Principles of
Knowledge Representation and Reasoning

3. Modal Logics

3.2 Semantics and Proof Systems

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Non-validity: Example

**Proposition.** $\Diamond \top$ is not $K$-valid.

**Proof.** A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto T)\} \rangle.$$

Apparently, we have $\mathcal{I}, w \not\models \Diamond \top$, because there is no world $u$ such that $wRu$.

**Proposition** $\square \varphi \rightarrow \varphi$ is not $K$-valid.

**Proof.** A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto F)\} \rangle.$$

Apparently, we have $\mathcal{I}, w \models \square a$, but $\mathcal{I}, w \not\models a$. 
Non-validity: Another Example

**Proposition.** $\Box \varphi \rightarrow \Box \Box \varphi$ is not $K$-valid.

**Proof.** A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{u, v, w\}, \{(u, v), (v, w)\}, \pi \rangle,$$

with

$$
\begin{align*}
\pi(u) & = \{a \mapsto T\} \\
\pi(v) & = \{a \mapsto T\} \\
\pi(w) & = \{a \mapsto F\}
\end{align*}
$$

This means $\mathcal{I}, u \models \Box a$, but $\mathcal{I}, u \not\models \Box \Box a$. 
Let us consider the following axiom schemata:

**T**: $\square \varphi \rightarrow \varphi$ (knowledge axiom)

**4**: $\square \varphi \rightarrow \square \square \varphi$ (positive introspection)

**5**: $\Diamond \varphi \rightarrow \square \Diamond \varphi$ (negative introspection: equivalently $\neg \square \varphi \rightarrow \square \neg \square \varphi$)

**B**: $\varphi \rightarrow \square \Diamond \varphi$

**D**: $\square \varphi \rightarrow \Diamond \varphi$ (disbelief in the negation, equivalently $\square \varphi \rightarrow \neg \square \neg \varphi$)

...and the following classes of frames, for which the accessibility relation is restricted as follows:
T: reflexive \((wRw \text{ for each world } w)\),

4: transitive \((wRu \text{ and } uRv \text{ implies } wRv)\),

5: euclidian \((wRu \text{ and } wRv \text{ implies } uRv)\),

B: symmetric \((wRu \text{ implies } uRw)\),

D: serial (for each \(w\) there exists \(v\) with \(wRv\))
Connection between Accessibility Relations and Axiom Schemata (1)

Theorem. Axiom schema $T(4, 5, B, D)$ is $T$-valid ($4$-, $5$-, $B$-, or $D$-valid, respectively).

Proof for $T$ and $T$. Let $F$ be a frame from class $T$. Let $I$ be an interpretation based on $F$ and let $w$ be an arbitrary world in $I$. If $\Box \phi$ is not true in a world $w$, then axiom $T$ is true in $w$. If $\Box \phi$ is true in $w$, then $\phi$ is true in all accessible worlds. Since the accessibility relation is reflexive, $w$ is among the accessible worlds, i.e., $\phi$ is true in $w$. This means that also in this case $T$ is true in $w$. This means, $T$ is true in all worlds in all interpretations based on $T$-frames, which we wanted to show.
Connection between Accessibility Relations and Axiom Schemata (2)

**Theorem.** If $T (4, 5, B, D)$ is valid in a frame $\mathcal{F}$, then $\mathcal{F}$ is a $T$-Frame (4-, 5-, B-, or D-frame, respectively).

**Proof** for $T$ and $T$. Assume that $\mathcal{F}$ is not a $T$-frame. We will construct an interpretation based on $\mathcal{F}$ that falsifies $T$.

Because $\mathcal{F}$ is not a $T$-frame, there is a world $w$ such that not $wRw$.

Construct an interpretation $\mathcal{I}$ such that $w \not\models p$ and $v \models p$ for all $v$ such that $wRv$.

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. 
Different Modal Logics

<table>
<thead>
<tr>
<th>Name</th>
<th>Property</th>
<th>Axiom schema</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$\vdash \phi \rightarrow \psi \rightarrow (\Box \phi \rightarrow \Box \psi)$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>reflexivity</td>
<td>$\Box \phi \rightarrow \phi$</td>
</tr>
<tr>
<td>$4$</td>
<td>transitivity</td>
<td>$\Box \phi \rightarrow \Box \Box \phi$</td>
</tr>
<tr>
<td>$5$</td>
<td>euclidicity</td>
<td>$\Diamond \phi \rightarrow \Box \Diamond \phi$</td>
</tr>
<tr>
<td>$B$</td>
<td>symmetry</td>
<td>$\phi \rightarrow \Box \Diamond \phi$</td>
</tr>
<tr>
<td>$D$</td>
<td>seriality</td>
<td>$\Box \phi \rightarrow \Diamond \phi$</td>
</tr>
</tbody>
</table>

Some basic modal logics:

\[
\begin{align*}
K & \equiv K + \Box \Box \\
KT4 & = S4 \\
KT5 & = S5 \\
\vdots
\end{align*}
\]
### Different Modal Logics

<table>
<thead>
<tr>
<th>logics</th>
<th>□</th>
<th>◇ = ¬□¬</th>
<th>K</th>
<th>T</th>
<th>4</th>
<th>5</th>
<th>B</th>
<th>D</th>
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<tbody>
<tr>
<td>aletic</td>
<td>necessarily</td>
<td>possibly</td>
<td>Y</td>
<td>Y</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>Y</td>
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<td>Y</td>
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<td>N</td>
<td>Y</td>
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<tr>
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<td>Y</td>
<td>N</td>
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<td>permitted</td>
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<td>?</td>
<td>?</td>
<td>Y</td>
</tr>
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<td>sometimes</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
</table>
Proof Methods

• How can we show that a formula is $\mathcal{C}$-valid?

• In order to show that a formula is not $\mathcal{C}$-valid, one can construct a counterexample (= an interpretation that falsifies it.)

• When trying out all ways of generating a counterexample without success, this counts as a proof of validity.

$\leadsto$ method of (analytic/semantic) tableaux
A **tableau** is a tree with nodes marked as follows:

- $w \models \varphi$, 
- $w \not\models \varphi$, and 
- $w R v$.

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is **closed**. All other branches are **open**. If all branches are closed, the tableau is closed.

A tableau is constructed by using the **tableau rules**.
Tableau Rules for the Propositional Logic

\[
\begin{align*}
\frac{w \models \varphi \lor \psi}{w \models \varphi} \quad & \quad \frac{w \models \varphi \lor \psi}{w \models \psi} \\
\frac{w \models \varphi \land \psi}{w \models \varphi} \quad & \quad \frac{w \models \varphi \land \psi}{w \models \psi} \\
\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi} \quad & \quad \frac{w \not\models \varphi \rightarrow \psi}{w \models \varphi}
\end{align*}
\]
Additional Tableau Rules for the Modal Logic K

\[
\begin{align*}
\frac{w \models \Box \varphi}{v \models \varphi} & \quad \text{if } wRv \text{ is on the branch already} \\
\frac{w \not\models \Box \varphi}{wRv} & \quad \text{for new } v \\
\frac{w \models \Diamond \varphi}{wRv} & \quad \text{for new } v \\
\frac{w \not\models \Diamond \varphi}{v \not\models \varphi} & \quad \text{if } wRv \text{ is on the branch already}
\end{align*}
\]
Properties of K Tableaux

**Proposition.** If a $K$-tableau is closed, the truth condition at the root cannot be satisfied.

**Theorem** (Soundness). If a $K$-tableau with root $w \nvDash \varphi$ is closed, then $\varphi$ is $K$-valid.

**Theorem** (Completeness). If $\varphi$ is $K$-valid, then there is a closed tableau with root $w \nvDash \varphi$.

**Proposition** (Termination). There are strategies for constructing $K$-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.
Tableau Rules for Other Modal Logics

For restricted classes of frames there are more tableau rules.

→ For reflexive (T) frames we may extend any branch with \( wRw \).

→ For transitive (4) frames we need one additional rule:
  ◦ If there are \( wRv \) and \( vRu \) on one branch, we can extend this branch by \( wRu \).

→ For serial (D) frames we need the following rule:
  ◦ If there is \( w \models \ldots \) or \( w \not\models \ldots \) on a branch, then add \( wRv \) for a new world \( v \).

• Similar rules for other properties...
Testing Logical Consequence with Tableaux

• Let $\Theta$ be a set of formulas. When does a formula $\varphi$ follow from $\Theta$: $\Theta \models_X \varphi$?

→ I.e., test whether in all interpretations on $X$-frames in which $\Theta$ is true, also $\varphi$ is true.

• Wouldn’t there be a deduction theorem we could use?

→ Example: $a \models_K \Box a$ holds, but $a \to \Box a$ is not $K$-valid.

∼ There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!
For testing logical consequence, we can use the following tableau rule:

• If \( w \) is a world on a branch and \( \psi \in \Theta \), then we can add \( w \models \psi \) to our branch.

\( \leadsto \) Soundness is obvious

\( \leadsto \) Completeness is non-trivial
Embedding Modal Logics in the Predicate Logic (1)

1. \( \tau(p, x) = p(x) \) for propositional variables \( p \)

2. \( \tau(\neg \phi, x) = \neg \tau(\phi, x) \)

3. \( \tau(\phi \lor \psi, x) = \tau(\phi, x) \lor \tau(\psi, x) \)

4. \( \tau(\phi \land \psi, x) = \tau(\phi, x) \land \tau(\psi, x) \)

5. \( \tau(\Box \phi, x) = \forall y(R(x, y) \rightarrow \tau(\phi, y)) \) for some new \( y \)

6. \( \tau(\Diamond \phi, x) = \exists y(R(x, y) \land \tau(\phi, y)) \) for some new \( y \)
Embedding Modal Logics in the Predicate Logic (2)

**Theorem.** \( \phi \) is K-valid if and only if \( \forall x \tau(\phi, x) \) is valid in the predicate logic.

**Theorem.** \( \phi \) is T-valid if and only if in the predicate logic the logical consequence \( \{ \forall x R(x, x) \} \models \forall x \tau(\phi, x) \) holds.

**Example.**

\[
((\Box p) \land \Diamond (p \to q)) \to \Diamond q
\]

is K-valid because

\[
\forall x ((\forall x'(R(x, x') \to p(x'))) \land \exists x'(R(x, x') \land (p(x') \to q(x'))))
\to \exists x'(R(x, x') \land q(x'))
\]

is valid in the predicate logic.
Outlook

We only looked at some basic propositional modal logics. There are also

• modal first order logics (with quantification $\forall$ and $\exists$, and predicates)

• multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents

• temporal and dynamic logics (modalities that refer to time or programs, respectively)


**Textbooks:**

