CHARACTERISTIC NUMBERS OF MULTIPLE-POINT
MANIFOLDS

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Abstract
We give a general method for expressing various characteristic numbers of the multiple-point
manifolds of an immersion.

1. Introduction
All manifolds in this paper are understood to be smooth. Given a generic (ie.
self-transverse) immersion \( f : M^m \to N^{m+k} \) where \( M \) is a closed manifold we
can try to determine the cobordism class of its multiple-point manifolds. (For the
definition of these see Section 2.1 below.) As the cobordism class depends only on
the characteristic numbers of a manifold, it suffices to calculate these numbers.
Instead of evaluating a characteristic class of the multiple-point manifold on its
fundamental class we can take its pushforward into \( M \) and evaluate it on \([M]\).
(From now on \([M]\) will denote the fundamental class of the manifold \(M\), unless
otherwise mentioned.) So our goal now is to express these pushforwards in terms
of \( M, N \) and \( f \). To do so we set up formulas involving the pushforwards.

The first such formula was stated by Lashof and Smale [5] but it turned out
to be partially false. It was corrected by Herbert [3] and later Ronga [7] gave
a simple proof. This formula expresses the relationship between the cohomology
classes represented by the multiple-point manifolds in \( M \). (In our terminology this
class is simply the pushforward of the 1 element of the cohomology ring.)

Szűcs [9] used the Herbert-Ronga formula in the oriented case for double-point
manifolds in K-theory and translated it to ordinary cohomology via the Chern
character to obtain a sequence of formulas involving pushforwards of Pontrjagin
classes of the double-point manifold.

Kamata [4] used the Herbert-Ronga formula in the unoriented cobordism coho-
mology and translated it via the Boardman homomorphism to obtain a sequence of
formulas involving the pushforwards of the Stiefel-Whitney classes of the multiple-
point manifolds.

Later in [8] Szűcs investigated the case of oriented manifolds immersed in Euclid-
ean space. Using a filtration on the multiple-point manifold he could calculate its
Pontrjagin numbers without pushing them forward to \( M \).

In this paper we show a simple method that gives a general result containing all
three above results at the same time and which avoids complicated homological cal-

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culations or the use of natural transformations between extraordinary cohomology theories and ordinary cohomology.

Szűcs used his original formulas to show that there are cobordism classes of manifolds that do not contain double-point manifolds. Here we carry out similar calculations for multiple-point manifolds of arbitrary multiplicity.

Our results compute characteristic classes numbers of multiple-point manifolds of a generic immersion. In Section 2.2 we recall a formula of Ronga on multiple-point manifolds of 

\[ f : M^{n-k} \to \mathbb{R}^n \]

representing an element \( \alpha \in \pi_n^S MO(k) \) by means of the Pontrjagin-Thom construction. Eccles showed in [2] that the bordism class of every multiple-point manifold of \( f \) is determined by the Hurewicz image of \( \alpha \) in \( H_r(\Omega^\infty S^\infty MO(k)) \). In [1] it is shown using detailed homological inspection of \( \Omega^\infty S^\infty MO(k) \) how the Stiefel-Whitney numbers of the multiple-point manifolds of \( f \) can be read off from the Hurewicz image of \( \alpha \).

The paper is organized as follows: In Section 2.1 we define the multiple-point manifolds on which our further calculations are based. We prove our main formula in Section 3 and give applications in Section 4.

2. Preliminaries

2.1. Definition of the multiple-point manifolds

Consider the generic immersion \( f : M^m \to N^{m+k} \). The \( r \)-fold points of \( f \) are those points in \( N \) whose preimage consists exactly of \( r \) different points. This is not a closed set in \( N \). Its closure \( N_r \) consists of those points that have at least \( r \) distinct preimages. Set \( M_r = f^{-1}(N_r) \) to obtain the closure of \( r \)-fold points of \( f \) in the source manifold.

The sets \( M_r \) and \( N_r \) are generally not submanifolds of \( M \) and \( N \), but they are images of (non-generic) immersions of manifolds. Here we recall the well-known construction to fix the notation: Let

\[ \tilde{M}_r(f) = \{(x_1, \ldots, x_r) \in M^{(r)} : f(x_1) = \cdots = f(x_r), (i \neq j) \Rightarrow (x_i \neq x_j)\} \subset M^{(r)}. \]

The symmetric group \( S_r \) acts on this set freely in the obvious way. Let \([x_1, \ldots, x_r]\) denote the equivalence class of \((x_1, \ldots, x_r)\). On the other hand \( S_{r-1} \) also acts freely on the last \( r-1 \) coordinates. Here the equivalence class of \((x_1, \ldots, x_r)\) is denoted by \((x_1, [x_2, \ldots, x_r])\). The sets of equivalence classes are denoted by

\[ \tilde{N}_r(f) = \tilde{M}_r(f)/S_r, \]

\[ \tilde{M}_r(f) = \tilde{M}_r(f)/S_{r-1}. \]

There are obvious mappings

\[ g_r : \tilde{N}_r(f) \to N \quad g_r([x_1, \ldots, x_r]) := f(x_1) \]

\[ f_r : \tilde{M}_r(f) \to M \quad f_r(x_1, [x_2, \ldots, x_r]) := x_1 \]

\[ s_r : \tilde{M}_r(f) \to \tilde{N}_r(f) \quad s_r(x_1, [x_2, \ldots, x_r]) := [x_1, \ldots, x_r]. \]

The images of \( g_r \) and \( f_r \) are clearly \( N_r \) and \( M_r \) and they are bijective to the
characteristic numbers of multiple-point manifolds

points that have multiplicity exactly r. On the other hand s_r is clearly an r-sheeted covering.

The sets  \( \tilde{N}_r(f) \) and  \( \tilde{M}_r(f) \) are called the r-fold multiple-point manifolds of \( f \) in the target and source respectively. They are indeed manifolds. Let

\[
\Delta_r(V) = \{ (x_1, \ldots, x_r) \in V^{(r)} : (\exists i, j)(x_i = x_j) \}
\]

\[
\delta_r(V) = \{ (x, \ldots, x) \in V^{(r)} \}
\]

denote the fat and the narrow diagonals of \( V^{(r)} \) for any manifold \( V \). Consider the r-fold product \( f^{(r)} : M^{(r)} \to N^{(r)} \). Clearly

\[
\tilde{M}_r(f) = (f^{(r)})^{-1}(\delta_r(N)) \setminus \Delta_r(M).
\]

Since \( f \) is a generic immersion \( f^{(r)} \) is transversal to \( \delta_r(N) \) and thus \( \tilde{M}_r(f) \) is a manifold of dimension \( m - (r - 1)k \). So after factoring out with the free group actions we still get manifolds.

2.2. Ronga’s formula

We need the notion of sub-cartesian diagram and two lemmas from [7]. In the sequel \( \nu \) always denotes the normal bundle of an immersion:

**Lemma 1.** \( f_r \) and \( g_r \) are proper immersions with normal bundles \( \nu_{g_r} = (\nu_{f_r}^{(r)}|_{\tilde{M}_r(f)})/S_r \) and \( \nu_{f_r} = (0 \times \nu_{f_r}^{(r-1)})|_{\tilde{M}_r(f)})/S_{r-1} \).

**Definition 1.** A commutative diagram of proper immersions:

\[
\begin{array}{ccc}
Z & \xrightarrow{f_B} & B \\
\downarrow{f_A} & & \downarrow{\beta} \\
A & \xrightarrow{\alpha} & X
\end{array}
\]

is said to be sub-cartesian if

(i) \( f_A \times f_B : Z \to A \times B \) is an embedding onto \( \{ (a, b) \in A \times B : \alpha(a) = \beta(b) \} \).

(ii) the following sequence is exact:

\[
0 \to TZ \xrightarrow{d(f_A \times f_B)} f_A^*TA \times f_B^*TB \xrightarrow{(d\alpha - d\beta)} f_A^*T \alpha T X
\]

The first condition of the definition says that \( Z \) is the intersection of \( A \) and \( B \) where multiple intersections (that is: points that are multiple for \( \alpha \) or \( \beta \)) are counted with appropriate multiplicity. The second condition says that the intersection is clean in the terminology of Quillen (cf [6]), that is, the tangent space of the intersection manifold is locally the same as the intersection of the two tangent spaces.

**Definition 2.** In a sub-cartesian diagram \( E = \text{coker}(d\alpha - d\beta) \) is called the excess vector bundle.
Let $h^*$ denote any generalized cohomology theory with products and let $e_h$ denote the Euler class of a vector bundle in $h^*$.

**Lemma 2.** If $A, B, X$ and $Z$ are h-orientable then $\alpha^* \beta(c) = f_A(e_h(E) \cdot f_B(c))$ holds for any $c \in h^*(B)$.

We would like to apply Lemma 2 with $X = N, A = M, B = \tilde{N}_{r-1}(f), \alpha = f, \beta = g_{r-1}$ since the r-tuple points are in the intersection of the $r-1$-tuple points with the image of $f$. It is easily seen that in this case we have to set $Z = \tilde{M}_r(f) \cup \tilde{M}_{r-1}(f)$, that is the disjoint union of the $r$-tuple point manifold and the $r-1$-tuple point manifold. The maps $f_A$ and $f_B$ have to be defined as below to make the diagram a pull-back diagram:

$$f_A|_{\tilde{M}_r(f)} := f_r \quad \quad f_B|_{\tilde{M}_r(f)} := p_r$$

$$f_A|_{\tilde{M}_{r-1}(f)} := f_{r-1} \quad \quad f_B|_{\tilde{M}_{r-1}(f)} := s_{r-1}$$

Here $p_r : \tilde{M}_r(f) \to \tilde{N}_{r-1}(f)$ is the projection map defined by the formula

$$p_r((x_1, [x_2, \ldots, x_r]) = [x_2, \ldots, x_r].$$

This way we get a sub-cartesian diagram. The genericity of $f$ implies that the excess vector bundle over $\tilde{M}_r(f)$ is the zero bundle and Lemma 1 implies that it is $f_{r-1}^* \nu_f$ over $\tilde{M}_{r-1}(f)$.

If all manifolds involved are h-orientable we can apply Lemma 2 to an element $c \in h^*(\tilde{N}_{r-1}(f))$ to get

$$f^*(g_{r-1}(c)) = f_r(p_r^*(c)) + f_{r-1}(e_h(f_{r-1}^* \nu_f) \cdot s_{r-1}^*(c))$$

$$= f_r(p_r^*(c)) + e_h(\nu_f) \cdot f_{r-1}(s_{r-1}^*(c)). \quad (2.1)$$

3. *The general multiple-point formula*

Let $\gamma$ denote a multiplicative characteristic class in $h^*$. That is, for any bundle $\xi : E \to B$ there is a class $\gamma(\xi) \in h^*(B)$ such that this class is natural with respect to induced bundles and $\gamma(\xi_1 \oplus \xi_2) = \gamma(\xi_1) \cdot \gamma(\xi_2)$. This definition is valid for bundles with any given structure group. Well-known examples of such a $\gamma$ are the total Stiefel-Whitney class when $h^* = H(\cdot, \mathbb{Z}_2)$ or the Euler class. We shall allow $\gamma(\xi)$ to be an infinite sum but we will assume that there is an other multiplicative characteristic class $\beta$ such that $\gamma \cdot \beta \equiv 1$ (so the Euler class is excluded now).

Let us choose $c = \gamma(\nu_{g_{r-1}})$. Then $f_B^*(c) = f_B(\gamma(\nu_{g_{r-1}})) = \gamma(f_B^*(\nu_{g_{r-1}}))$. We compute the two parts $f_B|_{\tilde{M}_r(f)}(c)$ and $f_B|_{\tilde{M}_{r-1}(f)}(c)$ separately.

First notice that since $\tilde{M}_r(f)$ is the transversal intersection of $M$ and $\tilde{N}_{r-1}(f)$ it follows that $f_B^*(\nu_{g_{r-1}})|_{\tilde{M}_r(f)} = \nu_{f_A|_{\tilde{M}_r(f)}} = \nu_{f_r}$. Thus

$$f_r(f_B^*(c)) = f_r(\gamma(f_B^*(\nu_{g_{r-1}}))) = f_r(\gamma(\nu_{f_r})) \quad (3.1)$$

To calculate the other part first we make a trivial remark that we will use later.

**Remark.** For any immersion $g : V \to W$ we have $\nu_g \oplus TV = g^*TW$ so $\gamma(\nu_g) = g^*\gamma(TW)$.

Using the standard formula $f(f^*x \cdot y) = x \cdot f(y)$ this implies that

$$g_*(\gamma(\nu_g)) = g_*(g^*\gamma(TW) \cdot \beta(TV)) = \gamma(TW) \cdot g_*(\beta(TV)) \quad (3.2)$$
It is easy to see that \( f_B |_\tilde{\mathcal{M}}_{r-1}(f) \) is an \( r-1 \)-sheeted covering of \( \tilde{N}_{r-1}(f) \). This implies that the bundle \( f_B^* \nu_{g_{r-1}} |_\tilde{\mathcal{M}}_{r-1}(f) \) is equal to the normal bundle of the composite map \( g_{r-1} \circ f_B |_\tilde{\mathcal{M}}_{r-1}(f) \) which is in turn equal to \( f \circ f_{r-1} \) since our sub-cartesian diagram is by definition commutative. Thus we have the following sequence of equations:

\[
f_{r-1}(f_B^*(c)) = f_{r-1}(\gamma(f_B^*(\nu_{g_{r-1}}))) = f_{r-1}(\gamma(\nu_{f_{r-1}})) = f_{r-1}(f_{r-1}(\gamma(\nu_{f_{r-1}}))) = f_{r-1}(\gamma(\nu_{f_{r-1}})) (3.3)
\]

(We used the the fact that \( \nu_{f_{r-1}} = \nu_{f_{r-1}} \oplus f_B^*(\nu_f) \) and the standard formula which we also used in the previous remark.)

**Lemma 3.** Let \( h^* \) be a generalized cohomology theory with products. Then for any invertible multiplicative characteristic class \( \gamma \) taking values in \( h^* \) and any generic immersion \( f : M \to N \) for which all the arising manifolds are \( h \)-orientable we have

\[
f^* g_{r-1}(\gamma(\nu_{g_{r-1}})) = f_r(\gamma(\nu_f)) + e_h(\nu_f) \cdot \gamma(\nu_f) \cdot f_{r-1}(\gamma(\nu_{f_{r-1}}))
\]

**Proof.** Plug (3.1) and (3.3) into (2.1). \( \square \)

The class \( \gamma \) of the normal bundles of \( f_r, f_{r-1} \) and \( g_{r-1} \) are hard to evaluate directly and so we write them in the terms of classes of the tangent bundles of the multiple-point manifolds. To this end we use (3.2). We get

\[
f^* g_{r-1}(\gamma(\nu_{g_{r-1}})) = f^*(\gamma(TN)) \cdot f^* g_{r-1}(\beta(T\tilde{N}_{r-1}(f)))
\]
\[
f_r(\gamma(\nu_f)) = f_r(\gamma(TM) \cdot f_r(\beta(T\tilde{M}_r(f)))
\]
\[
e_h(\nu_f) \cdot \gamma(\nu_f) \cdot f_{r-1}(\gamma(\nu_{f_{r-1}})) = e_h(\nu_f) \cdot f^*(\gamma(TN)) \cdot \beta(TM) \cdot \gamma(TM) \cdot f_{r-1}(\beta(T\tilde{M}_{r-1}(f)))
\]

Combining these formulas with Lemma 3 and dividing by \( \gamma(TM) \) we get

\[
f_{r-1}(\beta(T\tilde{M}_r(f))) = \frac{f^*(\gamma(TN))}{\gamma(TM)} \cdot \left( f^* g_{r-1}(\beta(T\tilde{N}_{r-1}(f))) - e_h(\nu_f) \cdot f_{r-1}(\beta(T\tilde{M}_{r-1}(f))) \right)
\]

We can think of this formula as a recursion which expresses an invariant of the \( r \)-tuple point manifold in terms of invariants of the \( r-1 \)-tuple point manifolds. Let us denote by \( m_r = f_r(\beta(T\tilde{M}_r(f))) \) and by \( n_{r-1} = g_r(\beta(T\tilde{N}_r(f))) \) the quantities we are interested in.

**Main formula.**

\[
m_r = \gamma(\nu_f) \cdot (f^* n_{r-1} - e_h(\nu_f) \cdot m_{r-1}) \quad (3.4)
\]

The difficulty in applying this formula is that a priori we know nothing about \( n_{r-1} \). But in favorable cases we can relate it to \( m_{r-1} \) thereby obtaining a real recursion-formula on the \( m_r \).

**Lemma 4.** \( f_r(m_r) = p \cdot n_r \) where \( p \in h^0(\tilde{N}_r(f)) \) is a cohomology class such that \( s_{r+1}[\tilde{M}_r(f)] = p \cdot [\tilde{N}_r(f)] \).
Proof. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{M}_r(f) & \xrightarrow{s_r} & \tilde{N}_r(f) \\
\downarrow f_r & & \downarrow g_r \\
M & \xrightarrow{f} & N
\end{array}
\]

As \(s_r\) is an \(r\)-sheeted covering, the tangent bundle of \(\tilde{M}_r(f)\) is induced from the tangent bundle of \(\tilde{N}_r(f)\) by \(s_r\). By Poincaré duality \(p = s_r(1) = s_r s_r^*(1) \in h^0(\tilde{N}_r(f))\). Thus:

\[
f_r(m_r) = f_r s_r(\beta(T\tilde{M}_r(f))) = g_r s_r(\beta(s_r^*(T\tilde{N}_r(f)))) = g_r(s_r s_r^*(\beta(T\tilde{N}_r(f))))
= g_r(\beta(T\tilde{N}_r(f)) \cdot s_r s_r^*(1)) = g_r(p \cdot \beta(T\tilde{N}_r(f))) = p \cdot n_r. \qed
\]

Now if \(p \in h^0(\tilde{N}_r(f))\) is an invertible element then it follows that \(f_r(m_r)\) is divisible by \(p\) and we can rewrite Lemma 4 in the form \(n_r = \frac{h^0(m_r)}{p}\). This is the case for example if we take cohomology \(\mathbb{Q}\) coefficients and restrict ourselves to oriented manifolds:

**THEOREM 1.** Let \(f : M^m \to N^{m+k}\) be a generic immersion of even codimension between oriented manifolds and choose a cohomology theory with coefficient ring \(h^0(pt) \cong \mathbb{Q}\). Then we have

\[
m_r = \gamma(\nu_f) \cdot \left( \frac{f^* f_r(m_{r-1})}{r-1} - e_h(\nu_f) m_{r-1} \right)
\]

Proof. Since \(M\) and \(N\) are oriented, so is \(\tilde{M}_r(f)\). The action of the symmetric groups \(S_r\) and \(S_{r-1}\) are orientation preserving since \(k\) is even. So both \(\tilde{M}_r(f)\) and \(\tilde{N}_r(f)\) are oriented and \(s_r : \tilde{M}_r(f) \to \tilde{N}_r(f)\) is also orientation preserving. This means that \(p = r\) in Lemma 4. Thus we are done if we combine the Main Formula and Lemma 4. \(\Box\)

**REMARK.** If \(k\) is odd then the \(S_r\) action contains orientation reversing involutions on \(\tilde{M}_r(f)\) and so either \(\tilde{N}_r(f)\) is unorientable or its components have no preferred orientation.

To make explicit calculations with the formula of Theorem 1 one has to deal with the term \(f^* f_r\). One way to do this is to suppose that \(f^* = 0\) in positive dimensions. We are going to use this approach in Section 4. However there is an other case when we can resolve it, and that is when there is a bundle \(\xi\) over \(N\) such that \(TM = f^*(\xi)\).

**THEOREM 2.** Let \(f\) and \(h^*\) be as in Theorem 1. Further assume that there is a bundle \(\xi : E \to N\) such that \(TM = f^*\xi\), and that \(e(\nu_f) = f^*(y)\) for an \(y \in h^*(N)\). Then for every \(r \geq 1\) there is a cohomology class \(k_r \in h^*(N)\) such that \(m_r = f^*(k_r)\).
and the following simple recursion formula holds:

\[ k_r = \gamma(TN)\beta(\xi) \left( \frac{c}{r-1} - y \right) \cdot k_{r-1} \]

where \( c = f_1 f^*(1) \in h^*(N) \).

**Proof.** The statement easily follows by induction on \( r \). For \( r = 1 \) we have \( m_1 = \beta(TM) = \beta(f^*(\xi)) = f^*(\beta(\xi)) \) so we can choose \( k_1 = \beta(\xi) \).

For the inductive step notice that \( \gamma(\nu_f) = \frac{f^*(\gamma(TN))}{f^*(\gamma(TM))} = f^*(\gamma(TN)\beta(\xi)) \). So by the inductive hypothesis and Theorem 1

\[ m_r = \gamma(\nu_f) \cdot \left( f_1 f_1 (m_{r-1}) \frac{c}{r-1} - e_h(\nu_f) m_{r-1} \right) = f^* \left( \gamma(TN)\beta(\xi) \left( \frac{f_1 f^*(k_{r-1})}{r-1} - y \cdot k_{r-1} \right) \right). \]

Thus we may choose

\[ k_r := \gamma(TN)\beta(\xi) \left( \frac{f_1 f^*(k_{r-1})}{r-1} - y \cdot k_{r-1} \right) = \gamma(TN)\beta(\xi) \left( \frac{c}{r-1} - y \right) \cdot k_{r-1} \]

which finishes the proof. \( \square \)

4. Applications

4.1. The unoriented case

Now we shall show how the results of Kamata [4] follow from our method.

Let our cohomology theory \( h^*(X) = H^*(X, \mathbb{Z}_2) \geq H^*(X, \mathbb{Z}_2) \) be the ring of formal power-series of infinite variables over \( H^*(X, \mathbb{Z}_2) \). For an \( n \)-dimensional bundle \( \xi : E \to B \) let us define

\[ w_t(\xi) = \prod_{i=1}^n (1 + \alpha_i t_1 + \alpha_i^2 t_2 + \alpha_i^3 t_3 + \cdots) \]

where the total Stiefel-Whitney class of \( \xi \) is expanded by the splitting principle as

\[ w(\xi) = (1 + \alpha_1) \cdots (1 + \alpha_n). \]

Since \( w_t(\xi) \) is symmetric in the variables \( \alpha_i \), it is really a characteristic class. Its multiplicativity and naturality easily follow from that of \( w(\xi) \). It is also invertible since \( w_t(\xi) \) always starts with \( 1 + \cdots \). Thus we may choose \( \gamma = w_t \). It is clear that \( e_h(\xi) = e(\xi) \in h^*(B, \mathbb{Z}_2) \) where \( e(\xi) \) is just the ordinary Euler class of \( \xi \).

**Theorem (Kamata).** Let \( f : M^m \to N^{m+k} \) be a self-transverse immersion for which \( f^* \) is the constant map in positive dimension. (This is satisfied if for example \( f \) is null-homotopic.) Then

\[ f_{rt} \left( \frac{1}{w_t(TM_r(f))} \right) = e(\nu_f)^{r-1} \cdot \left( \frac{1}{w_t(TM)} \right)^r \]
in $H^*(M, \mathbb{Z}_2)[[t_1, t_2, \ldots]]$.

**Proof.** Let us look at the Main Formula with the choice of $\gamma = \omega_t$ and $\beta = \frac{1}{\omega_t}$. Since $f^* = 0$ and we are working with $\mathbb{Z}_2$ coefficients, the formula simplifies to $m_r = \omega_t(\nu f) e(\nu f) m_{r-1}$. Thus by induction we have $m_r = (\omega_t(\nu f) e(\nu f))^{r-1} m_1$. Here $\omega_t(\nu f) = f^*(\omega_t(TN)) \cdot \beta(TM) = \beta(TM)$. On the other hand $f_1$ is just the identity map of $M$ so $m_1 = \beta(TM)$. Thus $m_r = \beta(TM)^r \cdot e(\nu f)^{r-1}$ and this is exactly what we wanted to prove.

Evaluating both sides on $[M]$ and noticing that $(f_1(x), [M]) = \langle x, [\tilde{M}_r(f)] \rangle$

we get

**Corollary (Kamata).** $\langle \frac{1}{\omega_t(TM_r(f))}, [\tilde{M}_r(f)] \rangle = \langle e(\nu f)^{r-1} \cdot \left( \frac{1}{\omega_t(TM)} \right)^r, [M] \rangle$.

**Remark.** It would have been simpler to use $\beta(\xi) = \omega_t(\xi)$ since this way we get a formula for the pushforward of $\omega_t(TM_r(f))$ instead of its reciprocal. Then instead of the above corollary we would get $\langle \omega_t(TM_r(f)), [\tilde{M}_r(f)] \rangle = \langle e(\nu f)^{r-1} \cdot (\omega_t(TM))^r, [M] \rangle$.

Though this form is better for any application, we wanted to state the theorem exactly as Kamata stated it in [4].

This formula then implies:

**Corollary (Kamata).** If a self-transverse immersion $f : M^m \to N^{m+k}$ is null-homotopic and $M$ is null-cobordant then so are all the multiple-point manifolds $\tilde{M}_r(f)$.

**Proof.** The idea of the proof is the following. The single equation of the above corollary actually implies equation of every coefficient in the formal power series. The coefficients on the left hand side are all the characteristic (Stiefel-Whitney) numbers of the multiple-point manifold. Similarly on the right hand side the coefficients are Stiefel-Whitney numbers of $M$. If $M$ is null-cobordant, then all its Stielfel-Whitney numbers are zero. Thus the same holds for the multiple-point manifold, hence it is also null-cobordant. For more details see [4].

**Remark.** We cannot see any easy way to avoid the use of formal power series. If, for instance, we choose $\beta(\xi) = \omega(\xi)$ the total Stiefel-Whitney class, then the corollary will express only one Stielfel-Whitney polynomial in each dimension, instead of expressing all of them at the same time.

**4.2. The oriented case**

In this section our cohomology theory $h^*(X)$ will be $H^*(X, \mathbb{Q})[[t_1, t_2, \ldots]]$. Let $f : M^m \to N^{m+k}$ be a generic immersion where $M$ and $N$ are oriented and $k$
is even. In the case when $f^*$ is the zero homomorphism in positive dimensions, we will be able to express the pushforward of any Pontrjagin polynomial of the multiple-point manifolds in terms of the Pontrjagin classes of $M$ and $e(\nu_f)$.

We present the same result in two different forms. There seems to be no simple direct proof of the fact that the two forms are actually equivalent.

4.2.1. **Symmetric polynomials** As in the previous section, we have a multiplicative characteristic class which is defined for a bundle $\xi : E \to B$ with the formula

$$
\beta(\xi) := \prod_{i=1}^{n} (1 + y_i t_1 + y_i^2 t_2 + \cdots)
$$

where the total Pontrjagin class of $\xi$ is written as

$$
p(\xi) = (1 + y_1) \cdots (1 + y_n).
$$

Here the $y_i$ are 4-dimensional cohomology classes and the $j^{th}$ Pontrjagin class of $\xi$ is $p_j(\xi) = \sigma_j(y_1, \ldots, y_n)$, the $j^{th}$ symmetric polynomial in the variables $y_i$. Since $\beta$ is symmetric in the variables $y_i$ it is well-defined. It is also obviously natural and invertible. It is also multiplicative since the total Pontrjagin class is multiplicative modulo 2-torsion but with $\mathbb{Q}$ coefficients there is no 2-torsion.

Let us apply the Main Formula with $\gamma = 1/\beta$. As in the previous section the $f^* = 0$ assumption simplifies the formula and we get $m_r = -\beta(TM)e(\nu_f)m_{r-1}$. As $m_1 = \beta(TM)$, we have by induction that

$$
f_{r+1}(\beta(\Omega^*(f))) = (-e(\nu_f))^{r-1} \beta(TM)^r. \quad (4.1)
$$

This formula is actually not a single equation, since both sides are formal power series with infinite variables. Thus they can only be equal if the coefficients of all the corresponding monomials are the same. So we have an equation for every monomial of the form $y_1^{b_1} y_2^{b_2} \cdots y_n^{b_n}, s \geq 0, b_i \geq 0$. These equations contain all the information needed to calculate the pushforwards of the Pontrjagin polynomials of the multiple-point manifold. To extract this information we use the Hirzebruch base of symmetric polynomials.

For a partition $I = (a_1, \ldots, a_r)$ of $|I| = a_1 + \cdots + a_r$ let $x_I \in H^{4|I|}(B, \mathbb{Q})$ denote the smallest symmetric polynomial containing the monomial $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_r}$. For example

$$
x_{(1)} = y_1 + \cdots + y_n = p_1(\xi)
$$

$$
x_{(1,1)} = y_1 y_2 + y_1 y_3 + \cdots = p_2(\xi)
$$

$$
x_{(2)} = y_1^2 + \cdots + y_n^2 = p_1(\xi)^2 - 2p_2(\xi)
$$

If (by abuse of notation) we introduce the following operation on partitions:

$$
I(\mathbf{b}) = I(b_1, \ldots, b_s) := (1, \ldots, 1, 2, \ldots, 2, s, \ldots, s)
$$

then it is easy to see that

$$
\beta(\xi) = (1 + y_1 t_1 + y_1^2 t_2 + \cdots)(1 + y_2 t_1 + y_2^2 t_2 + \cdots) \cdots (1 + y_n t_1 + y_n^2 t_2 + \cdots)
$$

$$
= \sum_{(b_1, \ldots, b_s), s \geq 0, b_i \geq 0} x_{I(\mathbf{b})} t_1^{b_1} t_2^{b_2} \cdots t_s^{b_s} \quad (4.2)
$$

Now we can think of (4.1) as a formula that tells us the pushforward of any
characteristic polynomial $x_I, I = (a_1, \ldots, a_r)$ of the multiple-point manifold. It is exactly $(-e(\nu_I))^{r-1}$ times the coefficient of $t_1^{b_1}t_2^{b_2} \cdots t_N^{b_N}$ in $\beta(TM)$, where $b_i = |\{j : a_j = i\}|$.

As any $x_I$ is a polynomial of Pontrjagin classes and every such polynomial is a linear combination of the $x_I$, we get the pushforward of all the Pontrjagin polynomials. And finally the formula

$$\langle f_{r_1}(x), [M] \rangle = \langle x, [\tilde{M}_r(f)] \rangle$$

gives us all the Pontrjagin numbers of the multiple-point manifolds.

4.2.2. Pontrjagin polynomials
With the use of a different multiplicative characteristic class we can express pushforwards of Pontrjagin polynomials directly with Pontrjagin classes of $M$ and $e(\nu_I)$. The formula we are going to prove this way was first proved by Szűcs in [8] by a completely different method.

For an orientable bundle $\xi : E \to B$ let

$$\beta(\xi) = \prod_{i=1}^N (1 + p_1(\xi)t_i + p_2(\xi)t_i^2 + \cdots) \in H^*(B, \mathbb{Q})[[t_1, \ldots, t_N]]$$

where $N$ is a large number. As $p(\xi \oplus \eta) = p(\xi) \cdot p(\eta)$, an easy calculation shows that each factor of our class $\beta$ is indeed multiplicative. Naturality and invertibility of $\beta$ is obvious.

Just as in the previous section, formula (4.1) holds for our class $\beta$. On the left-hand side the coefficient of $t_1^{b_1} \cdots t_N^{b_N}$ is the pushforward of the Pontrjagin polynomial $p_1^{b_1} \cdots p_N^{b_N}$ of the multiple-point manifold. It is also easy to see the coefficient of the same monomial in the right hand side of (4.1). As

$$\beta(TM)^r = \prod_{i=1}^N (1 + p_1(TM)t_i + p_2(TM)t_i^2 + \cdots)^r,$$

the $t_1^{b_1}$ part comes from the first factor, the $t_2^{b_2}$ from the second, and so on. It is also easy to see that the coefficient of $t_1^{b_1} t_2^{b_2}$ in $(1 + p_1(TM)t_1 + p_2(TM)t_1^2 + \cdots)^r$ is exactly the $4b_1$ dimensional part of $p(TM)^r = (1 + p_1(TM) + p_2(TM) + \cdots)^r$.

Let us denote by $q_j$ the $4j$ dimensional part of $p(TM)^r$. For a partition $I = (b_1, b_2, \ldots, b_N)$ let us denote $p^r(TM)_I = q_{b_1} \cdots q_{b_N}$, and let the the usual Pontrjagin polynomial $p_I(TM_r(f)) = p_{s_1}p_{s_2} \cdots p_{s_N}(TM_r(f))$.

**Theorem 3.**

$$f_{r_1}(p_I(TM_r(f)) \rangle = (-e(\nu_I))^{r-1}p^r(TM)_I$$

**Corollary (Szűcs).**

$$\langle p_I(TM_r(f)), [\tilde{M}_r(f)] \rangle = \langle (-e(\nu_I))^{r-1}p^r(TM)_I, [M] \rangle$$

4.3. Numerical calculations

In this section we are going to use the machinery of the previous section to obtain numerical results on the cobordism classes of multiple-point manifolds. We will show that many cobordism classes do not contain manifolds that arise as the multiple-point manifold of an immersion $f : M^m \to N^{m+k}$ with $f^* = 0$. 
The 4 dimensional part of this is $CP^2 = \text{square of the Euler class of the canonical line bundle over } CP^n$. Then

$\langle p_1^{k_1 + \cdots + k_r}(V), [V] \rangle = \frac{(k_1 + \cdots + k_r)!}{k_1! \cdots k_r!} \prod_{i=1}^{s} (2k_i + 1)^{k_i}$.

**Proof.** It is well-known that $p(CP^n) = (1 + y)^{n + 1}$ where $y \in H^4(CP^n)$ is the square of the Euler class of the canonical line bundle over $CP^n$. So $p(V) = p(CP^{k_1}) \times p(CP^{k_2}) \times \cdots \times p(CP^{k_r}) = (1 + y)^{2k_1 + 1} \times \cdots \times (1 + y)^{2k_r + 1}$.

The 4 dimensional part of this is

$p_1(V) = \sum_{i=1}^{s} (2k_i + 1) \left( \prod_{i=1}^{s} (2k_i + 1) \right)$

and so

$p_1^{k_1 + \cdots + k_r} = (y^{k_1} \times \cdots \times y^{k_r}) \cdot \prod_{i=1}^{s} (2k_i + 1)^{k_i} \cdot \prod_{i=1}^{s} \left( k_i + \cdots + k_r \right)$.

Since $y^{k_1} \times \cdots \times y^{k_r}$ is the generator of $H^{4(k_1 + \cdots + k_r)}(V)$ this finishes the proof. □

**Lemma 6.** The greatest common divisor of all the numbers $\langle p_1^r(V), [V] \rangle$ where $V$ runs over the $4n$ dimensional oriented closed manifolds is 1 if $n \neq 1(3)$ and is 1 or 3 if $n \equiv 1(3)$.

**Proof.** The function $f(V) = \langle p_1^r(V), [V] \rangle$ is an $f : \Omega_{4n} \to \mathbb{Z}$ homomorphism. By the previous lemma we have

$f(CP^{2n}) = (2n + 1)^n = A$ and $f(CP^2 \times CP^{2n-2}) = \binom{n}{1} \cdot 3 \cdot (2n - 1)^{n-1} = B$.

It is clear that $n, 2n - 1$ and $2n + 1$ are pairwise coprime, so the greatest common divisor $(A, B)$ of $A$ and $B$ equals to $(3, 2n + 1)$. Thus $(A, B) = 1$ if $n \neq 1(3)$ else $(A, B) = 3$. Since $f : \Omega_{4n} \to \mathbb{Z}$ is a homomorphism, there is a manifold $V^{4n}$ such that $f(V) = (A, B)$ and this proves the lemma. □

**Theorem 4.** If a 4 dimensional oriented manifold $V^{4t}$ is the $r$-tuple-point manifold of an immersion $f : M^m \to N^{m+k}$ (i.e. $V = M_r(f)$) with $f^* = 0$ in positive dimension, then $\langle p_1^r(V), [V] \rangle$ is divisible by $r^t$.

**Proof.** Let us calculate $\langle p_1^r(TV), [V] \rangle$. By (4.3) we have

$\langle p_1^r(TV), [V] \rangle = \langle (-e(\nu_f))^{r-1} p_1^r(TM)_{(1, \ldots, 1)}, [M] \rangle$

$= \langle (-e(\nu_f))^{r-1} (r \cdot p_1(TM))_{(r, 1)}, [M] \rangle = r^t \langle (-e(\nu_f))^{r-1} p_1^r(TM), [M] \rangle$ □

Define the homomorphism $\Delta_r : Imm^{SO}(m, k) \to \Omega_{m+k}^{(-r-1)}$ from the cobordism group of immersions of oriented $m$-manifolds into $\mathbb{R}^{m+k}$ to the oriented cobordism group by

$\Delta_r(f) := [M_r(f)]$,

where $[\cdot]$ now denotes cobordism class. This homomorphism is well-defined when $m$ and $k$ are even. Combining the last theorem with the last lemma we get:
COROLLARY. If \( m - k(r-1) \) is divisible by four then \(|\text{coker}(\Delta_r)| \geq \frac{r^{m-k(r-1)}}{4^3} \) where \( \epsilon = 1 \) if \( 3|r \) and \( m - k(r-1) \equiv 1(3) \), else \( \epsilon = 0 \).

This is a generalization of Szűcs’s result obtained in [9].

References


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