Übungsaufgaben Gewöhnliche Differentialgleichungen
Serie 5

1.)

a) $\left(y'\right)^3 + y' - x = 0$

$$x = y'^3 + y' \qquad\qquad y' = \frac{dy}{dx} = p$$

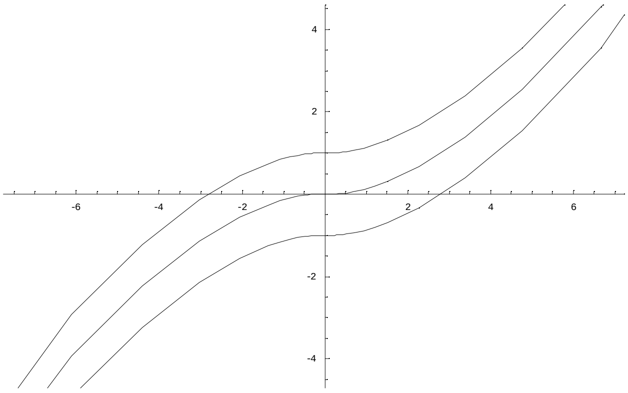
$$x = j\left(p\right) \qquad\qquad dy = pdx$$

$$dx = j'\left(p\right)dp$$

$$dy = pj'\left(p\right)dp$$

$$\begin{aligned} y &= \int pj'\left(p\right)dp \\ &= \int p\left(p^3 + p\right)dp \\ &= \frac{p^5}{5} + \frac{p^3}{3} + c \end{aligned}$$

$$\underline{\underline{x\left(p\right) = p^3 + p}} \qquad\qquad \underline{\underline{y\left(p\right) = \tfrac{1}{5}p^5 + \tfrac{1}{3}p^3 + c}}$$



b) $y = x^2 e^{y'} + xy' \qquad\qquad y\left(1\right) = 1$

$$y' = p$$

$$y = f\left(x, p\right) = x^2 e^p + xp$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = p$$

$$2xe^p + p + \left(x^2 e^p + x\right)\frac{dp}{dx} = p$$

$$2e^p + \left(xe^p + 1\right)\frac{dp}{dx} = 0$$

$$2e^p dx + \left(xe^p + 1\right)dp = 0$$

DG ist nicht exakt!

$$l\,2e^{p}dx + l\left(xe^{p}+1\right)dp = 0$$

$$l\,2e^{p} + 2e^{p}\,l' = ne^{p}$$

$$2e^{p}\,l' = -l\,e^{p}$$

$$\frac{l'}{l} = \frac{e^{p}}{2e^{p}} = -\tfrac{1}{2}$$

$$\ln l = -\tfrac{1}{2}p + c_1$$

$$l = ce^{-\frac{p}{2}}$$

$$e^{-\frac{p}{2}}\,2e^{p}dx + e^{-\frac{p}{2}}\left(xe^{p}+1\right)dp = 0$$

$$2e^{\frac{p}{2}}dx + \left(xe^{\frac{p}{2}} + e^{-\frac{p}{2}}\right)dp = 0 \qquad P_p = e^{\frac{p}{2}} = Q_x = e^{\frac{p}{2}}$$

$$\int 2e^{\frac{p}{2}}dx + \left(xe^{\frac{p}{2}} + e^{-\frac{p}{2}}\right)dp = 0 \qquad \text{Kurvenintegral 2. Art}$$

$$2e^{\frac{p0}{2}}\left(x-x_0\right) + \left[2xe^{\frac{p}{2}} - 2e^{-\frac{p}{2}}\right]_{p_0}^{p} = 0$$

$$2e^{\frac{p0}{2}}x - 2e^{\frac{p0}{2}}x_0 + 2xe^{\frac{p}{2}} - 2e^{-\frac{p}{2}} - 2e^{\frac{p0}{2}}x + 2e^{\frac{p0}{2}} = 0$$

$$2xe^{\frac{p}{2}} - 2e^{-\frac{p}{2}} = c_2$$

$$xe^{\frac{p}{2}} = c_2 + e^{-\frac{p}{2}}$$

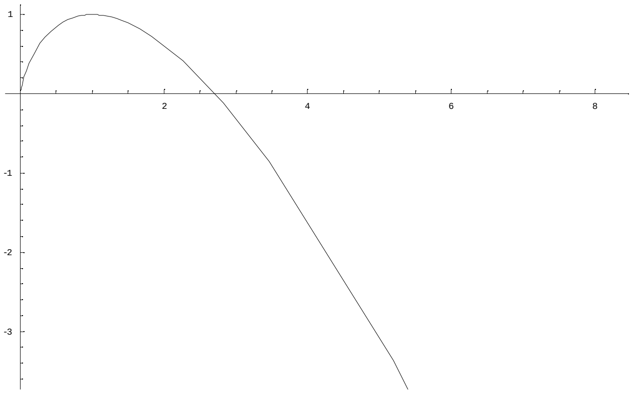
$$x = \underline{ce^{-\frac{p}{2}} + e^{-p}}$$

$$y = \left(ce^{-\frac{p}{2}} + e^{-p}\right)^2 e^{p} + \left(ce^{-\frac{p}{2}} + e^{-p}\right)p$$

$$= \underline{c^2 + 2ce^{-\frac{p}{2}} + e^{-p} + \left(ce^{-\frac{p}{2}} + e^{-p}\right)p}$$

Für c=0 stimmt die Anfangsbedingung $y(1)=1$:

$$x = \underline{\underline{e^{-p}}} \qquad y = \underline{\underline{e^{-p} + e^{-p}p}}$$



2.)

a)

$$y' = f_0(x)\,y^a + f_1(x)\,y$$

teilen durch $-y^a$

$$\frac{y'}{y^a} = -f_0(x) - f_1(x)\,y^{1-a}$$

Substitution $z = y^{1-a}$ $\qquad z' = -(a-1)\frac{y'}{y^a}$

$\rightarrow \quad \frac{z'}{a-1} + f_1(x)\,z = -f_0(x)$ lineare DG 1. Ordnung.

□

b)

AWP: $y' = -y^2 + \frac{1}{x}y, \quad y(1) = \frac{2}{3}$

$$-\frac{y'}{y^2} = 1 - \frac{1}{xy} \qquad\qquad z = \frac{1}{y} \qquad z' = -\frac{y'}{y^2}$$

$$z' = 1 - \frac{z}{x}$$

$$z' + \frac{z}{x} = 1$$

Lösung der hom. DG: $z' + \frac{z}{x} = 0$

$$\int \frac{\frac{dz}{dx}}{z}\,dx = -\int \tfrac{1}{x}\,dx$$

$$z = \frac{c}{x}$$

Variation der Konstanten:

$$z_i = \frac{c(x)}{x}$$

$$z_i' = \frac{c'(x)\,x - x(x)}{x^2}$$

$$\frac{c'(x)\,x - x(x)}{x^2} + \frac{c(x)}{x^2} = 1$$

$$\frac{c'(x)}{x} = 1$$

$$c(x) = \tfrac{1}{2}x^2$$

$$z_i = \tfrac{1}{2}x$$

Allg. Lösung:

$$z = \frac{c}{x} + \frac{x}{2} = \frac{2c + x^2}{2x}$$

$$y = \frac{2x}{2c + x^2}$$

mit AB $y(1) = \frac{2}{3}$: c=1

$$\underline{\underline{y = \frac{2x}{2 + x^2}}}$$

3.)

$$4y'\sin x = -y\left(1 + y^4\right) + y^5 \cos x$$

$$y' = -\frac{1}{4\sin x} y + \frac{\cos x - 1}{4\sin x} y^5$$

$$-\frac{y'}{y^5} = \frac{1}{4\sin x} y^{-4} + \frac{\cos x - 1}{4\sin x}$$

Subst. $z = \frac{1}{y^4}$ $\qquad z' = -4\frac{y'}{y^5}$

$$z' = \frac{z}{\sin x} - \frac{\cos x - 1}{\sin x} \qquad \rightarrow \text{lin. DG 1. Ordnung}$$

Lösung hom. DG $z' - \frac{z}{\sin x} = 0$

$$\int \frac{z_h'}{z_h} dx = \int \frac{1}{\sin x} dx$$

$$\ln z_h = \ln \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} + c_1$$

$$z_h = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} c = c \tan \tfrac{x}{2}$$

Var. d. Konst.:

$$z_i = c(x) \tan \tfrac{x}{2}$$

$$z_i' = \frac{c(x)}{2\cos^2 \frac{x}{2}} + c'(x) \tan \tfrac{x}{2}$$

$$\frac{c(x)}{2\cos^2 \frac{x}{2}} + c'(x)\tan \tfrac{x}{2} - \frac{c(x)\tan \frac{x}{2}}{\sin x} = -\frac{\cos x - 1}{\sin x}$$

$$\vdots$$

$$c'(x) = 1$$

$$c(x) = x$$

$$z_i = x \tan \tfrac{x}{2}$$

Allg. Lsg.:

$$z = c\tan\tfrac{x}{2} + x\tan\tfrac{x}{2} = \tan\tfrac{x}{2}\left(c+x\right)$$

$$y = \underline{\underline{\frac{1}{\sqrt[4]{\tan\frac{x}{2}\left(c+x\right)}}}}$$

4.)

Die allgemeine Lösung dieser DG ist, laut Aufgabe 4 der Serie 4:

$$y\left(x\right) = e^{-\int\limits_{x_0}^{x} f(t)dt}\left(y_0 + \int\limits_{x_0}^{x} g\left(t\right) e^{\int\limits_{x_0}^{t} f(t)dt} dt \right)$$

a)

Bei einer hom. DG ist $g\left(x\right) = 0$ und folglich

$$\left|y\left(x\right)\right| = e^{-\int\limits_{x_0}^{x} f(t)dt}\left|y_0\right| \le e^{-\int\limits_{x_0}^{x} a\,dt}\left|y_0\right| = \left|y_0\right| e^{-a\left(x-x_0\right)}$$

Daraus lässt sich leicht erkennen, dass $\lim\limits_{x\to\infty} e^{-\int\limits_{x_0}^{x} f(t)dt} y_0 = 0$

$\square$

b)

$$\begin{aligned} y\left(x\right) &= y_0 e^{-\int\limits_{x_0}^{x} f(t)dt} + \int\limits_{x_0}^{x} g\left(t\right) e^{\int\limits_{x_0}^{t} f(t)dt} e^{-\int\limits_{x_0}^{x} f(t)dt} dt \\ &= y_0 e^{-\int\limits_{x_0}^{x} f(t)dt} + \int\limits_{x_0}^{x} g\left(t\right) e^{\int\limits_{x_0}^{t} f(t)dt - \int\limits_{x_0}^{x} f(t)dt} dt \\ &= y_0 e^{-\int\limits_{x_0}^{x} f(t)dt} + \int\limits_{x_0}^{x} g\left(t\right) e^{-\int\limits_{t}^{x} f(t)dt} dt \qquad \left(x_0 < t < x\right) \end{aligned}$$

Abschätzung:

$$\begin{aligned} \left|y\right| &\le \left|y_0\right| e^{-\int\limits_{x_0}^{x} f(t)dt} + \int\limits_{x_0}^{x} \left|g\left(t\right)\right| e^{-\int\limits_{t}^{x} f(t)dt} dt \\ &\le \left|y_0\right| e^{-a\left(x-x_0\right)} + M\int\limits_{x_0}^{x} e^{-a\left(x-t\right)} dt \\ &\le \left|y_0\right| e^{-a\left(x-x_0\right)} + M\left[\frac{e^{-a\left(x-t\right)}}{a}\right]_{x_0}^{x} = \left|y_0\right| e^{-a\left(x-x_0\right)} + M\left(\frac{1}{a} - e^{-a\left(x-x_0\right)}\right) \end{aligned}$$

Betrachten wir das ganze nun für große x, so geht $e^{-a(x-x_0)}$ gegen 0.
Daraus folgt:

$$|y| \leq \frac{M}{a} + c \qquad c = \text{const}$$

□