Choice Logic

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1. A motivating example
2. Syntax and semantics
3. Computation
4. Applications
5. Alternative definitions of entailment
6. Conclusions

Motivation

- For applications like design or configuration intended properties of a particular problem solution must be represented
- Most often not all of these properties can be satisfied.
- One would like to express alternative, second (or third etc.) best options to be satisfied if the best option(s) are unavailable.
- Hotel booking: how to express that we prefer walking distance, if impossible transportation provided by the hotel, if impossible then at least public transportation.
- Propositional logic for representing choices; new logical connective $\times$: $A \times B$ says: if possible A, but if A is impossible then (at least) B.

Example

Preferences:

$walking \times hotel{-}transport \times public{-}transport$

4 hotels available out of which we have to pick one:

$hotel_1 \vee hotel_2 \vee hotel_3 \vee hotel_4$

information about the hotels:

$hotel_1 \rightarrow walking$

$hotel_2 \rightarrow \neg walking \wedge hotel{-}transport$

$hotel_3 \rightarrow \neg walking \wedge \neg hotel{-}transport \wedge public{-}transport$

$hotel_4 \rightarrow \neg walking \wedge \neg hotel{-}transport \wedge \neg public{-}transport$

Conclusion: $hotel_1$

we find out that $hotel_1$ is fully booked: $\neg hotel_1$

Conclusion: $hotel_2$

Syntax of choice logic

$$A_1 \times \ldots \times A_n$$

means: some A_j must be true, preferably A_1, but if this is not possible then A_2, if this is not possible A_3, etc.

Definition 1. Let A be a set of atoms. The set of well-formed formulas of choice logic is inductively defined as follows

1. every element of A is a well-formed formula,
2. if F_1 and F_2 are well-formed formulas, then $(\neg F_1)$, $(F_1 \vee F_2)$, $(F_1 \wedge F_2)$ and $(F_1 \times F_2)$ are well-formed formulas.

Semantics of choice logic

based on the degree of satisfaction of a formula in a (classical) model. Degree a measure of disappointment.

A and B atoms. If I contains A then $I \models_1 A \times B$. If I does not contain A but B, then $I \models_2 A \times B$.

Definition 2. The optionality of a choice logic formula is defined as follows:

$opt(A) = 1$ if A is an atom
$opt(\neg F) = 1$
$opt(F_1 \vee F_2) = max(opt(F_1), opt(F_2))$
$opt(F_1 \wedge F_2) = max(opt(F_1), opt(F_2))$
$opt(F_1 \times F_2) = opt(F_1) + opt(F_2)$

Intuitively, if the optionality of F is n, then there is a best way, a second best way, etc. and an n-th best way of satisfying F.

Satisfiability relation

Definition 3.

1. $I \models_1 A$ iff $A \in I$ (for propositional atoms A)
2. $I \models_k P \wedge Q$ iff $I \models_m P$ and $I \models_n Q$ and $k = max(m, n)$
3. $I \models_k P \vee Q$ iff
   $[I \models_k P$ and for no $j < k : I \models_j Q]$ or
   $[I \models_k Q$ and for no $j < k : I \models_j Q]$
4. $I \models_1 \neg P$ iff for no m: $I \models_m P$
5. $I \models_k P \times Q$ iff
   $I \models_k P$ or
   $I \models_1 \neg P$, $I \models_m Q$, and $k = m + opt(P)$.

Example

Let $F = A \vee (B \times C)$, A, B and C atoms.
Interpretations I containing A or containing B satisfy F with degree 1, that is, we have $I \models_1 F$.
Interpretations I which contain C but make A and B false satisfy F with degree 2, that is $I \models_2 F$.

Lemma 1. *$I \models_k F$ implies $k \leq opt(F)$.*

Proposition 1. *Let F be a choice logic formula, I an interpretation. There is a k such that $I \models_k F$ iff $I \models F^*$ where F^* is obtained from F by replacing each occurrence of $\times$ with standard disjunction ($\models$ is classical propositional consequence).*

Preferred models

Definition 4. Let T be a set of choice logic formulas. An interpretation I is a model of T if it satisfies each formula in T to some degree.

Degrees determine preferred models via lexicographical ordering:

Definition 5. Let T be a set of choice logic formulas and let $M^k(T)$ denote the subset of formulas of T satisfied by a model M to degree k.
A model M_1 is T-preferred over a model M_2 if there is a k such that $|M_1^k(T)| > |M_2^k(T)|$ and for all $j < k$: $|M_1^j(T)| = |M_2^j(T)|$.
M is a preferred model of T iff M is a maximally T-preferred model.

Consequence

Definition 6. Let T be a set of choice logic formulas, A a classical formula. $T \mid\!\sim A$ iff A is satisfied in all preferred models of T.

obviously nonmonotonic: consider $T = \{A \times B\}$. We have three models $\{A\}, \{A, B\}, \{B\}$ with satisfaction degree 1, 1, 2, respectively. This means that $\{A\}$ and $\{A, B\}$ are maximally preferred and $A \times B \mid\!\sim A$. If we add $\neg A$ the single (preferred) model is $\{B\}$. We thus obtain $\{A \times B, \neg A\} \mid\!\sim \neg A$. A is no longer a conclusion.

Proposition 2. *$\mid\!\sim$ satisfies cautious monotony, that is, $T \mid\!\sim A$ and $T \mid\!\sim B$ implies $T \cup \{A\} \mid\!\sim B$.*

Proposition 3. *$\mid\!\sim$ satisfies cumulative transitivity, that is, $T \mid\!\sim A$ and $T \cup \{A\} \mid\!\sim B$ implies $T \mid\!\sim B$.*

Computation

Idea: take a set of arbitrary choice logic formulas T and

1. translate T to $Norm(T)$, a strongly equivalent normal form where all formulas containing $\times$ are basic choice formulas
2. construct stratified knowledge base $Skb(Norm(T))$ such that F is lexicographically entailed by $Skb(Norm(T))$ iff $Norm(T) \mid\sim F$
3. use the techniques developed earlier to generate $Extract(Skb(Norm(T)))$, a classical propositional knowledge base whose consequences are exactly the formulas lexicographically entailed by $Skb(Norm(T))$.

Proposition 4. *$T \mid\sim A$ iff $Extract(Skb(Norm(T))) \vdash A$.*

Normal form

Definition 7. F_1 is equivalent to F_2, denoted $F_1 = F_2$, iff for all interpretations I and integers k we have $I \models_k F_1$ iff $I \models_k F_2$.

Definition 8. F_1 is strongly equivalent to F_2, denoted $F_1 \doteq F_2$, iff F_1 is equivalent to F_2 and $opt(F_1) = opt(F_2)$.

Proposition 5. *Ordered disjunction is associative:* $((F_1 \times F_2) \times F_3) \doteq (F_1 \times (F_2 \times F_3))$.

Definition 9. F is a basic choice formula if it is of the form $(A_1 \times \ldots \times A_n)$ where the A_i are classical and $n > 1$.

Strong equivalences

Proposition 6. *Let $A_i, B_j, C, \ldots$ be classical.*

1. $(A_1 \times \ldots \times A_n) \wedge (B_1 \times \ldots \times B_m) \doteq (C_1 \times \ldots \times C_k)$
   where $k = max(m, n)$ *and* $C_i =$
   $[(A_1 \vee \ldots \vee A_i) \wedge B_i] \vee [A_i \wedge (B_1 \vee \ldots \vee B_i)]$ *if* $i \leq min(m, n)$
   $[(A_1 \vee \ldots \vee A_n) \wedge B_i]$ *if* $n < i$
   $[A_i \wedge (B_1 \vee \ldots \vee B_m)]$ *if* $m < i$.

   Example:
   $(A_1 \times A_2) \wedge (B_1 \times B_2 \times B_3) \doteq$
   $A_1 \wedge B_1 \times$
   $[(A_1 \vee A_2) \wedge B_2] \vee [A_2 \wedge (B_1 \vee B_2)] \times$
   $[(A_1 \vee A_2) \wedge B_3]$

2. $(A_1 \times \ldots \times A_n) \vee (B_1 \times \ldots \times B_m) \doteq (A'_1 \vee B'_1) \times \ldots \times (A'_k \vee B'_k)$
where $k = max(m, n)$ *and*
$A'_i = A_i$ *if* $i \leq n$, *false otherwise, and*
$B'_i = B_i$ *if* $i \leq m$, *false otherwise.*

Example:
$(A_1 \times A_2) \vee (B_1 \times B_2 \times B_3) \doteq (A_1 \vee B_1) \times (A_2 \vee B_2) \times B_3$

3. $\neg(A_1 \times \ldots \times A_n) \doteq \neg(A_1 \vee \ldots \vee A_n)$

Proposition 7. *Every choice logic formula F can be translated to a strongly equivalent formula F' which is either classical or a basic choice formula.*

Normal form: every formula classical or basic choice, standardized (first options syntactically different)

Translation to stratified KB

Definition 10. A stratified knowledge base is a sequence $(S_0, \ldots, S_n)$ of sets S_i of classical propositional formulas.

Definition 11. Let $KB = (S_0, \ldots, S_n)$ be a stratified knowledge base. A maximal consistent subset S of $(S_0 \cup \ldots \cup S_n)$ is a lexicographically preferred subbase of KB iff

1. $S_0 \subseteq S$, and

2. if S' is a maximal consistent subset of $(S_0 \cup \ldots \cup S_n)$ containing S_0 and for some $k \in \{1, \ldots, n\} : |S' \cap S_k| > |S \cap S_k|$ then there is $j < k$ such that $|S \cap S_j| > |S' \cap S_j|$.

Definition 12. Let $KB = (S_0, \ldots, S_n)$. F is lexicographically entailed by KB, denoted $KB \vdash_{lex} F$, iff F is entailed by all lexicographically preferred subbases of KB

Definition 13. Let T be a choice logic base in normal form. The stratified knowledge base associated with T, denoted $Skb(T)$, is

$$Skb(T) = (T^*, T_1, \ldots, T_n)$$

where $n = max\{k \mid F \in T, opt(F) = k\}$
T^* is obtained from T by replacing each occurrence of $\times$ by $\vee$
$T_i = \{A_1 \vee \ldots \vee A_i \mid i \leq k, A_1 \times \ldots \times A_k \in T\}$.

Proposition 8. $T \mid\!\sim F$ *iff* $sub(T) \vdash_{lex} F$.

Compilation and applications

Compilation to a classical KB is described in Benferhat et al. IJCAI 2001.

Applications:

- design and configuration
  intended properties
- agent systems
  intended actions to be performed by agents together with backup actions covering situations where the standard actions are inappropriate
- database and web applications
  queries which describe suboptimal results: find a used convertible less than 2 years old in red, if red is unavailable then in black

Context dependent preferences

$$rain \to (cinema \times swimming)$$
$$\neg rain \to (swimming \times cinema)$$
$$cinema \to \neg swimming$$

Two preferred models, $\{rain, cinema\}$ and $\{swimming\}$. Given $rain$ we derive $cinema$, given $\neg rain$ we derive $swimming$ as intended.

Alternative definitions of entailment

entailment based on lexicographic ordering on models

$$A_1 \times A_2 \times A_3$$
$$B_1 \times B_2 \times B_3$$
$$C_1 \times C_2 \times C_3$$

M_1 satisfying A_1, B_1 and C_3 preferred over M_2 satisfying A_2, B_2 and C_1 because the number of formulas satisfied in M_1 with degree 1 is 2, the number in M_2 is 1.

Definition 14. Let $M^k(T)$ denote the set of formulas of T satisfied by a model M to degree k. A model M_1 of T is T-subset-preferred over a model M_2 if there is a k such that $M_1^k(T)$ is a strict superset of $M_2^k(T)$ and for all $j < k$: $M_1^j(T) = M_2^j(T)$. M is a subset-preferred model of T iff M is a maximally T-subset-preferred model of T.

Remarks

subset preference implies the cardinality based preference but not vice versa.
Thus in general more maximally subset-preferred models and fewer conclusions.
In the example above M_1 is not subset-preferred over M_2.

Further option:
add up the (dis)satisfaction degrees of all formulas in T for each model and prefer those models whose sum is minimal.

Conclusions

- developed a logic for representing ranked options
- interesting applications in different areas
- investigated computational aspects
- open questions:
  first order version
  application and computation of alternative inference notions
  application methodology