Principles of Knowledge Representation and Reasoning

3. Modal Logics

3.2 Semantics and Proof Systems

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Non-validity: Example

Proposition. $\Diamond\top$ is not **K**-valid.

Proof. A counterexample is the following interpretation:

$$\mathcal{I} = \langle\{w\}, \emptyset, \{w \longmapsto (a \longmapsto T)\}\rangle.$$

Apparently, we have $\mathcal{I}, w \not\models \Diamond\top$, because there is no world u such that wRu.

Proposition $\Box\varphi \rightarrow \varphi$ is not **K**-valid.

Proof. A counterexample is the following interpretation:

$$\mathcal{I} = \langle\{w\}, \emptyset, \{w \longmapsto (a \longmapsto F)\}\rangle.$$

Apparently, we have $\mathcal{I}, w \models \Box a$, but $\mathcal{I}, w \not\models a$.

Non-validity: Another Example

Proposition. $\Box\varphi \rightarrow \Box\Box\varphi$ is not **K**-valid.

Proof. A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{u, v, w\}, \{(u, v), (v, w)\}, \pi \rangle,$$

with

$$\begin{aligned} \pi(u) &= \{a \longmapsto T\} \\ \pi(v) &= \{a \longmapsto T\} \\ \pi(w) &= \{a \longmapsto F\} \end{aligned}$$

This means $\mathcal{I}, u \models \Box a$, but $\mathcal{I}, u \not\models \Box\Box a$.

Accessibility and Axiom Schemata

Let us consider the following axiom schemata:

T: $\Box\varphi \rightarrow \varphi$ (knowledge axiom)

4: $\Box\varphi \rightarrow \Box\Box\varphi$ (positive introspection)

5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ (negative introspection: equivalently $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$)

B: $\varphi \rightarrow \Box\Diamond\varphi$

D: $\Box\varphi \rightarrow \Diamond\varphi$ (disbelief in the negation, equivalently $\Box\varphi \rightarrow \neg\Box\neg\varphi$)

...and the following classes of frames, for which the accessibility relation is restricted as follows:

T: reflexive (wRw for each world w),

4: transitive (wRu and uRv implies wRv),

5: euclidian (wRu and wRv implies uRv),

B: symmetric (wRu implies uRw),

D: serial (for each w there exists v with wRv)

Connection between Accessibility Relations and Axiom Schemata (1)

Theorem. Axiom schema T $(4, 5, B, D)$ is **T**- valid (**4-, 5-, B-**, or **D**-valid, respectively).

Proof for T and **T**. Let $\mathcal{F}$ be a frame from class **T**. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let w be an arbitrary world in $\mathcal{I}$. If $\Box\varphi$ is not true in a world w, then axiom T is true in w. If $\Box\varphi$ is true in w, then φ is true in all accessible worlds. Since the accessibility relation is reflexive, w is among the accessible worlds, i.e., φ is true in w. This means that also in this case T is true w. This means, T is true in all worlds in all interpretations based on **T**-frames, which we wanted to show.

Connection between Accessibility Relations and Axiom Schemata (2)

Theorem. If T $(4, 5, B, D)$ is valid in a frame $\mathcal{F}$, then $\mathcal{F}$ is a **T**-Frame (**4-, 5-, B-**, or **D**-frame, respectively).

Proof for T and **T**. Assume that $\mathcal{F}$ is not a **T**-frame. We will construct an interpretation based on $\mathcal{F}$ that falsifies T.

Because $\mathcal{F}$ is not a **T**-frame, there is a world w such that not wRw.

Construct an interpretation $\mathcal{I}$ such that $w \not\models p$ and $v \models p$ for all v such that wRv.

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$.

Different Modal Logics

Name	Property	Axiom schema
K	$-$	$\Box(\varphi \longrightarrow \psi) \longrightarrow (\Box\varphi \longrightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \longrightarrow \varphi$
4	transitivity	$\Box\varphi \longrightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \longrightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \longrightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \longrightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{array}{rcl} K & & \\ KT4 & = & S4 \\ KT5 & = & S5 \\ \vdots & & \end{array}$$

Different Modal Logics

logics	$\Box$	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
aletic	necessarily	possibly	Y	Y	?	?	?	Y
epistemic	known	possible	Y	Y	Y	Y	N	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	N	?	?	Y
temporal	always (in future)	sometimes	Y	Y	Y	N	N	Y

Proof Methods

- How can we show that a formula is $\mathcal{C}$-valid?
- In order to show that a formula is **not $\mathcal{C}$-valid**, one can construct a counterexample (= an interpretation that falsifies it.)
- When trying out all ways of generating a counterexample without success , this counts as a proof of validity.

$\leadsto$ method of **(analytic/semantic) tableaux**

Tableau Method

A **tableau** is a tree with nodes marked as follows:

- $w \models \varphi$,
- $w \not\models \varphi$, and
- wRv.

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is **closed**. All other branches are **open**. If all branches are closed, the tableau is closed.

A tableau is constructed by using the **tableau rules**.

Tableau Rules for the Propositional Logic

$$\frac{w \models \varphi \vee \psi}{w \models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \vee \psi}{\begin{array}{c} w \not\models \varphi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \neg\varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \wedge \psi}{\begin{array}{c} w \models \varphi \\ w \models \psi \end{array}}$$

$$\frac{w \not\models \varphi \wedge \psi}{w \not\models \varphi \mid w \not\models \psi}$$

$$\frac{w \not\models \neg\varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \rightarrow \psi}{\begin{array}{c} w \models \varphi \\ w \not\models \psi \end{array}}$$

Additional Tableau Rules for the Modal Logic K

$$\frac{w \models \Box\varphi}{v \models \varphi}$$ if wRv is on the branch already

$$\frac{w \not\models \Box\varphi}{\begin{array}{c} wRv \\ v \not\models \varphi \end{array}}$$ for new v

$$\frac{w \models \Diamond\varphi}{\begin{array}{c} wRv \\ v \models \varphi \end{array}}$$ for new v

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi}$$ if wRv is on the branch already

Properties of K Tableaux

Proposition. If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness). If a K-tableau with root $w \not\models \varphi$ is closed, then φ is **K**-valid.

Theorem (Completeness). If φ is **K**-valid, then there is a closed tableau with root $w \not\models \varphi$.

Proposition (Termination). There are strategies for constructing **K**-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.

Tableau Rules for Other Modal Logics

For restricted classes of frames there are more tableau rules.

$\leadsto$ For reflexive (**T**) frames we may extend any branch with wRw.

$\leadsto$ For transitive (**4**) frames we need one additional rule :

- If there are wRv and vRu on one branch, we can extend this branch by wRu.

$\leadsto$ For serial (**D**) frames we need the following rule:

- If there is $w \models \ldots$ or $w \not\models \ldots$ on a branch, then add wRv for a new world v.

- Similar rules for other properties...

Testing Logical Consequence with Tableaux

- Let Θ be a set of formulas. When does a formula φ **follow** from Θ: $\Theta \models_{\mathbf{X}} \varphi$?

$\rightarrow$ I.e., test whether in all interpretations on $\mathbf{X}$-frames in which Θ is true, also φ is true.

- Wouldn't there be **a deduction theorem** we could use?

$\rightarrow$ **Example**: $a \models_{\mathbf{K}} \Box a$ holds, but $a \rightarrow \Box a$ is not **K**-valid.

$\rightsquigarrow$ There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!

Tableaus and Logical Implication

For testing logical consequence, we can use the following tableau rule:

- If w is a world on a branch and $\psi \in \Theta$, then we can add $w \models \psi$ to our branch.

$\leadsto$ Soundness is obvious

$\leadsto$ Completeness is non-trivial

Embedding Modal Logics in the Predicate Logic (1)

1. $\tau(p, x) = p(x)$ for propositional variables p

2. $\tau(\neg\phi, x) = \neg\tau(\phi, x)$

3. $\tau(\phi \vee \psi, x) = \tau(\phi, x) \vee \tau(\psi, x)$

4. $\tau(\phi \wedge \psi, x) = \tau(\phi, x) \wedge \tau(\psi, x)$

5. $\tau(\Box\phi, x) = \forall y(R(x, y) \rightarrow \tau(\phi, y))$ for some new y

6. $\tau(\Diamond\phi, x) = \exists y(R(x, y) \wedge \tau(\phi, y))$ for some new y

Embedding Modal Logics in the Predicate Logic (2)

Theorem. ϕ is K-valid if and only if $\forall x \tau(\phi, x)$ is valid in the predicate logic.

Theorem. ϕ is T-valid if and only if in the predicate logic the logical consequence $\{\forall x R(x,x)\} \models \forall x \tau(\phi, x)$ holds.

Example.

$$((\Box p) \wedge \Diamond(p \rightarrow q)) \rightarrow \Diamond q$$

is K-valid because

$$\begin{aligned} \forall x((\forall x'(R(x,x') \rightarrow p(x'))) \wedge \exists x'(R(x,x') \wedge (p(x') \rightarrow q(x')))) \\ \rightarrow \exists x'(R(x,x') \wedge q(x')) \end{aligned}$$

is valid in the predicate logic.

Outlook

We only looked at some basic propositional modal logics. There are also

- modal first order logics (with quantification $\forall$ and $\exists$, and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
- temporal and dynamic logics (modalities that refer to time or programs, respectively)

Literature

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