ABSOLUTE GRAPHS WITH PRESCRIBED ENDOMORPHISM MONOID

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ABSTRACT. We consider endomorphism monoids of graphs. It is well-known that any monoid can be represented as the endomorphism monoid M of some graph Γ with countably many colors. We give a new proof of this theorem such that the isomorphism between the endomorphism monoid $\mathrm{End}(\Gamma)$ and M is absolute, i.e. $\mathrm{End}(\Gamma) \cong M$ holds in any generic extension of the given universe of set theory. This is true if and only if $|M|, |\Gamma|$ are smaller than the first Erdős cardinal (which is known to be strongly inaccessible). We will encode Shelah's absolutely rigid family of trees [15] into Γ. The main result will be used to construct fields with prescribed absolute endomorphism monoids, see [8].

1. INTRODUCTION

In 1982 Shelah [15] constructed absolutely rigid families of 2-color trees for any cardinal $\lambda < \kappa(\omega)$, where $\kappa(\omega)$ denotes the *first ω-Erdős cardinal*, see [11, p. 302, Definition 17.28]. This is the smallest cardinal κ satisfying the uniformization principle $\kappa \to (\omega)^{<\omega}$, i.e. for every function f from the finite subsets of κ to 2 there exist an infinite subset $X \subset \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ for all finite subsets Y of X. The cardinal $\kappa(\omega)$ is (a large) strongly inaccessible cardinal [11, p. 303], and if it exists then it will also exist in Gödel's universe. In [15] is was also shown that the restriction to cardinals below $\kappa(\omega)$ is necessary when constructing families of absolutely rigid trees. Families of colored trees are called rigid, if there are no homomorphisms between two distinct members of this family, where a homomorphism is a mapping that preserves the ordering, height and color of the trees (see also Section 2). Moreover, such families are called absolutely rigid if rigidity is preserved when passing to generic extensions of the universe of set theory. On one hand, there are well-known criteria for absolute properties (see Levi [12]), on the other hand there are many statements (like being a powerset or an indecomposable $\aleph_1$-free abelian group of infinite rank) which are not absolute. Intuitively, if a property of a structure is absolute, for instance it remains true if in another model of set theory the structure has another cardinality, e.g. becomes countable. Thus it is interesting to check if certain algebraic results also hold absolutely. Here it is helpful to encode Shelah's trees into other algebraic structures. This way we can find new constructions which in addition

Date: December 5, 2006.

2000 *Mathematics Subject Classification.* Primary 03C55, 05C05, 05C60, 20M30 Secondary 03E75, 06A06.

Key words and phrases. absolute endomorphism monoids, action of semigroups on sets, colored graphs, absoluteness.

This work was supported by the project No. I-706-54.6/2001 of the German-Israeli Foundation for Scientific Research & Development and a fellowship within the Postdoc-Programme of the German Academic Exchange Service (DAAD).

have absolute properties. A first application is the existence of absolutely rigid families of abelian groups in [9] and more general of modules in [7]. In [8] we want to apply this idea to the construction of fields with absolutely prescribed endomorphism monoid. In this case we will follow a classical argument [6, 13] and want to encode graphs into fields. In order to do this absolutely, we will need an absoluteness result realizing monoids as endomorphism monoids of graphs. Again, there are well-known classical realization theorems for monoids as endomorphism monoids, but it is the aim of this paper to express monoids as endomorphism monoids of absolute graphs. Such results can only be derived if the involved cardinals are below $\kappa(\omega)$, as explained in Section 4. Also note that we cannot replace graphs in this paper by trees, because many groups (as well-known) are not automorphism groups of trees. We will give examples in Corollary 3.4. One of our main results is Theorem 2.5 showing that the classical realization of all monoids as endomorphism monoids of graphs with countably many colors carries over to absoluteness if the monoid and the graph have size $< \kappa(\omega)$.

2. Construction of graphs with prescribed endomorphisms

In this section we will present a new construction (in ZFC) of a family of colored graphs with an arbitrary prescribed monoid of endomorphisms M. As a consequence, all left-cancellative monoids arise as monoids of monomorphisms of some colored graph. The size of the colored graphs can be any cardinal $\geq |M|$.

To fix the terminology, we first recall some standard notations. Let $(A, (R_i)_{i<\omega})$ and $(B, (S_i)_{i<\omega})$ be two relational structures (with arities n_i of R_i and S_i respectively, for each $i < \omega$). A homomorphism is a map $f : A \to B$ such that $f(R_i) \subseteq S_i$ for each $i < \omega$, with f extended naturally to tuples of elements. A partially ordered set $(T, \leq)$ is called a *tree*, if $(T, \leq)$ is lower directed and for each $x \in T$, the set $\{t \in T \mid t < x\}$ is a well-ordered chain; we will call the ordinal type of this set the *height* of x, denoted by $\mathrm{ht}_T(x)$ or $\mathrm{ht}(x)$ if T is clear from the context. We denote the smallest element of T by $\perp$; thus $\mathrm{ht}(\perp) = 0$. Let $(T, \leq)$ be a tree in which each branch (= a maximal subchain of $(T, \leq)$) is finite. We will call the structure $(T, \leq, (H_i)_{i<\omega})$ where $H_i = \{x \in T \mid \mathrm{ht}(x) = i\}$ $(i < \omega)$ a *(height-) valuated tree*. If, moreover, $P_i \subseteq T$ for $i \in I$, then we call $\mathcal{T} = (T, \leq, (H_i)_{i<\omega}, (P_i)_{i\in I})$ a *colored valuated tree*; here, I is the set of colors, and if $x \in P_i$ we call i the *color* of x. Thus, a *homomorphism between two such colored valuated trees* is a mapping which preserves the order, the heights and the colors.

The following lemma summarizes the properties of Shelah's trees.

Lemma 2.1. (Shelah [15]) *Let $\lambda < \kappa(\omega)$ be a cardinal. Then there is a family of λ colored valuated trees $\{\mathcal{T}_\alpha \mid \alpha < \lambda\}$ with $|T_\alpha| = \lambda$ such that $\mathrm{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) = \emptyset$ holds absolutely for all $\alpha, \beta < \lambda$ with $\alpha \neq \beta$.*

Proof. See [15] or [8]. □

A *graph* is a structure (S, R) where $R \subseteq S \times S$. We will call a structure $\Gamma = (S, R, (Q_i)_{i<\omega})$ with $R \subseteq S \times S$ and $Q_i \subseteq S$ for each $i < \omega$ an *ω-colored graph*.

We want to present a rigid colored graph Γ with $\mathrm{End}(\Gamma) = \{id\}$. This result itself is well-known, see e.g. [14] However, the standard proofs involve limit ordinals (cf. [14, Ch. II] making them non-absolute. Here we will use the trees of Lemma 2.1 to ensure that our graph Γ will also be absolutely rigid for $|\Gamma| < \kappa(\omega)$.

Theorem 2.2. *Let λ be a cardinal. There exists an ω-colored graph*

$$\Gamma = (S, \leq, (F_j)_{j<\omega}, (H_j)_{j<\omega}, P_1, P_2)$$

of size λ such that $\mathrm{End}(\Gamma) = \{id\}$.

Proof. Let $\mathfrak{F} = \{\mathcal{T}_\alpha \mid \alpha \in I\}$ be the absolutely rigid family of colored valuated trees from Lemma 2.1 with

$$\mathcal{T}_\alpha = (T_\alpha, \leq_\alpha, (H_i^\alpha)_{i<\omega}, P_1^\alpha, P_2^\alpha)$$

and $|\, T_\alpha \,| = \lambda = |\, I \,|$ for each $\alpha \in I$.

We may assume that $T_\alpha \cap T_\beta = \emptyset$ for any $\alpha \neq \beta$ in I, and we define an ω-chain of trees

$$(S_0, \leq_0) \subseteq (S_1, \leq_1) \subseteq \dots$$

as follows. First, choose any tree $\mathcal{T}_0 \in \mathfrak{F}$ and put $(S_0, \leq_0) = (T_0, \leq)$. Next, split

$$\mathfrak{F} \setminus \{\mathcal{T}_0\} = \dot{\bigcup}_{1 \leq i < \omega} \mathfrak{F}_i,$$

a countable disjoint union, such that $|\, \mathfrak{F}_i \,| = \lambda$ for each $1 \leq i < \omega$. Now assume that $i < \omega$ and that we have constructed the tree $(S_i, \leq_i)$ such that $|\, S_i \,| = \lambda$ and each branch in $(S_i, \leq_i)$ is finite. Then the set F_i of all maximal elements in $(S_i, \leq_i)$ has size λ. We choose a bijection from F_i onto $\mathfrak{F}_{i+1}$ mapping $x \in F_i$ onto, say, $\mathcal{T}_x \in \mathfrak{F}_{i+1}$. Then let $S_{i+1} = S_i \cup \bigcup_{x \in F_i} T_x$, and we define the partial order $\leq_{i+1}$ on S_{i+1} such that it extends $\leq_i$, the order $\leq_x$ for each $x \in F_i$ and $x <_{i+1} y$ for each $x \in F_i$ and $y \in T_x$. Then $(S_{i+1}, \leq_{i+1})$ is again a tree with finite branches. Now let $(S, \leq) = \bigcup_{i<\omega} (S_i, \leq_i)$. Then $(S, \leq)$ is a tree of size λ in which each element has finite height and is not maximal. Moreover, let $H_i = \{s \in S \mid \mathrm{ht}(x) = i\}$ $(i < \omega)$ and $P_j = \bigcup_{\alpha \in I} P_j^\alpha$ for $j = 1, 2$, and put $\Gamma = (S, \leq, (F_j)_{j<\omega}, (H_j)_{j<\omega}, P_1, P_2)$.

Next, we prove that $\mathrm{End}(\Gamma) = \{id\}$. Let $h : \Gamma \to \Gamma$ be any color preserving homomorphism $h : \Gamma \to \Gamma$. We claim that $h = id$. Choose any $x \in S$ and let $x' = h(x)$. Then $x \in S_j$ for some $j < \omega$. If $i = \mathrm{ht}(x)$, then $x \in H_i$ and hence $x' = h(x) \in H_i$, so $\mathrm{ht}(x') = i$. Now choose $y \in F_j$ and $x \leq y$. We obtain $h(y) \in F_j$ and $y' = h(y) \geq h(x) = x'$. Note that $T_y = \{s \in S_{j+1} \mid y \leq_{j+1} s\}$ and $T_{y'} = \{s \in S_{j+1} \mid y' \leq_{j+1} s\}$. In Γ we have $S_{j+1} = \{s \in S \mid \exists z \in F_{j+1} \text{ with } s \leq z\}$. Since h is color preserving it follows that $h(F_{j+1}) \subseteq F_{j+1}$ and hence $h(S_{j+1}) \subseteq S_{j+1}$. Hence $T_y = \{s \in S \mid y \leq s \leq z \textit{ for some } z \in F_{j+1}\}$ and we obtain $h(T_y) \subseteq T_{y'}$. Since $\mathrm{ht}_S(y) = \mathrm{ht}_S(y')$, it follows that

$$h \restriction T_y : \mathcal{T}_y \to \mathcal{T}_{y'}$$

preserves the heights of the elements and thus is a homomorphism between $\mathcal{T}_y$ and $\mathcal{T}_{y'}$. By the rigidity property of our family of trees, we obtain $\mathcal{T}_y = \mathcal{T}_{y'}$, i.e. $y = y'$. Since $(S, \leq)$ is a tree, $x \leq y$, $x' \leq y$ and $\mathrm{ht}(x) = \mathrm{ht}(x')$, we get $x = x'$, proving the claim. □

We call any structure $(S, (R_i)_{i \in I})$ where S is a set and $R_i \subseteq S \times S$ for each $i \in I$ an *edge-colored graph*. If $(x, y) \in R_i$, then i is the color of the edge (x, y).

In the following simple lemma we represent a given monoid M as the homomorphism monoid of an edge-colored graph of arbitrary size. This is an obvious variation of Frucht's theorem and well-known. We include the proof for the sake of transparence of later arguments.

Lemma 2.3. *Let M be a monoid. Then there exists an edge-colored graph $\Gamma = (M, (R_i)_{i \in I})$ with $I = M \setminus \{1\}$ such that $M \cong \mathrm{End}(\Gamma)$. Moreover, M is left-cancellative if and only if $\mathrm{End}(\Gamma) = \mathrm{Mon}(\Gamma)$.*

Proof. Put $R_i = \{(m, mi) \mid m \in M\}$ for all $i \in I$. Define

$$\varphi : M \longrightarrow \mathrm{End}(\Gamma)$$
$$x \mapsto (f_x : \Gamma \to \Gamma \ \textit{by}\ m \mapsto xm).$$

We claim that φ is an isomorphism. It is clear that φ is a homomorphism. Now let $x, x' \in M$ with $x \neq x'$. Then $f_x(1) = x \neq x' = f_{x'}(1)$, hence φ is injective. Next, let $g \in \mathrm{End}(\Gamma)$ and put $x = g(1)$. We claim that $f_x = g$. Indeed, let $m \in M$. Then $(1, 1m) \in R_m$ and hence $(g(1), g(m)) \in R_m$ which implies $g(m) = g(1)m = xm = f_x(m)$ and our claim follows. Hence φ is an isomorphism.

Now assume that M is left-cancellative and let $g \in \mathrm{End}(\Gamma)$. By the above, $g = f_x$ for some $x \in M$. If $g(m) = g(m')$ for some $m, m' \in M$, then $xm = f_x(m) = f_x(m') = xm'$, so $m = m'$ by left-cancellation and hence g is injective. Conversely, assume that every $g \in \mathrm{End}(\Gamma)$ is injective. If $x, m, m' \in M$ with $xm = xm'$, then $f_x(m) = f_x(m')$, so $m = m'$ and M is left-cancellative. □

Note that the set I in Lemma 2.3 can be arbitrarily enlarged by adding empty relations R_i.

Observe that if the monoid M is uncountable, then Lemma 2.3 gives a realization result for M as $M \cong \mathrm{End}(\Gamma)$ by some edge-colored graph Γ using $|\,M\,|$ colors for the edges. The final graph in this paper will be applied in [8] for the construction of fields with prescribed endomorphism monoids and thus encoded into a field by adding roots of prime elements. Thus a realization result by ω-colored graphs is needed. This is done in our next result which uses the rigid colored valuated tree of Theorem 2.2. We will apply a simplified version of a construction of graphs from [14, pp. 72 ff.]

Theorem 2.4. *Let $\Gamma = (X, (R_i)_{i \in I})$ be an edge-colored graph with I infinite. Then there exists an ω-colored graph $\tilde{\Gamma} = (\tilde{X}, R, (C_i)_{i<\omega})$ of the same size such that* $\mathrm{End}(\Gamma) \cong \mathrm{End}(\tilde{\Gamma})$. *Moreover* $\mathrm{Mon}(\Gamma) \cong \mathrm{Mon}(\tilde{\Gamma})$.

Proof. We first define the graph $\tilde{\Gamma}$ and then prove its properties. Let

$$\Delta = (I, \leq, (F_n)_{n<\omega}, (H_n)_{n<\omega}, P_1, P_2)$$

be the rigid graph given by Theorem 2.2 on I with $\mathrm{End}(\Delta) = \{id\}$. Moreover, let

$$\tilde{\Gamma} = (\tilde{X}, R, C_1, C_2, C_3, (F_n)_{n<\omega}, (H_n)_{n<\omega}, P_1, P_2),$$

with

(i) $\tilde{X} = X \,\dot{\cup}\, I \,\dot{\cup}\, (X \times X)$,
(ii) $C_1 = X$, $C_2 = I$, and $C_3 = X \times X$,
(iii) $F_n, H_n, P_1, P_2 \subseteq I$ as given before $(n < \omega)$,

and $R \subseteq \tilde{X} \times \tilde{X}$ defined in the following way:

(a) $xR(x, y)$ for all $x, y \in X$,
(b) $(x, y)Ry$ for all $x, y \in X$,
(c) $(x, y)Ri :\Leftrightarrow xR_iy$ for all $x, y \in X, i \in I$,
(d) $iRj :\Leftrightarrow i \leq j$ for $i, j \in I$.

In Figure 1 the graph and its relations are sketched to illustrate the structure.

For each mapping $f : X \to X$ we define a mapping $\tilde{f} : \tilde{X} \to \tilde{X}$ by

$$\tilde{f} \restriction X = f,\ \tilde{f} \restriction I = id_I,\ \tilde{f}((x, y)) = (f(x), f(y)) \text{ for } (x, y) \in X \times X.$$

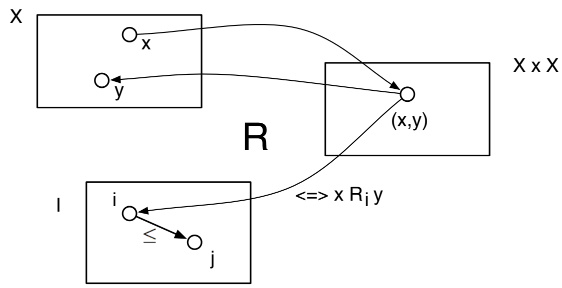

FIGURE 1. The graph $\tilde{\Gamma}$ and its relations

Now define

$$\varphi : \mathrm{End}(\Gamma) \to \mathrm{End}(\tilde{\Gamma}) \ by \ \varphi(f) = \tilde{f}.$$

We will prove that φ is an isomorphism.

Let $f \in \mathrm{End}(\Gamma)$. First we claim that $\tilde{f} \in \mathrm{End}(\tilde{\Gamma})$. For this, it suffices to consider $x, y \in \tilde{X}$ and $i \in I$ such that $(x,y)Ri$. Then xR_iy and since f is color-preserving we obtain $f(x)R_if(y)$. With $f(x) = \tilde{f}(x)$, $f(y) = \tilde{f}(y)$ this implies $(\tilde{f}(x), \tilde{f}(y))Ri$.

Moreover, $\varphi(fg) = \widetilde{fg} = \tilde{f}\tilde{g} = \varphi(f)\varphi(g)$ and $\varphi(id_X) = id_{\tilde{X}}$ which implies that φ is a homomorphism.

Since $\tilde{f} \upharpoonright X = f$ for each $f \in \mathrm{End}(\Gamma)$ it is clear that φ is injective.

Next we prove that φ is surjective. Choose any $g \in \mathrm{End}(\tilde{\Gamma})$. Then $g(X) \subseteq X$, $g(I) \subseteq I$ and $g(X \times X) \subseteq X \times X$. Put $f = g \upharpoonright X : X \to X$. We claim that $\tilde{f} = g$. As $g(I) \subseteq I$ it follows that $g \upharpoonright I \in \mathrm{End}(\Delta)$ and hence $g = id_I$. Now let $x, y \in X$. Then $xR(x,y)Ry$. We obtain $f(x) = g(x)Rg((x,y))Rg(y) = f(y)$ with $g((x,y)) \in X \times X$. Hence $g((x,y)) = (f(x), f(y)) = \tilde{f}((x,y))$ and we conclude $g = \tilde{f}$. It remains to prove that $f \in \mathrm{End}(\Gamma)$. Let $x, y \in X$, $i \in I$ such that xR_iy. Then $(x,y)Ri$ and hence $(f(x), f(y)) = g((x,y))Rg(i) = i$. We deduce $f(x)R_if(y)$. Hence φ is an isomorphism.

Next, we show that the restriction $\varphi \upharpoonright \mathrm{Mon}(\Gamma) : \mathrm{Mon}(\Gamma) \to \mathrm{Mon}(\tilde{\Gamma})$ is an isomorphism. If $f \in \mathrm{Mon}(\Gamma)$, then we claim that $\tilde{f} \in \mathrm{Mon}(\tilde{\Gamma})$. It is easily seen that $\tilde{f}$ is injective. By definition of $\tilde{f}$, it only remains to show that $\tilde{f}$ reflects R. First let $x \in X$ and $(z,y) \in X \times X$ such that $\tilde{f}(x)R\tilde{f}(z,y)$. We have to show that $xR(z,y)$.

Since $\tilde{f}(x)R\tilde{f}(z,y)$ it follows that $f(x)R(f(z), f(y))$ which implies that $f(x) = f(z)$. Now, as f is injective, we obtain $x = z$ and hence $xR(z,y)$. Similarly $\tilde{f}(z,y)R\tilde{f}(x)$ implies $(z,y)Rx$. Second, let $x, y \in X$ and $i \in I$ such that $\tilde{f}(x,y)R\tilde{f}(i)$ holds. Then we obtain $f(x)R_if(y)$ and hence xR_iy as f reflects R_i which implies $(x,y)Ri$. Hence $\tilde{f} \in \mathrm{Mon}(\tilde{\Gamma})$.

Now assume that $g \in \mathrm{Mon}(\tilde{\Gamma})$. Since φ is an isomorphism, we have $g = \tilde{f}$ for some $f \in \mathrm{End}(\Gamma)$. We claim that $f \in \mathrm{Mon}(\Gamma)$. As $\tilde{f}$ is injective, we immediately obtain that f is injective, too. So it remains to prove that f reflects R_i for all $i \in I$. Let $x, y \in X$, $i \in I$ such that $f(x)R_if(y)$. Then $\tilde{f}((x,y)) = (f(x), f(y))Ri = \tilde{f}(i)$. As $\tilde{f} \in \mathrm{Mon}(\tilde{\Gamma})$, we obtain $(x,y)Ri$, so xR_iy. Hence $\varphi \upharpoonright \mathrm{Mon}(\Gamma)$ is onto. □

The following theorem summarizes our results:

Main Theorem 2.5. *Let $\lambda \leq \kappa$ be two infinite cardinals, and let M be a monoid of size λ. Then there is an ω-colored graph $\Gamma = (X, R, (C_i)_{i<\omega})$ of size κ such that:*

(i) $\mathrm{End}(\Gamma) \cong M$.
(ii) *If M is left-cancellative, then* $\mathrm{Mon}(\Gamma) \cong M$.

Moreover, if $\kappa < \kappa(\omega)$, then the graph Γ can be constructed in an absolute way, i.e. such that properties (i) and (ii) are preserved under any generic extension of the underlying model of set theory.

Proof. Follows by Lemma 2.3, extending the index set I obtained if $\lambda < \kappa$, and Theorem 2.4. The absoluteness claim follows from the fact that all constructions are performed in an absolute way (see Burgess [2, pp. 408–412] for absolute arguments) involving trees. The construction of the trees involved is absolute by Lemma 2.1. □

In Section 4 we will see that $|\Gamma| < \kappa(\omega)$ in the absolute case is really the best we can achieve.

3. A fully rigid system of colored graphs

In this section we are going to generalize the main theorem of the preceding section by constructing a fully rigid family of colored graphs with prescribed endomorphism monoids. This is done in two steps. First we prove a generalized version of Theorem 2.2, which shows that there is a fully rigid family of colored graphs $\mathcal{S}$ such that $\mathrm{End}(S) = id$ for all $S \in \mathcal{S}$. Afterwards, we can directly prove the desired generalization of Main Theorem 2.5.

Theorem 3.1. *Let λ be any infinite cardinal. There exists a family $\mathcal{S}$ of ω-colored graphs Γ_X of size λ such that for all $X, Y \subseteq \lambda$ we have*

$$\mathrm{Hom}(\Gamma_X, \Gamma_Y) = \begin{cases} \{id\} \text{ if } X \subseteq Y \\ \emptyset \text{ if } X \not\subseteq Y. \end{cases}$$

Proof. Let $\mathfrak{F} = \{\mathcal{T}_\alpha \mid \alpha \in I\}$ be the absolutely rigid family of size λ of colored valuated trees from Lemma 2.1, each also of size λ. Split $\mathfrak{F} = \dot{\bigcup}_{\alpha \in I} \mathfrak{F}_\alpha$ such that $|\mathfrak{F}_\alpha| = \lambda$ for all $\alpha \in I$. We wish to obtain for each $\alpha \in I$ an ω-colored graph $\Gamma_\alpha = \big(S_\alpha, \leq_\alpha, (F_j^\alpha)_{j<\omega}, (H_j^\alpha)_{j<\omega}, P_1^\alpha, P_2^\alpha\big)$ of size λ such that $\mathrm{End}(\Gamma_\alpha) = \{id\}$. For this, we follow the proof of Theorem 2.2. Given $\alpha \in I$, in our first step we choose the tree $\mathcal{T}_\alpha \in \mathfrak{F}_\alpha$; then proceed as before (with $\mathfrak{F}$ replaced by $\mathfrak{F}_\alpha$ and $\mathcal{T}_0$ replaced by $\mathcal{T}_\alpha$) to obtain the ω-colored graph Γ_α as required. We denote the smallest element of $(S_\alpha, \leq_\alpha)$ by $\perp_\alpha$. Note that $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha \neq \beta$ in I.

Next we will construct for each subset $X \subseteq I$ an ω-colored graph

$$\Gamma_X = \big(S_X, \leq_X, (F_n^X)_{n<\omega}, (H_n^X)_{n<\omega}, P_1^X, P_2^X\big)$$

as follows. First choose a new element $\perp$ not contained in any S_α ($\alpha \in I$). We let $S_X = \{\perp\} \cup \bigcup_{\alpha \in X} S_\alpha$, and we define the partial order $\leq_X$ on S_X such that it extends each $\leq_\alpha$ ($\alpha \in X$), $\perp \leq_X s$ for each $s \in S_X$, and $x \not\leq_X y$ whenever $x \in S_\alpha, y \in S_\beta, \alpha, \beta \in X$ and $\alpha \neq \beta$. Hence $(S_X, \leq_X)$ is again a tree of size λ. For each $n < \omega$, let $H_n^X = \{x \in S_X \mid \mathrm{ht}_{S_X}(x) = n\}$. Note that if $x \in S_\alpha$ ($\alpha \in X$), then $\mathrm{ht}_{S_X}(x) = \mathrm{ht}_{S_\alpha}(x) + 1$, since we put $\perp$ below S_α. Furthermore, let $F_n^X = \bigcup_{\alpha \in X} F_n^\alpha$ for $n < \omega$ and $P_j^X = \bigcup_{\alpha \in X} P_j^\alpha$ for $j = 1, 2$.

Clearly, if $X \subseteq Y \subseteq I$, then $S_X \subseteq S_Y$ and $id : \Gamma_X \to \Gamma_Y$ is a homomorphism. Now let $X, Y \subseteq I$ and assume that $h : \Gamma_X \to \Gamma_Y$ is a homomorphism. We claim that $X \subseteq Y$

and $h = id_{S_X}$. Note that $H_0^X = \{\perp\} = H_0^Y$, so $h(\perp) = \perp$. Now choose any $\alpha \in X$. Since $H_1^X = \{\perp_x \mid x \in X\}$ and $h(H_1^X) \subseteq H_1^Y$, we have $h(\perp_\alpha) = \perp_\beta$ for some $\beta \in Y$. Since $S_\alpha = \{s \in S_X \mid \perp_\alpha \leq s\}$ and h is order-preserving, we obtain $h(S_\alpha) \subseteq S_\beta$. Note that $\mathcal{T}_\alpha = \{s \in S_\alpha \mid \exists z \in F_1^X, s \leq z\}$. Since $h(F_1^X) \subseteq F_1^Y$, we get $h(\mathcal{T}_\alpha) \subseteq \mathcal{T}_\beta$. As $h : \Gamma_X \to \Gamma_Y$ preserves heights, by the above formula so does $h \restriction \mathcal{T}_\alpha : \mathcal{T}_\alpha \to \mathcal{T}_\beta$. Using that $F_n^\alpha = \mathcal{T}_\alpha \cap F_n^X$ for $n < \omega$ and $P_j^\alpha = \mathcal{T}_\alpha \cap P_j^X$ for $j = 1, 2$, it follows that $h \restriction \mathcal{T}_\alpha : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ is a homomorphism. Thus by rigidity, $\alpha = \beta \in Y$, proving $X \subseteq Y$.

Similarly, we obtain that $h \restriction S_\alpha : \Gamma_\alpha \to \Gamma_\beta = \Gamma_\alpha$ is a homomorphism. Hence $h \restriction S_\alpha = id \restriction S_\alpha$. Since α was arbitrary and $h(\perp) = \perp$, we get $h = id_{S_X}$. □

Using the above theorem, now we can prove the corresponding generalization of Theorem 2.4.

Theorem 3.2. *Let λ be a cardinal, and let $\Gamma = (X, (R_i)_{i \in I})$ be an edge-colored graph of size λ. Then there exists a family $\mathfrak{F} = \{\tilde{\Gamma}_Z \mid Z \subseteq \lambda\}$ of ω-colored graphs $\tilde{\Gamma}_Z$ of size λ such that* $\mathrm{End}(\Gamma) \cong \mathrm{End}(\tilde{\Gamma}_Z)$ *and* $\mathrm{Mon}(\Gamma) \cong \mathrm{Mon}(\tilde{\Gamma}_Z)$ *for all $Z \subseteq \lambda$ and*

$$\mathrm{Hom}(\tilde{\Gamma}_Y, \tilde{\Gamma}_Z) = \begin{cases} \mathrm{End}(\tilde{\Gamma}_Y) \cong \mathrm{End}(\Gamma) \text{ if } Y \subseteq Z \\ \emptyset \text{ if } Y \not\subseteq Z \end{cases}$$

Proof. Let $\Gamma = (X, (R_i)_{i \in I})$. For every $Z \subseteq \lambda$ we define Γ_Z by using S_Z from Theorem 3.1. Put $\tilde{X}_Z = X \dot\cup S_Z \dot\cup (X \times X)$ and construct $\tilde{\Gamma}_Z$ as in Theorem 2.4. Note that, $\mathrm{End}(\tilde{\Gamma}_Z) \cong \mathrm{End}(\Gamma)$ for all $Z \subseteq \lambda$.

As before, for $Y \subseteq Z \subseteq \lambda$ it follows $S_Y \subseteq S_Z$. This implies $\tilde{\Gamma}_Y \subseteq \tilde{\Gamma}_Z$ and hence $\mathrm{End}(\tilde{\Gamma}_Y) \subseteq \mathrm{Hom}(\tilde{\Gamma}_Y, \tilde{\Gamma}_Z)$.

Now let $Y, Z \subseteq \lambda$ and assume that $h : \tilde{\Gamma}_Y \to \tilde{\Gamma}_Z$ is a homomorphism. Then $h \restriction S_Y : \Gamma_Y \to \Gamma_Z$ is a homomorphism and with Theorem 3.2 we obtain $Y \subseteq Z$, $h \restriction S_Y = id_{S_Y}$, and $h \in \mathrm{End}(\tilde{\Gamma}_Y)$ and hence the result. □

Corollary 3.3. *Let $\lambda \leq \kappa$ be infinite cardinals and let M be a monoid of size λ. Then there exists a family $\{\Gamma_Z \mid Z \subseteq \lambda\}$ of ω-colored graphs Γ_Z of size κ such that for all $Z \subseteq \lambda$:*

(1) If M is left-cancellative, then $\mathrm{Mon}(\Gamma_Z) \cong M$.
(2)

$$\mathrm{Hom}(\Gamma_Y, \Gamma_Z) = \begin{cases} \mathrm{End}(\Gamma_Y) \cong M \text{ if } Y \subseteq Z \\ \emptyset \text{ if } Y \not\subseteq Z \end{cases}$$

Moreover, if $\kappa < \kappa(\omega)$, then the graphs Γ_Z can be constructed in an absolute way, i.e. such that properties (1) and (2) are preserved under any generic extension of the underlying model of set theory.

Proof. We can proceed as in the argument of Theorem 2.5, using Lemma 2.3 and Theorem 3.2. □

It is well-known (cf. [1]) that a realization result as in Corollary 3.3 does not hold if we replace the graphs Γ_Z by a family of colored trees. Thus in [8] we can not work with trees and do need the graphs from Corollary 3.3. The following result shows that even automorphism groups of trees have a very restricted structure. We include its proof for the sake of completeness.

Observation 3.4. *Let $\mathcal{T} = (T, \leq)$ be a tree, and let $G = \mathrm{Aut}(\mathcal{T})$ contain an element of prime order p. Then G contains the symmetric group S_p as a subgroup.*

Proof. Let $g \in G$ have prime order p. Then for each $x \in T$ we have neither $x < xg$ nor $xg < x$ since otherwise g would have infinite order. Thus, since p is a prime, there is an orbit $A = \{a_1, ..., a_p\}$ of g in T with pairwise incomparable elements such that $a_i g = a_{i+1}$ for each $1 \leq i < p$ and $a_n g = a_1$. It follows that $z = \inf A$ satisfies $z = \inf\{a_i, a_{i+1}\} = \inf\{a_p, a_1\}$ for all $1 \leq i < p$. Hence the 'cones' $\uparrow a_i = \{x \in T : a_i \leq x\}$ $(1 \leq i \leq p)$ are pairwise order-isomorphic, and also 'side-cones' $\langle z, a_i \rangle = \{x \in T \mid$ either $z \leq x \leq a_i$, or else $z \leq x$ and x, a_i are incomparable$\}$ $(1 \leq i \leq p)$ are pairwise order-isomorphic. For both systems, we choose a minimal finite set of such isomorphisms between these posets such that the set forms a category, in particular it is closed under composition.

We claim that the symmetric group on A can be embedded into G. Indeed, let f be any permutation of A. Using the above isomorphisms, we can define a canonical extension of f to an automorphism f' of the poset $\uparrow z = \{x \in T \mid z \leq x\}$. Moreover, since the given isomorphisms between the cones resp. the side-cones were chosen to form a category, this extension can be done such that it gives an embedding of the symmetric group on A into $\mathrm{Aut}(\uparrow z)$. Clearly we can extend any automorphism of $\uparrow z$ trivially to an automorphism of $(T, \leq)$, proving the result. □

4. An upper bound for the size of the absolute graphs

In Section 2 we have shown how to construct colored graphs with prescribed endomorphism monoid of arbitrary size. But if we want this to hold in any generic extension of the underlying model of set theory, there is natural bound on the size, the Erdős cardinal $\kappa(\omega)$. The following result, which explains 'why', is an adaption of a more general theorem from model theory (see Eklof, Shelah [4]) to graphs. For the proof we use the following lemma which is a slightly simplified version of [4, Theorem 6]:

Lemma 4.1. *Let $(Q, \leq_Q)$ be a partial ordering of cardinality $< \kappa(\omega)$ and let $\mathcal{T} = \{T_i : i < \lambda\}$ be a family of Q-colored trees with $\lambda \geq \kappa(\omega)$. Then for some distinct $i, j < \lambda$ there exists an order-, height- and color-preserving map $\varphi : T_i \longrightarrow T_j$.*

Proof. See Eklof, Shelah [4] or Shelah [15]. □

Before continuing, we recall the following notions. Let $\Gamma = (X, (R_i)_{i<\omega})$ be an ω-colored graph, let $s = \langle s_0, \dots, s_{n_s} \rangle$ be a finite sequence of vertices in X with $n_s < \omega$, and let $\varphi(x_0, \dots, x_{n_s})$ be a quantifier-free formula built by finite conjunctions, disjunctions and negations of atomic formulas in Γ. We say $\varphi \in \mathrm{tp}_{qf}(s/\Gamma)$, if and only if $\varphi(s_0, \dots, s_{n_s})$ holds for Γ. Moreover, we define $Q_{cg} = \{\mathrm{tp}_{qf}(s/\Gamma) \mid \Gamma = (X, (R_i)_{i<\omega})$ *ω-(edge-)colored graph,* $s \in X^{(\omega)}\}$, the *types of colored graphs* which can be ordered by inclusion, i.e. $(Q_{cg}, \subseteq)$ is a partially ordered set. Since there are only countably many different formulas (as we use only ω colors) and Q_{cg} is a subset of the power set of all formulas, it follows that $|Q_{cg}| \leq 2^{\aleph_0} < \kappa(\omega)$, as $\kappa(\omega)$ is strongly inaccessible. Note that for $x, y \in X$ these quantifier-free formulas include the (easy) formula $\neg(x = y)$, as well as $x \in R_i$ (or $(x, y) \in R_i$) if R_i is a vertex-coloring (or an edge-coloring, respectively).

Theorem 4.2. *Let $\theta = \{\Gamma_i \mid i < \lambda\}$ be a family of ω-(edge-)colored graphs $\Gamma_i = (X_i, R_i)$ with $|\Gamma_i| \leq \lambda$ for all $i < \lambda$, and let $\lambda \geq \kappa(\omega)$. Then there exists a generic extension of the underlying model such that there exists a non-trivial, color-preserving embedding $f : \Gamma_i \to \Gamma_j$ for some distinct $i, j < \lambda$.*

Proof. Let T_i be the tree of finite sequences of elements of X_i for all $i < \lambda$. Moreover, for $s = \langle s_0, \ldots, s_{n_s} \rangle, t = \langle t_0, \ldots, t_{n_t} \rangle \in T_i$ we say $s \leq t$ if and only if $n_s \leq n_t$ and $t \restriction n_s = s$. For every $i < \lambda$ and $s \in X_i^\omega$ we consider the relation $R_{i,s} = \mathrm{tp}_{qf}(s/\Gamma_i)$ for the tree T_i. More clearly, for every finite sequence of elements s the relation $R_{i,s}$ consists of all quantifier-free $L_{\omega\omega}$-formulas which are valid for s in Γ_i. Since $|Q_{cg}| < \kappa(\omega)$, by Lemma 4.1 there exist distinct $i, j < \lambda$ and an order-, height- and color-preserving map $\varphi : T_i \to T_j$. Let $V[G]$ be a generic extension in which Γ_i is countable and let $\sigma : \omega \to X_i$ be an enumeration of X_i. We define the following map

$$f : \Gamma_i \to \Gamma_j$$
$$\text{by } f(\sigma(n)) := \varphi(\sigma \restriction n+1)(n).$$

First note that for $k \geq n+1$ we have that $\varphi(\sigma \restriction n+1)(n) = \varphi(\sigma \restriction k)(n)$ as φ is order-preserving. Hence, for every $x, y \in X_i$ there is (clearly) $k < \omega$ such that $x, y \in \mathrm{Range}(\sigma \restriction k)$ and $f(x), f(y) \in \mathrm{Range}(\varphi(\sigma \restriction k')))$. Note that k' could be chosen to be k as φ is height-preserving. Further, we have that $\Phi_i(\sigma \restriction k) \subseteq \Phi_j(\varphi(\sigma \restriction k))$ so that if '$(x, y) \in R_i$' is a formula which is valid for x, y in Γ_i then '$(f(x), f(y)) \in R_i$' is a formula which is valid for Γ_j and similarly for the formulas '$x \in R_i$' and '$\neg(x = y)$'. Hence $f : \Gamma_i \to \Gamma_j$ is a non-trivial, color-preserving embedding. □

Note that since φ is height- and order-preserving, this allows us to define $\varphi(\sigma) = \bigcup_{n<\omega} \varphi(\sigma \restriction n)$ which means that the infinite branch σ (read as sequence) in T_i is mapped to an infinite branch $\varphi(\sigma)$ in T_j. This is the property which is implicitly used in the above proof.

Using Theorem 4.2 we easily obtain the following two corollaries:

Corollary 4.3. *Let $\Gamma = (X, \leq, (C_i)_{i<\omega})$ be an ω-(edge-)colored graph with $|\Gamma| = \lambda \geq \kappa(\omega)$ and $\mathrm{End}(\Gamma) = \{id\}$. Then there exists a generic extension of the underlying model of set theory such that $\mathrm{End}(\Gamma) \neq \{id\}$ in the extension model.*

Proof. Let Γ be as above. We define the following family of ω-colored graphs:

$$\hat{\Gamma} = \{\Gamma_i \mid i < \lambda\} \text{ with } \Gamma_i = (X \dot{\cup} \{i\}, \leq, C_0, (C_{j+1})_{j<\omega}) \text{ and } C_0 = \{i\},$$

which means that we add a distinct node to every graph with its own color and shift all other colors. Note that in all Γ_i the added node has color 0. Now we can apply Theorem 4.2 to obtain an extension model and a non-trivial, color-preserving embedding $\varphi : \Gamma_i \to \Gamma_j$ for some distinct $i, j < \lambda$. As this map is color-preserving, it respects the order-relation as well as the C_i relations of the graph. Hence $i \in C_0$ implies that $\varphi(i) \in C_0$, thus $\varphi(i) = j$. We conclude that $\varphi \restriction \Gamma : \Gamma \to \Gamma$ is a non-trivial graph embedding and it follows $\mathrm{End}(\Gamma) \neq \{id\}$ which completes the proof. □

This proves that in Theorem 2.5 the condition $\lambda < \kappa(\omega)$ is necessary to ensure absoluteness of (i) and (ii). The next corollary establishes the same bound for fully rigid systems of ω-colored graphs. In fact we already get the non-existence from Corollary 4.3 since we cannot maintain $\mathrm{End}(\Gamma)$ as formulated in Section 3 but here we want to point out that $\mathrm{Hom}(\Gamma_Y, \Gamma_Z) \neq \emptyset$ in an extension model although $\mathrm{Hom}(\Gamma_Y, \Gamma_Z) = \emptyset$ in the base model.

Corollary 4.4. *Let* $\{\Gamma_X \mid X \subseteq \lambda\}$ *be a family of* ω*-colored graphs* Γ_X *with* $|\Gamma_X| = \lambda \geq \kappa(\omega)$ *for all* $X \subseteq \lambda$ *and*

$$\mathrm{Hom}(\Gamma_Y, \Gamma_Z) = \begin{cases} End(\Gamma_Y) \text{ if } Y \subseteq Z \\ \emptyset \text{ if } Y \not\subseteq Z \end{cases}.$$

Then $\mathrm{Hom}(\Gamma_Y, \Gamma_Z) \neq \emptyset$ *for some* $Y, Z \subseteq \lambda$ *with* $Y \not\subseteq Z$ *holds in some generic extension of the underlying model of set theory.*

Proof. Follows immediately from Theorem 4.2. □

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