation and scaling to be so that $c_{1,0}$ is rotated to the positive x-axis and scaled to unit magnitude. From

$$
c'_{1,0} = 1 \Leftrightarrow s_o e^{i\alpha_o} = \frac{s_i^3 e^{i\alpha_i}}{c_{1,0} - \frac{c_{0,0}\int_A z\,\mathrm{d}z}{\int_A \mathrm{d}z}} = \frac{s_i^3 e^{i\alpha_i}\int_A \mathrm{d}z}{c_{1,0}\int_A \mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z} \tag{36}
$$

and

$$
c'_{2,0} = 0 \Leftrightarrow t_i = \frac{\frac{c_{0,0}\int_A z^2\,\mathrm{d}z}{\int_A \mathrm{d}z} - c_{2,0}}{2(\frac{c_{0,0}\int_A z\,\mathrm{d}z}{\int_A \mathrm{d}z} - c_{1,0})} = \frac{c_{0,0}\int_A z^2\,\mathrm{d}z - c_{2,0}\int_A \mathrm{d}z}{2(c_{0,0}\int_A z\,\mathrm{d}z - c_{1,0}\int_A \mathrm{d}z)} \tag{37}
$$

we get

$$
\begin{aligned}
c'_{0,0} =&0,\\
c'_{1,0} =&1,\\
c'_{0,1} =&s_o e^{i\alpha_o} s_i^{-3} e^{i\alpha_i}\Big(c_{0,1} - \frac{c_{0,0}\int_A \bar{z}\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big)\\
=&\frac{s_i^3 e^{i\alpha_i}\int_A\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z} s_i^{-3} e^{i\alpha_i}\Big(c_{0,1} - \frac{c_{0,0}\int_A \bar{z}\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big)\\
=&e^{2i\alpha_i}\frac{c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z},\\
c'_{2,0} =&0,\\
c'_{0,2} =&s_o e^{i\alpha_o} s_i^{-4} e^{2i\alpha_i}\Big(-2\overline{t_i}\Big(c_{0,1} - \frac{c_{0,0}\int_A \bar{z}\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big) + c_{0,2} - \frac{c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big)\\
=&\frac{s_i^3 e^{i\alpha_i}\int_A\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z} s_i^{-4} e^{2i\alpha_i}\\
&\Big(-2\overline{t_i}\Big(c_{0,1} - \frac{c_{0,0}\int_A \bar{z}\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big) + c_{0,2} - \frac{c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{\int_A\,\mathrm{d}z}\Big)\\
=&s_i^{-1} e^{3i\alpha_i}\frac{-2\overline{t_i}\Big(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z\Big) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}.
\end{aligned}
\tag{38}
$$

Because $c'_{0,1}$ does no longer depend on s_i, we scale $c'_{0,2}$ to one using the right value for s_i

$$
|c'_{0,2}| = 1 \Leftrightarrow s_i = \Big|\frac{-2\overline{t_i}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big|
\tag{39}
$$

and also use its argument to find the right value for α_i

$$
\begin{aligned}
&c'_{0,2} \in \mathbb{R}^+\\
&\Leftrightarrow \alpha_i = -\frac{1}{3}\arg\Big(\frac{-2\overline{t_i}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big).
\end{aligned}
\tag{40}
$$

Instead one could also think of taking $c'_{0,1}$ to set α_i.

Theorem 2 *For an area A, its characteristic function $\chi_A(z)$, the function $g(z) = f(z)\chi_A(z) : \mathbb{C} \to \mathbb{C}$, and its complex moments $c_{0,0}, c_{1,0}, c_{0,1}, c_{2,0}, c_{0,2}$, we set*

$$
\begin{aligned}
t_i^c =& \frac{c_{0,0}\int_A z^2 \,\mathrm{d}z - c_{2,0}\int_A \,\mathrm{d}z}{2(c_{0,0}\int_A z \,\mathrm{d}z - c_{1,0}\int_A \,\mathrm{d}z)},\\
s_i^c =& \Big|\frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z + -c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big|,\\
\alpha_i^c =& -\frac{1}{3}\arg\Big(\frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big),\\
t_o^c =& -\frac{c_{0,0}}{\int_A \,\mathrm{d}z},\\
s_o^c =& \Big|\frac{(s_i^c)^3\int_A \,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big|,\\
\alpha_o^c =& \arg\Big(\frac{e^{i\alpha_i^c}\int_A \,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\Big).
\end{aligned}
\tag{41}
$$

Then, the normalized function

$$
g^c(z) = s_o^c e^{i\alpha_o^c}\Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c) + t_o^c\Big)\chi_A(s_i^c e^{i\alpha_i^c} z + t_i^c) \tag{42}
$$

is invariant with respect to inner and outer translation, rotation, and scaling of the form

$$
g'(z) = s_o e^{i\alpha_o}(f(s_i e^{i\alpha_i} z + t_i) + t_o)\chi_A(s_i e^{i\alpha_i} z + t_i), \tag{43}
$$

with arbitrary inner and outer scaling factors $s_i, s_o \in \mathbb{R}^+$, translational differences $t_i, t_o \in \mathbb{C}$, and rotation angles $\alpha_i, \alpha_o \in [-\pi, \pi]$.

PROOF. The assertion follows from straight calculation and can be found in Appendix B.

In praxis, not the function but only the moments are normalized. That way only few additions and multiplications are needed and no resampling and interpolation needs to be done. Summarizing, the normalized complex moments $c_{p,q}^c$ satisfy

$$
c_{0,0}^c = 0, \quad c_{1,0}^c = 1, \quad c_{2,0}^c = 0, \quad c_{0,2}^c = 1, \tag{44}
$$

and can be calculated from

$$
\begin{aligned}
c^c_{p,q} =& s^c_o e^{i\alpha^c_o} (s^c_i)^{-p-q-2} e^{i\alpha^c_i (q-p)} \Bigg(\sum_{k=0}^{p} \sum_{l=0}^{q} \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} q \\ l \end{pmatrix} \\
& (-t^c_i)^{p-k} (\overline{-t^c_i})^{q-l} \Big(c_{k,l} + t^c_o \int_A z^k \bar{z}^l \,\mathrm{d}z \Big) \Bigg).
\end{aligned} \tag{45}
$$

In this case, we had eight real degrees of freedom: two for the scaling, two for the rotation, and four for the translation. We eliminated them by setting fix four complex moments.

All transformations described in the introduction are special cases of Theorem 2. Depending on the specific application it may be useful not to normalize with respect to all the transforms. In order to customize the normalization to a specific problem, the superfluous parameters can simply be left out. For the scaling parameters, this means setting them to one, and for all the others, it means setting them to zero. The total transforms are generated by identifying the outer parameters with the negative of their inner counterparts. This will usually lead to a simplification of the remaining parameters.

As mentioned earlier, the choice of the moments that are predefined is free. If the underlying problem has a much smaller amount of degrees of freedom that coincide with the fixation of the second order moments, it might be useful for the sake of robustness to recalculate the normalization parameters. But the given setting is valid for all given kinds of transforms. We look closer at an example of customization in the coming section.

In case that the chosen moments should vanish, a recalculation of the normalization with non vanishing moments is also necessary. The details for flow fields can be found in [15].

5 Finding flow field patterns

Now, we want to focus on pattern matching and feature extraction [16,17] of flow fields. The problem is as follows. We have a relatively small pattern and want to decide where it appears in a larger vector field independent from its orientation, size, or position. As we depicted in the Figures 1, 2, and 3 this application is a special case of the general one treated in Theorem 2. We have to treat some parameters different than others.

In this application, the calculation of the inner translation and scaling can not be covered the same way as in the previous section because we do not compare the pattern to the whole field but only to parts of it. We have to choose smaller parts and cut off the surrounding information because otherwise it would disturb the comparison. That is why, we decide to look in all kinds of places and all kinds of scales in the big vector field. As a result, it is not useful to include these parameters in the calculation. There will always be a position where pattern and field match with respect to them. So, we leave out inner translation and scaling, which means $t_i = 0, s_i = 1$. Further, we will have to work with circular areas $A = B_r(0)$ about the center of coordinates to avoid values that move in or out during a rotation and could disturb the result. Without inner translation and scaling, the circles $B_r(0)$ satisfy

$$\chi_{B_r(0)'}(z) = \chi_{B_r(0)}(e^{i\alpha_i} z) = \chi_{B_r(0)}(z) \tag{46}$$

and

$$\forall p \neq q : \int_{B_r(0)} z^p \overline{z}^q \, \mathrm{d}z = 0. \tag{47}$$

The outer translation can be interpreted as a distortion of the pattern by some background flow. We would like to detect it despite this kind of disturbance, so we have to normalize with respect to outer translation t_o. When talking about flow fields, the outer scale represents the velocity of the flow. We want to detect the pattern independent from its speed. So, we have to normalize with respect to outer scaling s_o.

When it comes to rotation, we can not allow the field to be independently rotated from the inside and the outside, because that would essentially change the look of the pattern. We only want to normalize against total rotations $\alpha_o^c = -\alpha_i^c = \alpha^c$, because that does only change its alignment and not its shape, compare Figure 1.

All in all, the transforms of a function

$$g(z) = f(z)\chi_{B_r(0)}(z) : \mathbb{C} \to \mathbb{C}, \tag{48}$$

with respect to which we want to normalize take the shape

$$g'(z) = s_o e^{i\alpha} \left(f\left(e^{-i\alpha} z\right) + t_o \right) \chi_{B_r(0)}(z). \tag{49}$$

This is a special case of the results from the previous section. With the restrictions $t_i^c = 0, s_i^c = 1$, and $\alpha_o^c = -\alpha_i^c = \alpha^c$, the rules from (41) become

$$
\begin{aligned}
t_o^c =& -\frac{c_{0,0}}{\int_{B_r(0)} \mathrm{d}z},\\
s_o^c =& \Big|\frac{\int_{B_r(0)} \mathrm{d}z}{c_{1,0}\int_{B_r(0)} \mathrm{d}z - c_{0,0}\int_{B_r(0)} z\,\mathrm{d}z}\Big|\\
=& |c_{1,0}|^{-1},
\end{aligned}
\tag{50}
$$

and for α, we can either use

$$
\begin{aligned}
\alpha^c = -\alpha_i^c =& \frac{1}{3}\arg\Big(\frac{c_{0,2}\int_{B_r(0)} \mathrm{d}z - c_{0,0}\int_{B_r(0)} \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_{B_r(0)} \mathrm{d}z - c_{0,0}\int_{B_r(0)} z\,\mathrm{d}z}\Big)\\
=& \frac{1}{3}\arg\Big(\frac{c_{0,2}}{c_{1,0}}\Big),
\end{aligned}
\tag{51}
$$

or

$$
\begin{aligned}
\alpha^c = \alpha_o^c =& \arg\Big(\frac{e^{i\alpha_i^c}\int_{B_r(0)} \mathrm{d}z}{c_{1,0}\int_{B_r(0)} \mathrm{d}z - c_{0,0}\int_{B_r(0)} z\,\mathrm{d}z}\Big)\\
=& \alpha_i^c + \arg(c_{1,0}^{-1})\\
=& -\alpha^c - \arg(c_{1,0})\\
\overset{\alpha_o^c = -\alpha_i^c}{=}& -\frac{1}{2}\arg(c_{1,0}).
\end{aligned}
\tag{52}
$$

We choose the second, easier option.

Corollary 3 *Let* $c_{0,0}, c_{1,0}$ *be the complex moments of the flow field* $g(z) = f(z)\chi_{B_r(0)}(z) : \mathbb{C} \to \mathbb{C}$ *that vanishes outside the circle with radius* r *about the origin of coordinates. We set*

$$
\begin{aligned}
t^c =& -\frac{c_{0,0}}{\int_{B_r(0)} \mathrm{d}z},\\
s^c =& |c_{1,0}|^{-1},\\
\alpha^c =& -\frac{1}{2}\arg(c_{1,0}),
\end{aligned}
\tag{53}
$$

then, the normalized function

$$
g^c(z) = s^c e^{i\alpha^c}\Big(f(e^{-i\alpha^c} z) + t^c\Big)\chi_{B_r(0)}(z) \tag{54}
$$

is invariant with respect to outer translation and scaling and total rotation.

PROOF. The explicit derivation of the normalization parameters can be found in Appendix C.

Summarizing, the normalized complex moments $c^c_{p,q}$ satisfy

$$c^c_{0,0} = 0, \quad c^c_{1,0} = 1 \tag{55}$$

and can be calculated from the complex moments $c_{p,q}$ using the formula

$$c^c_{p,q} = s^c e^{i\alpha^c(p-q+1)} \Big(c_{p,q} + t^c \int_A z^p \overline{z}^q \, \mathrm{d}z \Big). \tag{56}$$

To underline our theoretical findings, we implemented the normalization and registered an example pattern.

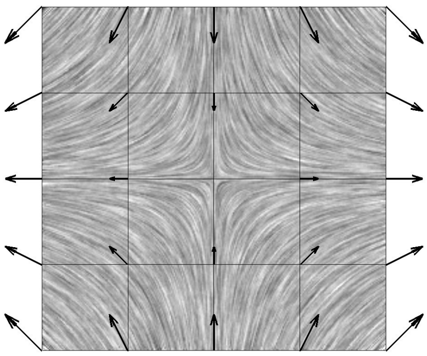


Fig. 4. The pattern, the saddle $f(z)$ described in (57).

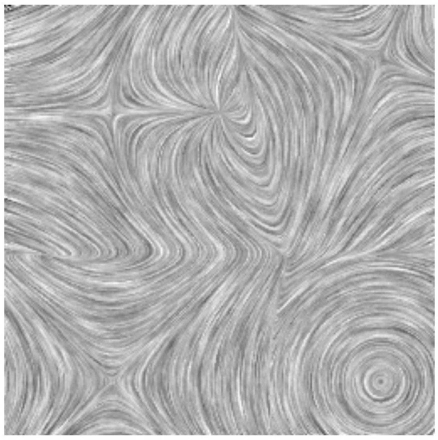


Fig. 5. The vector field $g(z)$ described in (58).

We try to find a saddle on the unit circle

$$f(z) = \begin{cases} \overline{z}, & \text{if } z \in B_1(0), \\ 0, & \text{else,} \end{cases} \tag{57}$$

compare Figure 4, in the larger vector field from Figure 5. This artificially designed field

$$\begin{aligned} g(z) = &(z-2i)\overline{(z+2-2i)}e^{-|z+2-2i|} - i\overline{(z+2+3i)}e^{-|z+2+3i|} \\ &+ i(z-3+3i)e^{-|z-3+3i|} + \sqrt{2}e^{-\frac{i\pi}{4}}\overline{(z-3-3i)}e^{-|z-3-3i|} \end{aligned} \tag{58}$$

consists of four parts: a linear saddle at $-2-3i$ and one at $3+3i$, a vortex at $3-3i$ and a saddle source combination of higher order at $-2+2i$. Our pattern appears in several distorted ways. We used Line Integral Convolution (LIC) [18] for the visualization of the flow patterns in order to make the results comparable. At equidistant positions $z_0 \in [-4-4i, 4+4i]$, we compare the normalized moments of $f(z)$ to the normalized moments

$$c_{p,q}(z_0) = \int_{B_1(z_0)} z^p \overline{z}^q g(z)\,\mathrm{d}z \tag{59}$$

of $g(z)$ calculated over the area $B_1(z_0)$. As similarity measure, we choose the reciprocal of the Euclidean distance of the moments of up to the third order

$$\Big(\sum_{p+q\leq 3} \big|c_{p,q} - c_{p,q}(z_0)\big|\Big)^{-1}. \tag{60}$$

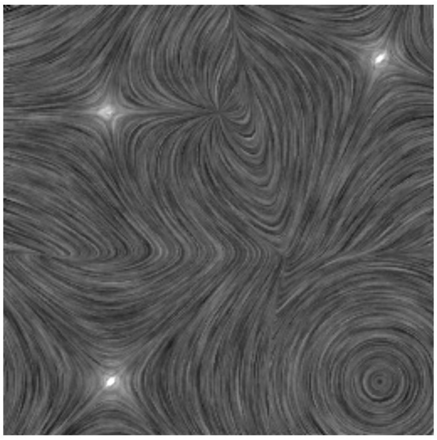


Fig. 6. The similarity of the flow field $g(z)$ (58) to the pattern $f(z)$ (57). Brighter colors resemble higher the similarity.

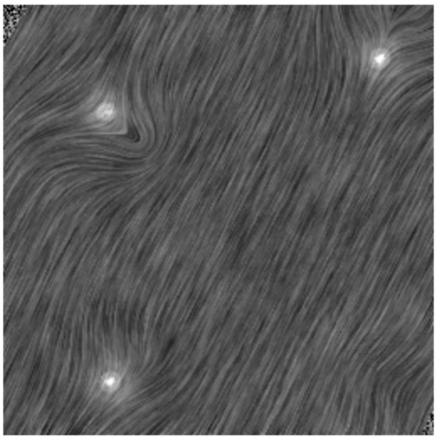


Fig. 7. The similarity of the flow field with higher velocity and added background flow (61) to the pattern $f(z)$ from (57).

The result is visualized in Figure 6. The color map resembles the similarity of each position in the vector field with the pattern. High similarity values (ca. 10) are depicted in white and low values (about zero) in dark gray. The two linear saddles in the upper right and lower left corner of $g(z)$ are clearly detected. The pattern in the left upper corner is of higher order. Since it differs from the pattern saddle not only in orientation, the similarity to the linear saddle is lower. The color map is not as bright there.

The vector field $g'(z)$ in Figure 7 is the one from Figure 6 with three times the velocity and an added background flow

$$g'(z) = 3g(z) + 1 + 2i. \tag{61}$$

The constant background flow can be interpreted as a global movement of the field, for example, the air between two carts of a moving train. The patterns that could be seen for someone on the train are invisible for someone looking from outside because of the dominant background flow. As expected, the results are invariant to the constant background flow and the velocity. This can be seen in Figure 7. Even though the pattern can not be recognized by the human eye any more, the original positions of the saddles are still detected accurately, because the moments are invariant to these kinds of changes.

In addition to the analytic example, we also tested our method on a real world data set. We look for a drawn out vortex on the unit circle on different scales

$$f(z) = \begin{cases} 0.75iz - 0.25i\overline{z}, & \text{if } z \in B_1(0), \\ 0, & \text{else,} \end{cases} \tag{62}$$

depicted in Figure 8 in the larger flow field from Figure 9. The latter is the velocity field of a swirling jet entering a fluid at rest.

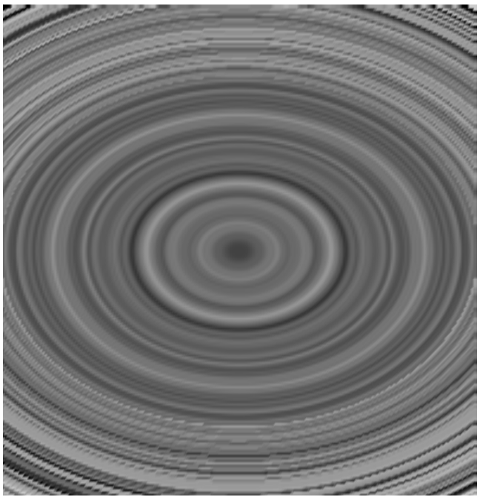


Fig. 8. The pattern, the vortex $f(z)$ described in (62).

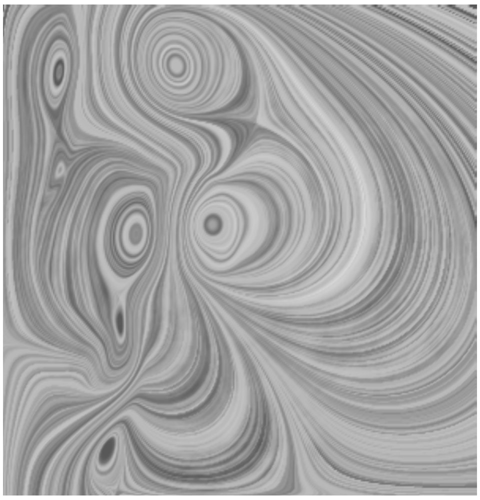


Fig. 9. The field, a swirling jet entering a fluid at rest.

In contrast to the analytic field, the vortices in the real world data set are no perfect matches to the pattern. The pattern is a very simple linear field. The vortices in the data set are not linear. As a result, we can not find exact peaks in Figures 10 and 11 and the similarity is generally lower. But one can very nicely distinguish the vortices on the left side of the field from the ones on the right because they differ in their swirling orientation. In contrast to the line integral convolution (LIC) the moments are sensitive with respect to the orientation. So this behavior can be detected. It would be easily possible to also construct invariance with respect to reflections, which would result in an overall vortex detection. But we chose to keep the sensitivity for this demonstration.

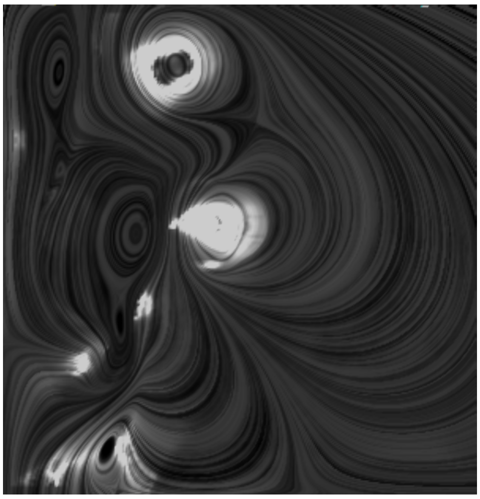


Fig. 10. The similarity of the flow field from Figure 9 to the pattern $f(z)$ (62). Brighter colors resemble higher similarity.

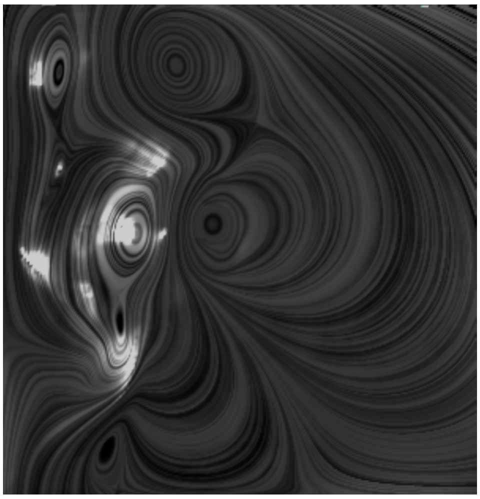


Fig. 11. The similarity of the flow field from Figure 9 to the pattern $-f(z)$ with $f(z)$ from (62), which is the vortex with opposite flow orientation.

6 Conclusions and Outlook

The requirements for invariants of vector fields or complex functions differ from the ones of the well analyzed real-valued functions depending on their meaning and application.

In Theorem 2, we showed how moments have to be normalized such that they are invariant with respect to inner and outer translation, rotation and scaling. This general result can be customized to specific problems by leaving out the superfluous parameters. Representative for all possible applications, Corollary 3 presents the configuration for the special case of moment invariants that can be used to find flow field patterns.

In our future work, we will concentrate on how thee-dimensional vector fields can be normalized using moment invariants.

Acknowledgements

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A Proof of Theorem 1

The function

$$g(z) = f(z)\chi_A(z) \tag{A.1}$$

has the normalized function

$$g^c(z) = f^c(z)\chi_{A^c}(z), \tag{A.2}$$

with

$$\begin{aligned} f^c(z) =& f(s^c e^{i\alpha^c} z + t^c), \\ \chi_{A^c}(z) =& \chi_A(s^c e^{i\alpha^c} z + t^c). \end{aligned} \tag{A.3}$$

The transformed function

$$g'(z) = f'(z)\chi_{A'}(z), \tag{A.4}$$

with

$$\begin{aligned} f'(z) =& f(se^{i\alpha} z + t), \\ \chi_{A'}(z) =& \chi_A(se^{i\alpha} z + t) \end{aligned} \tag{A.5}$$

has the normalized function

$$g'^c(z) = f'^c(z)\chi_{A'^c}(z), \tag{A.6}$$

with

$$
\begin{aligned}
f'^c(z) =& f'(s'^c e^{i\alpha'^c} z + t'^c)\\
=& f(se^{i\alpha}(s'^c e^{i\alpha'^c} z + t'^c) + t)\\
=& f(se^{i\alpha} s'^c e^{i\alpha'^c} z + se^{i\alpha} t'^c + t)\\
=& f(s'^c s e^{i(\alpha'^c+\alpha)} z + se^{i\alpha} t'^c + t),\\
\chi_{A'^c}(z) =& \chi_{A'}(s'^c e^{i\alpha'^c} z + t'^c)\\
=& \chi_A(s'^c s e^{i(\alpha'^c+\alpha)} z + se^{i\alpha} t'^c + t).
\end{aligned}
\tag{A.7}
$$

Since the lower degree moments of g' satisfy

$$
\begin{aligned}
c'_{0,0} =& s^{-2} c_{0,0},\\
c'_{1,0} =& s^{-3} e^{-i\alpha}(-tc_{0,0} + c_{1,0}),\\
c'_{2,0} =& s^{-4} e^{-2i\alpha}(t^2 c_{0,0} - 2tc_{1,0} + c_{2,0}),
\end{aligned}
\tag{A.8}
$$

the normalizing parameters of g' take the shapes

$$
\begin{aligned}
t'^c =& \frac{c'_{1,0}}{c'_{0,0}}\\
=& \frac{s^{-3} e^{-i\alpha}(-tc_{0,0} + c_{1,0})}{s^{-2} c_{0,0}}\\
=& \frac{1}{se^{i\alpha}}\left(\frac{c_{1,0}}{c_{0,0}} - t\right)\\
=& \frac{1}{se^{i\alpha}}(t^c - t),\\
s'^c =& \sqrt{c'_{0,0}}\\
=& \sqrt{s^{-2} c_{0,0}}\\
=& \frac{s^c}{s},\\
\alpha'^c =& \frac{1}{2}\arg\Big(\frac{c'_{2,0}}{(c'_{0,0})^2} - \frac{(c'_{1,0})^2}{(c'_{0,0})^3}\Big)\\
=& \frac{1}{2}\arg\Big(\frac{s^{-4} e^{-2i\alpha}(t^2 c_{0,0} - 2tc_{1,0} + c_{2,0})}{(s^{-2} c_{0,0})^2} - \frac{(s^{-3} e^{-i\alpha}(-tc_{0,0} + c_{1,0}))^2}{(s^{-2} c_{0,0})^3}\Big)\\
=& \frac{1}{2}\arg\Big(e^{-2i\alpha}\big(\frac{t^2 c_{0,0} - 2tc_{1,0} + c_{2,0}}{(c_{0,0})^2} - \frac{(-tc_{0,0} + c_{1,0})^2}{(c_{0,0})^3}\big)\Big)\\
=& \frac{1}{2}\arg(e^{-2i\alpha}) + \frac{1}{2}\arg\Big(\frac{t^2 c_{0,0} - 2tc_{1,0} + c_{2,0}}{(c_{0,0})^2} - \frac{t^2 c_{0,0}^2 - 2tc_{0,0}c_{1,0} + c_{1,0}^2}{(c_{0,0})^3}\Big)\\
=& -\alpha + \frac{1}{2}\arg\Big(\frac{c_{2,0}}{(c_{0,0})^2} - \frac{c_{1,0}^2}{(c_{0,0})^3}\Big)\\
=& \alpha^c - \alpha.
\end{aligned}
\tag{A.9}
$$

Insertion into (A.7) leads to

$$\begin{aligned}
f'^c(z) =& f(s'^c s e^{i(\alpha'^c+\alpha)} z + s e^{i\alpha} t'^c + t) \\
=& f(\frac{s^c}{s} s e^{i(\alpha^c-\alpha+\alpha)} z + s e^{i\alpha} \frac{1}{s e^{i\alpha}} (t^c - t) + t) \\
=& f(s^c e^{i\alpha^c} z + t^c) \\
=& f^c(z), \\
\chi_{A'^c}(z) =& \chi_{A^c}(z)
\end{aligned} \tag{A.10}$$

and therefore

$$g^c(z) = g'^c(z), \tag{A.11}$$

which proves the theorem.

B Proof of Theorem 2

The function

$$g(z) = f(z)\chi_A(z) \tag{B.1}$$

has the normalized function

$$g^c(z) = f^c(z)\chi_{A^c}(z), \tag{B.2}$$

with

$$\begin{aligned}
f^c(z) =& s_o^c e^{i\alpha_o^c} \Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c) + t_o^c \Big), \\
\chi_{A^c}(z) =& \chi_A(s_i^c e^{i\alpha_i^c} z + t_i^c).
\end{aligned} \tag{B.3}$$

The transformed function

$$g'(z) = f'(z)\chi_{A'}(z), \tag{B.4}$$

with

$$\begin{aligned}
f'(z) =& s_o e^{i\alpha_o} (f(s_i e^{i\alpha_i} z + t_i) + t_o), \\
\chi_{A'}(z) =& \chi_A(s_i e^{i\alpha_i} z + t_i)
\end{aligned} \tag{B.5}$$

has the normalized function

$$g'^c(z) = f'^c(z)\chi_{A'^c}(z), \tag{B.6}$$

with

$$
\begin{aligned}
f'^c(z) =& s_o'^c e^{i\alpha_o'^c}\Big(f'(s_i'^c e^{i\alpha_i'^c} z + t_i'^c) + t_o'^c\Big)\\
=& s_o'^c e^{i\alpha_o'^c}\bigg(s_o e^{i\alpha_o}\Big(f(s_i e^{i\alpha_i}(s_i'^c e^{i\alpha_i'^c} z + t_i'^c) + t_i) + t_o\Big) + t_o'^c\bigg)\\
=& s_o'^c e^{i\alpha_o'^c} s_o e^{i\alpha_o}\bigg(f(s_i e^{i\alpha_i} s_i'^c e^{i\alpha_i'^c} z + s_i e^{i\alpha_i} t_i'^c + t_i) + t_o + \frac{t_o'^c}{s_o e^{i\alpha_o}}\bigg)\\
=& s_o'^c s_o e^{i(\alpha_o'^c + \alpha_o)}\Big(f(s_i'^c s_i e^{i(\alpha_i'^c + \alpha_i)} z + s_i e^{i\alpha_i} t_i'^c + t_i) + t_o + \frac{t_o'^c}{s_o e^{i\alpha_o}}\Big),\\
\chi_{A'^c}(z) =& \chi_{A'}(s_i'^c e^{i\alpha_i'^c} z + t_i'^c)\\
=& \chi_A(s_i'^c s_i e^{i(\alpha_i'^c + \alpha_i)} z + s_i e^{i\alpha_i} t_i'^c + t_i).
\end{aligned}
\tag{B.7}
$$

To keep calculations clear and of manageable length, we will show the invariance with respect to the transforms one by one.

Outer translation: Let g' be an outer translated copy of g

$$
g'(z) = (f(z) + t_o)\chi_A(z), \tag{B.8}
$$

that means $s_o = s_i = 1, \alpha_o = \alpha_i = t_i = 0$, then the lower order moments (33) satisfy

$$
\begin{aligned}
c'_{0,0} =& c_{0,0} + t_o \int_A \mathrm{d}z,\\
c'_{0,1} =& c_{0,1} + t_o \int_A \overline{z}\,\mathrm{d}z,\\
c'_{1,0} =& c_{1,0} + t_o \int_A z\,\mathrm{d}z,\\
c'_{0,2} =& c_{0,2} + t_o \int_A \overline{z}^2\,\mathrm{d}z,\\
c'_{2,0} =& c_{2,0} + t_o \int_A z^2\,\mathrm{d}z.
\end{aligned}
\tag{B.9}
$$

then the standardizing parameters satisfy

$$
\begin{aligned}
t_i'^c =& \frac{c'_{0,0}\int_A z^2\,\mathrm{d}z - c'_{2,0}\int_A\,\mathrm{d}z}{2(c'_{0,0}\int_A z\,\mathrm{d}z - c'_{1,0}\int_A\,\mathrm{d}z)} \\
=& \frac{(c_{0,0}+t_o\int_A\,\mathrm{d}z)\int_A z^2\,\mathrm{d}z - (c_{2,0}+t_o\int_A z^2\,\mathrm{d}z)\int_A\,\mathrm{d}z}{2((c_{0,0}+t_o\int_A\,\mathrm{d}z)\int_A z\,\mathrm{d}z - (c_{1,0}+t_o\int_A z\,\mathrm{d}z)\int_A\,\mathrm{d}z)} \\
=& \frac{c_{0,0}\int_A z^2\,\mathrm{d}z + t_o\int_A\,\mathrm{d}z\int_A z^2\,\mathrm{d}z - c_{2,0}\int_A\,\mathrm{d}z - t_o\int_A z^2\,\mathrm{d}z\int_A\,\mathrm{d}z}{2(c_{0,0}\int_A z\,\mathrm{d}z + t_o\int_A\,\mathrm{d}z\int_A z\,\mathrm{d}z - c_{1,0}\int_A\,\mathrm{d}z - t_o\int_A z\,\mathrm{d}z\int_A\,\mathrm{d}z)} \\
=& \frac{c_{0,0}\int_A z^2\,\mathrm{d}z - c_{2,0}\int_A\,\mathrm{d}z}{2(c_{0,0}\int_A z\,\mathrm{d}z - c_{1,0}\int_A\,\mathrm{d}z)} \\
=& t_i^c,
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
s_i'^c =& \left|\frac{-2\overline{t_i'^c}(c'_{0,1}\int_A\,\mathrm{d}z - c'_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c'_{0,2}\int_A\,\mathrm{d}z - c'_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c'_{1,0}\int_A\,\mathrm{d}z - c'_{0,0}\int_A z\,\mathrm{d}z}\right| \\
=& \Big|\Big(-2\overline{t_i^c}(c_{0,1}+t_o\int_A \bar{z}\,\mathrm{d}z)\int_A\,\mathrm{d}z + 2\overline{t_i^c}(c_{0,0}+t_o\int_A\,\mathrm{d}z)\int_A \bar{z}\,\mathrm{d}z \\
&+ (c_{0,2}+t_o\int_A \bar{z}^2\,\mathrm{d}z)\int_A\,\mathrm{d}z - (c_{0,0}+t_o\int_A\,\mathrm{d}z)\int_A \bar{z}^2\,\mathrm{d}z\Big) \\
&\Big((c_{1,0}+t_o\int_A z\,\mathrm{d}z)\int_A\,\mathrm{d}z - (c_{0,0}+t_o\int_A\,\mathrm{d}z)\int_A z\,\mathrm{d}z\Big)^{-1}\Big| \\
=& \left|\frac{-2\overline{t_i^c}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\right| \\
=& s_i^c,
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
\alpha_i'^c =& -\frac{1}{3}\arg\left(\frac{-2\overline{t_i'^c}(c'_{0,1}\int_A\,\mathrm{d}z - c'_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c'_{0,2}\int_A\,\mathrm{d}z - c'_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c'_{1,0}\int_A\,\mathrm{d}z - c'_{0,0}\int_A z\,\mathrm{d}z}\right) \\
=& -\frac{1}{3}\arg\left(\frac{-2\overline{t_i^c}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\right) \\
=& \alpha_i^c,
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
t_o'^c =& -\frac{c'_{0,0}}{\int_A\,\mathrm{d}z} \\
=& -\frac{c_{0,0}+t_o\int_A\,\mathrm{d}z}{\int_A\,\mathrm{d}z} \\
=& -\frac{c_{0,0}}{\int_A\,\mathrm{d}z} - t_o \\
=& t_o^c - t_o,
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
s_o'^c =& |\frac{(s_i'^c)^3 \int_A \,\mathrm{d}z}{c_{1,0}' \int_A \,\mathrm{d}z - c_{0,0}' \int_A z \,\mathrm{d}z}| \\
=& |\frac{(s_i^c)^3 \int_A \,\mathrm{d}z}{(c_{1,0} + t_o \int_A z \,\mathrm{d}z) \int_A \,\mathrm{d}z - (c_{0,0} + t_o \int_A \,\mathrm{d}z) \int_A z \,\mathrm{d}z}| \\
=& |\frac{(s_i^c)^3 \int_A \,\mathrm{d}z}{c_{1,0} \int_A \,\mathrm{d}z - c_{0,0} \int_A z \,\mathrm{d}z}| \\
=& s_o^c,
\end{aligned} \tag{B.14}
$$

$$
\begin{aligned}
\alpha_o'^c =& \arg(\frac{e^{i\alpha_i'^c} \int_A \,\mathrm{d}z}{c_{1,0}' \int_A \,\mathrm{d}z - c_{0,0}' \int_A z \,\mathrm{d}z}) \\
=& \arg(\frac{e^{i\alpha_i^c} \int_A \,\mathrm{d}z}{(c_{1,0} + t_o \int_A z \,\mathrm{d}z) \int_A \,\mathrm{d}z - (c_{0,0} + t_o \int_A \,\mathrm{d}z) \int_A z \,\mathrm{d}z}) \\
=& \arg(\frac{e^{i\alpha_i^c} \int_A \,\mathrm{d}z}{c_{1,0} \int_A \,\mathrm{d}z - c_{0,0} \int_A z \,\mathrm{d}z}) \\
=& \alpha_o^c.
\end{aligned} \tag{B.15}
$$

Insertion into (B.7) with $s_o = s_i = 1, \alpha_o = \alpha_i = t_i = 0$ leads to

$$
\begin{aligned}
f'^c(z) =& s_o'^c s_o e^{i(\alpha_o'^c + \alpha_o)} \Big(f(s_i'^c s_i e^{i(\alpha_i'^c + \alpha_i)} z + s_i e^{i\alpha_i} t_i'^c + t_i) + t_o + \frac{t_o'^c}{s_o e^{i\alpha_o}} \Big) \\
=& s_o'^c e^{i\alpha_o'^c} \Big(f(s_i'^c e^{i\alpha_i'^c} z + t_i'^c) + t_o + t_o'^c \Big) \\
=& s_o^c e^{i\alpha_o^c} \Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c) + t_o + t_o^c - t_o \Big) \\
=& f^c(z), \\
\chi_{A'^c}(z) =& \chi_{A^c}(z)
\end{aligned} \tag{B.16}
$$

and therefore

$$
g'^c(z) = g^c(z), \tag{B.17}
$$

which shows that the normalized function is invariant to outer translation.

Inner translation: Let g' be an inner translated copy of g

$$
g'(z) = f(z + t_i) \chi_A(z + t_i), \tag{B.18}
$$

that vanishes outside the area A' with the characteristic function $\chi_{A'}(z) = \chi_A(z + t_i)$. In this case we have $s_o = s_i = 1, \alpha_o = \alpha_i = t_o = 0$, so the lower order moments (33) satisfy

$$
\begin{aligned}
c'_{0,0} =& c_{0,0}, \\
c'_{1,0} =& c_{1,0} - t_i c_{0,0}, \\
c'_{0,1} =& c_{0,1} - \overline{t_i} c_{0,0}, \\
c'_{2,0} =& c_{2,0} - 2t_i c_{1,0} + t_i^2 c_{0,0}, \\
c'_{0,2} =& c_{0,2} - 2\overline{t_i} c_{0,1} + \overline{t_i}^2 c_{0,0}.
\end{aligned} \tag{B.19}
$$

and the standardizing parameters

$$
\begin{aligned}
t_i'^c =& \frac{c'_{0,0} \int_{A'} z^2 \,\mathrm{d}z - c'_{2,0} \int_{A'} \,\mathrm{d}z}{2(c'_{0,0} \int_{A'} z \,\mathrm{d}z - c'_{1,0} \int_{A'} \,\mathrm{d}z)} \\
=& \frac{c_{0,0} \int_{\mathbb{C}} z^2 \chi_A(z+t_i) \,\mathrm{d}z - (c_{2,0} - 2t_i c_{1,0} + t_i^2 c_{0,0}) \int_{\mathbb{C}} \chi_A(z+t_i) \,\mathrm{d}z}{2(c_{0,0} \int_{\mathbb{C}} z \chi_A(z+t_i) \,\mathrm{d}z - (c_{1,0} - t_i c_{0,0}) \int_{\mathbb{C}} \chi_A(z+t_i) \,\mathrm{d}z)} \\
\overset{z=z+t_i}{=}& \frac{c_{0,0} \int_{\mathbb{C}} (z-t_i)^2 \chi_A(z) \,\mathrm{d}z - (c_{2,0} - 2t_i c_{1,0} + t_i^2 c_{0,0}) \int_{\mathbb{C}} \chi_A(z) \,\mathrm{d}z}{2(c_{0,0} \int_{\mathbb{C}} (z-t_i) \chi_A(z) \,\mathrm{d}z - (c_{1,0} - t_i c_{0,0}) \int_{\mathbb{C}} \chi_A(z) \,\mathrm{d}z)} \\
=& \frac{c_{0,0} \int_A (z-t_i)^2 \,\mathrm{d}z - (c_{2,0} - 2t_i c_{1,0} + t_i^2 c_{0,0}) \int_A \,\mathrm{d}z}{2(c_{0,0} \int_A z - t_i \,\mathrm{d}z - (c_{1,0} - t_i c_{0,0}) \int_A \,\mathrm{d}z)} \\
=& \Big(c_{0,0} \int_A z^2 \,\mathrm{d}z - 2t_i c_{0,0} \int_A z \,\mathrm{d}z + t_i^2 c_{0,0} \int_A \,\mathrm{d}z \\
& - c_{2,0} \int_A \,\mathrm{d}z + 2t_i c_{1,0} \int_A \,\mathrm{d}z - t_i^2 c_{0,0} \int_A \,\mathrm{d}z \Big) \\
& \Big(2(c_{0,0} \int_A z \,\mathrm{d}z - t_i c_{0,0} \int_A \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z + t_i c_{0,0} \int_A \,\mathrm{d}z) \Big)^{-1} \\
=& \frac{c_{0,0} \int_A z^2 \,\mathrm{d}z - 2t_i c_{0,0} \int_A z \,\mathrm{d}z - c_{2,0} \int_A \,\mathrm{d}z + 2t_i c_{1,0} \int_A \,\mathrm{d}z}{2(c_{0,0} \int_A z \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z)} \\
=& \frac{c_{0,0} \int_A z^2 \,\mathrm{d}z - c_{2,0} \int_A \,\mathrm{d}z}{2(c_{0,0} \int_A z \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z)} - \frac{2t_i(c_{0,0} \int_A z \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z)}{2(c_{0,0} \int_A z \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z)} \\
=& t_i^c - t_i,
\end{aligned} \tag{B.20}
$$

$$
\begin{aligned}
s'^c_i =& |\frac{-2\overline{t'^c_i}(c'_{0,1}\int_{A'}\,\mathrm{d}z - c'_{0,0}\int_{A'}\bar{z}\,\mathrm{d}z) + c'_{0,2}\int_{A'}\,\mathrm{d}z - c'_{0,0}\int_{A'}\bar{z}^2\,\mathrm{d}z}{c'_{1,0}\int_{A'}\,\mathrm{d}z - c'_{0,0}\int_{A'} z\,\mathrm{d}z}| \\
=& |\Big(-2\overline{t^c_i}c_{0,1}\int_A\,\mathrm{d}z + 2\overline{t^c_i t_i}c_{0,0}\int_A\,\mathrm{d}z + 2\overline{t^c_i}c_{0,0}\int_A\overline{z}\,\mathrm{d}z - 2\overline{t^c_i t_i}c_{0,0}\int_A\,\mathrm{d}z \\
&+ 2\overline{t_i}c_{0,1}\int_A\,\mathrm{d}z - 2\overline{t_i t_i}c_{0,0}\int_A\,\mathrm{d}z - 2\overline{t_i}c_{0,0}\int_A\overline{z}\,\mathrm{d}z + 2\overline{t_i t_i}c_{0,0}\int_A\,\mathrm{d}z \\
&+ c_{0,2}\int_A\,\mathrm{d}z - 2\overline{t_i}c_{0,1}\int_A\,\mathrm{d}z + \overline{t_i}^2 c_{0,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A\overline{z}^2\,\mathrm{d}z \\
&+ 2\overline{t_i}c_{0,0}\int_A\overline{z}\,\mathrm{d}z - \overline{t_i}^2 c_{0,0}\int_A\,\mathrm{d}z\Big) \\
&\Big(c_{1,0}\int_A\,\mathrm{d}z - t_i c_{0,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z + t_i c_{0,0}\int_A\,\mathrm{d}z\Big)^{-1}| \\
=& |\frac{-2\overline{t^c_i}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A\bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A\bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}| \\
=& s^c_i,
\end{aligned}
\tag{B.21}
$$

$$
\begin{aligned}
\alpha'^c_i =& -\frac{1}{3}\arg(\frac{-2\overline{t'^c_i}(c'_{0,1}\int_A\,\mathrm{d}z - c'_{0,0}\int_A\bar{z}\,\mathrm{d}z) + c'_{0,2}\int_A\,\mathrm{d}z - c'_{0,0}\int_A\bar{z}^2\,\mathrm{d}z}{c'_{1,0}\int_A\,\mathrm{d}z - c'_{0,0}\int_A z\,\mathrm{d}z}) \\
=& -\frac{1}{3}\arg(\frac{-2\overline{t^c_i}(c_{0,1}\int_A\,\mathrm{d}z - c_{0,0}\int_A\bar{z}\,\mathrm{d}z) + c_{0,2}\int_A\,\mathrm{d}z - c_{0,0}\int_A\bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}) \\
=& \alpha^c_i,
\end{aligned}
\tag{B.22}
$$

$$
\begin{aligned}
t'^c_o =& -\frac{c'_{0,0}}{\int_A\,\mathrm{d}z} \\
=& -\frac{c_{0,0}}{\int_A\,\mathrm{d}z} \\
=& t^c_o,
\end{aligned}
\tag{B.23}
$$

$$
\begin{aligned}
s'^c_o =& |\frac{(s'^c_i)^3\int_{A'}\,\mathrm{d}z}{c'_{1,0}\int_{A'}\,\mathrm{d}z - c'_{0,0}\int_{A'} z\,\mathrm{d}z}| \\
=& |\frac{(s^c_i)^3\int_A\,\mathrm{d}z}{(c_{1,0} - t_i c_{0,0})\int_A\,\mathrm{d}z - c_{0,0}\int_A z - t_i\,\mathrm{d}z}| \\
=& |\frac{(s^c_i)^3\int_A\,\mathrm{d}z}{c_{1,0}\int_A\,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}| \\
=& s^c_o,
\end{aligned}
\tag{B.24}
$$

$$
\begin{aligned}
\alpha_o'^c =& \arg\Big(\frac{e^{i\alpha_i'^c}\int_{A'}\,\mathrm{d}z}{c_{1,0}'\int_{A'}\,\mathrm{d}z - c_{0,0}'\int_{A'} z\,\mathrm{d}z}\Big)\\
=& \arg\Big(\frac{e^{i\alpha_i^c}\int_{A}\,\mathrm{d}z}{c_{1,0}\int_{A}\,\mathrm{d}z - c_{0,0}\int_{A} z\,\mathrm{d}z}\Big)\\
=& \alpha_o^c.
\end{aligned}
\tag{B.25}
$$

Insertion into (B.7) with $s_o = s_i = 1, \alpha_o = \alpha_i = t_o = 0$ leads to

$$
\begin{aligned}
f'^c(z) =& s_o'^c s_o e^{i(\alpha_o'^c+\alpha_o)}\Big(f(s_i'^c s_i e^{i(\alpha_i'^c+\alpha_i)} z + s_i e^{i\alpha_i} t_i'^c + t_i) + t_o + \frac{t_o'^c}{s_o e^{i\alpha_o}}\Big)\\
=& s_o'^c e^{i\alpha_o'^c}\Big(f(s_i'^c e^{i\alpha_i'^c} z + t_i'^c + t_i) + t_o'^c\Big)\\
=& s_o^c e^{i\alpha_o^c}\Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c - t_i + t_i) + t_o^c\Big)\\
=& f^c(z),\\
\chi_{A'^c}(z) =& \chi_{A^c}(z),
\end{aligned}
\tag{B.26}
$$

and therefore

$$
g'^c(z) = g^c(z), \tag{B.27}
$$

which shows that the normalized function is invariant to inner translation.

Inner rotation and scaling: Let g' be an inner rotated and scaled copy of g

$$
g'(z) = f(s_i e^{i\alpha_i} z)\chi_A(s_i e^{i\alpha_i} z), \tag{B.28}
$$

that vanishes outside the area A' with the characteristic function $\chi_{A'}(z) = \chi_A(s_i e^{i\alpha_i} z)$. In this case we have $s_o = 1, \alpha_o = t_i = t_o = 0$, so the lower order moments (33) satisfy

$$
\begin{aligned}
c_{0,0}' =& s_i^{-2} c_{0,0},\\
c_{1,0}' =& s_i^{-3} e^{-i\alpha_i} c_{1,0},\\
c_{0,1}' =& s_i^{-3} e^{i\alpha_i} c_{0,1},\\
c_{2,0}' =& s_i^{-4} e^{-2i\alpha_i} c_{2,0},\\
c_{0,2}' =& s_i^{-4} e^{2i\alpha_i} c_{0,2}.
\end{aligned}
\tag{B.29}
$$

and the standardizing parameters

$$
\begin{aligned}
t_i'^c =& \frac{c'_{0,0}\int_{A'} z^2 \,\mathrm{d}z - c'_{2,0}\int_{A'} \,\mathrm{d}z}{2(c'_{0,0}\int_{A'} z \,\mathrm{d}z - c'_{1,0}\int_{A'} \,\mathrm{d}z)} \\
=& \frac{s_i^{-2} c_{0,0}\int_{\mathbb{C}} z^2 \chi_A(s_i e^{i\alpha_i} z) \,\mathrm{d}z - s_i^{-4} e^{-2i\alpha_i} c_{2,0}\int_{\mathbb{C}} \chi_A(s_i e^{i\alpha_i} z) \,\mathrm{d}z}{2(s_i^{-2} c_{0,0}\int_{\mathbb{C}} z\chi_A(s_i e^{i\alpha_i} z) \,\mathrm{d}z - s_i^{-3} e^{-i\alpha_i} c_{1,0}\int_{\mathbb{C}} \chi_A(s_i e^{i\alpha_i} z) \,\mathrm{d}z)} \\
\overset{z=s_i e^{i\alpha_i} z}{=}& \frac{s_i^{-2} c_{0,0}\int_{\mathbb{C}} (s_i^{-1} e^{-i\alpha_i} z)^2 \chi_A(z) s_i^{-2} \,\mathrm{d}z - s_i^{-4} e^{-2i\alpha_i} c_{2,0}\int_{\mathbb{C}} \chi_A(z) s_i^{-2} \,\mathrm{d}z}{2(s_i^{-2} c_{0,0}\int_{\mathbb{C}} (s_i^{-1} e^{-i\alpha_i} z)\chi_A(z) s_i^{-2} \,\mathrm{d}z - s_i^{-3} e^{-i\alpha_i} c_{1,0}\int_{\mathbb{C}} \chi_A(z) s_i^{-2} \,\mathrm{d}z)} \\
=& \frac{s_i^{-6} e^{-2i\alpha_i} c_{0,0}\int_A z^2 \,\mathrm{d}z - s_i^{-6} e^{-2i\alpha_i} c_{2,0}\int_A \,\mathrm{d}z}{2(s_i^{-5} e^{-i\alpha_i} c_{0,0}\int_A z \,\mathrm{d}z - s_i^{-5} e^{-i\alpha_i} c_{1,0}\int_A \,\mathrm{d}z)} \\
=& \frac{t_i^c}{s_i e^{i\alpha_i}},
\end{aligned}
\tag{B.30}
$$

$$
\begin{aligned}
s_i'^c =& \left| \frac{-2\overline{t_i'^c}(c'_{0,1}\int_{A'} \,\mathrm{d}z - c'_{0,0}\int_{A'} \bar{z} \,\mathrm{d}z) + c'_{0,2}\int_{A'} \,\mathrm{d}z - c'_{0,0}\int_{A'} \bar{z}^2 \,\mathrm{d}z}{c'_{1,0}\int_{A'} \,\mathrm{d}z - c'_{0,0}\int_{A'} z \,\mathrm{d}z} \right| \\
=& \Big| \Big(-2 s_i^{-1} e^{i\alpha_i} \overline{t_i^c} (s_i^{-3} e^{i\alpha_i} c_{0,1} \int_A s_i^{-2} \,\mathrm{d}z - s_i^{-2} c_{0,0} \int_A (\overline{s_i^{-1} e^{-i\alpha_i} z}) s_i^{-2} \,\mathrm{d}z) \\
& + s_i^{-4} e^{2i\alpha_i} c_{0,2} \int_A s_i^{-2} \,\mathrm{d}z - s_i^{-2} c_{0,0} \int_A (\overline{s_i^{-1} e^{-i\alpha_i} z})^2 s_i^{-2} \,\mathrm{d}z \Big) \\
& \Big(s_i^{-3} e^{-i\alpha_i} c_{1,0} \int_A s_i^{-2} \,\mathrm{d}z - s_i^{-2} c_{0,0} \int_A s_i^{-1} e^{-i\alpha_i} z s_i^{-2} \,\mathrm{d}z \Big)^{-1} \Big| \\
=& \Big| \Big(-2 s_i^{-6} e^{2i\alpha_i} \overline{t_i^c} (c_{0,1} \int_A \,\mathrm{d}z - c_{0,0} \int_A \bar{z} \,\mathrm{d}z) \\
& + s_i^{-6} e^{2i\alpha_i} c_{0,2} \int_A \,\mathrm{d}z - s_i^{-6} e^{2i\alpha_i} c_{0,0} \int_A \bar{z}^2 \,\mathrm{d}z \Big) \\
& \Big(s_i^{-5} e^{-i\alpha_i} c_{1,0} \int_A \,\mathrm{d}z - s_i^{-5} e^{-i\alpha_i} c_{0,0} \int_A z \,\mathrm{d}z \Big)^{-1} \Big| \\
=& \left| \frac{s_i^{-1} e^{3i\alpha_i} \Big(-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z} \,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2 \,\mathrm{d}z \Big)}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z \,\mathrm{d}z} \right| \\
=& \frac{s_i^c}{s_i},
\end{aligned}
\tag{B.31}
$$

$$
\begin{aligned}
\alpha_i'^c =& -\frac{1}{3}\arg(\frac{-2\overline{t_i'^c}(c'_{0,1}\int_A \,\mathrm{d}z - c'_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c'_{0,2}\int_A \,\mathrm{d}z - c'_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c'_{1,0}\int_A \,\mathrm{d}z - c'_{0,0}\int_A z\,\mathrm{d}z}) \\
=& -\frac{1}{3}\arg(\\
& \frac{s_i^{-1}e^{3i\alpha_i}\Big(-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z\Big)}{c_{1,0}\int_A \,\mathrm{d}z - t_i c_{0,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z + t_i c_{0,0}\int_A \,\mathrm{d}z}) \\
=& -\frac{1}{3}\arg(e^{3i\alpha_i}) - \frac{1}{3}\arg(\\
& \frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}) \\
=& \alpha_i^c - \alpha_i,
\end{aligned}
\tag{B.32}
$$

$$
\begin{aligned}
t_o'^c =& -\frac{c'_{0,0}}{\int_{A'} \,\mathrm{d}z} \\
=& -\frac{s_i^{-2}c_{0,0}}{\int_A s_i^{-2}\,\mathrm{d}z} \\
=& t_o^c,
\end{aligned}
\tag{B.33}
$$

$$
\begin{aligned}
s_o'^c =& |\frac{(s_i'^c)^3\int_{A'} \,\mathrm{d}z}{c'_{1,0}\int_{A'} \,\mathrm{d}z - c'_{0,0}\int_{A'} z\,\mathrm{d}z}| \\
=& |\frac{(s_i^c)^3 s_i^{-3}\int_A s_i^{-2}\,\mathrm{d}z}{s_i^{-3}e^{-i\alpha_i}c_{1,0}\int_A s_i^{-2}\,\mathrm{d}z - s_i^{-2}c_{0,0}\int_A s_i^{-1}e^{-i\alpha_i}z s_i^{-2}\,\mathrm{d}z}| \\
=& |\frac{(s_i^c)^3 s_i^{-5}\int_A \,\mathrm{d}z}{s_i^{-5}e^{-i\alpha_i}(c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z)}| \\
=& |\frac{(s_i^c)^3\int_A \,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}| \\
=& s_o^c,
\end{aligned}
\tag{B.34}
$$

$$
\begin{aligned}
\alpha_o'^c =& \arg(\frac{e^{i\alpha_i'^c}\int_{A'} \,\mathrm{d}z}{c'_{1,0}\int_{A'} \,\mathrm{d}z - c'_{0,0}\int_{A'} z\,\mathrm{d}z}) \\
=& \arg(\frac{e^{i\alpha_i^c}e^{-i\alpha_i}\int_A s_i^{-2}\,\mathrm{d}z}{s_i^{-3}e^{-i\alpha_i}c_{1,0}\int_A s_i^{-2}\,\mathrm{d}z - s_i^{-2}c_{0,0}\int_A s_i^{-1}e^{-i\alpha_i}z s_i^{-2}\,\mathrm{d}z}) \\
=& \arg(\frac{e^{i\alpha_i^c}\int_A \,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}) \\
=& \alpha_o^c.
\end{aligned}
\tag{B.35}
$$

Insertion into (B.7) with $s_o = 1, \alpha_o = t_i = t_o = 0$ leads to

$$
\begin{aligned}
f'^c(z) =& s'^c_o s_o e^{i(\alpha'^c_o+\alpha_o)} \Big(f(s'^c_i s_i e^{i(\alpha'^c_i+\alpha_i)} z + s_i e^{i\alpha_i} t'^c_i + t_i) + t_o + \frac{t'^c_o}{s_o e^{i\alpha_o}} \Big) \\
=& s'^c_o e^{i\alpha'^c_o} \Big(f(s'^c_i s_i e^{i(\alpha'^c_i+\alpha_i)} z + s_i e^{i\alpha_i} t'^c_i) + t'^c_o \Big) \\
=& s'^c_o e^{i\alpha'^c_o} \Big(f(\frac{s^c_i}{s_i} s_i e^{i(\alpha^c_i-\alpha_i+\alpha_i)} z + s_i e^{i\alpha_i} \frac{t^c_i}{s_i e^{i\alpha_i}}) + t^c_o \Big) \\
=& s'^c_o e^{i\alpha'^c_o} \Big(f(s^c_i e^{i\alpha^c_i} z + t^c_i) + t^c_o \Big) \\
=& f^c(z), \\
\chi_{A'^c}(z) =& \chi_{A^c}(z),
\end{aligned}
\tag{B.36}
$$

and therefore

$$
g'^c(z) = g^c(z), \tag{B.37}
$$

which shows that the normalized function is invariant to inner rotation and scaling.

Outer rotation and scaling: Let g' be an outer rotated and scaled copy of g, that means $s_i = 1, \alpha_i = t_i = t_o = 0$, so the lower order moments (33) satisfy

$$
g'(z) = s_o e^{i\alpha_o} f(z) \chi_A(z), \tag{B.38}
$$

then the lower order moments satisfy

$$
\begin{aligned}
c'_{0,0} =& s_o e^{i\alpha_o} c_{0,0}, \\
c'_{1,0} =& s_o e^{i\alpha_o} c_{1,0}, \\
c'_{0,1} =& s_o e^{i\alpha_o} c_{0,1}, \\
c'_{2,0} =& s_o e^{i\alpha_o} c_{2,0}, \\
c'_{0,2} =& s_o e^{i\alpha_o} c_{0,2}.
\end{aligned}
\tag{B.39}
$$

and the standardizing parameters

$$
\begin{aligned}
t'^c_i =& \frac{c'_{0,0} \int_A z^2 \,\mathrm{d}z - c'_{2,0} \int_A \,\mathrm{d}z}{2(c'_{0,0} \int_A z \,\mathrm{d}z - c'_{1,0} \int_A \,\mathrm{d}z)} \\
=& \frac{s_o e^{i\alpha_o} c_{0,0} \int_A z^2 \,\mathrm{d}z - s_o e^{i\alpha_o} c_{2,0} \int_A \,\mathrm{d}z}{2(s_o e^{i\alpha_o} c_{0,0} \int_A z \,\mathrm{d}z - s_o e^{i\alpha_o} c_{1,0} \int_A \,\mathrm{d}z)} \\
=& \frac{c_{0,0} \int_A z^2 \,\mathrm{d}z - c_{2,0} \int_A \,\mathrm{d}z}{2(c_{0,0} \int_A z \,\mathrm{d}z - c_{1,0} \int_A \,\mathrm{d}z)} \\
=& t^c_i,
\end{aligned}
\tag{B.40}
$$

$$
\begin{aligned}
s_i'^c =& \left|\frac{-2\overline{t_i'^c}(c_{0,1}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A z\,\mathrm{d}z}\right| \\
=& \Big|\Big(- 2\overline{t_i^c}(s_o e^{i\alpha_o} c_{0,1}\int_A \,\mathrm{d}z - s_o e^{i\alpha_o} c_{0,0}\int_A \bar{z}\,\mathrm{d}z) \\
& + s_o e^{i\alpha_o} c_{0,2}\int_A \,\mathrm{d}z - s_o e^{i\alpha_o} c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z\Big) \\
& \Big(s_o e^{i\alpha_o} c_{1,0}\int_A \,\mathrm{d}z - s_o e^{i\alpha_o} c_{0,0}\int_A z\,\mathrm{d}z\Big)^{-1}\Big| \\
=& \left|\frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\right| \\
=& s_i^c,
\end{aligned}
\tag{B.41}
$$

$$
\begin{aligned}
\alpha_i'^c =& -\frac{1}{3}\arg\left(\frac{-2\overline{t_i'^c}(c_{0,1}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A z\,\mathrm{d}z}\right) \\
=& -\frac{1}{3}\arg\left(\frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - t_i c_{0,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z + t_i c_{0,0}\int_A \,\mathrm{d}z}\right) \\
=& -\frac{1}{3}\arg\left(\frac{-2\overline{t_i^c}(c_{0,1}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}\,\mathrm{d}z) + c_{0,2}\int_A \,\mathrm{d}z - c_{0,0}\int_A \bar{z}^2\,\mathrm{d}z}{c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z}\right) \\
=& \alpha_i^c,
\end{aligned}
\tag{B.42}
$$

$$
\begin{aligned}
t_o'^c =& -\frac{c_{0,0}'}{\int_A \,\mathrm{d}z} \\
=& -\frac{s_o e^{i\alpha_o} c_{0,0}}{\int_A \,\mathrm{d}z} \\
=& s_o e^{i\alpha_o} t_o^c,
\end{aligned}
\tag{B.43}
$$

$$
\begin{aligned}
s_o'^c =& \left|\frac{(s_i'^c)^3\int_A \,\mathrm{d}z}{c_{1,0}'\int_A \,\mathrm{d}z - c_{0,0}'\int_A z\,\mathrm{d}z}\right| \\
=& \left|\frac{(s_i^c)^3\int_A \,\mathrm{d}z}{s_o e^{i\alpha_o} c_{1,0}\int_A \,\mathrm{d}z - s_o e^{i\alpha_o} c_{0,0}\int_A z\,\mathrm{d}z}\right| \\
=& \left|\frac{(s_i^c)^3\int_A \,\mathrm{d}z}{s_o e^{i\alpha_o}(c_{1,0}\int_A \,\mathrm{d}z - c_{0,0}\int_A z\,\mathrm{d}z)}\right| \\
=& \frac{s_o^c}{s_o},
\end{aligned}
\tag{B.44}
$$

$$
\begin{aligned}
\alpha_o'^c =& \arg\Big(\frac{e^{i\alpha_i'^c} \int_A \,\mathrm{d}z}{c_{1,0}' \int_A \,\mathrm{d}z - c_{0,0}' \int_A z \,\mathrm{d}z}\Big) \\
=& \arg\Big(\frac{e^{i\alpha_i^c} \int_A \,\mathrm{d}z}{s_o e^{i\alpha_o}(c_{1,0} \int_A \,\mathrm{d}z - c_{0,0} \int_A z \,\mathrm{d}z)}\Big) \\
=& \arg\Big(\frac{e^{i\alpha_i^c} \int_A \,\mathrm{d}z}{c_{1,0} \int_A \,\mathrm{d}z - c_{0,0} \int_A z \,\mathrm{d}z}\Big) - \arg(e^{i\alpha_o}) \\
=& \alpha_o^c - \alpha_o.
\end{aligned} \tag{B.45}
$$

Insertion into (B.7) with $s_i = 1, \alpha_i = t_i = t_o = 0$ leads to

$$
\begin{aligned}
f'^c(z) =& s_o'^c s_o e^{i(\alpha_o'^c + \alpha_o)} \Big(f(s_i'^c s_i e^{i(\alpha_i'^c + \alpha_i)} z + s_i e^{i\alpha_i} t_i'^c + t_i) + t_o + \frac{t_o'^c}{s_o e^{i\alpha_o}} \Big) \\
=& s_o'^c s_o e^{i(\alpha_o'^c + \alpha_o)} \Big(f(s_i'^c e^{i\alpha_i'^c} z + t_i'^c) + \frac{t_o'^c}{s_o e^{i\alpha_o}} \Big) \\
=& \frac{s_o^c}{s_o} s_o e^{i(\alpha_o^c - \alpha_o + \alpha_o)} \Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c) + \frac{s_o e^{i\alpha_o} t_o^c}{s_o e^{i\alpha_o}} \Big) \\
=& s_o^c e^{i\alpha_o^c} \Big(f(s_i^c e^{i\alpha_i^c} z + t_i^c) + t_o^c \Big) \\
=& f^c(z), \\
\chi_{A'^c}(z) =& \chi_{A^c}(z)
\end{aligned} \tag{B.46}
$$

and therefore

$$
g'^c(z) = g^c(z), \tag{B.47}
$$

which shows that the normalized function is invariant to outer rotation and scaling.

Putting the four parts together shows, that the normalized function is invariant to inner and outer translation, rotation, and scaling, which proves Theorem 2.

C Proof of Corollary 3

The function

$$
g(z) = f(z)\chi_{B_r(0)}(z) \tag{C.1}
$$

has the normalized function

$$g^c(z) = f^c(z)\chi_{B_r(0)}(z), \tag{C.2}$$

with

$$f^c(z) = s^c e^{i\alpha^c}(f(e^{-i\alpha^c}z) + t^c). \tag{C.3}$$

The transformed function

$$g'(z) = f'(z)\chi_{B_r(0)}(z), \tag{C.4}$$

with

$$f'(z) = se^{i\alpha}(f(e^{-i\alpha}z) + t) \tag{C.5}$$

has the normalized function

$$g'^c(z) = f'^c(z)\chi_{B_r(0)}(z), \tag{C.6}$$

with

$$\begin{aligned} f'^c(z) =& s'^c e^{i\alpha'^c}\Big(f'(e^{-i\alpha'^c}z) + t'^c\Big) \\ =& s'^c e^{i\alpha'^c}\Big(se^{i\alpha}\Big(f(e^{-i\alpha}e^{-i\alpha'^c}z) + t\Big) + t'^c\Big) \\ =& s'^c s e^{i(\alpha'^c+\alpha)}\Big(f(e^{-i(\alpha+\alpha'^c)}z) + t + \frac{t'^c}{se^{i\alpha}}\Big). \end{aligned} \tag{C.7}$$

The lower order moments of g' satisfy by (33)

$$\begin{aligned} c'_{0,0} =& se^{i\alpha}(c_{0,0} + t\int_{B_r(0)} \mathrm{d}z), \\ c'_{1,0} =& se^{2i\alpha}c_{1,0} \end{aligned} \tag{C.8}$$

and the normalizing parameters of (50) and (52)

$$
\begin{aligned}
t'^c =& - \frac{c'_{0,0}}{\int_{B_r(0)} \,\mathrm{d}z} \\
=& - \frac{se^{i\alpha}(c_{0,0} + t \int_{B_r(0)} \,\mathrm{d}z)}{\int_{B_r(0)} \,\mathrm{d}z} \\
=& se^{i\alpha}(- \frac{c_{0,0}}{\int_{B_r(0)} \,\mathrm{d}z} - t) \\
=& se^{i\alpha}(t^c - t), \\
s'^c =& |c'_{1,0}|^{-1} \\
=& |se^{2i\alpha} c_{1,0}|^{-1} \\
=& s^{-1} |c_{1,0}|^{-1} \\
=& \frac{s^c}{s}, \\
\alpha'^c =& - \frac{1}{2} \arg(c'_{1,0}) \\
=& - \frac{1}{2} \arg(se^{2i\alpha} c_{1,0}) \\
=& - \frac{1}{2} \arg(e^{2i\alpha}) - \frac{1}{2} \arg(c_{1,0}) \\
=& \alpha^c - \alpha.
\end{aligned}
\tag{C.9}
$$

Insertion into (C.7) leads to

$$
\begin{aligned}
f'^c(z) =& s'^c se^{i(\alpha'^c + \alpha)} \Big(f(e^{-i(\alpha + \alpha'^c)} z) + t + \frac{t'^c}{se^{i\alpha}} \Big) \\
=& \frac{s^c}{s} se^{i(\alpha^c - \alpha + \alpha)} \Big(f(e^{-i(\alpha + \alpha^c - \alpha)} z) + t + \frac{se^{i\alpha}(t^c - t)}{se^{i\alpha}} \Big) \\
=& s^c e^{i\alpha^c} (f(e^{-i\alpha^c} z) + t^c) \\
=& f^c(z), \\
\chi_{B_r(0)'^c}(z) =& \chi_{B_r(0)^c}(z)
\end{aligned}
\tag{C.10}
$$

and therefore

$$
g'^c(z) = g^c(z), \tag{C.11}
$$

which shows that the normalized function is invariant to outer translation and scaling and total rotation and therefore proves Corollary 3.