Towards an Ontology of Space for GFO

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Abstract. Space and time are basic categories of any top-level ontology. They account for fundamental assumptions of the modes of existence of those individuals that are said to be in space and time. The present paper is devoted to GFO-Space, the ontology of space in the General Formal Ontology (GFO). This ontology is introduced by a set of axioms formalized in first-order logic and further elucidated by consequences of the axiomatization.

The theory is based on four primitives: the category of space regions, the relations of being a spatial part and being a spatial boundary, as well as the relation of spatial coincidence. The presence of boundaries and the notion of coincidence witness an inspiration of the ontology by well-motivated ideas of Franz Brentano on space, time and the continuum. Taking up a line of prior investigations of his approach, the present work contributes a further step in establishing a corresponding ontology of space, employing rigorous logical methods.

Keywords. top-level ontology, theory of space, theory of boundaries

1. Introduction

Space and time are very basic notions for many domains. Accordingly, their ontological analysis is a subject matter for top-level ontologies. Theories of space and time account for fundamental assumptions underlying the modes of existence of those individuals that are said to be *in space and time*. This paper contributes to the overarching aim of providing ontologies of space, in the form of axiomatic theories, for the top-level ontology General Formal Ontology (GFO) [1]. These space ontologies will complement the time ontologies of GFO presented in [2]. They rest on the same principles, which originate from ideas on space, time and the continuum by Franz Brentano [3]. The investigation and exploration of Brentano's ideas on space and time began about twenty years ago by work of Roderick M. Chisholm [4] as well as Barry Smith and Achille C. Varzi, cf. e.g. [5]. The present paper takes up this research and makes a further step in establishing an ontology of space along these lines, proposed as a basic module for GFO. This module results from the continuation as well as a substantial reworking of earlier theories devoted to applying Brentano's ideas to space [6].

We argue that space has a double nature. On the one hand, space is generated and determined by material objects and the relations that hold between them. This space appears to the mind and provides the frame for visual and tactile experience. We call it the *phenomenal space* of material objects and claim its subject-dependence. On the other

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hand, any material object has a subject-independent disposition, called *extension space*, which unfolds in the mind/subject as phenomenal space.[2] The basic entities of phenomenal space are called *space regions* which are abstracted from aggregates of material objects generating them. Hence, phenomenal space can be understood as a category the instances of which are space regions.

The present paper is devoted to the investigation and axiomatization of the category of phenomenal space. This axiomatization of GFO-Space – expounded and specified as a first-order theory $\mathcal{BS}$ – is inspired by ideas on space that are set forth by Brentano in [3], such that we may speak of a Brentano ontology of Space[3]. We believe that Brentano's ideas on space and continuum correspond to our experience of sense data.

The paper proceeds as follows. We start with a brief survey on related work in section 2. In section 3 basics on the relation between material entities and space are outlined, connected with the motivation for this work. The main section 4 of the paper presents an outline of an axiomatic system that describes GFO-Space in a formal way. Some consequences of this theory are presented at the end of the section. Section 5 sketches possible applications, followed by conclusions and future work in section 6.

2. Related Work

There is a vast literature on space and spatial theories, as there are diverse approaches to understanding space. They can very broadly be classified into two general kinds. First, there are *container space* approaches, like the foundation adopted by Isaac Newton in [9]. The second kind accounts for *relational space*, e.g. in the sense of Gottfried Wilhelm Leibniz (cf. [10] for a dispute on these opposing approaches). More recent valuable and detailed analyses of different space ontologies are expounded in [11,12].

In connection with GFO-Space as presented in section 4, we restrict attention to mereotopological theories, for which [13] and [14] are overviews of value. Existing theories vary in their expressiveness and ontological decisions. We focus on three features of high relevance in relation to GFO-Space:

1. the introduction and definition of mereological and topological relations,
2. the treatment of boundaries and dimensionality of entities and
3. the degree of specification of the intended model(s).

A well-known source on mereological theories is [15]. Enriching such a theory by topological primitives like *self-connectedness* [16] or *disconnection* [17] is one common way to specify a mereotopological theory. Different kinds of connectedness are defined and discussed in [18], a.o. regarding theories with vs. without boundaries. Another possibility is to define mereological relations like parthood in terms of topological primitives. One well-studied representative is the *Region Connection Calculus* (RCC) [19], based on the definability of *parthood* in terms of *connection* [20]. The RCC theory postulates only two axioms, namely reflexivity and symmetry of the connection relation. Apart from the usefulness of RCC for qualitative spatial representation and reasoning, this weak axiomatization is a shortcoming with respect to the third criterion above.

[2] To a certain extent the distinction between the extension space of material entities and the phenomenal space relates to the one between spatiality and space, as considered by Nicolai Hartmann in [7,8].

[3] This motivates the acronym $\mathcal{BS}$, in line with the notation in [2].

It is a common restriction of many mereotopological theories to include only equal-dimensional entities in a single model, e.g. in [16,19]. This restriction may be explained by additional issues that arise when space entities of different dimensions are mixed [21], e.g., "Has a spherical volume its surface as a part?". Nevertheless, some approaches highlight the utility of space entities of different dimensions and account for them, e.g. [21,22,23]. In particular, [23] is a recent proposal of an axiomatic spatial theory that modularly combines a theory of spatial dimensions with a dimension-independent mereotopological account. The latter is grounded in *containment* as its mereological primitive, which generalizes parthood by being applicable across dimensions.

3. Motivation and Conceptual Overview

Many intuitions and motivations underlying especially [21] and [23] in the previous section are applicable to phenomenal space, which we intend to capture. In addition, more particular aspects of GFO-Space arise from the notion of continuum as inspired by Brentano [3]. Continua have two sources, phenomenal time [2] and phenomenal space. Both can be accessed through introspection. Phenomenal space is related to the extension of a material body, presented to the mind through the occupation of space. Space continua are wholly present at time points. We argue that space and time cannot be conflated into a homogeneous, four-dimensional system [24]. Yet there is a natural integration of both types of continua in GFO through its law of object-process integration [25].

Every continuum exhibits certain basic features: it has a boundary, it does not possess atomic parts, and for any division of it into two non-overlapping parts each of them possesses a boundary, such that these boundaries are distinct and coincide. It follows that any continuum is connected; there are no jumps or gaps in it. Moreover, in our framework continua cannot be adequately modeled by sets of points (e.g., of the Euclidean space $\mathbb{R}^3$). Space continua are classified into being 1-dimensional, called lines, 2-dimensional, being connected surfaces, and 3-dimensional, called topoids.

We believe that the features of continua are needed to achieve an adequate modeling of material objects and their relation to space. A boundary of a material object, or more briefly, a *material boundary* occurs if the object is demarcated from its environment. We hold that a material boundary is a cognitive construction. It does not consist of matter itself and depends on the granularity of the view. Note in this regard that the present theory is oriented towards the mesoscopic world of phenomenal objects.

Boundaries of space regions must be distinguished from material boundaries. *Natural boundaries*, called bona fide boundaries in [5], are specific material boundaries that exhibit a discontinuity. There are several classical puzzles that pertain to material boundaries, among them Leonardo da Vinci's problem of "What is it that divides the atmosphere from the water? Is it air or is it water?" [26], and Brentano's problem of "What color is the line of the demarcation between a red surface and blue surface being in contact with each other?" [3]. In our opinion, the following is a natural solution to these problems, which we discuss for the surface problem (also called flag problem). The red and the blue surfaces are surfaces of material objects. Any of these material surfaces has its own material boundary, called blue boundary and red boundary. These boundaries are distinct and touch along certain parts, where there is nothing in between the touching parts of them. How can this phenomenon be adequately modeled? First, we assume that the considered material surfaces R and B occupy corresponding *closed* spatial surfaces R' and B'. This implies that the material boundaries r of R and b of B occupy corresponding spatial boundaries r' of R' and b' of B'. The features of continua include the coincidence of space boundaries. Thus we may state that the material boundaries r and b touch if the corresponding space boundaries r' and b' coincide. The distinction between material boundary and space boundary is important here: material boundaries may touch, but cannot coincide, whereas space boundaries may coincide. Clearly, the considered situation cannot be modeled within the Euclidean space $\mathbb{R}^3$, because the surfaces R' and B' cannot both be represented as closed subsets of $\mathbb{R}^3$.

Conceptual Overview Considerations of the kind above motivated us to develop GFO-Space, an axiomatic theory for the description of the phenomenal space. We assume that the phenomenal space is uniquely determined, although our knowledge about it is limited. Our source for the justification of these axioms is the pure apperception and daily experience, but also analogies to classical mathematical theories of manifolds.

Let us preview major conceptual constituents for the axiomatization, which in parallel illuminates its four primitives: the category 'space region' and the relations 'spatial part of', 'spatial boundary of' and 'coincidence'. The (intended) universe of discourse for GFO-Space is the category of space entities. It can be divided into four pairwise disjoint categories, namely *space regions*, *surface regions*, *line regions* and *point regions*. They correspond to three-, two-, one-, and zero-dimensional space entities, respectively, and the latter three are jointly referred to as lower-dimensional entities. All four categories accommodate entities sui generis, i.e. higher-dimensional entities cannot not be defined as a set or sum of lower-dimensional entities.

An important subclass of lower-dimensional space entities are *spatial boundaries*. A spatial boundary cannot exist independently and is always the *boundary of* a higher-dimensional space entity. Importantly, if B is a boundary of E that does not mean that B "fully covers" E. If E is a spatial cube, for example, solely its upper side U is a boundary of E, but also the overall surface S of E is (more technically speaking, S is the greatest boundary of E). U is a part of S, accordingly.

Another important property of spatial boundaries is the ability to *coincide*. Two surface/line/point regions coincide if they are compatible and co-located, which means, intuitively speaking, that they are "congruent" and there is no distance between them. In contrast to classical topology, two coinciding boundaries may be distinct. Consider, for example, two spatial cubes, one on top of the other. Each cube seen individually has its own two-dimensional spatial boundaries. The upper side of the lower cube and the lower side of the upper cube coincide, but they are different. Coincidence is stipulated to be an equivalence relation.

The mereological sum of coincident spatial boundaries yields examples of *extraordinary* entities, which have distinct parts that coincide, but do not overlap. Entities without such parts are *ordinary*. Ordinary three-, two- and one-dimensional space entities that are additionally connected are called *topoids*, *surfaces* and *lines*, respectively. The co-dimension between a spatial boundary and the corresponding space entity it bounds is 1. That means, for instance, that a line region cannot be a spatial boundary of a space region and point regions cannot have boundaries. We postulate that every space region has a greatest spatial boundary, whereas this cannot be justified for lower-dimensional entities, e.g., realizing that a circle has no boundary at all.

Spatial parthood is the final primitive notion of GFO-Space. It is assumed to exhibit a partial ordering. Only space entities of equal dimension may be related by spatial parthood. Hence, a surface cannot be spatial part of a space region. However, it may be a *hyper part*, which captures the connection between a space entity "inside" another of co-dimension at least 1. Thus, containment in [23] means spatial part or hyper part here.

4. Outline of an Axiomatization of GFO-Space

GFO-Space is axiomatized as a theory $\mathcal{BS}$ in first-order predicate logic with equality over a signature of four predicates for primitives, see B1–B4 below.[4] In the overall effort, we follow the principles of the Onto-Axiomatic Method presented in [2, sect. 2]. Due to the page limit, though focusing on the axiomatization problem, we cannot present the overall theory. Thus we select and discuss a substantial, coherent subset of crucial definitions and axioms that suffices to show the consequences gathered in section 4.6.

4.1. Space Regions, Spatial Boundaries and Lower-Dimensional Entities

Space regions form the most fundamental category of GFO-Space, in the sense that all other entities under consideration can be derived from space regions. The most important relation for such derivations is that of one entity being a spatial boundary of another, where an entity is always of one dimension higher than its boundaries. Parthood and spatial coincidence relate entities within the same dimension. Note that parthood applies to all space entities, whereas only spatial boundaries are subject to coincidence (A22).

We begin expounding the formal theory by introducing these four relations as predicates that form the primitives of $\mathcal{BS}$, which are independent of each other. For their intended interpretation beyond the short phrases we refer to the conceptual overview in section 3. All remaining categories and relations of $\mathcal{BS}$ can be defined ultimately on their basis. This and the subsequent sections present these definitions in thematic groups, thereby introducing predicates in parallel.

B1. $SReg(x)$ (x is a space region)

B2. $sb(x, y)$ (x is a spatial boundary of y)

B3. $scoinc(x, y)$ (x and y are coincident)

B4. $spart(x, y)$ (x is a spatial part of y)

The relation sb lends itself to a natural definition of the category SB of spatial boundaries. Moreover, we distinguish spatial boundaries $2DB$, $1DB$, and $0DB$ at the different dimensions by the kind of entity that they bound. Corresponding binary relations $2db$, $1db$, and $0db$ prove widely useful, not only immediately below. The ($spart$-)greatest boundary of an entity is defined because the parts of a boundary are boundaries of the same entity (A24). Note that the theory accommodates boundaryless entities, such as a surface of a ball, as well as bounded entities without a greatest boundary.

On a technical note, certain symbols and definitions follow a common scheme for different dimensions. Then we use schemas with the dimension parameter **d** as a symbol component. It can only be replaced by 2, 1, and 0. Unless revoked, all three values apply to **d**. For example, D5 stands for three definitions, namely of $2DB$, $1DB$, and $0DB$, e.g., $2DB(x) := \exists y\ 2db(x, y)$. In formula remarks we follow the order $2, 1, 0$ by convention.

[4] From here on we may refer to constituents of $\mathcal{BS}$ plainly by their labels.

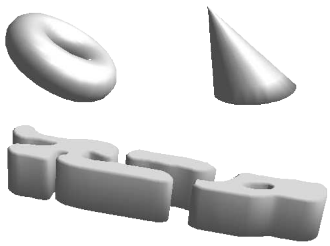

Figure 1. Space Regions

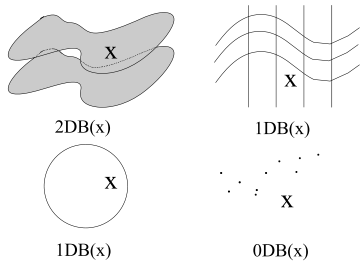

Figure 2. Spatial Boundaries

D1. $2db(x, y) := SReg(y) \wedge sb(x, y)$ (x is a 2-dim. boundary of y)

D2. $1db(x, y) := 2DB(y) \wedge sb(x, y)$ (x is a 1-dim. boundary of y)

D3. $0db(x, y) := 1DB(y) \wedge sb(x, y)$ (x is a 0-dim. boundary of y)

D4. $SB(x) := \exists y\ sb(x, y)$ (x is a spatial boundary)

D5. $\mathbf{d}DB(x) := \exists y\ \mathbf{d}db(x, y)$ (x is a 2-dim./1-dim./0-dim. boundary)

D6. $grsb(x, y) := sb(x, y) \wedge \forall z\ (sb(z, y) \rightarrow spart(z, x))$ (x is the greatest boundary of y)

A direct effect of these definitions is that any spatial boundary can be "traced back" via sb to a space region. However, we consider additional entities in the domain that may not be boundaries of other entities themselves, but which only consist of boundaries. Thus a more relaxed notion of lower-dimensional space entity is defined. By reflexivity of spatial parthood (A4, sect. 4.5) any spatial boundary is a lower-dimensional entity. Furthermore, space regions and lower-dimensional entities cover all entities in the domain (A2).

D7. $LDE(x) := \exists y\ (SB(y) \wedge spart(y, x))$ (x is a lower dimensional space entity)

D8. $\mathbf{d}DE(x) := \exists y\ (\mathbf{d}DB(y) \wedge spart(y, x))$ (x is a surface/line/point region)

D9. $eqdim(x, y) := (SReg(x) \wedge SReg(y)) \vee$ (x and y have the same dimension)
$(2DE(x) \wedge 2DE(y)) \vee (1DE(x) \wedge 1DE(y)) \vee (0DE(x) \wedge 0DE(y))$

4.2. Mereological Notions – Parts, Hyper Parts and Inner vs. Tangential Parts

While proper parthood and spatial overlap are defined as usual, the definitions of mereological sum, intersection, relative complement and partition are introduced as schemata, allowing for compact notation. If clear from the context we may drop the subscript. Sum, intersection, and relative complement can be shown to be functional in the last argument, i.e. the x is uniquely determined by the x_i.

D10. $sppart(x, y) := spart(x, y) \wedge x \neq y$ (x is a proper spatial part-of y)

D11. $sov(x, y) := \exists z\ (spart(z, x) \wedge spart(z, y))$ (spatial overlap)

D12. $sum_n(x_1, \ldots, x_n, x) := \forall y\ (sov(y, x) \leftrightarrow \bigvee_{i=1}^{n} sov(y, x_i))$
(x is the mereological sum of $x_1, \ldots, x_n$, $n \geq 2$)

D13. $intsect_n(x_1, \ldots, x_n, x) := \forall y\ (spart(y, x) \leftrightarrow \bigwedge_{i=1}^{n} spart(y, x_i))$
(x is the mereological intersection of $x_1, \ldots, x_n$, $n \geq 2$)

D14. $rcompl_n(x_1, \ldots, x_n, x) :=$ (x is the rel. complement of x_n and $x_1, \ldots, x_{n-1}$, $n \geq 2$)
$\bigwedge_{1 \leq i < j \leq n} eqdim(x_i, x) \wedge \forall y\ (spart(y, x) \leftrightarrow \bigwedge_{i=1}^{n-1} \neg sov(y, x_i) \wedge spart(y, x_n))$

D15. $partition_n(x_1, \ldots, x_n, x) := sum_n(x_1, \ldots, x_n, x) \wedge \bigwedge_{1 \leq i < j \leq n} \neg sov(x_i, x_j)$
($x_1, \ldots, x_n$ partition x, $n \geq 2$)

A spatial part of a space entity has the same dimension as the entity itself. The term hyper part is used for "parts" with co-dimension greater than or equal to 1. More intuitively, lower dimensional entities "inside" a space entity are its hyper parts.

Figure 3 illustrates hyper parts (in each case the entity x in dark gray) of all three lower dimensions with respect to a cylindric space region (y in light gray). A shadow (in black) of the actual hyper part x is shown at the bottom of the cylinder in order to improve the visual apperception.

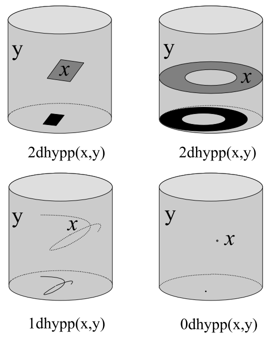

Figure 3. Two-, One- and Zero-Dimensional Hyper Parts

D16. $2dhypp(x, y) := \exists z\ (spart(z, y) \wedge 2db(x, z))$ (x is a 2-dim. hyper part of y)

D17. $1dhypp(x, y) := \exists z\ ((spart(z, y) \vee 2dhypp(z, y)) \wedge 1db(x, z))$ (x is a 1-dim. hyper part of y)

D18. $0dhypp(x, y) := \exists z\ ((spart(z, y) \vee 1dhypp(z, y)) \wedge 0db(x, z))$ (x is a 0-dim. hyper part of y)

D19. $hypp(x, y) := 2dhypp(x, y) \vee 1dhypp(x, y) \vee 0dhypp(x, y)$ (x is a hyper part of y)

Spatial as well as hyper parts of a whole are tangential if they touch the boundary of the whole, whereas they are inner parts if they do not. In the presence of the notion of coincidence, two space entities touch each other if parts of their boundaries coincide or have coincident hyper parts. This leads us to the definitions of tangential and inner parts. In the special case of boundaryless entities, no boundary can be touched by parts or hyper parts, such that the latter are all inner parts.

D20. $tangpart(x, y) := (spart(x, y) \vee hypp(x, y)) \wedge \exists x'zz'\ ((spart(x', x) \vee hypp(x', x)) \wedge sb(z, y) \wedge (spart(z', z) \vee hypp(z', z)) \wedge scoinc(x', z'))$ (x is a tangential part of y)

D21. $inpart(x, y) := (spart(x, y) \vee hypp(x, y)) \wedge \neg tangpart(x, y)$ (x is an inner part of y)

4.3. Ordinariness and Connectedness

The ability to coincide is exclusive for spatial boundaries. Furthermore, spatial boundaries are *ordinary* entities, i.e. they do not possess two non-overlapping parts that coincide. *Extraordinary* space entities result from mereological summation of distinct coincident spatial boundaries. Figure 4 illustrates a case of extraordinary entities that is less abstract, based on material entities and their ability to occupy space. Imagine, for example, a solid rubber sleeve, which is cut through vertically at one position. Assume further that both ends are in contact. We argue that the greatest material boundary of this rubber sleeve occupies an extraordinary surface region, because the occupied surface regions of both material ends coincide, but they are not identical.[5]

[5] Note that the material ends do not consist of substance. Rather, they are cognitive constructions of the mind. The ability to coincide is a feature of spatial boundaries, whereas material boundaries cannot coincide.

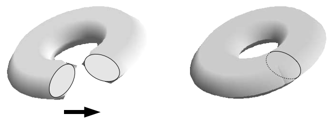

Figure 4. Extraordinary Surface Region

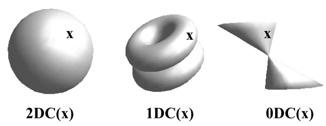

Figure 5. Spatial Connectedness

D22. $ExOrd(x) := \exists yz\ (spart(y,x) \wedge spart(z,x) \wedge \neg sov(y,z) \wedge scoinc(y,z))$ (x is an extraordinary space entity)

D23. $Ord(x) := \neg ExOrd(x)$ (x is an ordinary space entity)

Spatial *connectedness* is an important distinguishing feature of space entities. Figure 5 shows three types of connectedness that we distinguish: two-, one- and zero-dimensional connectedness are jointly defined by the schema D24. The basic idea behind it is that a connected space entity x cannot be partitioned into y and z such that any two hyper parts y' of y and z' of z do not coincide. Rephrased more positively, each partition of x must have at least two coinciding hyper parts.

D24. $\mathbf{d}DC(x) := \exists uv\ (partition(u,v,x)) \wedge$ (x is 2-dim./1-dim./0-dim. connected)
$\forall yz\ (partition(y,z,x) \rightarrow \exists y'z'(\mathbf{d}dhypp(y',y) \wedge \mathbf{d}dhypp(z',z) \wedge scoinc(y',z')))$

A space entity is connected if it is connected regarding at least one dimension. Two space entities are connected if their mereological sum is connected. Furthermore, two connected but non-overlapping entities are called external connected.

D25. $C(x) := 2DC(x) \vee 1DC(x) \vee 0DC(x)$ (x is connected)

D26. $c(x,y) := \exists z\ (sum(x,y,z) \wedge C(z))$ (x and y are connected)

D27. $exc(x,y) := c(x,y) \wedge \neg sov(x,y)$ (x and y are external connected)

Based on connectedness, we introduce the notion of (*the number of*) *connected components*. It comes in three flavors, each of which classifies a given entity according to counting its building blocks based on two-, one- or zero-dimensional connectedness. This notwithstanding, the definitions apply to entities of any dimension. The parametrized schemata D28–D29 capture the corresponding notions inductively.

D28. $1_{\mathbf{d}}CC(x) := \mathbf{d}DC(x)$ (x has one 2-dim./1-dim./0-dim. connected component)

D29. $n_{\mathbf{d}}CC(x) := \bigwedge_{i=1}^{n-1} \neg i_{\mathbf{d}}CC(x) \wedge \exists x_1 ... x_n (partition_n(x_1, ..., x_n, x) \wedge \bigwedge_{i=1}^{n} 1_{\mathbf{d}}CC(x_i))$
(x has n 2-dim./1-dim./0-dim. connected components)

4.4. Common Spatial Categories

Most space entities that are occupied by material entities can be observed to be connected and ordinary entities. This leads us to defining categories associated with well-known terms on the basis of connectedness and ordinariness. The common terminology of *point*, *line* and *surface* is employed, whereas a connected three-dimensional object is called a *topoid*. We argue that their definitions are adequate in the context of the present theory $\mathcal{BS}$ to the extent that they reflect intuitive assumptions that can be stated using the given vocabulary. From a formal point of view, within $\mathcal{BS}$ some of the definitions could be further simplified, e.g., ordinariness could be ommitted in D30, because only lower-dimensional entities can have coincident parts (via A7, A22), such that all space regions are ordinary entities.

D30. $Top(x) := SReg(x) \wedge Ord(x) \wedge 2DC(x)$ (x is a topoid)

D31. $2D(x) := 2DE(x) \wedge Ord(x) \wedge 1DC(x)$ (x is a surface)

D32. $1D(x) := 1DE(x) \wedge Ord(x) \wedge 0DC(x)$ (x is a line)

D33. $0D(x) := 0DE(x) \wedge Ord(x) \wedge \neg\exists y\ sppart(y, x)$ (x is a point)

4.5. Axioms of the Theory $\mathcal{BS}$

Having laid out the terminology and definitions for $\mathcal{BS}$, its axiomatization follows in this section. The axioms are presented in manageable thematic groups.

Basic Taxonomy and Existence In order to avoid a trivial theory we assume that there is at least one space region. The overall domain of space entities is divided into four mutually disjoint classes, namely space regions, surface regions, line regions, and point regions. In interaction with other axioms it follows from A1 that each category is not only non-empty, but each has infinitely many instances. Each category accounts for entities *sui generis*. That means, a space entity cannot be captured by a set of lower-dimensional entities and, in particular, it cannot be equated with the set of its of hyperparts.

A1. $\exists x\ SReg(x)$ (existence of a space region)

A2. $LDE(x) \leftrightarrow \neg SReg(x)$ (exhaustive and mutually exclusive)

A3. $\neg\exists x\ (\ (2DE(x) \wedge 1DE(x)) \vee (2DE(x) \wedge 0DE(x)) \vee (1DE(x) \wedge 0DE(x))\ )$
(three mutually disjoint classes)

Mereological Considerations Following established spatial theories, it is assumed that spatial parthood satisfies the conditions of a partial ordering. Two entities in parthood relation must be of equal dimension. This leads to equal-dimensionality of the arguments of all mereological relations defined in D10–D15 ($sppart$, sov, sum, etc.).

Mereological systems differ in their basic assumptions about supplementation, atomicity and extensibility. We deem the strong supplementation principle (A8) adequate for space entities, in spite of criticism if A8 is considered for material or abstract entities, cf. [15, sect. 3.3]. Atomic space entities are entities without spatial proper parts. We postulate that at least one proper inner part is possessed by all space entities except for points, which have no proper parts by definition. This axiom may constitute a difference between theories of spatial and material entities, since it is reasonable to assume that there are atomic material entities. Vice versa, all space entities are spatial proper parts of another one (A10). With further axioms this implies the existence of an infinite increasing sequence of topoids, linearly ordered by the relation $sppart$. However, the mereological sum of the components of such a sequence and the sum of infinitary constructions in general does not necessarily determine a space region, e.g. if the construction is unbounded in extension. The absence of a general fusion axiom is thus deliberate.

A4. $spart(x, x)$ (reflexivity)

A5. $spart(x, y) \wedge spart(y, x) \rightarrow x = y$ (antisymmetry)

A6. $spart(x, y) \wedge spart(y, z) \rightarrow spart(x, z)$ (transitivity)

A7. $spart(x, y) \rightarrow eqdim(x, y)$ (domain restriction)

A8. $\neg spart(y, x) \rightarrow \exists z\ (spart(z, y) \wedge \neg sov(z, x))$ (strong supplementation principle)

A9. $\neg 0D(x) \rightarrow \exists y\ (sppart(y, x) \wedge inpart(y, x))$ (only points are atomic)

A10. $\exists y\ sppart(x, y)$ (extensibility)

Existence of Space Entities We require that each space region can be embedded in a single connected region, i.e. a topoid. Furthermore, space regions are necessarily bounded entities. For two- and one-dimensional entities, however, we can only claim that there is at least a spatial part with a boundary. For example, a circle has clearly no boundary – but any spatial proper part does. The existence of a greatest spatial boundary must be guaranteed for bounded, ordinary entities. The final three axioms in this group postulate the existence of entities due to standard mereological operations. Note that the latter are constrained to at least equal-dimensional entities.

A11. $SReg(x) \rightarrow \exists y\ (Top(y) \wedge spart(x, y))$ (embedding topoid for space regions)

A12. $SReg(x) \rightarrow \exists y\ sb(y, x)$ (unrestricted existence of boundaries)

A13. $2DE(x) \rightarrow \exists yz\ (sppart(y, x) \wedge sb(z, y))$ (restricted existence of boundaries)

A14. $1DE(x) \rightarrow \exists yz\ (sppart(y, x) \wedge sb(z, y))$ (restricted existence of boundaries)

A15. $Ord(x) \wedge \exists y\ sb(y, x) \rightarrow \exists z\ grsb(z, x)$ (restricted existence of a greatest boundary)

A16. $eqdim(x, y) \rightarrow \exists z\ sum(x, y, z)$ (existence of sum)

A17. $sov(x, y) \rightarrow \exists z\ intsect(x, y, z)$ (existence of mereological intersection)

A18. $eqdim(x, y) \wedge \neg spart(y, x) \rightarrow \exists z\ rcompl(x, y, z)$ (existence of relative complement)

Spatial boundaries Coincidence is a key feature of spatial boundaries in GFO-Space. Roughly speaking, two spatial boundaries coincide if they are literally not spatially distant from each other. Accordingly, we stipulate that coincidence is an equivalence relation on every class of equal-dimensional spatial boundaries.

We claim further that ordinariness inherits from a space entity to its boundaries, which we see as an intuitively accepted regularity among ordinary space entities. In combination with further axioms this entails that any spatial boundary is ordinary. Consequently, extraordinary space entities witness the fact that spatial boundaries form a proper subcategory of lower dimensional entities. It remains future work to study which further conditions are necessary to guarantee being a spatial boundary.

Axiom A25 says that boundaries of tangential parts that coincide with boundaries of the respective whole are likewise boundaries of that whole. In this sense, A25 claims that tangential parts do not generate new boundaries. Beyond that, combination with further axioms entails that it is actually one and the same boundary, of the part and the whole.

A19. $SB(x) \rightarrow scoinc(x, x)$ (reflexivity)

A20. $scoinc(x, y) \rightarrow scoinc(y, x)$ (symmetry)

A21. $scoinc(x, y) \wedge scoinc(y, z) \rightarrow scoinc(x, z)$ (transitivity)

A22. $scoinc(x, y) \rightarrow eqdim(x, y) \wedge SB(x) \wedge SB(y)$ (domain)

A23. $sb(x, y) \wedge Ord(y) \rightarrow Ord(x)$ (ordinariness and spatial boundaries)

A24. $sb(y, z) \wedge spart(x, y) \rightarrow sb(x, z)$ (parts of boundaries are boundaries)

A25. $\forall xx'yy'\ (tangpart(x, y) \wedge sb(x', x) \wedge sb(y', y) \wedge scoinc(x', y') \rightarrow sb(x', y))$
(there are no new boundaries)

A26. $sb(x, y) \rightarrow (2DB(x) \wedge SReg(y)) \vee (1DB(x) \wedge 2DE(y)) \vee (0DB(x) \wedge 1DE(y))$
(domain of spatial boundary)

Spatial Coincidence and Distinct Hyper Parts The first three axioms of this final group formalize the intuition of “congruence” of coincident space entities, in that these have corresponding coincident parts, hyper parts and boundaries. Moreover, it appears natural that equal-dimensional, non-overlapping entities cannot share any hyper parts.

A27. $scoinc(x, y) \wedge spart(x', x) \rightarrow \exists y' \ (spart(y', y) \wedge scoinc(x', y'))$ (existence of coincident spatial parts)

A28. $scoinc(x, y) \wedge hypp(x', x) \rightarrow \exists y' \ (hypp(y', y) \wedge scoinc(x', y'))$ (existence of coincident hyper parts)

A29. $scoinc(x, y) \wedge sb(x', x) \rightarrow \exists y' \ (sb(y', y) \wedge scoinc(x', y'))$ (existence of coincident spatial boundaries)

A30. $eqdim(x, y) \wedge \neg sov(x, y) \wedge hypp(x', x) \wedge hypp(y', y) \rightarrow x' \neq y'$ (distinct hyper parts)

4.6. *Theorems of the Theory $\mathcal{BS}$*

This section gathers a number of consequences of the axiomatization above. All proofs are omitted due to space limitations. Let us start with entailed identity conditions.

Identity Principles The initial three identity principles result mainly from the basic mereological axioms A4–A6 in combination with the strong supplementation principle A8. These axioms yield an extensional mereology. More specific to $\mathcal{BS}$, an analogous identity criterion to T3 turns out to hold for hyper parts. The most surprising identity principle for ourselves is T5, which requires only the definitions of hyper parts D16–19 and axioms A5, A8, A12–14 and A30 to be derived. It reveals that any space entity can be identified based on the set of its hyper parts that are points. At a closer look, this does *not* conflict with Brentano's position of not equating space entities with certain sets, e.g. of points. Identifying an entity *based on* the set of its point hyper parts does not entail the identification of the entity *with* that set – indeed, equating these two must be clearly rejected. Similarly (and if it exists at all[6]), the mereological sum of all point hyper parts of an entity does typically not yield that entity, as summation cannot transcend dimensions.

T1. $\forall z \ (spart(z, x) \leftrightarrow spart(z, y)) \leftrightarrow x = y$ (extensionality, identity by equal parts)

T2. $\forall z \ (spart(x, z) \leftrightarrow spart(y, z)) \leftrightarrow x = y$ (identity by equal wholes)

T3. $\exists u \ (sppart(u, x) \vee sppart(u, y)) \rightarrow (\forall z \ (sppart(z, x) \leftrightarrow sppart(z, y)) \leftrightarrow x = y)$ (identity by equal proper parts)

T4. $\exists u \ (hypp(u, x) \vee hypp(u, y)) \rightarrow (\forall z \ (hypp(z, x) \leftrightarrow hypp(z, y)) \leftrightarrow x = y)$ (identity by equal hyper parts)

T5. $\exists u \ (0dhypp(u, x) \vee 0dhypp(u, y)) \rightarrow (\forall z \ (0dhypp(z, x) \leftrightarrow 0dhypp(z, y)) \leftrightarrow x = y)$ (identity by equal points as hyper parts)

No Least and Greatest Elements, Embeddings Especially existence postulates like requiring an inner proper part for all entities except for points or stipulating embedding topoids for space regions (A11) contribute to the consequence regarding the spatial part and hyper part relations that there is neither a greatest space entity ("space" as a whole), nor a least one (nothing is part or hyper part of every space entity). Moreover, A11 is instrumental in deriving further embedding properties, up to the general embedding result that any two space entities, possibly of distinct dimension, have a common framing topoid. Loosely speaking, there are no separate "areas of space" with unsurmountable gaps between their entities.

T6. $\neg \exists x \forall y \ (spart(x, y) \vee hypp(x, y))$ (no least space entity)

T7. $\neg \exists x \forall y \ (spart(y, x) \vee hypp(y, x))$ (no greatest space entity)

T8. $\exists y \ (Top(y) \wedge (spart(x, y) \vee hypp(x, y)))$ (embedding for arbitrary single entities)

[6] The lack of any general fusion principle, cf. [27], is briefly discussed in section 4.5.

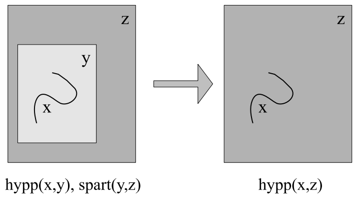

Figure 6. Upward Compatibility

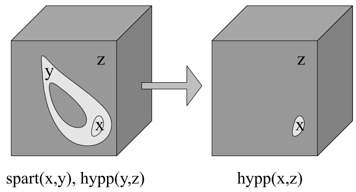

Figure 7. Downward Compatibility

T9. $\exists z\ (Top(z) \wedge ((spart(x,z) \vee hypp(x,z)) \wedge (spart(y,z) \vee hypp(y,z)))$ (general embedding)

Hyper Part Transitivity and Compatibility The proof of T9 relies on various further axioms in addition to A11, of course. In particular, the following three theorems are employed as lemmas and constitute interesting properties of the hyper part relation.

T10. $hypp(x,y) \wedge hypp(y,z) \rightarrow hypp(x,z)$ (transitivity of hyper part)

T11. $hypp(x,y) \wedge spart(y,z) \rightarrow hypp(x,z)$ (upward compatibility of hyper part)

T12. $spart(x,y) \wedge hypp(y,z) \rightarrow hypp(x,z)$ (downward compatibility of hyper part)

Connectedness Interrelations The notions of connectedness and of connected components are each present in three dimension-based variants in $\mathcal{BS}$. The final set of results provides insights on subsumptions among the variants of connectedness as well as on interconnections of the numbers of connected components of different kinds.

T13. $2DC(x) \rightarrow 1DC(x)$ (subsumption of two- by one-dim. connectedness)

T14. $1DC(x) \rightarrow 0DC(x)$ (subsumption of one- by zero-dim. connectedness)

T15. $n_2CC(x) \rightarrow \bigvee_{i=0}^{n-1}(n-i)_1CC(x)$ (limit of one-dim. connected components)

T16. $n_1CC(x) \rightarrow \bigvee_{i=0}^{n-1}(n-i)_0CC(x)$ (limit of zero-dim. connected components)

Proposition 1 *The numbers of connected components of any space entity can only decrease along the dimensionality under consideration: for any model $\mathcal{A} \models \mathcal{BS}$ such that $\mathcal{A} \models n_2CC(x) \wedge l_1CC(x) \wedge k_0CC(x)$, it is the case that $n \geq l \geq k$.*

5. Applications

An ontology of space is a necessary prerequisite for the development of an ontology of material objects, the modeling and representation of which leads to a number of practical applications. First, the space ontology of GFO, extended by the ontology of material entities, closes some gaps of earlier theories of boundaries. For example, in [5,15] it is stated and it follows from the axioms in [5] that bona fide boundaries (see section 3) cannot be in contact. This is not plausible, as also criticized in [13,28,29]. The GFO approach to space and material objects clearly distinguishes material and spatial boundaries, restricting coincidence to the latter. This allows for a new solution: two material boundaries are in contact if the corresponding occupied space boundaries coincide.

A related field of potential applications links to anatomy and to the Foundational Model of Anatomy (FMA) [30], a very large ontology of anatomic categories and relations. Referring to FMA, [31] outlines ideas on an anatomical information science. Clearly, this must be grounded on a coherent and consistent ontology of space and of material objects. The four upper-level FMA categories anatomical structure, anatomical substance, anatomical space and anatomical boundary, and others like body spaces suggest the applicability of GFO-Space. Not only in view of the problem of boundaries and contact above, which conveys to boundaries in [31], we argue that GFO-Space lends itself to providing alternatives for understanding, e.g., anatomical structure, bona fide and fiat boundary, connectedness, continuity, as well as part, containment and location. Another aspect in this connection is the strength of the axiomatization available.

Overall, as a mereotopological theory, especially when integrated in a wider framework such as GFO, the presented ontology can be applied in various areas, cf. the survey of corresponding fields in [14, sect. 9]. For example, geography and Geographical Information Science preserve their actuality in this respect, because many geographical objects are intrinsically related to space, and, in particular, their boundaries play a decisive role. Finally, we expect that also qualitative spatial reasoning can benefit from an axiomatic foundation of space entities that is adequate to cognition.

6. Conclusions and Future Work

As the main contribution of this paper we provide the theory GFO-Space as a basic axiomatization of the phenomenal space of material objects. This notion is inspired by the work of Franz Brentano [3] and we hold the view that the resulting theory is compatible with our visual experience. GFO-Space further constitutes the ontology of space of the top-level ontology General Formal Ontology (GFO) [1] and as such it complements the ontology of time of GFO [2], as well as both share some principles. The theory is developed in first-order logic and is outlined herein by means of selected definitions, axioms and an initial set of noteworthy consequences. Another reason for the term ‘outlined’ is that we have not presented a proof of consistency. While an unpublished proof sketch exists, its completion to a detailed proof is an immediate next step in the metatheoretic analysis of the theory. Subsequently, a proper comparison with closely related theories, e.g. those in [21,23], is another promising task.

We believe that GFO-Space already documents the conceptual richness of the domain of space. Nevertheless, the investigation of phenomenal space and its relations to material objects remains in an initial stage. Hence, there are a number of open ontological problems whose further investigation may be interesting and fruitful. We sketch some of these issues, which are related to pure space as well as material objects.

Morphology of Pure Space Entities Space entities exhibit forms, therefore a further step in our work is the ontological investigation of morphological structures. Forms cannot be captured by the principles of pure mereotopology. Our idea is to introduce only few additional primitives and to remain mainly in the framework of mereotopology. For this purpose, we may introduce certain standard forms, for example, the ball or the cube, formally by adding predicates $ball(x)$ and $cube(x)$. Then we can try to grasp intuitions about these forms axiomatically. For example, the following axiom may be employed to characterize the category of balls (cf. [32]): If x and y are balls, then their mereological relative complement is connected.

Mereotopology and Morphology of Material Objects The investigation of material objects with respect to their mereotopological and morphological properties opens a new field of research, because phenomena of essentially new character occur. One basic insight, already discussed herein, is the fact that material boundaries must be distinguished from spatial boundaries. The relation between these different types of boundaries is that of occupation. Furthermore, in connection with material boundaries the notion of granularity must be taken into account, whereas granularity plays only a minor role for space entities. Initial ideas on these topics are already indicated in [1].

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