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Algorithms for Computing Complete Deterministic Fuzzy Automata via Invariant Fuzzy Quasi-Orders

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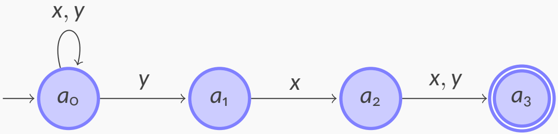
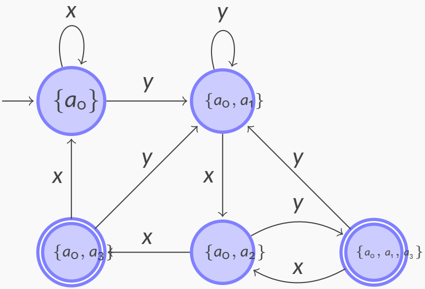
Content

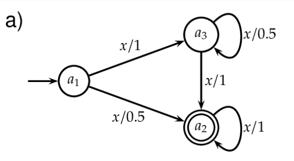
- Determinization is a procedure of finding a language equivalent deterministic finite automaton (DFA, for short) for a given nondeterministic finite automaton (NFA, for short)
- It is a *fundamental problem* in automata theory
- Contrary to NFAs, for any DFA there is a trivial linear time, constant space, online algorithm to check whether an input sequence is accepted or not
- For this reason, in many applications it pays to convert NFAs into DFAs
- Particularly significant determinization methods are those that provide a minimal DFA equivalent to the original NFA, called canonization methods.
- For every NFA there exists a DFA that is language equivalent
- The well-known accessible subset construction is simple, but *exponential* algorithm for determinization
  - a state in the determinized automaton is a subset of states of the input automaton

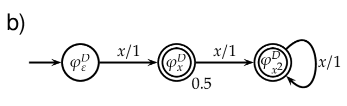
NFA:



DFA:







$$\varphi^D_\varepsilon = \begin{bmatrix} 1 & 0 & 0.5 \end{bmatrix}$$
$$\varphi^D_x = \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix}$$
$$\varphi^D_{x^2} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Figure (a): $\mathcal{A}$ - a fuzzy automaton over $X = \{x\}$ and the product (Gougen) structure
Figure (b): Minimal complete deterministic fuzzy automaton (CDFA) of $\mathcal{A}$

- Unlike NFAs, which can always be determinized, not all *fuzzy finite automata* FFAs, can be determinized
- Some well-known algorithms cannot compute CDFA, even if it exists.
- Another problem: How to find a minimal CDFA equivalent to $\mathcal{A}$?
- *Note*: In general, a minimal CDFA is not unique (there may be several equivalent CDFAs with the same number of states, but with different membership values in their transitions, fuzzy initial and fuzzy terminal states).

Our motivation and goal

- To provide **improvements** of the determinization and canonization methods for fuzzy automata via ***factorization of fuzzy states*** previously studied in

  J. R. G. de Mendívil, J. R. Garitagoitia, ***Determinization of fuzzy automata via factorization of fuzzy states***, Information Sciences, 283 (2014) 165–179.

  J. R. G. de Mendívil, ***Conditions for minimal fuzzy deterministic finite automata via Brzozowski's procedure***, IEEE Transactions on Fuzzy Systems (2017)

- Our methods are based on the usage of the ***right and left invariant fuzzy quasi-orders*** - originally used to model the indistinguishability of states of FFAs and employed in the state reduction of fuzzy automata

  A. Stamenković, M. Ćirić, J. Ignjatović, ***Reduction of fuzzy automata by means of fuzzy quasi-orders***, Information Sciences 275 (2014) 168–198.

- Our methods basically “combine” the determinization and the state reduction problems, similarly as in

  Z. Jančić, I. Micić, J. Ignjatović, M. Ćirić, ***Further improvement of determinization methods for fuzzy finite automata***, Fuzzy Sets and Systems 301(2016) 79–102.

- Our methods always result in a **smaller or equal** CDFA compared to the one obtained by the factorization of fuzzy states;
- It can even result in a **finite** CDFA in some cases when the factorization of fuzzy states results in an **infinite** CDFA;

Complete residuated lattices

Structure of truth values – ***(complete) residuated lattice*** $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$

- $(L, \wedge, \vee, 0, 1)$ - (complete) lattice with the least element 0 and the greatest element 1,
- $(L, \otimes, 1)$ - commutative monoid with the unit 1,
- *adjunction property*: for every $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z.$$

Examples of complete residuated lattices

$L = [0,1]$, $\vee = \max$, $\wedge = \min$, and

- Product (Goguen) operations:

$$a \otimes b = a \cdot b, \quad a \to b = \begin{cases} 1, & a \leqslant b \\ b/a, & \text{otherwise} \end{cases}$$

- Gödel operations:

$$a \otimes b = \min\{a, b\}, \quad a \to b = \begin{cases} 1, & a \leqslant b \\ b, & \text{otherwise} \end{cases}$$

- Łukasiewicz operations:

$$a \otimes b = \max\{a + b - 1, 0\}, \quad a \to b = \min\{1 - a + b, 1\},$$

Fuzzy sets and fuzzy relations

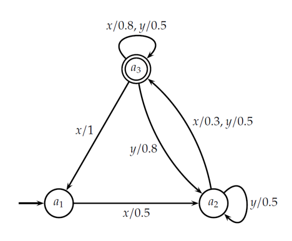
- ***fuzzy subset*** of A – function $\alpha : A \to L$
  L^A – all fuzzy subsets of A
  (*fuzzy vectors* of length $|A|$ with entries in $\mathcal{L}$, A-finite)
- ***fuzzy relation*** between A and B – function $\varphi : A \times B \to L$
  $L^{A\times B}$ – all fuzzy relations between A and B
  (*fuzzy matrices* of type $|A| \times |B|$ with entries in $\mathcal{L}$, A,B-finite)
- Fuzzy relations $\varphi \in L^{A\times B}$, $\phi \in L^{B\times C}$
  Composition: $\varphi \circ \phi \in L^{A\times C}$ - the matrix product
- Fuzzy relation $\varphi \in L^{A\times B}$ and fuzzy sets $\alpha \in L^A$, $\beta \in L^B$
  Composition: $\alpha \circ \varphi \in L^B$, $\varphi \circ \beta \in L^A$ - the vector-matrix products
- Fuzzy sets: $\alpha_1, \alpha_2 \in L^A$
  Composition: $\alpha_1 \circ \alpha_2 \in L$ - the scalar (dot) product

Fuzzy automata

$\mathcal{L}$ – complete residuated lattice, X – finite alphabet, X^* – free monoid, $\varepsilon \in X^*$ – ***empty word***

$\mathcal{A} = (A, \sigma, \delta, \tau)$ – ***fuzzy automaton***

- A – ***set of states*** (not necessarily finite)
- $\sigma \in L^A$ – ***fuzzy set of initial states***, $\tau \in L^A$ – ***fuzzy set of terminal (final) states***, $\delta \in L^{A\times X\times A}$ – ***fuzzy transition function***
- $\delta \equiv \{\delta_x\}_{x\in X}$ – $\delta_x \in L^{A\times A}$, $\delta_x(a,b) = \delta(a,x,b)$



$\mathcal{L}$ – product structure

$$A = \{a_1, a_2, a_3\},$$

$$\sigma = [1\ 0\ 0],$$

$$\tau = [0\ 0\ 1]^{-1},$$

$$\delta_x = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 1 & 0 & 0.8 \end{bmatrix}, \quad \delta_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.8 & 0.5 \end{bmatrix}.$$

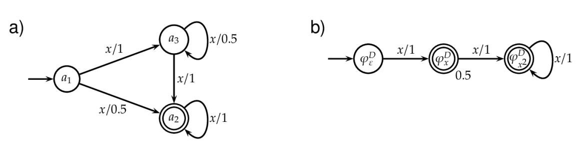
Fuzzy languages

- ***fuzzy language*** – $f : X^* \to L$ – fuzzy subset of the free monoid X^*
- ***fuzzy language recognized by*** $\mathcal{A}$ – $[\![\mathcal{A}]\!](u) = \sigma \circ \delta_u \circ \tau$
  (***degree to which*** $\mathcal{A}$ ***accepts the word*** u)

CDFAs

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over $\mathcal{L}$ and X. Then $\mathcal{A}$ is called:

- **complete**, if
$$(\forall x \in X)(\forall a \in A)(\exists b \in A)\delta_x(a, b) > 0$$
- **deterministic**, if
  - $(\exists! a \in A)\sigma(a) > 0$
  - $(\forall x \in X)(\forall a, b_1, b_2 \in A)\delta_x(a, b_1) > 0 \text{ and } \delta_x(a, b_2) > 0 \Rightarrow b_1 = b_2$
- If $\mathcal{A}$ is simultaneously complete and deterministic, then it is called **complete deterministic fuzzy automaton** (**CDFA** for short).



Factorization of A

Let $\mathcal{A}$ be a fuzzy automation over X and $\mathcal{L}$. A ***factorization of automation*** $\mathcal{A}$, or a ***factorization*** of A, is an ordered pair $D = (f, g)$ of functions $g : L^A \to L$ and $f : L^A \to L^A$ that satisfy the following properties

- $\sigma = g(\sigma) \otimes f(\sigma), \quad$ for every $\sigma \in L^A$;
- $g(\emptyset) = 1$;

Factorizations of fuzzy states

The idea of factorizations was firstly introduced by

D. Kirsten, L. Mäurer, ***On the determinization of weighted automata***, J. Automata, Lang. Combin. 10 (2005) 287–312.

in the determinization algorithm for weighted finite automata (WFAs) over semirings.

Trivial factorization

The factorization $D_N = (f_N, g_N)$ of A, where $g_N(\sigma) = 1$ and $f_N(\sigma) = \sigma$ for every $\sigma \in L^A$ is called the ***trivial factorization***.

Mohri's factorization

Mohri's factorization: the ordered pair $D_M = (f_M, g_M)$, where $f_M : L^A \to L^A$ and $g_M : L^A \to L$, for every $\sigma \in L^A$ and $a \in A$ by

$$g_M(\sigma) = \begin{cases} 1, & \sigma = \emptyset \\ \bigvee_{a\in A} \sigma(a), & \text{otherwise} \end{cases}, \quad \textit{and} \quad f_M(\sigma)(a) = g_M(\sigma) \to \sigma(a);$$

Maximal factorization

A factorization $D = (f, g)$ of A is called ***maximal*** if

- $g(a \otimes \sigma) = a \otimes g(\sigma)$,
- $f(a \otimes \sigma) = f(\sigma)$,

for all $a \in L$ and $\sigma \in L^A$ such that $a \otimes \sigma \neq \emptyset$.
Maximal factorizations play a *crutial* role in the determinization of fuzzy automata.

Weakly right and left invariant fuzzy relations

$\mathcal{A} = (A, \sigma, \delta, \tau)$ - fuzzy automaton, φ - reflexive fuzzy relation on A. Then φ is:

- ***weakly right invariant***, if $\varphi \circ \tau_u = \tau_u$, for each $u \in X^*$.
- ***weakly left invariant***, if $\sigma_u \circ \varphi = \sigma_u$, for each $u \in X^*$.

Right and left invariant fuzzy relations

$\mathcal{A} = (A, \sigma, \delta, \tau)$ - fuzzy automaton, φ - fuzzy quasi-order on A. Then φ is:

- ***right invariant***, if

$$\varphi \circ \delta_x \circ \varphi = \delta_x \circ \varphi, \qquad \text{for each } x \in X,$$
$$\varphi \circ \tau = \tau,$$

- ***left invariant***, if

$$\varphi \circ \delta_x \circ \varphi = \varphi \circ \delta_x, \qquad \text{for each } x \in X,$$
$$\sigma \circ \varphi = \sigma.$$

- φ is right (left) invariant $\Rightarrow \varphi$ is weakly right (left) invariant

A determinization algorithm for fuzzy automata

Construction of family of fuzzy sets

- $\mathcal{A} = (A, \sigma, \delta, \tau)$ – fuzzy automaton;
- $X = \{x_1, x_2, \ldots, x_m\}$ – alphabet;
- φ – fuzzy relation on A;
- We define a family $\{\varphi_u^D\}_{u \in X^*}$ of fuzzy sets over A inductively by

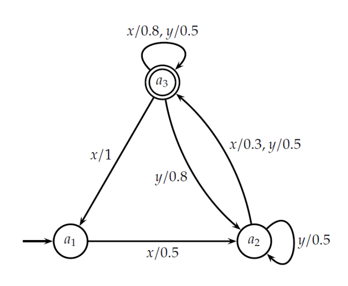
$$\begin{aligned} \varphi_\varepsilon^D &= f(\sigma \circ \varphi), \\ \varphi_{ux}^D &= f(\varphi_u^D \circ \delta_x \circ \varphi), \text{ for every } u \in X^* \text{ and } x \in X. \end{aligned}$$

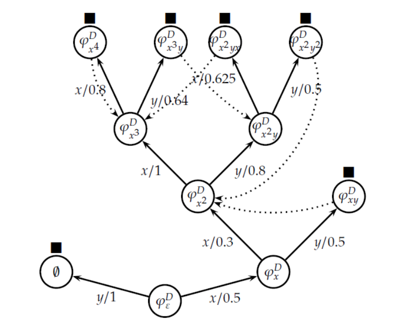
Construction of CDFA

- Set $A_\varphi^D = \{\varphi_u^D | u \in X^*\}$;
- Define fuzzy sets $\delta_\varphi^D : A_\varphi^D \times X \times A_\varphi^D \to L$, $\sigma_\varphi^D : A_\varphi^D \to L$ and $\tau_\varphi^D : A_\varphi^D \to L$ with

$$\delta_\varphi^D(\alpha, x, \beta) = \begin{cases} g(\varphi_u^D \circ \delta_x \circ \varphi), & \text{if } \alpha = \varphi_u^D \text{ and } \beta = \varphi_{ux}^D, \quad u \in X^* \\ 0, & \text{otherwise} \end{cases};$$

$$\sigma_\varphi^D(\alpha) = \begin{cases} g(\sigma \circ \varphi), & \text{if } \alpha = \varphi_\varepsilon^D \\ 0, & \text{otherwise} \end{cases}; \qquad \tau_\varphi^D(\alpha) = \alpha \circ \tau, \text{ for every } \alpha \in A_\varphi^D.$$





$$\varphi = \varphi^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\sigma \circ \varphi = [1\ 0\ 0] = g(\sigma \circ \varphi) \otimes f(\sigma \circ \varphi) = 1 \otimes [100] = 1 \otimes \varphi^D_\varepsilon$$

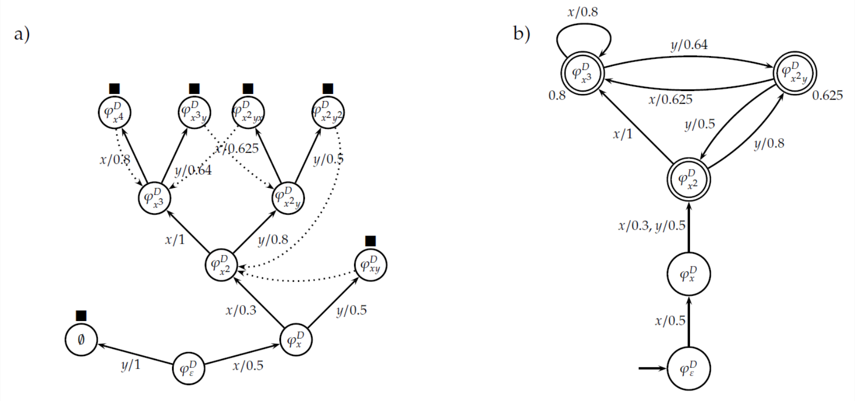
$$\varphi^D_\varepsilon \circ \delta_x \circ \varphi = [0.3\ 0.5\ 0] = g(\varphi^D_\varepsilon \circ \delta_x \circ \varphi) \otimes f(\varphi^D_\varepsilon \circ \delta_x \circ \varphi) = 0.5 \otimes [0.6 1 0] = 0.5 \otimes \varphi^D_x$$

$$\varphi^D_\varepsilon \circ \delta_y \circ \varphi = [0\ 0\ 0] = g(\varphi^D_\varepsilon \circ \delta_y \circ \varphi) \otimes f(\varphi^D_\varepsilon \circ \delta_y \circ \varphi) = 1 \otimes [000] = 1 \otimes \varphi^D_y$$

$$\varphi^D_x \circ \delta_x \circ \varphi = [0.3\ 0.3\ 0.3] = g(\varphi^D_x \circ \delta_x \circ \varphi) \otimes f(\varphi^D_x \circ \delta_x \circ \varphi) = 0.3 \otimes [111] = 0.3 \otimes \varphi^D_{x^2}$$

A determinization algorithm for fuzzy automata

Example (cont.)



A determinization algorithm for fuzzy automata
(cont.)

Lemma

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ be a factorization of A, and φ a fuzzy relation on A. Then $\mathcal{A}^D_\varphi = (A^D_\varphi, \sigma^D_\varphi, \delta^D_\varphi, \tau^D_\varphi)$ is a CDFA.

Theorem

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ a factorization of A, φ a ***reflexive weakly right invariant fuzzy relation*** on A, and let $\mathcal{A}^D_\varphi = (A^D_\varphi, \sigma^D_\varphi, \delta^D_\varphi, \tau^D_\varphi)$. Then $[\![\mathcal{A}]\!] = [\![\mathcal{A}^D_\varphi]\!]$.

Theorem

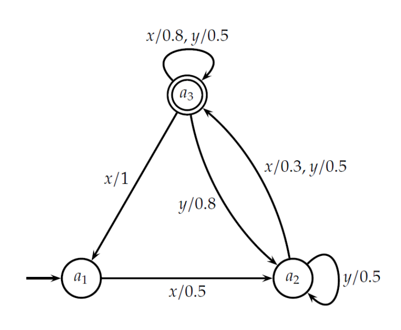
Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ a ***maximal*** factorization of A, and let φ and ϕ be ***right invariant fuzzy quasi-orders*** on A. Moreover, let $\mathcal{A}^D_\varphi = (A^D_\varphi, \sigma^D_\varphi, \delta^D_\varphi, \tau^D_\varphi)$ and $\mathcal{A}^D_\phi = (A^D_\phi, \sigma^D_\phi, \delta^D_\phi, \tau^D_\phi)$ be the corresponding CDFAs of $\mathcal{A}$. If $\varphi \leqslant \phi$ holds, then $|\mathcal{A}^D_\phi| \leqslant |\mathcal{A}^D_\varphi|$ follows.

Example 1

Consider a fuzzy automation $\mathcal{A}$ over the alphabet $X = \{x, y\}$ and the product (Gougen) structure

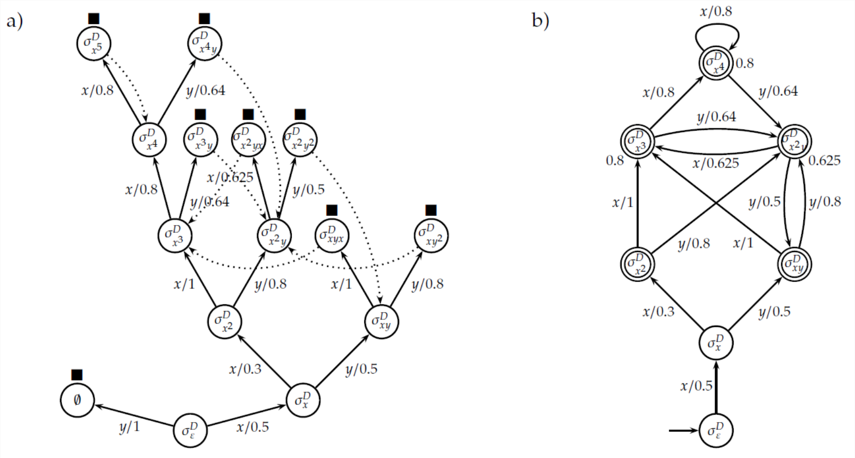
$$a \otimes b = a \cdot b, \quad a \to b = \begin{cases} 1, & a \leqslant b \\ b/a, & \text{otherwise} \end{cases}$$

with the transition graph shown in the next Figure.



Example 1
(cont.)

The transition tree (a) and the transition graph (b) of the CDFA $\mathcal{A}^D$ of the fuzzy automaton $\mathcal{A}$:

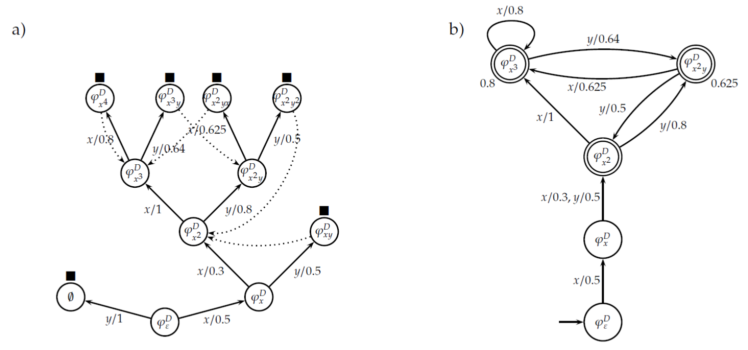


Example 1
(cont.)

On the other hand, the greatest right invariant fuzzy quasi-order on A is given by

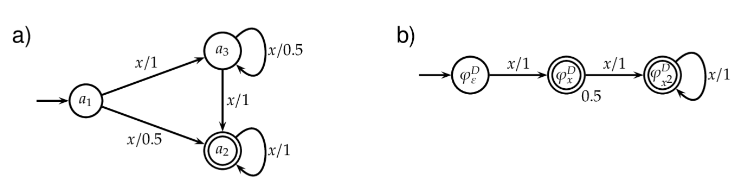
$$\varphi = \varphi^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The transition tree (a) and the transition graph (b) of the CDFA $\mathcal{A}^D_\varphi$ of the fuzzy automaton $\mathcal{A}$:



Example 2

Let $\mathcal{A}$ be an automaton over the one-element alphabet $X = \{x\}$ and the product structure given by the transition graph shown in Figure a).



The Nerode automaton of $\mathcal{A}$ and the CDFA $\mathcal{A}_D$ of $\mathcal{A}$ have **infinitely many states**.
The greatest right invariant fuzzy quasi-order is equal to

$$\varphi = \varphi^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0.5 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The resulting CDFA $\mathcal{A}^D_\varphi$ is shown in Figure b).

A canonization algorithm for fuzzy automata

- If the previous algorithm yields the finite CDFA, then it is not necessarily a minimal CDFA equivalent to the starting fuzzy automaton $\mathcal{A}$
- Previous method can be adapted to a double-reverse canonization method for fuzzy finite automata over complete residuated lattices
- The idea of constructing the double-reverse canonization algorithm is not new in the literature: Brzozowski's original idea was to construct, for a given nondeterministic automaton $\mathcal{A}$, the minimal deterministic automaton $d(r(d(r(\mathcal{A}))))$ equivalent to $\mathcal{A}$, where by $r(\mathcal{A})$ we mean the reverse, and by $d(\mathcal{A})$ we mean the determinization process (for example, the well-known subset construction)
- We adapt the Brzozowski's procedure and perform twice the construction of the CDFA that recognizes the reverse fuzzy language of the starting fuzzy automaton

Construction of family of fuzzy sets

- $\mathcal{A} = (A, \sigma, \delta, \tau)$ – fuzzy automaton; $X = \{x_1, x_2, \dots, x_m\}$ – alphabet;
- φ – fuzzy relation on A;
- We define a family $\{\tilde{\varphi}_u^D\}_{u \in X^*}$ of fuzzy sets over A inductively by

$$\tilde{\varphi}_\varepsilon^D = f(\varphi \circ \tau),$$
$$\tilde{\varphi}_{xu}^D = f(\varphi \circ \delta_x \circ \tilde{\varphi}_u^D).$$

Construction of CDFA equivalent to $\overline{\mathcal{A}}$

- Set $\tilde{A}_\varphi^D = \{\tilde{\varphi}_u^D\}_{u \in X^*}$
- Define fuzzy sets $\tilde{\delta}_\varphi^D : \tilde{A}_\varphi^D \times X \times \tilde{A}_\varphi^D \to L$ by, $\tilde{\sigma}_\varphi^D : \tilde{A}_\varphi^D \to L$ and $\tilde{\tau}_\varphi^D : \tilde{A}_\varphi^D \to L$ with

$$\tilde{\delta}_\varphi^D(\alpha, x, \beta) = \begin{cases} g(\varphi \circ \delta_x \circ \tilde{\varphi}_u^D), & \alpha = \tilde{\varphi}_u^D \text{ and } \beta = \tilde{\varphi}_{xu}^D, \quad u \in X^* \\ 0, & \text{otherwise} \end{cases};$$

$$\tilde{\sigma}_\varphi^D(\alpha) = \begin{cases} g(\varphi \circ \tau), & \alpha = \tilde{\varphi}_\varepsilon^D \\ 0, & \text{otherwise} \end{cases}, \qquad \tilde{\tau}_\varphi^D(\alpha) = \sigma \circ \alpha, \text{ for every } \alpha \in \tilde{A}_\varphi^D.$$

Lemma

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ be a factorization of A, and φ a fuzzy relation on A. Then $\tilde{\mathcal{A}}^D_\varphi = (\tilde{A}^D_\varphi, \tilde{\sigma}^D_\varphi, \tilde{\delta}^D_\varphi, \tilde{\tau}^D_\varphi)$ is a CDFA.

Theorem

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ a factorization of A, φ a ***reflexive weakly left invariant fuzzy relation*** on A, and let $\tilde{\mathcal{A}}^D_\varphi = (\tilde{A}^D_\varphi, \tilde{\sigma}^D_\varphi, \tilde{\delta}^D_\varphi, \tilde{\tau}^D_\varphi)$. Then $[\![\overline{\mathcal{A}}]\!] = [\![\tilde{\mathcal{A}}^D_\varphi]\!]$.

Theorem

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X, $D = (f, g)$ a ***maximal factorization*** of A, and let φ and ϕ be ***left invariant fuzzy quasi-orders*** on A. Let $\tilde{\mathcal{A}}^D_\varphi = (\tilde{A}^D_\varphi, \tilde{\sigma}^D_\varphi, \tilde{\delta}^D_\varphi, \tilde{\tau}^D_\varphi)$ and $\tilde{\mathcal{A}}^D_\phi = (\tilde{A}^D_\phi, \tilde{\sigma}^D_\phi, \tilde{\delta}^D_\phi, \tilde{\tau}^D_\phi)$ be the corresponding CDFAs equivalent to $\overline{\mathcal{A}}$. If $\varphi \leqslant \phi$ holds, then $|\tilde{\mathcal{A}}^D_\phi| \leqslant |\tilde{\mathcal{A}}^D_\varphi|$ follows.

- Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X and $\mathcal{L}$, $D = (f, g)$ a maximal factorization of A, and φ a left invariant fuzzy quasi-order on A.
- By constructing the fuzzy automaton $\tilde{\mathcal{A}}^D_\varphi = (\tilde{A}^D_\varphi, \tilde{\sigma}^D_\varphi, \tilde{\delta}^D_\varphi, \tilde{\tau}^D_\varphi)$, we obtain CDFA that is equivalent to the reverse fuzzy automaton $\overline{\mathcal{A}}$ of the initial fuzzy automaton $\mathcal{A}$.
- Given the maximal factorization $D_1 = (f_1, g_1)$ of $\tilde{A}^D_\varphi$ and the left invariant fuzzy quasi-order φ_1 on $\tilde{A}^D_\varphi$, we ask if the fuzzy automaton $\widetilde{\left(\tilde{\mathcal{A}}^D_\varphi\right)}^{D_1}_{\varphi_1}$ is equivalent to $\mathcal{A}$:
  - The obtained fuzzy automaton is a minimal CDFA equivalent to the starting fuzzy automaton,
  - Fuzzy relation φ_1 does not play a role in the second reverse process and can be replaced by the identity relation.

A canonization algorithm for fuzzy automata
(cont.)

Theorem

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a fuzzy automaton over X and $\mathcal{L}$, $D = (f, g)$ a maximal factorization of A, φ a left invariant fuzzy quasi-order on A, and $D_1 = (f_1, g_1)$ a maximal factorization of A_φ^D. Define a fuzzy automaton $\mathcal{B}$ as

$$\mathcal{B} = \widetilde{(\tilde{\mathcal{A}}_\varphi^D)}^{D_1}.$$

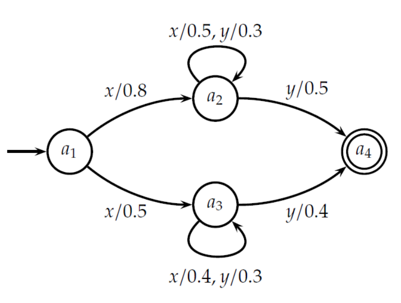
Then $\mathcal{B}$ is a minimal CDFA equivalent to $\mathcal{A}$.

Example 1

Consider a fuzzy automation $\mathcal{A}$ over the alphabet $X = \{x, y\}$ and the product (Gougen) structure

$$a \otimes b = a \cdot b, \quad a \to b = \begin{cases} 1, & a \leqslant b \\ b/a, & \text{otherwise} \end{cases}$$

with the transition graph shown in the next Figure.

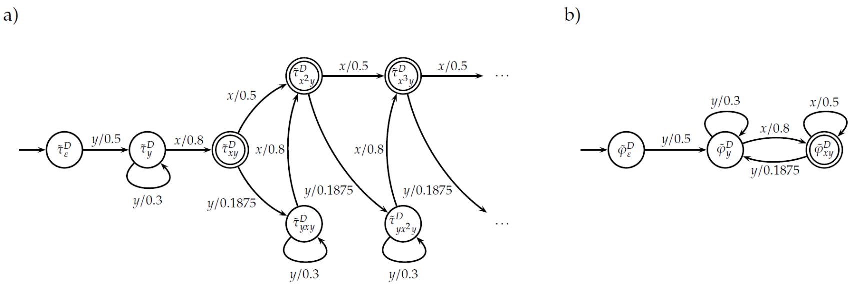


Example 1
(cont.)

The reverse CDFA $\tilde{\mathcal{A}}^D$ is infinite, and part of its transition graph is shown in Figure (a). On the other hand, we have that the greatest left invariant fuzzy quasi-order is equal to

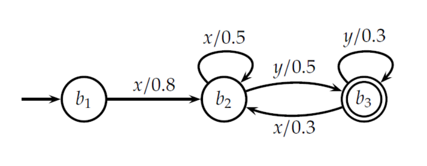
$$\varphi = \varphi^{\mathrm{li}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix}.$$

The automaton $\tilde{\mathcal{A}}^D_\varphi$ is finite, and its transition graph is shown in Figure (b).



Example 1
(cont.)

The transition graph of the fuzzy automaton $\mathcal{B} = \widetilde{(\tilde{\mathcal{A}}_{\varphi}^{D})}^{D_1}$ is shown in the following figure.

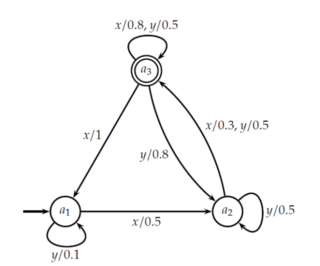


Example 2

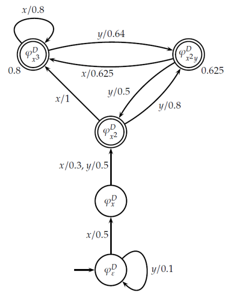
Let $\mathcal{A}$ be an automaton over the alphabet $X = \{x, y\}$ and the product (Gougen) structure with the transition graph shown in Figure (a).
The greatest right invariant and the greatest left invariant fuzzy quasi-orders on A are respectively equal to

$$\varphi = \varphi^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \varphi^{\mathrm{li}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Brzozowski fuzzy automaton $\mathcal{B}$ cannot be obtained by means of the Algorithm.
On the other hand, the resulting CDFA $\mathcal{A}_\varphi^D$ is shown in Figure (b).



a)



b)

Thank you for your attention!