Flexible Moment Invariant Bases for 2D Scalar and Vector Fields

Roxana Bujack
Los Alamos National Laboratory
P.O. Box 1663
USA, 87544 Los Alamos, NM
bujack@lanl.gov

Jan Flusser
Institute of Information Theory and Automation
Pod Vodarenskou vezi 4
Czech Republic, 182 08 Praha 8
flusser@utia.cas.cz

ABSTRACT

Complex moments have been successfully applied to pattern detection tasks in two-dimensional real, complex, and vector valued functions.

In this paper, we review the different bases of rotational moment invariants based on the generator approach with complex monomials. We analyze their properties with respect to independence, completeness, and existence and present superior bases that are optimal with respect to all three criteria for both scalar and vector fields.

Keywords

Pattern detection, moment invariants, scalar fields, vector fields, flow fields, generator, basis, complex, monomial

1 INTRODUCTION

Pattern detection is an important tool for the generation of expressive scientific visualizations. The datasets, scientists are studying get bigger and bigger, the bandwidth of the human visual channel remains constant. Pattern detection algorithms allow to reduce the abundance of information to features, i.e. the areas in the domain, the scientist is actually interested in.

Physical phenomena expressed in coordinates usually come with some degrees of freedom. Especially the underlying feature is present no matter how it is oriented. Also the exact position or the scale in which a pattern occurs should not change whether or not it is detected. Using pattern detection algorithms that are independent with respect to these coordinate transformations can therefore gravely accelerate the process.

A very popular and successful class of such algorithms is based on moment invariants. These are characteristic descriptors of functions that do not change under certain transformations. They can be constructed from moments in two different ways, namely the generator approach and normalization. Moments are the projections of a function onto a function space basis.

During the normalization, certain moments are put into a predefined standard position. Then, the remaining moments are automatically invariant with respect to this transformation. In contrast to that, the generator approach uses algebraic relations to explicitly define a set of moment invariants that are constructed from the moments through addition, multiplication, or other arithmetic operations.

Each approach comes with its own advantages and disadvantages. Depending on the application, one may be superior to the other. In this paper, we will concentrate on the generator approach. We will recap the so far suggested generators for two-dimensional scalar and vector fields, demonstrate their differences and shortcomings, and present a flexible basis, which is able to overcome them. For a more detailed introduction to moment invariants, we recommend [1].

A set of moment invariants should have the following three important qualities:

Completeness: The set is complete if any arbitrary moment invariant can be constructed from it.

Independence.: The set is independent if none of its elements can be constructed from its other elements.

Existence: The set is existent, in other words flexible, if it is generally defined[1] without requiring any specific moments[2] to be non zero.

Completeness ensures that the set has the power to discriminate two objects that differ by something other than only a rotation. Independence accelerates the de-

[1] We use the arithmetic meaning of defined. For example, the operation $1/x$ is defined for $x \neq 0$ and undefined if $x = 0$.

[2] As a counter example, the so far suggested basis for real valued functions requires at least one moment to be non zero that suffices $p_0 - q_0 = 1$.

tection, because it prevents us from comparing redundant values. Finally, existence guarantees that the set can detect any pattern and does not have restrictions to its specific form, like having a non vanishing linear component.

In the real-valued case, a complete and independent set of moment invariants was proposed by Flusser in [2]. We build upon his results to construct a basis that generally exists. Since our basis is flexible, it can be adapted, which makes it robust even if all moments that correspond to rotational non-symmetric complex monomials are close to zero. Further, it is automatically suitable for the detection of symmetric patterns without prior knowledge of the specific symmetry.

In the vector field case, generators have been proposed by Schlemmer [3] and Flusser et al. [1]. We proof that the generator suggested by Schlemmer is neither complete nor independent, show that the basis by Flusser satisfies these desirable properties, and further introduce a novel basis, which exceeds all so far suggested ones. Like in the real-valued case, our suggested basis is independent, complete, solves the inverse problem, and generally exists.

2 RELATED WORK

In 1962 Moment invariants were introduced to the image processing society by Hu [4]. He used a set of seven rotation invariants.

The use of complex moments has been advocated by Teague [5] and Mostafa and Psaltis [6]. It particularly simplifies the construction of rotation invariants, because in this setting, rotations take the simple form of products with complex exponentials.

In 2000 Flusser [2] presented a calculation rule to compute a complete and independent basis of moment invariants of arbitrary order for 2D scalar functions. He also showed that the invariants by Hu [4] are not independent and that his basis solves the inverse problem [7]. Building on Flusser's work, Schlemmer et al. [8, 9] provided five invariants for vector fields in 2007. Later, in his thesis, Schlemmer also provided a general rule for moments of arbitrary order [3].

Apart from the use of complex numbers, moment tensors are the other popular framework for the construction of moment invariants. They were suggested by Dirilten and Newman in 1977 [10]. The principal idea is that tensor contractions to zeroth order are naturally invariant with respect to rotation. It is more difficult to answer questions of completeness or independence in the tensor setting [11], but in contrast to the complex appproach, it generalizes more easily to three-dimensional functions. Pinjo et al. [12], for example, estimated 3D orientations from the contractions to first order, which behave like vectors. Another path that has been successfully taken uses spherical harmonics [13, 14, 15, 16] and their irreducible representation of the rotation group.

A generalization of the tensor approach to vector fields was suggetsed by Langbein and Hagen [17].

In contrast to the derivation of explicit calculation rules that generate invariants, normalization can be used. A description of normalization for scalar fields can be found in [1]. Bujack et al. followed the normalization approach to construct moment invariants for two-dimensional [18] and three-dimensional [19] vector fields. Another publication that is worth mentioning in this context is [20]. Even though Liu and Ribeiro do not call it moment normalization, they follow a very similar approach.

For a more detailed introduction to the theory of moment invariants we recommend [1]. An overview on feature-based flow visualization can be found in [21].

3 REAL-VALUED FUNCTIONS

Two-dimensional real valued functions $\mathbb{R}^2 \to \mathbb{R}$ are often embedded into the complex plane $\mathbb{C} \sim \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{C}$ to make use of the easy representation of rotations in the setting of complex numbers. We briefly revisit the foundation of moment invariant bases of complex monomials. A more detailed introduction can be found in [1].

For a function $f : \mathbb{C} \to \mathbb{C}$ and $p, q \in \mathbb{N}$, the complex moments $c_{p,q}$ are defined by

$$c_{p,q} = \int_{\mathbb{C}} z^p \bar{z}^q f(z)\, \mathrm{d}z. \tag{3.1}$$

Let $f'(z) : \mathbb{C} \to \mathbb{C}$ differ from f by an inner rotation by the angle $\alpha \in (-\pi, \pi]$

$$f'(z) = f(e^{-i\alpha} z), \tag{3.2}$$

then, the moments $c'_{p,q}$ of f' are related to the moments of f by

$$c'_{p,q} = e^{i\alpha(p-q)} c_{p,q}. \tag{3.3}$$

Starting with (3.3), Flusser [2] shows that a rotational invariant can be constructed by choosing $n \in \mathbb{N}$ and for $i = 1, ..., n$ non negative integers $k_i, p_i, q_i \in \mathbb{N}_0$. If they satisfy

$$\sum_{i=1}^{n} k_i (p_i - q_i) = 0, \tag{3.4}$$

then, the expression

$$I = \prod_{i=1}^{n} c_{p_i,q_i}^{k_i} \tag{3.5}$$

is invariant with respect to rotation. From this formula infinitely many rotation invariants can be generated, but most of them are redundant. In order to minimize redundancy Flusser constructs a basis of independent invariants. The following definitions and the theorem stem from [2].

Definition 3.1. An invariant J of the shape (3.5) is considered to be dependent on a set $I_1,...,I_k$ if there is a function F containing the operations multiplication, involution with an integer exponent and complex conjugation, such that $J = F(I_1,...,I_k)$.

Definition 3.2. A basis of a set of rotation invariants is an independent subset such that any other element depends on this subset.

3.1 Flusser's Basis

The following basis was suggested by Flusser in [2], where the proof of the theorem can be found.

Theorem 3.3. *Cited from [2]. Let M be a set of complex moments of a real-valued function, $\bar{M}$ the set of their complex conjugates and $c_{p_0,q_0} \in M \cup \bar{M}$ such that $p_0 - q_0 = 1$ and $c_{p_0,q_0} \neq 0$. Let $\mathscr{I}$ be the set of all rotation invariants created from the moments of $M \cup \bar{M}$ according to (3.5) and $\mathscr{B}$ be constructed by*

$$\forall p,q, p \geq q \wedge c_{p,q} \in M \cup \bar{M} : \phi(p,q) := c_{p,q} c_{q_0,p_0}^{p-q} \in \mathscr{B}, \tag{3.6}$$

then $\mathscr{B}$ is a basis of $\mathscr{I}$.

This basis satisfies another important property, it solves the inverse problem. That means up to the one degree of freedom, which comes from the rotational invariance, the original moments can be unambiguously reconstructed from the basis [7].

In certain situations, it happens that no non zero moment with $p_0 - q_0 = 1$, which is required for Theorem 3.3, can be found. Then, Flusser's basis is not defined. Actually, it is sufficient for c_{q_0,p_0} to have a value close to zero to make the produced invariants unstable and therefore unusable.

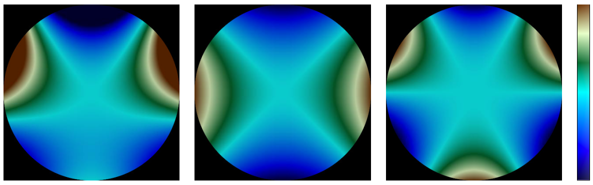
Example 3.4. The function

$$f(x,y) = (-y^3 + 3x^2y + x^2 - y^2)\chi(x^2 + y^2 \leq 1) \tag{3.7}$$

with χ corresponding to the characteristic function, has the complex moments

$$\begin{aligned} c_{2,0} &= \frac{\pi}{6}, & c_{0,2} &= \frac{\pi}{6}, \\ c_{3,0} &= \frac{i\pi}{8}, & c_{0,3} &= -\frac{i\pi}{8}, \\ c_{3,1} &= \frac{\pi}{8}, & c_{1,3} &= \frac{\pi}{8}. \end{aligned} \tag{3.8}$$

All other moments up to fourth order are zero. There is no $p_0 - q_0 = 1$ with $c_{p_0,q_0} \neq 0$. Therefore, the basis from Theorem 3.3 does not exist. Still, it would be possible to construct moment invariants for f, for example, $c_{3,1}c_{0,2} = \frac{\pi^2}{48}$.



Function (3.7) without rotational symmetry. | Its quadratic part with two-fold symmetry. | Its cubic part with three-fold rotational symmetry.

Figure 1: The function (3.7) from Example 3.4 and its components visualized using the height colormap.

It should be noted that the situation of vanishing moments always occurs with symmetric functions. In this case Flusser et al. [22] provide a different basis, which is tailored toward the specific n-fold rotational symmetry, which needs to be known in advance. But, as can be seen in Example 3.4, all moments with $p_0 - q_0 = 1$ can be zero for non-symmetric functions, too.

3.2 Flexible Basis

Motivated by Example 3.4, we suggest the following basis. Since it is adaptive, it exists for any pattern.

Theorem 3.5. *Let $M = \{c_{p,q}, p+q \leq o\}$ be the set of complex moments of an arbitrary real-valued function $f : \mathbb{R}^2 \to \mathbb{R}$ up to a given order $o \in \mathbb{N}$. If there is a $0 \neq c_{p_0,q_0} \in M$ with $p_0 - q_0 < 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \leq o, p \geq q\}$ with*

$$\phi(p,q) := c_{p,q} c_{p_0,q_0}^{-\frac{p-q}{p_0-q_0}}, \tag{3.9}$$

and otherwise by $\mathscr{B} := \{c_{p,p}, p+p \leq o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M, which is generally existent independent of f.

Before proving the theorem, we would like to insert some remarks that may aid understanding of the proof.

Please note that the basis is tailored toward a given function. Different functions, may result in different bases, and a basis that exists for one function may not exist for another function.

In order to maximize stability, we suggest to choose the lowest order moment as c_{p_0,q_0} that has a magnitude that is above average, i.e.

$$|c_{p_0,q_0}| \geq \frac{\sum_{p+q<o} |c_{p,q}|}{\sum_{p+q<o}}. \tag{3.10}$$

The fraction in the exponent of (3.9) corresponds to a root of a complex number, which has $|p_0 - q_0|$ solutions. It is not necessary to store the invariants for all complex roots, but only for a single arbitrary but consistent one. But during the comparison step with the pattern, we need to take the ambiguity into account and compare the arbitrary root of the function to each of the multiple roots of the pattern. We do not need to store the multiple roots of the pattern either, because we can compute the missing ones if we know just one invariant $\phi(p,q)$ and the chosen p_0, q_0 from (3.10) using the rule

$$\phi(p,q)e^{\frac{2i\pi k}{p_0-q_0}} \tag{3.11}$$

for $k = 1, ..., p_0 - q_0$. Please note though that it is crucial, that all elements $\phi(p,q)$ of the set of stored invariants were generated using the same complex root. We show in detail why it is necessary to work with this ambiguity in Subsection 3.3.

Proof. This proof consists of four parts.

Invariance. We can see from (3.5) and (3.4) that the elements $\phi(p,q)$ are rotation invariant, because of $1(p-q) + (p_0 - q_0)(-\frac{p-q}{p_0-q_0}) = 0$. The elements $c_{p,p}$ are naturally invariant with respect to arbitrary rotations, because of (3.2).

Completeness. We will solve the inverse problem. The assertion then follows from the fundamental theorem of moment invariants [23]. Analogously to [7], we can pick one orientation to remove the degree of freedom that comes from the rotation invariance. We assume $c_{p_0,q_0} \in \mathbb{R}^+$. Firstly, since $c_{p_0,q_0} \in \mathbb{R}^+$, it coincides with its absolute value, which can be constructed from $\phi(q_0, p_0)$ via

$$\begin{aligned} c_{p_0,q_0} &= |c_{p_0,q_0}| = \sqrt{\overline{c_{p_0,q_0}} c_{p_0,q_0}} = \sqrt{c_{q_0,p_0} c_{p_0,q_0}} \\ &= \sqrt{c_{q_0,p_0} c_{p_0,q_0}^{-\frac{q_0-p_0}{p_0-q_0}}} = \sqrt{\phi(q_0,p_0)} \end{aligned} \tag{3.12}$$

because real valued functions suffice

$$c_{p,q} = \overline{c_{q,p}}. \tag{3.13}$$

Please note that the invariant $\phi(q_0, p_0)$ is part of the basis, because from the restriction on the normalizer $p_0 - q_0 < 0$ follows the restriction for the elements of the basis $p > q$ with $p = q_0, q = p_0$. Secondly, for all $p > q$, the original moment $c_{p,q}$ can be reconstructed from any of the possibly multiple $\phi(p,q)$ using the calculation rule

$$c_{p,q} = \phi(p,q) c_{p_0,q_0}^{\frac{p-q}{p_0-q_0}}. \tag{3.14}$$

Then, for all $p < q$, the original moments can afterwards be reconstructed from $c_{q,p}$ using the relation (3.13). Finally, for $p = q$, the moments are already part of the basis.

Existence. If all moments with $p_0 - q_0 \neq 0$ are zero, the basis reduces to $\{c_{p,p}, p+p \leq o\}$. It is known from [22] that this is a basis for circular symmetric functions[3]. For all other functions, a non zero non symmetric moment c_{p_0,q_0} with $p_0 - q_0 \neq 0$ can be chosen. If it should suffice $p_0 - q_0 > 0$, then we automatically know from (3.13), that $c_{q_0,p_0} \neq 0$, too. It satisfies the constraint $q_0 - p_0 < 0$ and the basis exists as defined.

Independence. We use the polar representation $c_{p_0,q_0} = re^{i\phi}$ of the normalizer of a function f to construct the new function

$$f'(z) := r^{\frac{1}{p_0-q_0}} f(e^{\frac{i\phi}{p_0-q_0}} z). \tag{3.15}$$

Using (3.2), we see that moments of f' suffice $c'_{p,q} = c_{p,q} c_{p_0,q_0}^{-\frac{p-q}{p_0-q_0}}$ and therefore coincide with the basis elements $\phi(p,q)$ of f. Since the moments of f' are independent, so is the basis. If no normalizer c_{p_0,q_0} can be found, the basis consists solely of moments and is therefore independent, too. □

Example 3.6. The flexible basis exists for the function (3.7) from Example 3.4 and Figure 1. In agreement with (3.10) among the moments up to fourth order, we pick $p_0 = 0, q_0 = 2$. Then, the non zero elements of the basis are

$$\begin{aligned} \phi(2,0) &= c_{2,0} c_{0,2} = \frac{\pi^2}{36}, \\ \phi(3,0) &= c_{3,0} c_{0,2}^{\frac{3}{2}} = \pm \frac{i\pi\sqrt{\pi}^3}{8\sqrt{6}^3}, \\ \phi(3,1) &= c_{3,1} c_{0,2} = \frac{\pi^2}{48}. \end{aligned} \tag{3.16}$$

Pleas note that during the pattern recognition task, the flexible basis that is tailored toward the pattern will be evaluated on the field, where the chosen normalizer c_{p_0,q_0} may vanish. The moment invariants always become unstable if the moment c_{p_0,q_0} is close to zero , which leads to very high values in the invariants. But because of3.10 these areas must be very different from the pattern. So this kind of instability does not influence the result of the pattern matching.

3.3 Multiple Complex Roots

In this subsection, we will show, why the suggested treatment of the multiple complex roots is necessary in order to guarantee independence, invariance, completeness, and existence. It may be skipped on first reading.

Invariance. If, we restrict the basis from Theorem 3.5 to one representative of the possibly multiple complex roots, the resulting set is no longer invariant with respect to rotation. Without loss of generality, let us choose the root with the lowest non negative angle to

[3] We call a function circular symmetric or completely rotationally symmetric if its rotated version coincides with the original function independently from the rotation angle α, that means it suffices $\forall \alpha \in [0, 2\pi) : f(z) = R_\alpha f(z)$. One could say, it is n-fold symmetric with $n = \infty$.

the positive real axis. Then, using function f from (3.7) as in Example 3.6, we would pick $\sqrt{\frac{\pi}{6}}$ as the representative complex root of $c_{0,2} = \frac{\pi}{6}$. The generated set would have the form

$$\begin{aligned} \phi(2,0) &= c_{2,0}c_{0,2} = \frac{\pi^2}{36}, \\ \phi(3,0) &= c_{3,0}c_{0,2}^{\frac{3}{2}} = \frac{i\pi\sqrt{\pi}^3}{8\sqrt{6}^3}, \\ \phi(3,1) &= c_{3,1}c_{0,2} = \frac{\pi^2}{48}. \end{aligned} \tag{3.17}$$

Let f' be f if we rotate it by π, then the moments of

$$f'(x,y) = (y^3 - 3x^2y + x^2 - y^2)\chi(x^2 + y^2 \leq 1) \tag{3.18}$$

suffice are the same as in (3.8), only the ones of odd order in the middle row change their sign. As a result, the chosen representative root of $c_{0,2}$ is still $\sqrt{\frac{\pi}{6}}$, and the new generated set would differ from (3.17), because $\phi(3,0) = c_{3,0}c_{0,2}^{\frac{3}{2}} = -\frac{i\pi\sqrt{\pi}^3}{8\sqrt{6}^3}$ has opposite sign.

Completeness. In many applications, the full discriminative power of a complete basis is not necessarily required. In these cases, we can replace $\phi(p,q)$ from Theorem 3.5 by the easier formula

$$\phi'(p,q) := c_{p,q}^{p_0-q_0} c_{p_0,q_0}^{-(p-q)}. \tag{3.19}$$

The resulting generator $\mathscr{B}$ can be used instead of the basis from Theorem 3.5. It has only one unique element for each p,q because it does not contain complex roots. But please note that this set is not generally complete. To proof that, we revisit the function from Example 3.6 with moments calculated up to fourth order. If we use the basis from (3.9), the invariant $c_{3,1}c_{0,2} = \frac{\pi^2}{48}$ is part of the basis and can therefore be constructed from the basis trivially.

But if we use $\phi'(p,q)$ from (3.19), we get the generator

$$\begin{aligned} \phi'(2,0) &= c_{2,0}^2 c_{0,2}^2 = \frac{\pi^4}{6^4}, \\ \phi'(3,0) &= c_{3,0}^2 c_{0,2}^3 = -\frac{\pi^5}{8^2 6^3}, \\ \phi'(3,1) &= c_{3,1}^2 c_{0,2}^2 = \frac{\pi^4}{8^2 6^2}, \end{aligned} \tag{3.20}$$

from which $c_{3,1}c_{0,2}$ can not be constructed. We can only use $\phi'(3,1) = (c_{3,1}c_{0,2})^2$, which does not contain the more detailed information that $c_{3,1}c_{0,2} = \frac{\pi^2}{48}$ was actually positive.

As an example, the function

$$\begin{aligned} g(x,y) = &(31(x^2 - y^2) - 40(x^4 - y^4) - y^3 + 3x^2y) \\ &\chi(x^2 + y^2 \leq 1) \end{aligned} \tag{3.21}$$

shown in Figure 2 has the moments

$$\begin{aligned} c_{2,0} &= \frac{\pi}{6}, & c_{0,2} &= \frac{\pi}{6}, \\ c_{3,0} &= -\frac{i\pi}{8}, & c_{0,3} &= \frac{i\pi}{8}, \\ c_{3,1} &= -\frac{\pi}{8}, & c_{1,3} &= -\frac{\pi}{8}. \end{aligned} \tag{3.22}$$

The basis from Theorem 3.5 clearly shows the difference between g and f, because here $\phi_g(3,1) = c_{3,1}c_{0,2} = -\frac{\pi^2}{48}$ has opposite sign than $\phi_f(3,1) = \frac{\pi^2}{48}$ in (3.16). In contrast to that, the generator defined in (3.19) assumes the exact same values $\phi'_g(3,1) = c_{3,1}^2 c_{0,2}^2 = \frac{\pi^4}{8^2 6^2} = \phi'_f(3,1)$ for g as for f, compare (3.20).

Existence. If we restrict ourselves to moments that have no symmetry with respect to rotation whatsoever, i.e. $p_0 - q_0 = 1$, then we have no complex roots and get one unique solution for each p,q. In this case, the basis reduces to the one suggested by Flusser and it may not exist even for non-symmetric functions as could already be seen in Example 3.4.

Independence. Considering the multiplcity of the complex roots does not violate the independence if we interpret them in the following way. The multiple roots of an invariant are not independent invariants themselves, but merely manifestations of the same invariant. We do not have to store them separately, because we can construct all roots from one representative using formula (3.11).

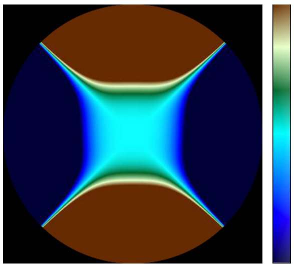


Figure 2: The function $g(x,y)$ from (3.21) visualized using the height color map. The generator (3.19) produces the same invariants as for $f(x,y)$ from Figure 1, even though they are clearly different.

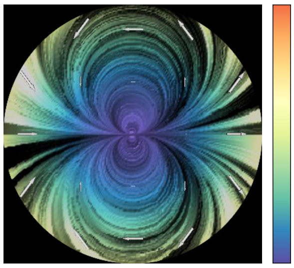


Figure 3: Arrow glyphs and line integral convolution (LIC) [24] of the function (5.8) from Example 5.2. Color and size of the arrows represent the speed. The generator (5.5) does not exist for this pattern.

4 COMPLEX FUNCTIONS

The bases from the previous section were tailored towards real valued functions. Since they satisfy $c_{p,q} = \overline{c_{q,p}}$, it was sufficient to only include $\phi(p,q)$ for $p > q$. Analogously to Theorem 3.5, a flexible basis for arbitrary complex functions that behave under rotations as given in (3.3) can be constructed using the following theorem.

Theorem 4.1. *Let $M = \{c_{p,q}, p+q \leq o\}$ be the set of complex moments of a complex function up to a given order $o \in \mathbb{N}$. If there is a $0 \neq c_{p_0,q_0} \in M$ with $p_0 - q_0 \neq 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \leq o\} \setminus \{\phi(p_0,q_0)\} \cup \{|c_{p_0,q_0}|\}$ with*

$$\phi(p,q) := c_{p,q} c_{p_0,q_0}^{-\frac{p-q}{p_0-q_0}}, \tag{4.1}$$

and otherwise by $\mathscr{B} := \{c_{p,p}, p+p \leq o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M that exists for any arbitrary complex function.

Proof. The proof works analogously to the proof of Theorem 3.5. □

5 FLOW FIELDS

We can interpret a complex function $f : \mathbb{C} \to \mathbb{C}$ as a two-dimensional vector field by means of the isomorphism between the complex and the Euclidean plane. Analogously to scalar functions, we can make use of the complex moments $c_{p,q}$ as defined in (3.1).

In contrast to the scalar case, flow fields transform by a total rotation. Therefore, we assume that $f'(z) : \mathbb{C} \to \mathbb{C}$ suffices

$$f'(z) = e^{i\alpha} f(e^{-i\alpha} z). \tag{5.1}$$

In this case, the moments $c'_{p,q}$ of f' are related to the moments of f by

$$c'_{p,q} = e^{i\alpha(p-q+1)} c_{p,q}. \tag{5.2}$$

A proof can, for example be found in [18].

Schlemmer and Heringer [8] showed that analogously to (3.5), any expression of the shape

$$I = \prod_{i=1}^{n} c_{p_i,q_i}^{k_i} \tag{5.3}$$

with $n \in \mathbb{N}$ and for $i = 1,...,n : k_i, p_i, q_i \in \mathbb{N}_0$ is invariant to total rotation, if

$$\sum_{i=1}^{n} k_i(p_i - q_i + 1) = 0, \tag{5.4}$$

because of (5.2).

5.1 Schlemmer's Generator

The first moment invariants for vector fields were suggested by Schlemmer et al. in 2007 [8]. In that paper, instead of presenting a rule for the generation of moment invariants of arbitrary order, a set of five invariants was explicitly stated. Two years later in his thesis [3], Schlemmer provided the general formula with which invariants of arbitrary order can be produced. The five moments from [8] are exactly the invariants that are produced from this formula if the maximal order of the moments is restricted to two. We therefore assume that Schlemmer at al. already used this formula in their 2007 paper [8], but did not explicitly write it down.

Theorem 5.1. *Cited from [3]. Let M be the set or a subset of all complex moments $c_{p,q}$ of order $(p+q) \in \{0,...,o\}$, $o \geq 2$. Let $\mathscr{I}$ be the set of all moment invariants being constructed according to (3.5) from the elements of M. Let $c_{\dot{p},\dot{q}}$ and $c_{\ddot{p},\ddot{q}} \in M$, with $\dot{p} - \dot{q} = \ddot{q} - \ddot{p} = 2$ and $c_{\dot{p},\dot{q}}$ as well as $c_{\ddot{p},\ddot{q}} \neq 0$ If the set $\mathscr{B}$ is constructed as follows:*

$$\mathscr{B} = \{\phi(p,q) := c_{p,q} c_{\dot{p},\dot{q}}^{a_{p-q}} c_{\ddot{p},\ddot{q}}^{b_{p-q}}, c_{p,q} \in M\}, \tag{5.5}$$

with

$$a_m = \begin{cases} 0, & \text{if } m \geq -1 \\ ((|m|+1) \operatorname{div} 3, & \text{if } m \leq -2 \end{cases} \tag{5.6}$$

and

$$b_m = \begin{cases} m+1, & \text{if } m \geq -1 \\ (m+1) \bmod 3, & \text{if } m \leq -2 \end{cases} \tag{5.7}$$

then $\mathscr{B}$ is a basis of $\mathscr{I}$.

This theorem in fact happens to be incorrect. Schlemmer's generator is neither independent nor complete and therefore no basis in the sense of Definition 3.2.

Completeness. This generator is not complete because the magnitudes $|c_{\dot{p},\dot{q}}|$ and $|c_{\ddot{p},\ddot{q}}|$ can not be reconstructed from its elements.

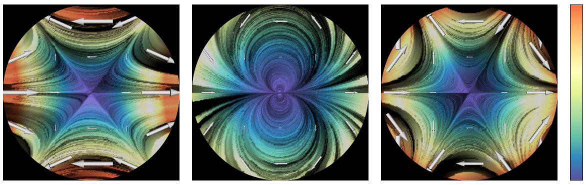
Independence. This generator is not independent because the invariant $\phi(\dot{p},\dot{q})$ and $\phi(\ddot{p},\ddot{q})$ are identical.

The generator can be transformed into a basis via $\mathscr{B} \setminus \{\phi(\dot{p},\dot{q})\} \cup \{|c_{\dot{p},\dot{q}}|\}$. But even with this correction, the basis is not chosen very well. For once, it is unnecessarily complicated, because it requires evaluation of the two auxiliary functions (5.6) and (5.7) and each element can consist of up to thee factors. Further, it does not exist for functions that do not have non zero $c_{\dot{p},\dot{q}} \neq 0$ as well as $c_{\ddot{p},\ddot{q}} \neq 0$ with $\dot{p} - \dot{q} = \ddot{q} - \ddot{p} = 2$. This situation is similar to the one in Subsection 3.1. But in this case even two non vanishing moments of specific orders need to be present, which increases the number of cases in which the generator does not exist.

Example 5.2. The vector field given by the function

$$f(z) = z^2 \chi(|z| \leq 1) \tag{5.8}$$

has only one non-zero moment up to third order $c_{0,2} = \frac{\pi}{3}$. It is visualized in Figure 3. Even though it is not symmetric, Schlemmer's generator does not exist, because $c_{\dot{p},\dot{q}} \neq 0$ can not be found to suffice $\dot{p} - \dot{q} = 2$.



The function has no symmetry. | Its two-fold symmetric part. | Its three-fold symmetric part.

Figure 4: Arrow glyphs and LIC of the function (5.9) from Example 5.3 and its components. The color and the size of the arrows represent the speed of the flow.

Example 5.3. The vector field given by the function

$$f(z) = (z^2 + 2\bar{z}^2)\chi(|z| \leq 1), \tag{5.9}$$

with χ being the characteristic function, is visualized in Figure 4. It has two non-zero moments up to third order

$$c_{0,2} = \frac{\pi}{3}, \quad c_{2,0} = \frac{2\pi}{3}. \tag{5.10}$$

Here, Schlemmer's generator does exist, because we can choose $c_{\dot{p},\dot{q}} = c_{2,0}$ and $c_{\ddot{p},\ddot{q}} = c_{0,2}$, but it contains only the redundant information

$$\begin{aligned} \phi(0,2) &= c_{0,2}c_{2,0}^{a_{-2}}c_{0,2}^{b_{-2}} = c_{0,2}c_{2,0}^{1}c_{0,2}^{2} = 2(\frac{\pi}{3})^4, \\ \phi(2,0) &= c_{2,0}c_{2,0}^{a_2}c_{0,2}^{b_2} = c_{2,0}c_{2,0}^{0}c_{0,2}^{3} = 2(\frac{\pi}{3})^4, \end{aligned} \tag{5.11}$$

from which we can not reconstruct the magnitudes of the moments.

5.2 Flusser et al.'s Basis

A straight forward approach to generate a basis of moment invariants for vector fields was suggested by Flusser et al. in [1].

Theorem 5.4. *Let M be the set of moments up to the order $o \in \mathbb{N}$ and $c_{p_0,q_0} \neq 0$ satisfying $p_0 - q_0 = -2$. Further let $\mathscr{I}$ be the set of all rotation invariants created from the moments of M according to (5.3) and $\mathscr{B}$ be constructed by*

$$\forall p,q,p+q \leq o : \phi(p,q) := c_{p,q}c_{p_0,q_0}^{(p-q+1)} \in \mathscr{B}, \tag{5.12}$$

then $\mathscr{B} \setminus \{\phi(p_0,q_0)\} \cup \{|\phi(p_0,q_0)|\}$ is a basis of $\mathscr{I}$.

That does not only produce an independent and complete set, but is also more flexible than Schlemmer's generator because it does not need two specific non-zero moments, but only a single one. Further, it is simpler and more intuitive because it does not need any additional series, like (5.6) and (5.7).

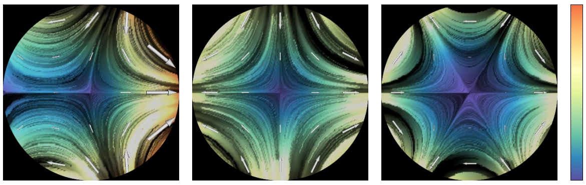
Example 5.5. Flusser's basis exists for the vector field given by the function (5.8) from Example 5.2 visualized in Figure 3. It has one non zero element $|c_{0,2}| = 2\frac{\pi}{3}$.

Example 5.6. Flusser's basis exists for the vector field given by the function (5.9) from Example 5.3 visualized in Figure 4 and up to one degree of freedom, the moments can be reconstructed from the basis

$$\begin{aligned} |c_{0,2}| &= \frac{2\pi}{3}, \\ \phi(2,0) &= c_{2,0}c_{0,2}^3 = 8(\frac{\pi}{3})^4. \end{aligned} \tag{5.13}$$

To show that, we fix the rotational degree of freedom by setting $c_{0,2} \in \mathbb{R}^+$ and get

$$\begin{aligned} c_{0,2} &= |c_{0,2}| = 2\frac{\pi}{3}, \\ c_{2,0} &= \phi(2,0)c_{0,2}^{-3} = \frac{\pi}{3}. \end{aligned} \tag{5.14}$$



The function has no rotational symmetry. | Its linear part with two-fold symmetry. | Its quadratic part with three-fold symmetry.

Figure 5: Arrow glyphs and LIC of the function (5.15) from Example 5.7. The color and the size of the arrows represent the speed of the flow.

Example 5.7. The vector field given by the function

$$f(z) = (\bar{z} + \bar{z}^2)\chi(|z| \leq 1) \tag{5.15}$$

has three non-zero moments up to third order

$$c_{1,0} = \frac{\pi}{2}, \quad c_{2,0} = \frac{\pi}{3}, \quad c_{2,1} = \frac{\pi}{4} \tag{5.16}$$

It is visualized in Figure 5. Here, Flusser's basis does not exist, because we can not find any $c_{p_0,q_0} \neq 0$ with $p_0 - q_0 = -2$, even though, the function is not symmetric.

5.3 Flexible Basis

Analogously to the scalar case, we can also derive a robust basis even for patterns that do not have a numerically significant moment of one-fold symmetry.

Theorem 5.8. *Let $M = \{c_{p,q}, p+q \leq o\}$ be the set of complex moments of a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ up to a given order $o \in \mathbb{N}$. If there is a $0 \neq c_{p_0,q_0} \in M$ with $p_0 - q_0 + 1 \neq 0$, we define the set $\mathscr{B}$ by $\mathscr{B} := \{\phi(p,q), p+q \leq o\} \setminus \{\phi(p_0,q_0)\} \cup \{|c_{p_0,q_0}|\}$ with*

$$\phi(p,q) := \phi(p,q) := c_{p,q}c_{p_0,q_0}^{-\frac{p-q+1}{p_0-q_0+1}}, \tag{5.17}$$

and otherwise by $\mathscr{B} := \{c_{p,p+1}, p+p+1 \leq o\}$. Then $\mathscr{B}$ is a basis of all rotation invariants of M, which generally exists independent of f.

Proof. The proof works analogously to the proof of Theorem 3.5. □

Remark 5.9. This last basis of invariants is equivalent to the normalization approach proposed by Bujack et al. [25].

Example 5.10. The flexible basis exists for the vector field (5.15) from Example 5.7 visualized in Figure 5. Any of the three non-zero moments up to third order

$$c_{1,0} = \frac{\pi}{2}, \quad c_{2,0} = \frac{\pi}{3}, \quad c_{2,1} = \frac{\pi}{4} \tag{5.18}$$

can be chosen as normalizer c_{p_0,q_0}. In order to maximize stability, the suggested algorithm would choose $c_{p_0,q_0} = c_{1,0}$, which would result in two solutions of the complex square root $c_{1,0}^{-1/2} = \pm\sqrt{\frac{\pi}{2}}$ and the basis

$$\begin{aligned} |c_{1,0}| &= \frac{\pi}{2}, \\ \phi(2,0) &= c_{2,0}c_{1,0}^{-\frac{1}{2}} = \pm\frac{\sqrt{2\pi}}{3}, \\ \phi(2,1) &= c_{2,1}c_{1,0}^{-1} = \frac{1}{2}. \end{aligned} \tag{5.19}$$

6 EXPERIMENT

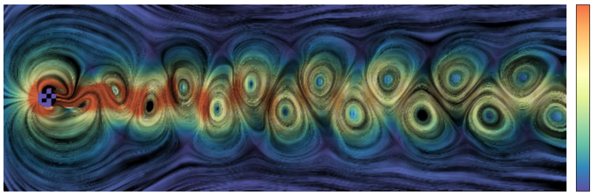
We apply the different vector field bases to a pattern detection task in a vector field. The used dataset is the result of a computational fluid dynamics simulation of the flow behind a cylinder. The characteristic pattern of the fluid is called the von Kármán vortex street. A visualization of the vortices with removed average flow can be found in Figure 6a. The direction of the flow is visualizaed using line integral convolution [24] and the speed is color coded using the colormap from Figure 7.

In our experiments, we consider moments up to first order in Figure 6 and moments up to second order in Figure 8. Please note that the basis suggested by Schlemmer [3] from Theorem 5.1 and the one suggested by Flusser [1] from Theorem 3.3 do not exist for moments calculated only up to first order, because a moment c_{p_0,q_0} with $p_0 - q_0 = -2$ can not be found using only $c_{0,0}, c_{1,0}$, and $c_{0,1}$. For moments up to second order, there is only one potential moment $c_{p_0,q_0} = c_{0,2}$ satisfying $p_0 - q_0 = -2$, which is why there is only one basis configuration for these two approaches. They coincide for the moments up to second order, except for the magnitude of the normalizer $|c_{0,2}|$. The remaining moment invariants are

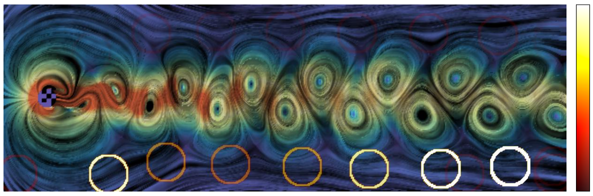
$$c_{0,0}c_{0,2}, \quad c_{0,1}, \quad c_{1,0}c_{0,2}^2, \quad c_{1,1}c_{0,2}, \quad c_{2,0}c_{0,2}^3, \tag{6.1}$$

as already presented in [8].

The as long as the normalizer $c_{0,2}$ is numerically non zero, all three bases will produce stable and identical results up to minor numerical differences. To show the



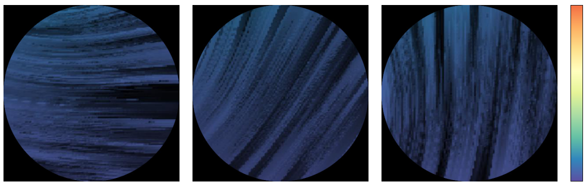
The non flexible bases do not exist for moments up to first order. The algorithm does not produce any output.



The flexible basis does exist with normalizer $c_{1,0}$. The pattern from Figure 7 and its repetitions are correctly detected.

Figure 6: Result of the pattern detetction task using only moments up to first order. The speed of the flow is encoded using the colorbar on the top, the similarity of the field to the pattern using the colorbar on the bottom.

difference between the flexible and non-flexible bases, we therefore use the pattern from Figure 7a, which satisfies $|c_{0,2}| < 0.01$. It was cut out from the dataset itself. Its position in the von Kármán vortex street can be found in the lower, rightmost circle of Figure 6b. Since the only element, which differs in the two non-flexible bases is close to zero, the results of the two are almost identical. The differences are numerically small and can not be perceived by the human eye. To save space, we plot only the instance that corresponds to Schlemmer's basis. The other one is identical.

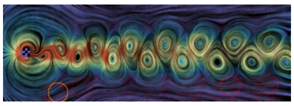


(a) Pattern cut out from the dataset. (b) The pattern rotated by $\pi/3$. (c) The pattern rotated by $\pi/2$.

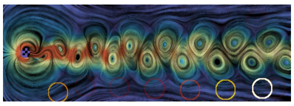
Figure 7: The pattern in different orientations. It was cut out from the dataset at the position of the lower right-most white circle in Figure 6b.

The output of our pattern detection algorithm are circles that indicate the position, the size and the similarity of the matches. The similarity is encoded in the colormap in the bottom row of Figure 6. The higher the similarity, the brighter the color of the corresponding circle. The color white is applied to all matches that have a Euclidean distance of all the moment invariants of less than 0.02. A more detailed description of the algorithm and the visualization can be found in [18].

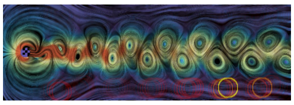
In Figure 6b, we can see that the flexible basis exists even for this pattern and that it correctly finds the pattern's original position. It further detects the pattern's similar occurrences as it repeats itself in the periodic von Kármán street. As expected, the further we move towards the obstacle, the repetitions become less similar in general, which can be seen in the decreasing brightness of the circles.



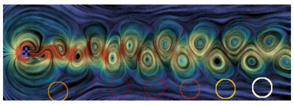
Non flexible bases for the pattern oriented as in Figure 7a.



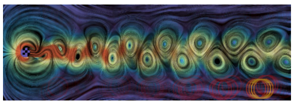
Flexible bases for the pattern oriented as in Figure 7a.



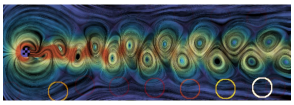
Non flexible bases for the pattern oriented as in Figure 7b.



Flexible bases for the pattern oriented as in Figure 7b.



Non flexible bases for the pattern oriented as in Figure 7c.



Flexible bases for the pattern oriented as in Figure 7c.

Figure 8: Result of the pattern detetction task using moments up to second order. The result of the algorithm using the non flexible bases is unstable (left). It depends on the orientation of input pattern. In contrast to that, the flexible basis produces consistent results (right).

Figure 8 compares the output of the algorithm using the flexible basis from Theorem 5.8 and the two non-flexible bases for moments up to second order. To show the instability of the non-flexible bases, we used three different instances of the pattern. They differ solely by their orientation. Theoretically, the invariants of all three bases should be invariant with respect to this degree of freedom and produce the same results for all three instances. But as can be seen in the left column of Figure 8, this is not true for the non-flexible bases. Depending on the orientation of the pattern, the similarity of the exact location of the pattern in the field is rather low. Sometimes its position is not the match with the highest similarity, or multiple fuzzy matches occur. On the right side, we can see that the flexible basis produces coherent, stable, and correct results independent from the orientation of the pattern.

7 DISCUSSION

We have reviewed the different bases of moment invariants built from complex monomials using the generator approach and compared their behavior with respect to three important qualities such a basis should suffice: independence, completeness and general existence.

For scalar fields, the basis suggested by Flusser [2] is complete and independent, but it does only exist, if the pattern has a non zero moment that is not rotationally symmetric. We have extended his basis to one that always exists, no matter how the values of the moments of a function are distributed.

For vector fields, the first generator approach was suggested by Schlemmer [3]. We show that his set of moment invariants is nor complete neither independent and therefore does not satisfy the properties of a basis. As a result Flusser et al. [1] were the first ones to provide a basis of moment invariants for vector fields using the generator approach. Like in the scalar case, their basis is complete and independent, but requires a non zero moment that has no rotational symmetry. We have derived an extension that exists for arbitrary vector fields and found it to coincide with the normalization approach by Bujack et al. [18]. Some showcase examples and experiments using a fluid dynamics simulation show the superiority of the flexible basis.

One of the most interesting observations in this work is the equivalence of the optimal generator approach with the optimal normalization approach. Assuming that this fact should also be true for three-dimensional fields, it might be used for the study of 3D moment invariants. The 3D situation is much more complicated and neither the generator nor the normalization approach have so far resulted in a set of moment invariants that is complete, independent, and generally existing. Assuming equivalence might guide future research to improve both methods.

In future work, we want to generalize the concept of flexible bases to bases ones built from orthogonal polynomials, like for example, Zernike polynomials or Hermite polynomials.

8 ACKNOWLEDGEMENTS

We would like to thank the FAnToM development group from Leipzig University for providing the environment for the visualization of the presented work, especially Sebastian Volke. Further, we thank Professor Mario Hlawitschka for providing the dataset used in this publication. This work is published under LA-UR-17-20144. It was funded by the National Nuclear Security Administration (NNSA) Advanced Simulation and Computing (ASC) Program and by the Czech Science Foundation under Grant GA15-16928S.

9 REFERENCES

[1] J. Flusser, B. Zitova, and T. Suk. *2D and 3D Image Analysis by Moments*. Wiley, 2016.

[2] Jan Flusser. On the independence of rotation moment invariants. *Pattern Recognition*, 33(9):1405–1410, 2000.

[3] Michael Schlemmer. *Pattern Recognition for Feature Based and Comparative Visualization*. PhD thesis, Universität Kaiserslautern, Germany, 2011.

[4] Ming-Kuei Hu. Visual pattern recognition by moment invariants. *IRE Transactions on Information Theory*, 8(2):179–187, 1962.

[5] Michael Reed Teague. Image analysis via the general theory of moments∗. *Journal of the Optical Society of America*, 70(8):920–930, 1980.

[6] Yaser S. Abu-Mostafa and Demetri Psaltis. Recognitive Aspects of Moment Invariants. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, pages 698–706, 1984.

[7] Jan Flusser. On the inverse problem of rotation moment invariants. *Pattern Recognition*, 35:3015–3017, 2002.

[8] Michael Schlemmer, Manuel Heringer, Florian Morr, Ingrid Hotz, Martin Hering-Bertram, Christoph Garth, Wolfgang Kollmann, Bernd Hamann, and Hans Hagen. Moment Invariants for the Analysis of 2D Flow Fields. *IEEE Transactions on Visualization and Computer Graphics*, 13(6):1743–1750, 2007.

[9] Michael Schlemmer, Ingrid Hotz, Bernd Hamann, and Hans Hagen. Comparative Visualization of Two-Dimensional Flow Data Using Moment Invariants. In *Proceedings of Vision, Modeling, and Visualization (VMV'09)*, volume 1, pages 255–264, 2009.

[10] Hudai Dirilten and Thomas G Newman. Pattern matching under affine transformations. *Computers, IEEE Transactions on*, 100(3):314–317, 1977.

[11] Tomas Suk and Jan Flusser. Tensor Method for Constructing 3D Moment Invariants. In *Computer Analysis of Images and Patterns*, volume 6855 of *Lecture Notes in Computer Science*, pages 212–219. Springer Berlin, Heidelberg, 2011.

[12] Ziha Pinjo, David Cyganski, and John A Orr. Determination of 3-D object orientation from projections. *Pattern Recognition Letters*, 3(5):351–356, 1985.

[13] C.H. Lo and H.S. Don. 3-D Moment Forms: Their Construction and Application to Object Identification and Positioning. *IEEE Trans. Pattern Anal. Mach. Intell.*, 11(10):1053–1064, 1989.

[14] Gilles Burel and Hugues Henocq. 3D Invariants and their Application to Object Recognition. *Signal procesing*, 45(1):1–22, 1995.

[15] Michael Kazhdan, Thomas Funkhouser, and Szymon Rusinkiewicz. Rotation Invariant Spherical Harmonic Representation of 3D Shape Descriptors. In *Symposium on Geometry Processing*, 2003.

[16] Nikolaos Canterakis. Complete moment invariants and pose determination for orthogonal transformations of 3D objects. In *Mustererkennung 1996, 18. DAGM Symposium, Informatik aktuell*, pages 339–350. Springer, 1996.

[17] Max Langbein and Hans Hagen. A generalization of moment invariants on 2d vector fields to tensor fields of arbitrary order and dimension. In *International Symposium on Visual Computing*, pages 1151–1160. Springer, 2009.

[18] Roxana Bujack, Ingrid Hotz, Gerik Scheuermann, and Eckhard Hitzer. Moment Invariants for 2D Flow Fields via Normalization in Detail. *IEEE Transactions on Visualization and Computer Graphics (TVCG)*, 21(8):916–929, Aug 2015.

[19] Roxana Bujack, Jens Kasten, Ingrid Hotz, Gerik Scheuermann, and Eckhard Hitzer. Moment Invariants for 3D Flow Fields via Normalization. In *IEEE Pacific Visualization Symposium, PacificVis 2015 in Hangzhou, China*, 2015.

[20] Wei Liu and Eraldo Ribeiro. Scale and Rotation Invariant Detection of Singular Patterns in Vector Flow Fields. In *IAPR International Workshop on Structural Syntactic Pattern Recognition (S-SSPR)*, pages 522–531, 2010.

[21] Frits H. Post, Benjamin Vrolijk, Helwig Hauser, Robert S. Laramee, and Helmut Doleisch. The State of the Art in Flow Visualisation: Feature Extraction and Tracking. *Computer Graphics Forum*, 22(4):775–792, 2003.

[22] Jan Flusser. Moment Invariants in Image Analysis. In *World Academy of Science, Engineering and Technology 11*, pages 196–201, 2006.

[23] Thomas H. Reiss. The revised fundamental theorem of moment invariants. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(8):830–834, 1991.

[24] Brian Cabral and Leith Casey Leedom. Imaging vector fields using line integral convolution. In *Proceedings of the 20th annual conference on Computer graphics and interactive techniques*, SIGGRAPH '93, pages 263–270. ACM, 1993.

[25] Roxana Bujack, Ingrid Hotz, Gerik Scheuermann, and Eckhard Hitzer. Moment Invariants for 2D Flow Fields Using Normalization. In *IEEE Pacific Visualization Symposium, PacificVis 2014 in Yokohama, Japan*, pages 41–48, 2014.