# The Equivalence Zoo for Dung-style Semantics

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**Abstract**

Notions of equivalence which are stronger than standard equivalence in the sense that they also take potential modifications of the available information into account have received considerable interest in nonmonotonic reasoning. In this paper we focus on equivalence notions in argumentation. More specifically, we establish a number of new results about the relationships among various equivalence notions for Dung argumentation frameworks which are located between strong equivalence [16] and standard equivalence. We provide the complete picture for this variety of equivalence relations (which we call the *equivalence zoo*) for stable, preferred, admissible and complete semantics.

**keywords:** argumentation, expansion equivalence

## 1 Introduction

Notions of equivalence which are stronger than standard equivalence in the sense that they also take potential modifications of the available information into account have received considerable interest in nonmonotonic reasoning, and in particular in logic programming [15, 14]. In this paper we focus on equivalence notions in argumentation. Formal argumentation has developed into a highly active field within Artificial Intelligence over the last decades. For a very good overview see [10]. Dung's abstract argumentation frameworks (AFs) [13] play a dominant role in the area. In AFs arguments and attacks among them are treated as abstract entities. The focus is on conflict resolution and argument acceptability. Various semantics for AFs have been defined, each of them specifying acceptable sets of arguments, so-called *extensions*, in a particular way. In a nutshell (cf. [8, Section 1.1.1] for more details), the typical use of AFs can be characterized as follows: starting from some knowledge base expressed in a potentially rich KR language, one constructs arguments, that is structures containing a proposition together with reasons for accepting them, and conflicts among them (so-called attacks). The arguments, viewed as abstract entities, are then evaluated using an AF. The accepted propositions then are those which are supported by an argument which is accepted under the chosen AF semantics.

Argumentation is an inherently dynamic process. It is thus apparent that equivalence notions which guarantee mutual replaceability of two AFs - without any loss of information - in specific dynamic argumentation scenarios, that is, even under potential expansions of the current AF, are highly significant. For this reason, the study of various such equivalence notions has become an active and fruitful research line over the last years. Standard equivalence of two AFs $\mathcal{F}$ and $\mathcal{G}$, i.e. both possess the same extensions, guarantees that all queries w.r.t. credulously or skeptically accepted arguments are answered identically. However, this does not mean that $\mathcal{F}$ can simply be replaced by $\mathcal{G}$: the same modification applied to both $\mathcal{F}$ and $\mathcal{G}$ may lead to AFs which are not equivalent in any reasonable sense.

Strong equivalence [16], in contrast, even guarantees that both AFs possess the same extensions *under arbitrary expansions*. As its name suggests, this is a very (and often unrealistically) strong notion of equivalence. In many argumentation scenarios the type of modification which may potentially occur can be anticipated. For this reason, in [5] the middle ground between these two extremes was investigated, i.e. various intermediate equivalence notions taking into account specific anticipated types of expansions reflecting the nature of various argumentation scenarios were defined and characterized. Furthermore, in [6] the notion of minimal change equivalence between two AFs was introduced which guarantees that the minimal effort needed to convince the participants of a certain opinion $E$ (a set of arguments) is identical.

In this paper we present a number of new results about the relationships among the mentioned equivalence notions. Our results provide a complete picture about the relationships among these notions for some of the most relevant semantics of Dung-style AFs, namely stable, preferred, admissible and complete semantics. It turns out that minimal change equivalence naturally fits into this equivalence zoo, although its definition includes a graph-theoretical distance function and therefore an arithmetic aspect, in contrast to standard or strong equivalence, or any of the other considered intermediate variants. Furthermore we clarify an open question concerning the characterizations for preferred, admissible and complete semantics w.r.t. weak expansions.

The rest of the paper is organized as follows. Sect. 2 reviews the necessary background. Sect. 3 presents our results. We first give a characterization of weak expansion equivalence (Sect. 3.1) and then provide the complete picture about the relationships among 10 different equivalence notions for stable (Sect. 3.2), preferred (Sect. 3.3), admissible (Sect. 3.4) and complete (Sect. 3.5) semantics. Sect. 4 concludes the paper.

# 2 Background

An *argumentation framework* $\mathcal{F}$ is a pair $(A, R)$, where $A$ is a non-empty finite set whose elements are called *arguments* and $R \subseteq A \times A$ a binary relation, called the *attack relation*. The set of all AFs is denoted by $\mathscr{A}$. If $(a, b) \in R$ holds we say that $a$ *attacks* $b$, or $b$ is *defeated* by $a$ in $\mathcal{F}$. An argument $a \in A$ is *defended* by a set $A' \subseteq A$ in $\mathcal{F}$ if for each $b \in A$ with $(b,a) \in R$, $b$ is defeated by some $a' \in A'$ in $\mathcal{F}$. Furthermore, we say that a set $A' \subseteq A$ is *conflict-free* in $\mathcal{F}$ if there are no arguments $a, b \in A'$ such that $a$ attacks $b$. The set of all conflict-free sets of an AF $\mathcal{F}$ is denoted by $cf(\mathcal{F})$. For an AF $\mathcal{F} = (B, S)$ we use $A(\mathcal{F})$ to refer to $B$ and $R(\mathcal{F})$ to refer to $S$. Finally, we introduce the union of two AFs as usual, namely $\mathcal{F} \cup \mathcal{G} = (A(\mathcal{F}) \cup A(\mathcal{G}), R(\mathcal{F}) \cup R(\mathcal{G}))$.

Semantics determine acceptable sets of arguments for a given AF $\mathcal{F}$, so-called *extensions*. The set of all extensions of $\mathcal{F}$ under semantics $\sigma$ is denoted by $\mathcal{E}_\sigma(\mathcal{F})$. We consider the semantics already introduced by Dung [13], namely stable ($st$), preferred ($pr$), complete ($co$) as well as admissible ($ad$) semantics.

**Definition 1** (**Semantics**). *Given an AF $\mathcal{F} = (A, R)$ and $E \subseteq A$. $E$ is a*

1. *stable extension ($E \in \mathcal{E}_{st}(\mathcal{F})$) iff*
   *$E \in cf(\mathcal{F})$ and each $a \in A \backslash E$ is defeated by some $e \in E$,*

2. *admissible extension*[1] *($E \in \mathcal{E}_{ad}(\mathcal{F})$) iff*
   *$E \in cf(\mathcal{F})$ and each $e \in E$ is defended by $E$ in $\mathcal{F}$,*

3. *complete extension ($E \in \mathcal{E}_{co}(\mathcal{F})$) iff*
   *$E \in \mathcal{E}_{ad}(\mathcal{F})$ and for each $e \in A$ defended by $E$ in $\mathcal{F}$, $e \in E$,*

4. *preferred extension ($E \in \mathcal{E}_{pr}(\mathcal{F})$) iff*
   *$E \in \mathcal{E}_{ad}(\mathcal{F})$ and for each $E' \in \mathcal{E}_{ad}(\mathcal{F})$, $E \notin E'$ and*

There are several relations between the considered semantics. In particular, for any AF $\mathcal{F}$, $\mathcal{E}_{st}(\mathcal{F}) \subseteq \mathcal{E}_{pr}(\mathcal{F}) \subseteq \mathcal{E}_{co}(\mathcal{F}) \subseteq \mathcal{E}_{ad}(\mathcal{F})$ [13]. The converses do not hold in general but if, for instance, the considered AFs are SCC-symmetric and self-loop-free, then stable and preferred semantics coincide (compare [2]).

In [9] several types of expansions were introduced. They will be our object of investigation since they represent reasonable types of dynamic argumentation scenarios.

**Definition 2** (**Expansions**). *An AF $\mathcal{F}^*$ is an expansion of AF $\mathcal{F} = (A, R)$ (for short, $\mathcal{F} \preceq_E \mathcal{F}^*$) iff $\mathcal{F}^* = (A \cup A^*, R \cup R^*)$ where $A^* \cap A = R^* \cap R = \varnothing$. An expansion is*

1. *normal ($\mathcal{F} \preceq_N \mathcal{F}^*$) iff $\forall ab \ ((a,b) \in R^* \to a \in A^* \vee b \in A^*)$,*

2. *strong ($\mathcal{F} \preceq_S \mathcal{F}^*$) iff $\mathcal{F} \preceq_N \mathcal{F}^*$ and $\forall ab \ ((a,b) \in R^* \to \neg(a \in A \wedge b \in A^*))$,*

3. *weak ($\mathcal{F} \preceq_W \mathcal{F}^*$) iff $\mathcal{F} \preceq_N \mathcal{F}^*$ and $\forall ab \ ((a,b) \in R^* \to \neg(a \in A^* \wedge b \in A))$,*

4. *local ($\mathcal{F} \preceq_L \mathcal{F}^*$) iff $A^* = \varnothing$.*

In a nutshell, normal expansions add new arguments and possibly new attacks which concern at least one of the fresh arguments. Strong (weak) expansions are normal and only add *strong (weak) arguments*, i.e. the added

[1] Note that it is more common to speak about admissible sets instead of the admissible semantics. For reasons of unified notation we used the uncommon version.

arguments never are attacked by (attack) former arguments. Normal expansions naturally occur in case of instantiation-based argumentation. If one adds a new piece of information to the underlying knowledge base, then only new arguments which may interact with the previous ones arise.

As usual $\mathcal{F} \prec_X \mathcal{F}^*$ for $X \in \{E, N, S, W, L\}$ stands for $\mathcal{F} \preceq_X \mathcal{F}^*$ and $\mathcal{F} \neq \mathcal{F}^*$. To simplify notation we will later on often use $X$ to refer to $\preceq_X$. Whenever infix notation is used we stick to $\preceq_X$, though.

The *minimal change problem* [6] is the problem of determining the minimal effort needed to transform a given argumentation framework, using a particular type of modifications $\Phi$, into a framework that possesses an $\sigma$-extension containing a specific set of arguments $C$. The effort is characterized by the $(\sigma, \Phi)$-*characteristic*:

**Definition 3** (**Characteristic**). *Given a semantics $\sigma$, a binary relation $\Phi \subseteq \mathscr{A} \times \mathscr{A}$ and an AF $\mathcal{F}$. The $(\sigma, \Phi)$-characteristic of a set $C \subseteq A(\mathcal{F})$ is a natural number or infinity defined by the following function*

$$N^{\mathcal{F}}_{\sigma,\Phi} : \wp(A(\mathcal{F})) \to \mathbb{N}_\infty$$

$$C \mapsto \begin{cases} 0, & \exists C' : C \subseteq C' \textit{ and } C' \in \mathcal{E}_\sigma(\mathcal{F}) \\ k, & k = \min\{d(\mathcal{F}, \mathcal{G}) \mid (\mathcal{F}, \mathcal{G}) \in \Phi, N^{\mathcal{G}}_{\sigma,\Phi}(C) = 0\} \\ \infty, & \textit{otherwise.} \end{cases}$$

Here $d(\mathcal{F}, \mathcal{G})$ is the number of added or removed attacks needed to transform $\mathcal{F}$ to $\mathcal{G}$, i.e. $d(\mathcal{F}, \mathcal{G}) = |R(\mathcal{F}) \,\Delta\, R(\mathcal{G})|$ where $\Delta$ is the well-known symmetric difference.

The following notions of equivalence have been studied in the literature [16, 5, 6]. For the sake of clarity and comprehensibility we use *expansion equivalence* instead of *strong equivalence* (the term originally coined in [16]) to indicate that arbitrary expansions are allowed.

**Definition 4** (**Equivalence**). *Given a semantics $\sigma$. Two AFs $\mathcal{F}$ and $\mathcal{G}$ are*

1. *standard equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma \mathcal{G}$) iff they possess the same extensions under $\sigma$, i.e. $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ holds,*

2. *expansion equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma_E \mathcal{G}$) iff for each AF $\mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^\sigma \mathcal{G} \cup \mathcal{H}$ holds,*

3. *normal expansion equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma_N \mathcal{G}$) iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_N \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^\sigma \mathcal{G} \cup \mathcal{H}$ holds,*

4. *strong expansion equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma_S \mathcal{G}$) iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_S \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_S \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^\sigma \mathcal{G} \cup \mathcal{H}$ holds,*

5. *weak expansion equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma_W \mathcal{G}$) iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_W \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_W \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^\sigma \mathcal{G} \cup \mathcal{H}$ holds,*

6. *local expansion equivalent w.r.t. $\sigma$ ($\mathcal{F} \equiv^\sigma_L \mathcal{G}$) iff for each AF $\mathcal{H}$, s.t. $A(\mathcal{H}) \subseteq A(\mathcal{F} \cup \mathcal{G})$, $\mathcal{F} \cup \mathcal{H} \equiv^\sigma \mathcal{G} \cup \mathcal{H}$ holds,*

7. *minimal change equivalent* ($\mathcal{F} \equiv^{\sigma,MC}_{\Phi} \mathcal{G}$) *w.r.t.* $\sigma$ *and a binary relation* $\Phi \subseteq \mathscr{A} \times \mathscr{A}$ *iff for all* $E$*, s.t.* $E \subseteq A(\mathcal{F})$ *or* $E \subseteq A(\mathcal{G})$*,* $N^{\mathcal{F}}_{\sigma,\Phi}(E) = N^{\mathcal{G}}_{\sigma,\Phi}(E)$.

# 3 Analyzing the Equivalence Zoo

In the recent literature many new equivalence relations were discussed (see Def. 4). Each of them captures different conditions for mutual replaceability in certain dynamic scenarios. In this section we want to shed light on the equivalence zoo by providing a complete analysis w.r.t. ten different notions of equivalence, namely those introduced in Def. 4 where the relation $\Phi$ in the definition of minimal change equivalence is instantiated by arbitrary, normal, weak and strong expansions. Besides the general case, i.e. considering arbitrary AFs, we also provide results for two special cases, namely the case where the AFs do not contain self-loops, i.e. attacks of the form $(a, a)$ for some argument $a$, and the case where two AFs have the same arguments.

## 3.1 Characterizing Weak Expansion Equivalence

In case of stable semantics it is already shown (cf. [4, Proposition 3]) that two AFs are weak expansion equivalent iff both do not possess stable extensions at all or if they share the same arguments and at the same time possess the same stable extensions.

In the following we present characterization theorems for preferred, complete and admissible semantics in case of weak expansions. For any considered semantics $\sigma$ it turns out that two AFs are weak expansion equivalent iff they share the same arguments, possess the same $\sigma$-extensions and furthermore, for any $\sigma$-extension $E$ the set of arguments which are not in the $\sigma$-extension without being refuted has to coincide in both AFs. The proof is mainly based on splitting results firstly published in [4].

**Theorem 1.** *Let* $\sigma \in \{pr, co, ad\}$*. For any two AFs* $\mathcal{F}$*,* $\mathcal{G}$ *we have* $\mathcal{F} \equiv^{\sigma}_{W} \mathcal{G}$ *iff* $A(F) = A(G)$*,* $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ *and for each* $E \in \mathcal{E}_\sigma(\mathcal{F}) : U^{\mathcal{F}}_{E} = U^{\mathcal{G}}_{E}$ *where* $U^{\mathcal{A}}_{E} = \{a \in A(\mathcal{A}) \mid a \notin E \wedge (E, a) \notin R(\mathcal{A})\}$.

*Proof.* ($\Leftarrow$) Given an AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_W \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_W \mathcal{G} \cup \mathcal{H}$. We have to show that $\mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$. If $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$, then $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$ since $A(\mathcal{F}) = A(\mathcal{G})$ is assumed. In consideration of $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ the assertion follows. Assume now that $\mathcal{F} \neq \mathcal{F} \cup \mathcal{H}$. Using splitting results [4, Theorem 2, statement 2] one may easily show that $E \in \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H})$ implies $E \in \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$ (and vice versa).
($\Rightarrow$) We will show the contrapositive, i.e. if $A(F) \neq A(G)$ or $\mathcal{E}_\sigma(\mathcal{F}) \neq \mathcal{E}_\sigma(\mathcal{G})$ or there exists an $E \in \mathcal{E}_\sigma(\mathcal{F})$, s.t. $U^{\mathcal{F}}_{E} \neq U^{\mathcal{G}}_{E}$, then $\mathcal{F} \not\equiv^{\sigma}_{W} \mathcal{G}$. Consider $E \in \mathcal{E}_\sigma(\mathcal{F})$ and $E \notin \mathcal{E}_\sigma(\mathcal{G})$. Consequently, $E \cup \{d\} \in \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H})$ and $E \cup \{d\} \notin \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$ where $\mathcal{H} = (\{d\}, \emptyset)$ and $d$ is a fresh argument, i.e. $d \notin A(\mathcal{F}) \cup A(\mathcal{G})$. Assume now $A(F) \neq A(G)$ and $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$. W.l.o.g. let $a \in A(F) \setminus A(G)$. Consequently, there is no $\sigma$-extension $E$, s.t. $a \in E$. If $\mathcal{H} = (\{a\}, \emptyset)$, then $\mathcal{F} \cup \mathcal{H} = \mathcal{F}$ and thus, there is no $E \in \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H})$, s.t. $a \in E$. On the other hand, $\{a\}$ is admissible

in $\mathcal{G} \cup \mathcal{H}$. Furthermore, since $a$ is unattacked in $\mathcal{G} \cup \mathcal{H}$ we deduce that $a$ ha s to be contained in the grounded extension and thus, $a$ is contained in at least one preferred extension of $\mathcal{G} \cup \mathcal{H}$. Altogether, $\mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H}) \neq \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$ is shown. Finally, we consider $A(F) = A(G)$ and $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ but there exists an $E \in \mathcal{E}_\sigma(\mathcal{F})$, s.t. $U_E^{\mathcal{F}} \neq U_E^{\mathcal{G}}$. W.l.o.g. let $a \in U_E^{\mathcal{G}} \setminus U_E^{\mathcal{F}}$. This means, $a \notin E$, $(E, a) \notin R(\mathcal{G})$ and $(E, a) \in R(\mathcal{F})$, i.e. $a$ is attacked by $E$ in $\mathcal{F}$. Consider now $\mathcal{H} = (\{a, b\}, \{(a, b)\})$ where $b$ is a fresh argument. One can easily see that $E \cup \{b\} \in \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H})$ (compare splitting results [4, Theorem 2, statement 1]) but $E \cup \{b\} \notin \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{H})$ since $b$ is not defended by $E$. Therefore, $E \cup \{b\} \notin \mathcal{E}_\tau(\mathcal{G} \cup \mathcal{H})$ where $\tau \in \{pr, co\}$. Altogether, $\mathcal{F} \not\equiv_W^\sigma \mathcal{G}$ is shown. □

## 3.2 Stable Semantics

The following proposition characterizes stable semantics in general. In the interest of readability we present our results not only in terms of propositions, but also graphically. Our graphics will contain boxes connected by directed arrows. The boxes contain the names of equivalence notions, separated by +. Notions within the same box are equivalent: if, say, a box contains $e$ and $e'$, then 2 argumentation frameworks $\mathcal{F}$ and $\mathcal{G}$ are $e$-equivalent iff they are $e'$-equivalent. Links between two boxes represent implication: if box $B_1$ is connected via a directed link to box $B_2$ and the former contains $e$, the latter $e'$, then whenever $\mathcal{F}$ and $\mathcal{G}$ are $e$-equivalent they are also $e'$-equivalent. Note that whenever there is a link representing an implication, the converse implication does *not* hold.

**Proposition 1.** *For stable semantics and arbitrary argumentation frameworks the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ local expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ standard equivalence,*
- *expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ MC equivalence with weak expansion $\subsetneq$ weak expansion equivalence.*

Figure 1 describes the results for stable semantics graphically. In case of stable semantics only local expansion equivalence and the family of minimal change equivalence relations are unrelated. For any other two equivalence relations we have at least one implication chain. In particular, the different forms of minimal change equivalence *are shown to be intermediate forms between strong expansion and weak expansion equivalence.*

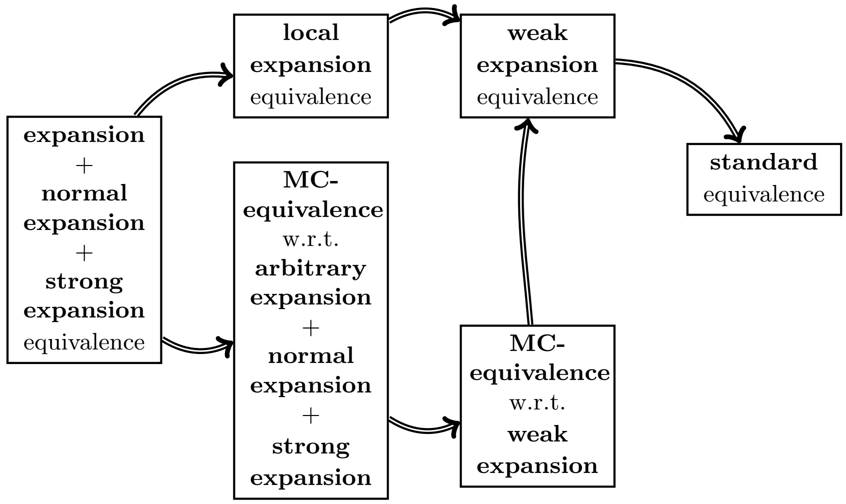



Figure 1: Stable semantics in general

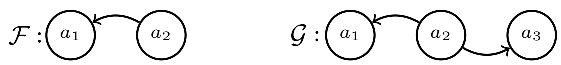
*Proof.* In [5] (Theorem 13) it was already shown that $\mathcal{F} \equiv_E^{st} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_N^{st} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_S^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv_L^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st} \mathcal{G}$. Since stable semantics satisfies regularity, i.e. expansion equivalent AFs have to share the same arguments (compare Definition 3, Theorem 1 in [16]), we conclude that $\mathcal{F} \equiv_E^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv_E^{st,MC} \mathcal{G}$ (Theorem 14 [6]). Furthermore, by applying Theorem 9, Definition 10 [6] we deduce $\mathcal{F} \equiv_E^{st,MC} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_N^{st,MC} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_S^{st,MC} \mathcal{G}$.

We will show now that $\mathcal{F} \equiv_S^{st,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{st,MC} \mathcal{G}$. Assume $\mathcal{F} \equiv_S^{st,MC} \mathcal{G}$ and $\mathcal{F} \not\equiv_W^{st,MC} \mathcal{G}$. Note that the first assumption implies that $A(\mathcal{F}) = A(\mathcal{G})$. The second assumption means that there is a set $E$, s.t. $N_{st,W}^{\mathcal{F}}(E) \neq N_{st,W}^{\mathcal{G}}(E)$. W.l.o.g. we assume $N_{st,W}^{\mathcal{F}}(E) = \infty$ and $N_{st,W}^{\mathcal{G}}(E) = 0$ (Theorem 6, Definition 10 in [6]). Since the characteristic w.r.t. strong expansions does not exceed the characteristic w.r.t. weak expansions we have $N_{st,S}^{\mathcal{G}}(E) = 0$ (Proposition 10 [6]). Consequently (first assumption), $N_{st,S}^{\mathcal{F}}(E) = 0$ in contradiction to $N_{st,W}^{\mathcal{F}}(E) = \infty$ which proves the claimed implication.

We show now that $\mathcal{F} \equiv_W^{st,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{st} \mathcal{G}$. Assume $\mathcal{F} \equiv_W^{st,MC} \mathcal{G}$ and $\mathcal{F} \not\equiv_W^{st} \mathcal{G}$. First, minimal change equivalence implies $A(\mathcal{F}) = A(\mathcal{G})$. In [4] (Proposition 3) it was shown that two AFs are weak expansion equivalent w.r.t. stable semantics iff i) $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ or ii) $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \varnothing$. Consequently, $\mathcal{E}_{st}(\mathcal{F}) \neq \mathcal{E}_{st}(\mathcal{G})$. Let $E \in \mathcal{E}_{st}(\mathcal{F})$ and $E \notin \mathcal{E}_{st}(\mathcal{G})$. Hence, $N_{st,W}^{\mathcal{F}}(E) = 0$. Since minimal change equivalence is assumed, $N_{st,W}^{\mathcal{G}}(E) = 0$. Since we assumed $E \notin \mathcal{E}_{st}(\mathcal{G})$ there has to be a proper superset $E'$ of $E$, s.t. $E' \in \mathcal{E}_{st}(\mathcal{G})$. Consequently, $N_{st,W}^{\mathcal{G}}(E') = 0$ and therefore $N_{st,W}^{\mathcal{F}}(E') = 0$. This means there is a superset $E''$ of $E'$, s.t. $E'' \in \mathcal{E}_{st}(\mathcal{F})$. This means, there are two stable extensions $E, E''$ of $\mathcal{F}$, s.t. $E \subset E''$. This is impossible because stable semantics satisfies the I-maximality principle [3]. Altogether, the claimed implications are shown.

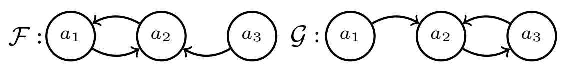
Now we present some counter-examples showing that the converse directions do not hold. It suffices to consider the following four cases. The other non-relations can be easily obtained by using the already shown relations presented in Figure 1.

1. $\mathcal{F} \equiv^{st} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{st}_{W} \mathcal{G}$.



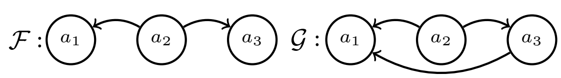

We have $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \{\{a_2\}\} \neq \varnothing$ and obviously, $A(\mathcal{F}) \neq A(\mathcal{G})$. In [4] (Proposition 3) it was shown that two AFs are weak expansion equivalent w.r.t. stable semantics iff i) $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ or ii) $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \varnothing$. Consequently, $\mathcal{F} \not\equiv^{st}_{W} \mathcal{G}$ and obviously, $\mathcal{F} \equiv^{st} \mathcal{G}$.

2. $\mathcal{F} \equiv^{st,MC}_{\Phi} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{st}_{L} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.



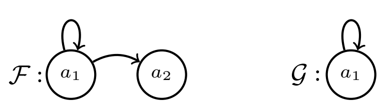

Both AFs share the same arguments. Furthermore, $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \{\{a_1, a_3\}\}$. Applying Definition 8 and Theorem 9 in [6] we conlude: First, for any $E \subseteq \{a_1, a_3\}$, we have $N^{\mathcal{F}}_{st,S}(E) = N^{\mathcal{G}}_{st,S}(E) = 0$. Second, $N^{\mathcal{F}}_{st,S}(\{a_2\}) = N^{\mathcal{G}}_{st,S}(\{a_2\}) = 1$ and third, for all not mentioned subsets $C$ of $A(\mathcal{F})$, $N^{\mathcal{F}}_{st,S}(C) = N^{\mathcal{G}}_{st,S}(C) = \infty$ because they contain at least one conflict. This verifies $\mathcal{F} \equiv^{st,MC}_{\Phi} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$ (Theorem 9, Theorem 13 in [6]). Consider the AFs $\mathcal{H} = (\{a_2, a_3\}, \{(a_2, a_3)\})$. We observe that $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) = \{\{a_1, a_3\}, \{a_2\}\} \neq \{\{a_1, a_3\}\} = \mathcal{E}_{st}(\mathcal{G}) = \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$. Thus, $\mathcal{F} \not\equiv^{st}_{L} \mathcal{G}$.

3. $\mathcal{F} \equiv^{st,MC}_{W} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{st,MC}_{\Phi} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.




Both AFs share the same arguments and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \{\{a_2\}\}$. Thus, $N^{\mathcal{F}}_{st,W}(\varnothing) = N^{\mathcal{G}}_{st,W}(\varnothing) = N^{\mathcal{F}}_{st,W}(\{a_2\}) = N^{\mathcal{G}}_{st,W}(\{a_2\}) = 0$. Furthermore, for any other subset $C$ of $A(\mathcal{F})$, $N^{\mathcal{F}}_{st,W}(C) = N^{\mathcal{G}}_{st,W}(C) = \infty$ because they are not contained in an extension (Definition 7, Theorem 6 in [6]). Consequently, $\mathcal{F} \equiv^{st,MC}_{W} \mathcal{G}$. On the other hand, $N^{\mathcal{F}}_{st,S}(\{a_1\}) = 1 \neq 2 = N^{\mathcal{G}}_{st,S}(\{a_1\})$ (compare Definition 8, Theorem 9 in [6]). This means, $\mathcal{F} \not\equiv^{st,MC}_{\Phi} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.

4. $\mathcal{F} \equiv^{st}_{L} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{st,MC}_{W} \mathcal{G}$.




Since minimal change equivalence implies sharing the same arguments we state $\mathcal{F} \not\equiv^{st,MC}_{W} \mathcal{G}$. Furthermore, it can be easily checked that for any AF $\mathcal{H}$, s.t.

$A(\mathcal{H}) \subseteq \{a_1, a_2\}$, we have $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$. Hence, $\mathcal{F} \equiv_L^{st} \mathcal{G}$. □

How does the situation change if we restrict our considerations to AFs possessing the same arguments? It turns out that the equivalence zoo collapses to only 3 distinct classes, and in contrast to the general case local expansion equivalence and the different forms of minimal change equivalence become comparable.

**Proposition 2.** *For stable semantics and argumentation frameworks with the same arguments the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *weak expansion equivalence = standard equivalence = MC equivalence with weak expansion,*
- *expansion equivalence ⊊ MC equivalence with arbitrary expansion ⊊ weak expansion equivalence.*

Here is the graphical representation of the result:

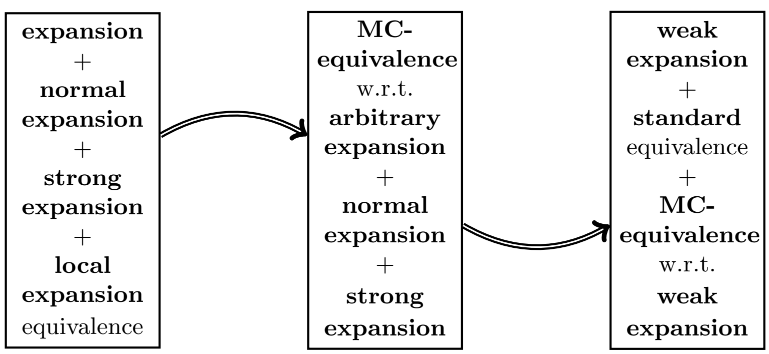



Figure 2: Stable semantics in case of $A(\mathcal{F}) = A(\mathcal{G})$

*Proof.* For this proof we consider AFs sharing the same arguments, i.e. $A(\mathcal{F}) = A(\mathcal{G})$. Using the results presented in Figure 1 it suffices to show the following three implications.

At first we will show that $\mathcal{F} \equiv_L^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv_E^{st} \mathcal{G}$. In [16] (Theorem 9) it is proven that two AFs are local expansion equivalent iff i) $\mathcal{F} \equiv_E^{st} \mathcal{G}$ or ii) $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ and there is an argument $a \in (A(\mathcal{F}) \setminus A(\mathcal{G})) \cup (A(\mathcal{F}) \setminus A(\mathcal{G}))$ satisfying certain properties. Since $A(\mathcal{F}) = A(\mathcal{G})$ is assumed, $\mathcal{F} \equiv_E^{st} \mathcal{G}$ follows because there are no arguments in $(A(\mathcal{F}) \setminus A(\mathcal{G})) \cup (A(\mathcal{F}) \setminus A(\mathcal{G}))$.

We will show now that $\mathcal{F} \equiv^{st} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{st} \mathcal{G}$. In [4] (Proposition 3) it was shown that two AFs are weak expansion equivalent w.r.t. stable semantics iff

i) $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ or ii) $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \emptyset$. Consequently, standard equivalence, i.e. $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ together with the assumption $A(\mathcal{F}) = A(\mathcal{G})$ implies weak expansion equivalence, i.e. $\mathcal{F} \equiv^{st}_{W} \mathcal{G}$ is shown.

Finally, we show that $\mathcal{F} \equiv^{st}_{W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st,MC}_{W} \mathcal{G}$. Assume $\mathcal{F} \equiv^{st}_{W} \mathcal{G}$ and $\mathcal{F} \not\equiv^{st,MC}_{W}$. Using the characterization theorem in [4] (Proposition 3) we deduce $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$. Since we assumed that $\mathcal{F}$ and $\mathcal{G}$ are not minimal change equivalent we deduce $N^{\mathcal{F}}_{st,W}(E) \neq N^{\mathcal{G}}_{st,W}(E)$ for some $E \subseteq A(\mathcal{F})(= A(\mathcal{G}))$. W.l.o.g. we assume $N^{\mathcal{F}}_{st,W}(E) = 0$ and $N^{\mathcal{G}}_{st,W}(E) = \infty$ (Theorem 6, Definition 10 in [6]). This means there is a superset $E'$ of $E$, s.t. $E' \in \mathcal{E}_{st}(\mathcal{F})$. Consequently, $E' \in \mathcal{E}_{st}(\mathcal{G})$ in contradiction to $N^{\mathcal{G}}_{st,W}(E) = \infty$.

In consideration of the counter-examples 2 and 3 it follows that the converse directions do not hold because the considered AFs share the same arguments. □

The role of self-loops is somewhat controversial in the literature. It is sometimes argued such self-attacks are necessary as they model paradoxical statements. On the other hand, it was shown (see Theorem 4.13 in [11]) that self-attacking arguments do not occur if Dung-style AFs are considered as instantiations of classical logic-based frameworks. At least in such contexts investigating AFs without self-loops is of interest. For this reason we present the equivalence zoo restricted to self-loop-free AFs. In contrast to the general case, local expansion equivalence coincides with expansion, normal expansion and strong expansion equivalence and thus, the equivalence zoo becomes totally ordered.

**Proposition 3.** *For stable semantics and argumentation frameworks without self-loops the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ MC equivalence with weak expansion $\subsetneq$ weak expansion equivalence $\subsetneq$ standard equivalence.*

Here is again the graphical representation of the result:

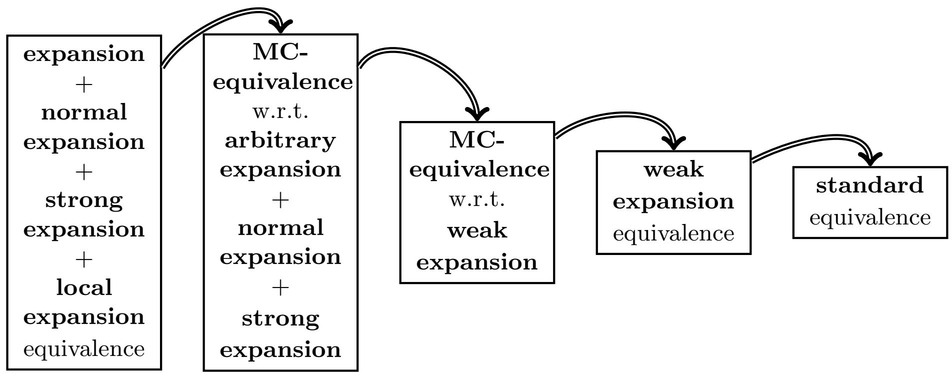
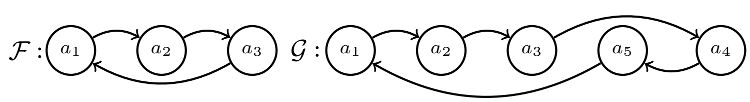



Figure 3: Stable semantics in case of self-loop-free AFs

*Proof.* In consideration of the counter-examples given in the proof for stable semantics without restrictions we observe that only the fourth example showing that $\mathcal{F} \equiv_L^{st} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_W^{st,MC} \mathcal{G}$ contains self-loops. This is not a coincidence because local expansion equivalence coincides with expansion equivalence in case of self-loop-free AFs. This follows immediately by Theorem 13 in [16] and Proposition 4 in [5].

Finally, we present a counter-example showing that $\mathcal{F} \equiv_W^{st} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_W^{st,MC} \mathcal{G}$.




Two AFs are weak expansion equivalent w.r.t. stable semantics iff i) $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ or ii) $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \varnothing$ [4, Proposition 3]. The second condition holds for the considered AFs. Furthermore, they are not minimal change equivalent w.r.t. weak expansions since they do not share the same arguments. □

## 3.3 Preferred Semantics

How does the equivalence zoo look if we turn to the more relaxed notion of preferred semantics? The following result presents the interrelations if we put no restriction on the considered AFs. We observe that as in the case of stable semantics there is no total ordering of the equivalence relations in the equivalence zoo. In particular, weak expansion equivalence is not comparable with strong expansion equivalence and minimal change equivalence w.r.t. arbitrary, normal and strong expansions. Furthermore, members of the family of minimal change equivalence relations *are shown to be intermediate forms between strong expansion and standard equivalence.* Interestingly, weak expansion equivalence and minimal change equivalence w.r.t. weak expansions change their position in comparison to stable semantics.

**Proposition 4.** *For preferred semantics and arbitrary argumentation frameworks the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ strong expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ MC equivalence with weak expansion $\subsetneq$ standard equivalence,*
- *expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ MC equivalence with weak expansion.*

Here is again the graphical representation of the result:

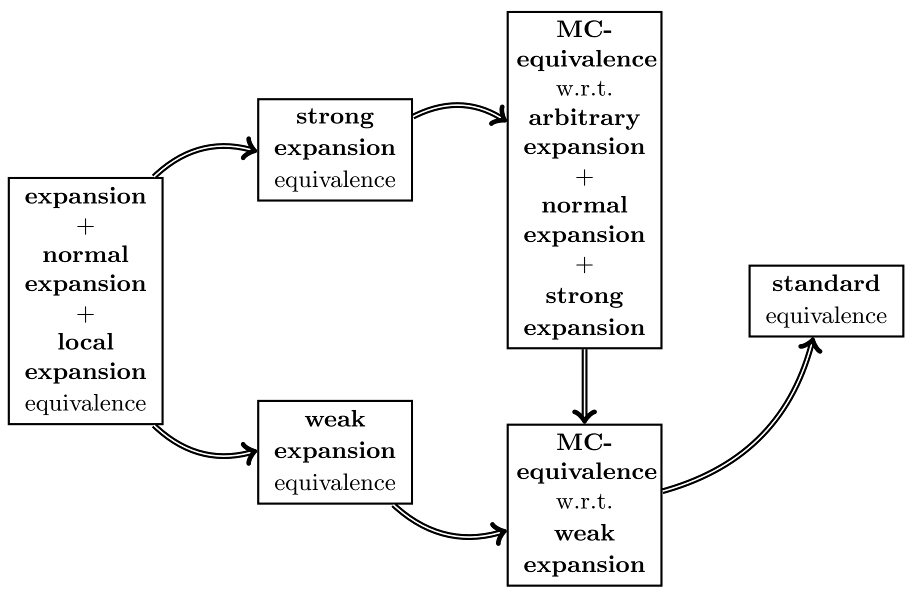



Figure 4: Preferred semantics in general

*Proof.* In order to save space we deal with preferred, admissible and complete semantics simultaneously. This can be done with less additional effort as we will see. In the following let $\sigma \in \{pr, ad, co\}$. In case of preferred semantics we have already shown that $\mathcal{F} \equiv_E^{pr} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_N^{pr} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_L^{pr} \mathcal{G} \Rightarrow \mathcal{F} \equiv_S^{pr} \mathcal{G}, \mathcal{F} \equiv_W^{pr} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{pr} \mathcal{G}$ [5, Theorem 13].

First, we will show that weak expansion equivalence implies minimal change equivalence w.r.t. weak expansions, i.e. $\mathcal{F} \equiv_W^{\sigma} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{\sigma,MC} \mathcal{G}$. Applying

Theorem 1 we deduce $A(F) = A(G)$ and $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$. If $\mathcal{F} \not\equiv_W^{\sigma,MC} \mathcal{G}$, then there is a set $E \subseteq A(F)$, s.t. $N_{\sigma,W}^{\mathcal{F}}(E) \neq N_{\sigma,W}^{\mathcal{G}}(E)$. W.l.o.g. we assume $N_{\sigma,W}^{\mathcal{F}}(E) = 0$ and $N_{\sigma,W}^{\mathcal{G}}(E) = \infty$ (compare [6, Definition 7, Theorem 6]). Hence, there is a superset $E'$ of $E$, s.t. $E' \in \mathcal{E}_\sigma(\mathcal{F})$ and $E' \notin \mathcal{E}_\sigma(\mathcal{G})$ in contradiction to $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$.

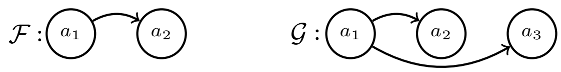
Since preferred semantics satisfies I-maximality [3], i.e. no extension can be a proper subset of another one, we conclude $\mathcal{F} \equiv_W^{pr,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{pr} \mathcal{G}$ ([6, Theorem 15]). Note that this implication does not hold in case of admissible or complete semantics.

We show now that $\mathcal{F} \equiv_S^\sigma \mathcal{G}$ implies $\mathcal{F} \equiv_\Phi^{\sigma,MC} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$. We have already shown that minimal change equivalence w.r.t. arbitrary, normal and strong expansions coincide [6, Definition 10, Theorem 6]. Hence, it suffices to prove that $\mathcal{F} \equiv_S^\sigma \mathcal{G}$ and $\mathcal{F} \not\equiv_S^{\sigma,MC} \mathcal{G}$ yields a contradiction. Since strong expansion equivalence implies sharing the same arguments [5, Definition 7, Theorem 6] it follows the existence of a subset $E \subseteq A(\mathcal{F}) = A(\mathcal{G})$, s.t. $\mathcal{F} \not\equiv_S^{\sigma,E} \mathcal{G}$. Consequently, $N_{\sigma,S}^{\mathcal{F}}(E) \neq N_{\sigma,S}^{\mathcal{G}}(E)$. Let $N_{\sigma,S}^{\mathcal{F}}(E) = k_1 < k_2 = N_{\sigma,S}^{\mathcal{G}}(E)$ where $k_1, k_2 \in \mathbb{N}_\infty$. Note that $k_1 = 0$ yields a contradiction because strong expansion equivalence implies standard equivalence, i.e. $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ ([5, Proposition 3]). Assume $k_1 \neq 0$. Consequently, there are an AF $\mathcal{H}$ and a set $E' \subseteq A(\mathcal{H})$, s.t. $\mathcal{F} \preceq_S \mathcal{H}$, $d(\mathcal{F}, \mathcal{H}) = k_1$ and $E \subseteq E' \in \mathcal{E}_\sigma(\mathcal{H})$. W.l.o.g. there exists an AF $\mathcal{H}'$, s.t. $R(\mathcal{F}) \cap R(\mathcal{H}') = \emptyset$ and any attack in $R(\mathcal{H}')$ contains at least one fresh argument and $\mathcal{H} = \mathcal{F} \cup \mathcal{H}'$ (compare Definition 2). Since $\mathcal{F} \equiv_S^\sigma \mathcal{G}$ is assumed and $A(\mathcal{F}) = A(\mathcal{G})$ is already shown we conlude $\mathcal{G} \preceq_S \mathcal{G} \cup \mathcal{H}'$ and therefore $E' \in \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H}')$. It can be easily seen that $d(\mathcal{G}, \mathcal{G} \cup \mathcal{H}') = k_1$. Thus, $k_2 = N_{\sigma,S}^{\mathcal{G}}(E) = k_1$ in contradiction to $k_1 < k_2$. Consequently, $\mathcal{F} \equiv_S^\sigma \mathcal{G} \Rightarrow \mathcal{F} \equiv_\Phi^{\sigma,MC} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.

Finally, we will show that $\mathcal{F} \equiv_\Phi^{\sigma,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{\sigma,MC} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$. Again, it suffices to show that $\mathcal{F} \equiv_S^{\sigma,MC} \mathcal{G}$ and $\mathcal{F} \not\equiv_W^{\sigma,MC} \mathcal{G}$ yields a contradiction [6, Theorem 9, Definition 10]. The first assumption implies $A(\mathcal{F}) = A(\mathcal{G})$. The second assumption means that there is a set $E$, s.t. $N_{\sigma,W}^{\mathcal{F}}(E) \neq N_{\sigma,W}^{\mathcal{G}}(E)$. Let $N_{\sigma,W}^{\mathcal{F}}(E) = \infty$ and $N_{\sigma,W}^{\mathcal{G}}(E) = 0$ [6, Theorem 6, Definition 10]. Recalling that the characteristic w.r.t. strong expansions does not exceed the characteristic w.r.t. weak expansions [6, Proposition 10] we conclude $N_{\sigma,S}^{\mathcal{G}}(E) = 0$. Hence, applying the minimal change equivalence w.r.t. strong expansions we deduce $N_{\sigma,S}^{\mathcal{F}}(E) = 0$. This means, there is a superset $E'$ of $E$, s.t. $E' \in \mathcal{E}_\sigma(\mathcal{F})$. Consequently, $N_{\sigma,W}^{\mathcal{F}}(E) = \infty$ is impossible concluding the proof.

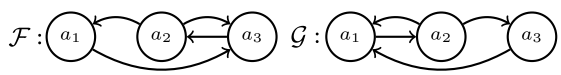
For the sake of completeness we will present some counterexamples showing that the converse directions do not hold. In case of preferred semantics it suffices to check the following four cases. The other non-relations can be easily obtained by using the already shown relations depicted in Figure 4. Again, we show the non-relations even for admissible and complete semantics. So, let $\sigma \in \{ad, pr, co\}$.

1. $\mathcal{F} \equiv^\sigma \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_W^{\sigma,MC} \mathcal{G}$.



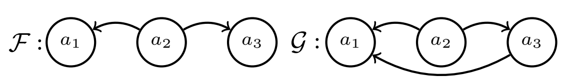

Obviously, $\mathcal{E}_{ad}(\mathcal{F}) = \mathcal{E}_{ad}(\mathcal{G}) = \{\varnothing, \{a_1\}\}$ and thus, $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G}) = \mathcal{E}_{co}(\mathcal{F}) = \mathcal{E}_{co}(\mathcal{G}) = \{\{a_1\}\}$. Furthermore, $\mathcal{F} \not\equiv_W^{\sigma,MC} \mathcal{G}$ since minimal change equivalence guarantees sharing the same arguments (compare Definition 10 in [6]) but $A(\mathcal{F}) \neq A(\mathcal{G})$.

2. $\mathcal{F} \equiv_\Phi^{\sigma,MC} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_S^\sigma \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.




It can be checked (Definition 8, Theorem 9 in [6]) that $N_{\sigma,S}^{\mathcal{F}}(\{a_1\}) = N_{\sigma,S}^{\mathcal{G}}(\{a_1\}) = N_{\sigma,S}^{\mathcal{F}}(\{a_3\}) = N_{\sigma,S}^{\mathcal{G}}(\{a_3\}) = 1$ and $N_{\sigma,S}^{\mathcal{F}}(\{a_2\}) = N_{\sigma,S}^{\mathcal{G}}(\{a_2\}) = N_{\sigma,S}^{\mathcal{F}}(\varnothing) = N_{\sigma,S}^{\mathcal{G}}(\varnothing) = 0$ and for any other $E \subseteq A(\mathcal{F}) = A(\mathcal{G})$ (thus, $E \notin cf(\mathcal{F}) = cf(\mathcal{G})$), we have $N_{\sigma,S}^{\mathcal{F}}(E) = N_{\sigma,S}^{\mathcal{G}}(E) = \infty$. Hence, $\mathcal{F} \equiv_S^{\sigma,MC} \mathcal{G}$. Furthermore, $\mathcal{F} \not\equiv_S^\sigma \mathcal{G}$ because $\mathcal{F}$ and $\mathcal{G}$ are self-loop-free but not syntactically identical (compare Proposition 4 in [5]).

3. $\mathcal{F} \equiv_W^\sigma \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_\Phi^{\sigma,MC} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.




We have $A(\mathcal{F}) = A(\mathcal{G})$, $\mathcal{E}_{ad}(\mathcal{F}) = \mathcal{E}_{ad}(\mathcal{G}) = \{\varnothing, \{a_2\}\}$ and hence, $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G}) = \mathcal{E}_{co}(\mathcal{F}) = \mathcal{E}_{co}(\mathcal{G}) = \{\{a_2\}\}$. Furthermore, $U_{\{a_2\}}^{\mathcal{F}} = \varnothing = U_{\{a_2\}}^{\mathcal{G}}$ and $U_\varnothing^{\mathcal{F}} = \{a_1, a_2, a_3\} = U_\varnothing^{\mathcal{G}}$. Consequently, $\mathcal{F} \equiv_W^\sigma \mathcal{G}$ (Theorem 1). Observe that $N_{\sigma,S}^{\mathcal{F}}(\{a_1\}) = 1 \neq 2 = N_{\sigma,S}^{\mathcal{G}}(\{a_1\})$ (compare Definition 8, Theorem 9 in [6]). Hence, $\mathcal{F} \not\equiv_S^{\sigma,MC} \mathcal{G}$.

4. $\mathcal{F} \equiv_S^\sigma \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_W^\sigma \mathcal{G}$.


Since $\mathcal{F}$ and $\mathcal{G}$ possess identical admissible-*-kernels as well as complete-*-kernels, namely $\mathcal{F} = \mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)} = \mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)}$ we deduce $\mathcal{F} \equiv_S^\sigma \mathcal{G}$ (compare Definitions 7 and 9, Theorems 6 and 8 in [5]). Furthermore, $\mathcal{E}_{ad}(\mathcal{F}) = \mathcal{E}_{ad}(\mathcal{G}) = \{\varnothing, \{a_1\}\}$ and $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G}) = \mathcal{E}_{co}(\mathcal{F}) = \mathcal{E}_{co}(\mathcal{G}) = \{\{a_1\}\}$. Observe that $U_{\{a_1\}}^{\mathcal{F}} = \{a_2\} \neq \varnothing = U_{\{a_1\}}^{\mathcal{G}}$. Hence, $\mathcal{F} \not\equiv_W^\sigma \mathcal{G}$ (cf. Theorem 1). □

Restricting our considerations to AFs sharing the same arguments does not have a big effect in comparison to the general case. We state a slight difference only, namely standard equivalence of two AFs becomes sufficient for their minimal change equivalence w.r.t. weak expansions.

**Proposition 5.** *For preferred semantics and argumentation frameworks with the same arguments the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *standard equivalence = MC equivalence with weak expansion,*
- *expansion equivalence $\subsetneq$ strong expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ standard equivalence,*
- *expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ standard equivalence.*

Graphically:

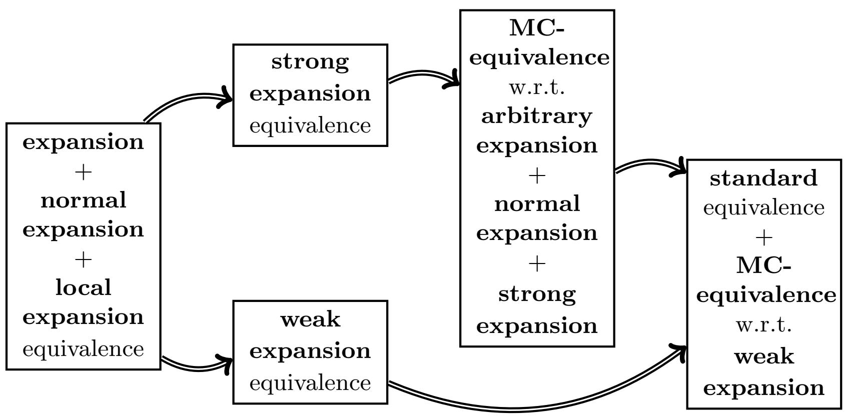



Figure 5: Preferred semantics in case of $A(\mathcal{F}) = A(\mathcal{G})$

*Proof.* Consider again the counter-examples given in the proof before showing that some relations do not hold. We observe that only the first counter-example (showing that $\mathcal{F} \equiv^{\sigma} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_{W}^{\sigma,MC} \mathcal{G}$) where $\sigma \in \{ad, pr, co\}$ deals with AFs which do not share the same arguments. This is not a coincidence as the following proof shows.

We assume $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{F} \equiv^{\sigma} \mathcal{G}$. Standard equivalence of two AFs, i.e. $\mathcal{E}_{\sigma}(\mathcal{F}) = \mathcal{E}_{\sigma}(\mathcal{G})$ together with the assumption guarentees that for any set $E \subseteq A(\mathcal{F})$, either $N_{\sigma,W}^{\mathcal{F}}(E) = N_{\sigma,W}^{\mathcal{G}}(E) = 0$ or $N_{pr,W}^{\mathcal{F}}(E) = N_{pr,W}^{\mathcal{G}}(E) = \infty$ (compare [6, Definition 7]). Consequently, in case of sharing the same arguments $\mathcal{F} \equiv_{W}^{\sigma,MC} \mathcal{G}$ is shown [6, Theorem 6]. □

Finally, we consider the class of self-loop-free AFs. It turns out that in this case stable and preferred semantics behave in a very similar manner. The only difference is the role (or better, position) of weak expansion equivalence and minimal change equivalence w.r.t. weak expansions.

**Proposition 6.** *For preferred semantics and argumentation frameworks without self-loops the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence ⊊ MC equivalence with arbitrary expansion ⊊ weak expansion equivalence ⊊ MC equivalence with weak expansion ⊊ standard equivalence.*

Graphically:

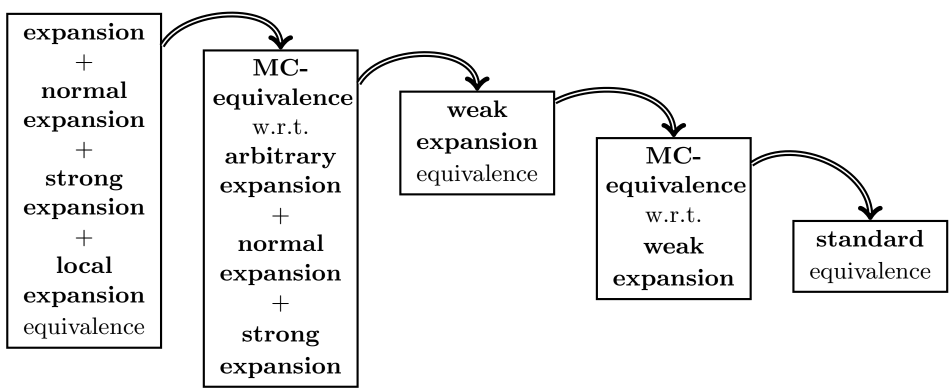


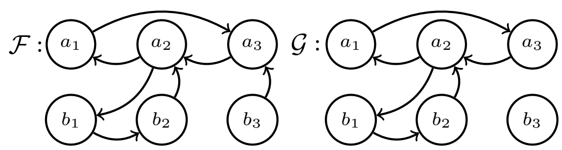

Figure 6: Preferred semantics in case of self-loop-free AFs

*Proof.* In this proof we assume that the considered AFs do not possess self-loops. Consequently, in consideration of the results presented in Figure 4 it suffices to show the following two relations. First, $\mathcal{F} \equiv_S^{pr} \mathcal{G} \Rightarrow \mathcal{F} \equiv_N^{pr} \mathcal{G}$ (already shown in [5, Proposition 4]) and second, $\mathcal{F} \equiv_S^{pr,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{pr} \mathcal{G}$.

So, given $\mathcal{F} \equiv_S^{pr,MC} \mathcal{G}$ we deduce $A(\mathcal{F}) = A(\mathcal{G})$ and $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G})$. Consequently, if $\mathcal{F} \not\equiv_W^{pr} \mathcal{G}$, then there is an extension $E \in \mathcal{E}_{pr}(\mathcal{F})$, s.t. $U_E^{\mathcal{F}} \neq U_E^{\mathcal{G}}$ (Theorem 1). W.l.o.g. let $a \in U_E^{\mathcal{F}} \setminus U_E^{\mathcal{G}}$. This means $a \notin E$, $(E,a) \notin R(\mathcal{F})$ and $(E,a) \in R(\mathcal{G})$. Consequently, $(a,E) \notin R(\mathcal{F})$ because $E$ is assumed to be preferred in $\mathcal{F}$. Furthermore, we deduce $E \cup \{a\} \in cf(\mathcal{F})$ since we consider self-loop-free AFs. In [6, Corollary 7] it was shown that whenever a set $C$ is conflict-free we may enforce $C$ in finitely many steps. This means, $N_{pr,S}^{\mathcal{F}}(E \cup \{a\}) < \infty$. On the other hand, we have $E \cup \{a\} \notin cf(\mathcal{G})$ because $(E,a) \in R(\mathcal{G})$ is already shown. Thus, $N^{\mathcal{F}}_{pr,S}(E \cup \{a\}) = \infty$ (Definition 8, Theorem 9 in [6]). This means, $\mathcal{F} \not\equiv^{pr,MC}_{S} \mathcal{G}$ is implied (in contradiction to the assumption) concluding the proof.

Consider again the counter-examples given in Proposition 4. We observe that the counter-examples 1-3 do not possess self-loops. Hence, these (non)-relations do not hold here either. One counter-example remains to be given.

1. $\mathcal{F} \equiv^{pr,MC}_{W} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{pr}_{W} \mathcal{G}$.




One may check that $\mathcal{E}_{ad}(\mathcal{F}) = \mathcal{E}_{ad}(\mathcal{G}) = \{\emptyset, \{b_3\}\}$ and hence, $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G}) = \{\{b_3\}\}$. Furthermore, $A(\mathcal{F}) = A(\mathcal{G})$. Consequently, for any set $E \subseteq A(\mathcal{F})$, either $N^{\mathcal{F}}_{\sigma,W}(E) = N^{\mathcal{G}}_{\sigma,W}(E) = 0$ or $N^{\mathcal{F}}_{\sigma,W}(E) = N^{\mathcal{G}}_{\sigma,W}(E) = \infty$ (compare Definition 7 in [6]). Hence, $\mathcal{F} \equiv^{\sigma,MC}_{W} \mathcal{G}$ is shown (Theorem 6 and Definition 10 in [6]). On the other hand, $U^{\mathcal{F}}_{\{b_3\}} = \{a_1, a_2, b_1, b_2\} \neq \{a_1, a_2, a_3, b_1, b_2\} = U^{\mathcal{G}}_{\{b_3\}}$. Thus, in consideration of Theorem 1, $\mathcal{F} \not\equiv^{\sigma}_{W} \mathcal{G}$. □

### 3.4 Admissible Semantics

We now present the results for admissible semantics. In contrast to stable and preferred semantics, the different forms of minimal change equivalence do not lie between expansion and standard equivalence. They are weaker than strong (resp. weak) expansion equivalence but incomparable to standard equivalence.

**Proposition 7.** *For admissible semantics and arbitrary argumentation frameworks the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence ⊊ strong expansion equivalence ⊊ MC equivalence with arbitrary expansion ⊊ MC equivalence with weak expansion,*
- *expansion equivalence ⊊ weak expansion equivalence ⊊ MC equivalence with weak expansion,*
- *strong (weak) expansion equivalence ⊊ standard equivalence.*

Here is again the graphical representation of the result:

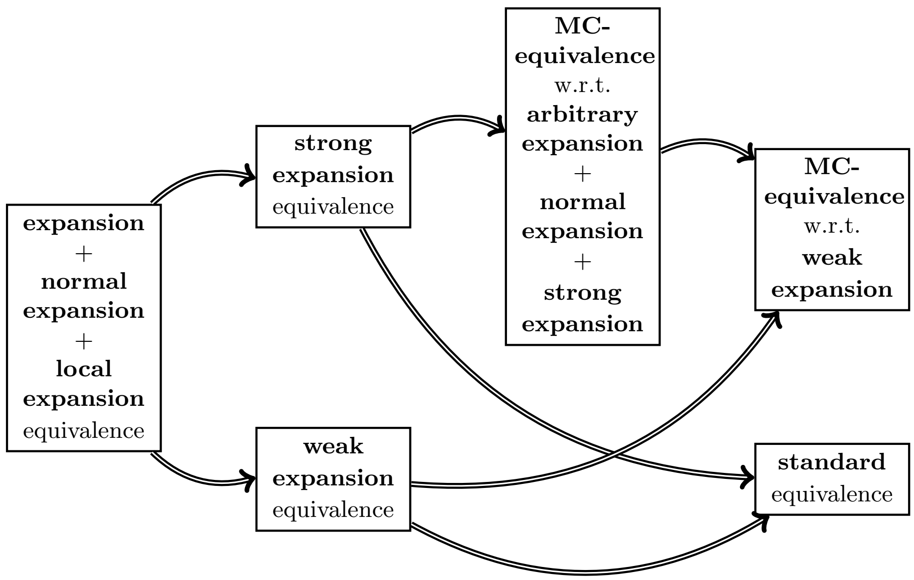
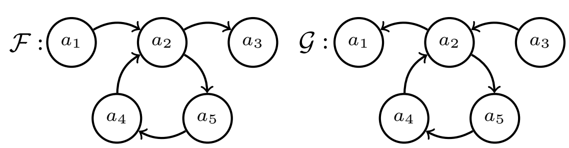



Figure 7: Admissible semantics in general

*Proof.* In case of admissible semantics we have already shown that $\mathcal{F} \equiv_E^{ad} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_N^{ad} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_L^{ad} \mathcal{G} \Rightarrow \mathcal{F} \equiv_S^{ad} \mathcal{G}, \mathcal{F} \equiv_W^{ad} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{ad} \mathcal{G}$ [5, Theorem 13]. Furthermore, in Proposition 4 we proved all proposed implications for admissible semantics. In contrast to preferred semantics we have not shown that $\mathcal{F} \equiv_W^{ad,MC} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{ad} \mathcal{G}$. Any attempt to prove this implication is doomed to failure as the following example shows. Note that the following counter-example implies the claimed non-implication.

1. $\mathcal{F} \equiv_\Phi^{ad,MC} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{ad} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.




Obviously, $\mathcal{E}_{ad}(\mathcal{F}) = \{\emptyset, \{a_1\}, \{a_1, a_3\}, \{a_1, a_3, a_5\}\}$ and furthermore, $\mathcal{E}_{ad}(\mathcal{G}) = \{\emptyset, \{a_3\}, \{a_3, a_1\}, \{a_3, a_1, a_5\}\}$. This means, $\mathcal{F} \not\equiv^{ad} \mathcal{G}$. Given these admissible extensions we observe that for any $E \subseteq \{a_1, a_3, a_5\}$ we have $N_{ad,\Phi}^{\mathcal{F}}(E) = N_{ad,\Phi}^{\mathcal{G}}(E) = 0$. Obviously, $N_{ad,\Phi}^{\mathcal{F}}(\{a_2\}) = N_{ad,\Phi}^{\mathcal{G}}(\{a_2\}) = 2$ and $N_{ad,\Phi}^{\mathcal{F}}(\{a_4\}) = N_{ad,\Phi}^{\mathcal{G}}(\{a_4\}) = 1$. For instance, $a_2$ does not counter-attack $a_1$ and $a_4$ in $\mathcal{F}$. Thus, $N_{ad,\Phi}^{\mathcal{F}}(\{a_2\}) = 2$. Furthermore, for any other conflicting set $E' \subseteq \{a_1, ..., a_5\}$ we have $N_{ad,\Phi}^{\mathcal{F}}(E') = N_{ad,\Phi}^{\mathcal{G}}(E') = \infty$ (compare Definition 8, Theorem 9 in [6]). Thus, $\mathcal{F} \not\equiv_\Phi^{ad,MC} \mathcal{G}$ (cf. Definition 10, Theorem 9 in [6]).

The other relevant counter-examples are given in Proposition 4. □

We now restrict our considerations to AFs sharing the same arguments. Minimal change equivalence w.r.t. weak expansions becomes the most coarse-grained equivalence relation. Interestingly, it is implied by standard equivalence and not vice versa.

**Proposition 8.** *For admissible semantics and argumentation frameworks sharing the same arguments the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ strong expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ MC equivalence with weak expansion,*
- *expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ standard equivalence $\subsetneq$ MC equivalence with weak expansion,*
- *strong expansion equivalence $\subsetneq$ standard equivalence.*

Graphically:

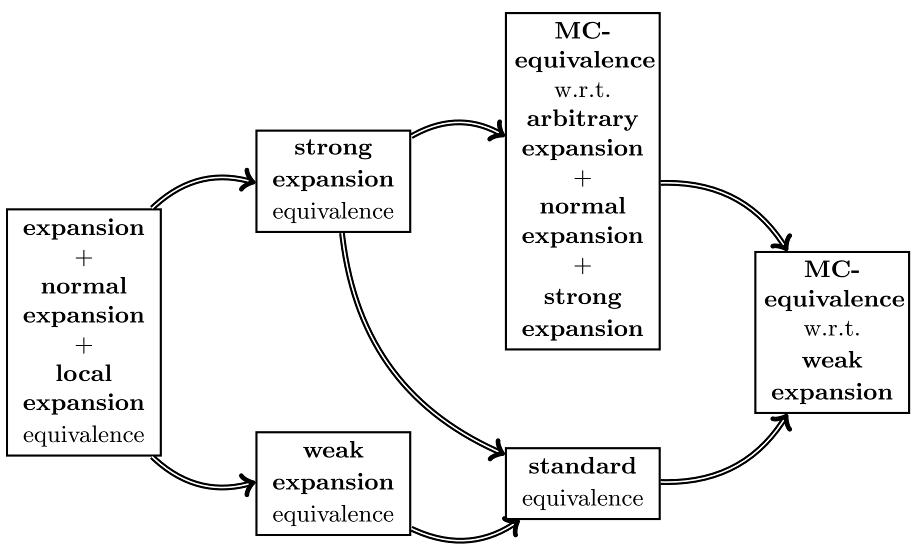



Figure 8: Admissible semantics in case of $A(\mathcal{F}) = A(\mathcal{G})$

*Proof.* Consider again the counter-examples given in Proposition 4 showing that some relations do not hold. We observe that only the first counter-example (showing that $\mathcal{F} \equiv^{\sigma} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_W^{\sigma,MC} \mathcal{G}$) where $\sigma \in \{ad, pr, co\}$ deals with AFs which do not share the same arguments. Furthermore, in Proposition 5 is shown that this is not a coincidence because $\mathcal{F} \equiv^\sigma \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma,MC}_W \mathcal{G}$ in case of sharing the same arguments. $\square$

Finally, we consider the class of self-loop-free AFs. In contrast to AFs sharing the same arguments minimal change and standard equivalence remain incomparable.

**Proposition 9.** *For admissible semantics and argumentation frameworks without self-loops the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ MC equivalence with weak expansion,*
- *expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ MC equivalence with weak expansion,*
- *weak expansion equivalence $\subsetneq$ standard equivalence.*

Graphically:

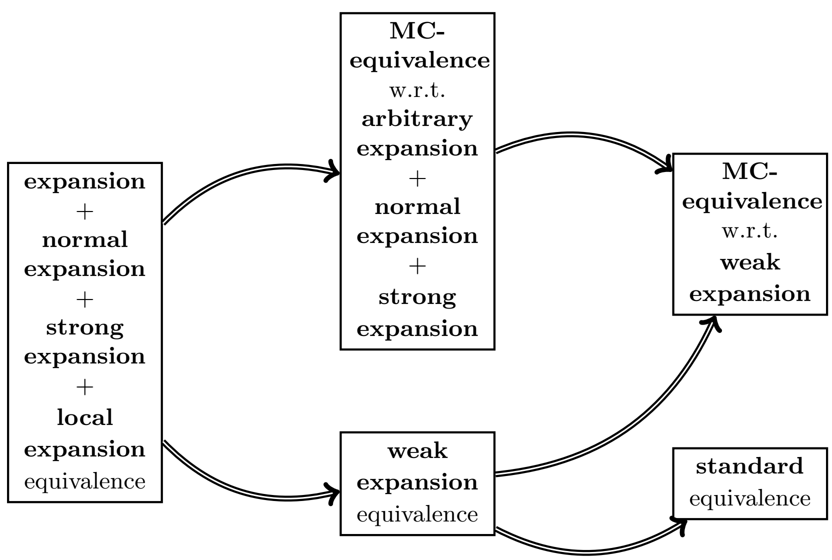



Figure 9: Admissible (and complete[2]) semantics in case of self-loop-free AFs

[2]In Proposition 12 we will show that the relations depicted in Figure 9 also hold for complete semantics.

*Proof.* In this proof we assume that the considered AFs do not possess self-loops. Consequently, in consideration of the results presented in Proposition 7 it suffices to argue for $\mathcal{F} \equiv_S^{pr} \mathcal{G} \Rightarrow \mathcal{F} \equiv_N^{pr} \mathcal{G}$. Fortunately, this implication is already proven in [5, Proposition 4].

We do not have to show further non-implications because the counter-examples 1-3 in Proposition 4 as well as counter-example 1 in Proposition 7 do not possess self-loops implying that these (non)-relations do not hold here either. □

## 3.5 Complete Semantics

Finally we present the results for complete semantics.

**Proposition 10.** *For complete semantics and arbitrary argumentation frameworks the following relationships hold:*

- *expansion equivalence = normal expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence ⊊ strong expansion equivalence ⊊ MC equivalence with arbitrary expansion ⊊ MC equivalence with weak expansion,*
- *expansion equivalence ⊊ local expansion equivalence ⊊ weak expansion equivalence ⊊ MC equivalence with weak expansion,*
- *strong (weak) expansion equivalence ⊊ standard equivalence,*
- *local expansion equivalence ⊊ MC equivalence with arbitrary expansion.*

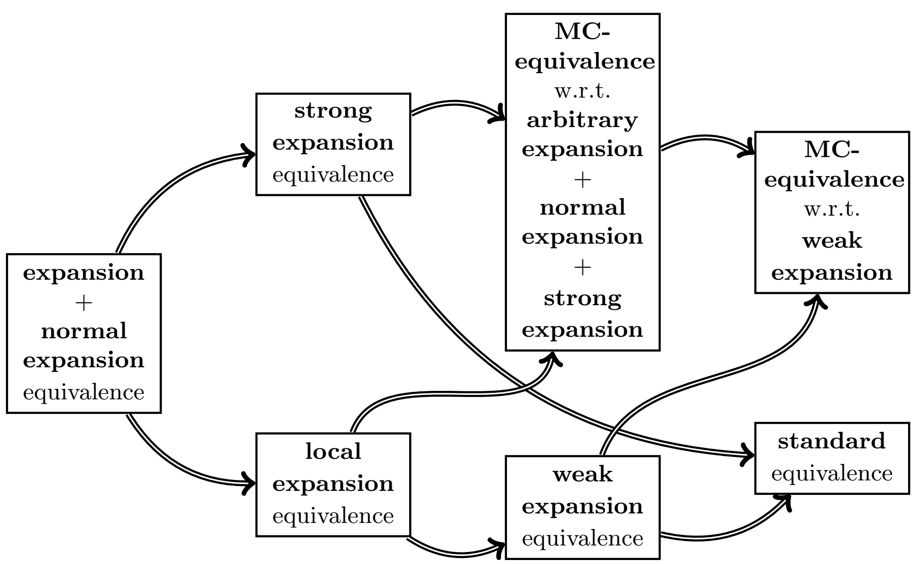
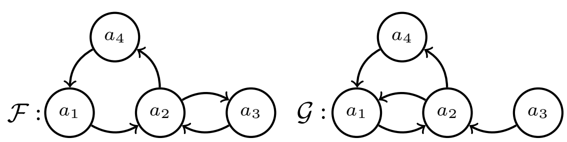



Figure 10: Complete semantics in general

*Proof.* In case of complete semantics we have already shown that $\mathcal{F} \equiv_E^{co} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_N^{co} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv_L^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_S^{co} \mathcal{G}, \mathcal{F} \equiv_W^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{co} \mathcal{G}$ [5, Theorem 13]. In Proposition 4 we proved all remaining implications except of $\mathcal{F} \equiv_L^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{co} \mathcal{G}$ and $\mathcal{F} \equiv_L^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_E^{co,MC} \mathcal{G}$.
So, given $\mathcal{F} \equiv_L^{co} \mathcal{G}$ we deduce $\mathcal{E}_{co}(\mathcal{F}) = \mathcal{E}_{co}(\mathcal{G})$ [5, Proposition 3] and furthermore, $A(\mathcal{F}) = A(\mathcal{G})$ [16, Theorem 11]. Consequently, if $\mathcal{F} \not\equiv_W^{co} \mathcal{G}$, then there is a (non-empty) complete extension $E \in \mathcal{E}_{co}(\mathcal{F})$, s.t. $U_E^{\mathcal{F}} \neq U_E^{\mathcal{G}}$ (Theorem 1). Without loss of generality let $a \in U_E^{\mathcal{F}} \smallsetminus U_E^{\mathcal{G}}$. This means $a \notin E$, $(E, a) \notin R(\mathcal{F})$ and $(E, a) \in R(\mathcal{G})$. Let $e \in E$ with $(e, a) \in R(\mathcal{G})$. Note that $(a, E) \in R(\mathcal{F})$ is impossible because complete extension are admissible. Furthermore, we observe that adding $(a, e)$ to $R(\mathcal{G})$ does not affect the completeness of $E$ because $(e, a) \in R(\mathcal{G})$ was assumed. Both observations contradict the assumption $\mathcal{F} \equiv_L^{co} \mathcal{G}$ and thus, $\mathcal{F} \equiv_L^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_W^{co} \mathcal{G}$ is shown.
The implication $\mathcal{F} \equiv_L^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_E^{co,MC} \mathcal{G}$ can be seen as follows: Given $\mathcal{F} \equiv_L^{co} \mathcal{G}$ we deduce $\mathcal{F} \equiv_L^{pr} \mathcal{G}$ [16, Theorem 12, statement 2]. By Proposition 4 we obtain $\mathcal{F} \equiv_E^{pr,MC} \mathcal{G}$ and finally $\mathcal{F} \equiv_E^{co,MC} \mathcal{G}$ due to [6, Theorems 9, 13].

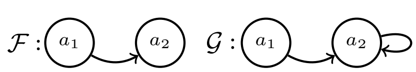
In Proposition 4 we have already shown that counter-examples 1, 3 and 4 even serve for complete semantics. It remains to show the following three non-implications.

1. $\mathcal{F} \equiv_\Phi^{co,MC} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv^{co} \mathcal{G}$ for each $\Phi \in \{E, N, S\}$.




Obviously, $\mathcal{E}_{co}(\mathcal{F}) = \{\varnothing, \{a_3, a_4\}\}$ and furthermore, $\mathcal{E}_{co}(\mathcal{G}) = \{\{a_3, a_4\}\}$. This means, $\mathcal{F} \not\equiv^{co} \mathcal{G}$. In consideration of complete extensions we observe, for any $E \subseteq \{a_3, a_4\}$ we have $N_{co,\Phi}^{\mathcal{F}}(E) = N_{co,\Phi}^{\mathcal{G}}(E) = 0$. Furthermore, $N_{co,\Phi}^{\mathcal{F}}(\{a_2\}) = N_{co,\Phi}^{\mathcal{G}}(\{a_2\}) = N_{co,\Phi}^{\mathcal{F}}(\{a_1\}) = N_{co,\Phi}^{\mathcal{G}}(\{a_1\}) = 1$. For any other conflicting set $E' \subseteq \{a_1, ..., a_4\}$ we have $N_{co,\Phi}^{\mathcal{F}}(E') = N_{co,\Phi}^{\mathcal{G}}(E') = \infty$ (compare Definition 8, Theorem 9 in [6]). Thus, $\mathcal{F} \not\equiv_\Phi^{ad,MC} \mathcal{G}$ (cf. Definition 10, Theorems 9, 13 in [6]).

2. $\mathcal{F} \equiv_W^{co} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_L^{co} \mathcal{G}$.




Obviously, $A(\mathcal{F}) = A(\mathcal{G})$, $\mathcal{E}_{co}(\mathcal{F}) = \mathcal{E}_{co}(\mathcal{G}) = \{\{a_1\}\}$ and $U_{\{a_1\}}^{\mathcal{F}} = \varnothing = U_{\{a_1\}}^{\mathcal{G}}$. Hence according to Theorem 1 we have, $\mathcal{F} \equiv_W^{co} \mathcal{G}$. Nevertheless, adding the attack $(a_2, a_1)$ proves $\mathcal{F} \not\equiv_L^{co} \mathcal{G}$.

3. $\mathcal{F} \equiv_L^{co} \mathcal{G} \not\Rightarrow \mathcal{F} \equiv_S^{co} \mathcal{G}$.


The example is taken from [16, Example 19]. Therein it is already shown that $\mathcal{F} \equiv_L^{co} \mathcal{G}$ and $\mathcal{F} \not\equiv_E^{co} \mathcal{G}$. Since $\mathcal{F} = \mathcal{F}^{k^*(co)}$ and $R(\mathcal{G}) \subset R(\mathcal{F})$ we obtain $\mathcal{F} \not\equiv_S^{co} \mathcal{G}$ (compare [5, Definition 9, Theorem 8]). □

We now restrict our considerations to AFs sharing the same arguments.

**Proposition 11.** *For complete semantics and argumentation frameworks sharing the same arguments the following relationships hold:*

- *expansion equivalence = normal expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence $\subsetneq$ strong expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion $\subsetneq$ standard equivalence $\subsetneq$ MC equivalence with weak expansion,*
- *expansion equivalence $\subsetneq$ local expansion equivalence $\subsetneq$ weak expansion equivalence $\subsetneq$ MC equivalence with weak expansion,*
- *strong expansion equivalence $\subsetneq$ standard equivalence,*
- *local expansion equivalence $\subsetneq$ MC equivalence with arbitrary expansion.*

Here is again the graphical representation of the result:

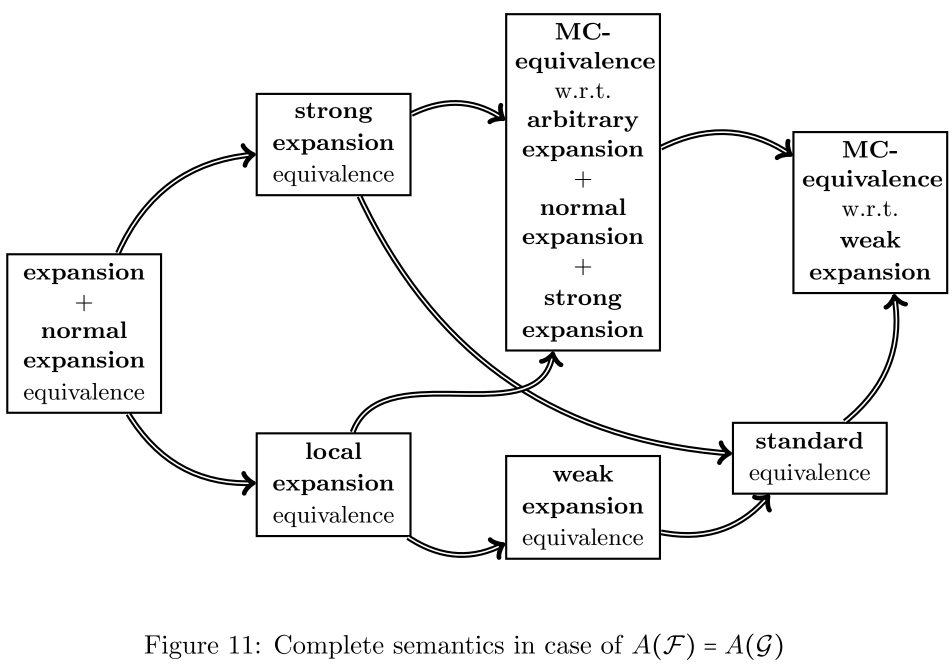



Figure 11: Complete semantics in case of $A(\mathcal{F}) = A(\mathcal{G})$

*Proof.* In Proposition 5 it is shown that $\mathcal{F} \equiv^{\sigma} \mathcal{G} \Rightarrow \mathcal{F} \equiv_{W}^{\sigma,MC} \mathcal{G}$ for any $\sigma \in \{ad, pr, co\}$ (counter-example 1 in Proposition 4 uses different sets of arguments). Furthermore, all remaining counter-examples, namely examples $1-3$ in Proposition 10 as well as examples $3,4$ in Proposition 4 share the same arguments and thus can be used to show that the converse directions do not hold. □

Finally, we consider the class of self-loop-free AFs. The results we obtain here are exactly the same as in case of admissible semantics. For this reason we refer to Figure 9 for the graphical representation.

**Proposition 12.** *For complete semantics and argumentation frameworks without self-loops the following relationships hold:*

- *expansion equivalence = normal expansion equivalence = strong expansion equivalence = local expansion equivalence,*
- *MC equivalence with arbitrary expansion = MC equivalence with normal expansion = MC equivalence with strong expansion,*
- *expansion equivalence* $\subsetneq$ *MC equivalence with arbitrary expansion* $\subsetneq$ *MC equivalence with weak expansion,*
- *expansion equivalence* $\subsetneq$ *weak expansion equivalence* $\subsetneq$ *MC equivalence with weak expansion,*
- *weak expansion equivalence* $\subsetneq$ *standard equivalence.*

*Proof.* In this proof we assume that the considered AFs do not possess self-loops. Consequently, in consideration of the results presented in Proposition 10 it suffices to argue for $\mathcal{F} \equiv_{S}^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_{N}^{co} \mathcal{G}$ as well as $\mathcal{F} \equiv_{L}^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv_{N}^{co} \mathcal{G}$. Fortunately, the first implication is already shown in [5, Proposition 4]. The second implication can be seen as follows. Given $\mathcal{F} \equiv_{L}^{co} \mathcal{G}$ we deduce $\mathcal{F} \equiv_{L}^{pr} \mathcal{G}$ [16, Theorem 12, statement 2]. By [16, Theorem 8] and [5, Propositions 3, 4] we obtain $\mathcal{F} = \mathcal{G}$ since self-loop-freeness is assumed. Obviously, $\mathcal{F} \equiv_{N}^{co} \mathcal{G}$ is implied.

We do not have to show further non-implications because the counter-examples $1-3$ in Proposition 4 as well as counter-example 1 in Proposition 10 do not possess self-loops implying that these (non)-relations do not hold here either. □

# 4 Conclusions

In this paper we continued earlier work of the first author [5]. We fully clarified the relationship among all equivalence notions for AFs so far discussed in the literature for stable, preferred, admissible and complete semantics. We provided an analysis for the whole class of AFs as well as for two important subclasses, namely AFs sharing the same arguments and self-loop-free AFs. The most relevant "take home" message following from our results is that the different notions of minimal change equivalence fit nicely into the global picture of other equivalence notions. More precisely, in case of stable and preferred semantics minimal change equivalence constitutes alternative notions in between strong expansion and standard equivalence. This property does not carry over to the more relaxed notions of complete and admissible semantics. Here, any considered form of minimal change equivalence is implied by strong expansion equivalence but incomparable with standard equivalence.

Our results are not only of theoretical interest, they can also be very useful in practice. For instance, it is easy to decide (in linear time) whether two AFs are strong expansion equivalent. In cases where strong expansion equivalence is established, minimal change equivalence - which is much more difficult to decide in general - immediately follows, as shown in this paper. In general, changing the underlying knowledge base cause an arbitrary modification, so-called *update* [7], on the abstract level. Since argument and attack construction is monotonic, adding a new piece of information to the underlying knowledge base does not rule out old arguments and attacks. Thus, notions like normal expansion equivalence allow us to simplify AFs adequately as they reflect this kind of dynamic scenarios. In brief, abstract equivalence notions can *detect* redundant attacks no matter what the underlying KR-language is. An equivalence notion for the special case where classical logic is used is defined in [1].

Some further work remains to be done. First of all, instead of considering a certain semantics one may alternatively look at general criteria sufficient and/or necessary for being in a particular interrelation. Examples of such criteria are *regularity* and *I-maximality* as it was shown in [6, Theorems 14,15]. Secondly, a further direction is the generalization of existing equivalence notions concerning AFs to ADFs first introduced in [12]. Finally, we consider to study in detail the link to instantiation-based argumentation. In particular, we want to use equivalence notions on the abstract level to identify redundant information on the underlying knowledge base. In particular, we will consider the ASPIC-system [17] since it "allows" for self-defeating arguments which are essential for most of the equivalence notions considered in this article.

# Acknowledgments

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