A Data Dependent Triangulation for Vector Fields

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Abstract

This article deals with dependencies of a piecewise linear vector field and the triangulation of the domain. It shows that the topology of the field may depend on the triangulation and gives a suitable choice to obtain a simple topology by changes of the triangulation. The main point is the appearance of pairs of critical points with positive and negative Poincaré index. Many of these occurences can be avoided by changing the grid. This is proved in the article. An algorithm is presented which uses these results to extract simpler topological skeletons than usual methods. Finally there are several examples comparing these algorithms to demonstrate the effects of the data-dependent triangulation.

1. Introduction

Scientific Data is often given on discrete positions of the domain. The data points build a structured or unstructured grid or may be scattered over the domain without any structure. In many cases one needs to find a description at every position by an approximation. One way in two-dimensional problems is to look for a triangulation of the domain with the data points as vertices. If there is a grid, one tends to keep the edges and chooses suitable diagonals of the cell to end up with a triangulation.

For scalar fields there are several approaches to the problem for different types of grids and scattered data. For curvilinear and cartesian grids one can take a consistent choice for all the diagonals. Another choice are the so called MinMax- and MaxMin-triangulations. The MinMax-triangulation tries to minimize the maximum angle in each triangle starting with the maximum angle of all triangles. The goal of the MaxMin-triangulation is to maximize the minimum angle in each triangle. For a nice overview one may look at [6]. Besides these ideas there are methods which take the given data into account instead of relying only on the position of the samples. This is based on the observation that the result of the approximation depends on the triangulation and the choice of the triangulation may help to accomplish certain requirements like a smoother approximation. A nice approach can be found in [2] where four different cost functions for an edge in a triangluation are defined. One checks the diagonals in convex quadrilaterals in the triangulation. If the diagonal in the triangulation is more expensive than the other possible choice, one switches the diagonals. In this way one can come to a local minimum.

For vector fields one could use similar approaches. But in our case we are interested in the topology of the field, so we base our decision about the triangulation on it. The basic idea is to keep the number of critical points low by avoiding saddle-node pairs in triangles made of a single, convex quadrilateral. The paper continues with basic notations and facts on triangulations. The theoretical background is worked out in section 3. Section 4 shows the algorithm and the last part deals with some examples.

2. Triangulations

We orient our description on the definitions made in [6]. Let $D \subset R^2$ be a closed and bounded subset of the plane and the boundary

$$\partial D = \bigcup_{j=1}^{M} B_j$$

a finite union of polygons B_j without self-intersections. Let

$$\mathcal{P} = \{P_i | i = 1, \ldots, N\} \subset D$$

be a collection of sampled points containing all vertices of the B_j.

Definition 1 *A* **triangle** T **over** $\mathcal{P}$ *is a triangle*

$$T = T_{ijk} = \{b_1 P_i + b_2 P_j + b_3 P_k | b_1 + b_2 + b_3 = 1\}$$

with $T \subseteq D$*;* P_i, P_j, P_k *different elements of* $\mathcal{P}$*.*

Definition 2 *A* **triangulation** $\mathcal{T}$ **over** $\mathcal{P}$ *is a set*

$$\mathcal{T} = \{T_{ijk} | T_{ijk} \text{ is a triangle over } \mathcal{P}\}$$

that fulfils the following five conditions :

(i) Every $P \in \mathcal{P}$ is a vertex of at least one $T \in \mathcal{T}$.

(ii) P_i, P_j and P_k are not collinear for all $T_{ijk} \in \mathcal{T}$, i. e.

$$Int(T_{ijk}) = \emptyset \qquad \forall T_{ijk} \in \mathcal{T}$$

(iii) The interior of any two triangles $T_{ijk}, T_{lmn} \in \mathcal{T}$ do not intersect, i. e.

$$Int(T_{ijk}) \cap Int(T_{lmn}) = \emptyset$$

(iv) The intersection of two triangles $T_{ijk}, T_{lmn} \in \mathcal{T}$ is an edge of both triangles.

(v) The union of all triangles $T \in \mathcal{T}$ is D.

$$\bigcup_{T \in \mathcal{T}} T = D$$

Figure 1 shows a regular triangulation of 28 vertices.

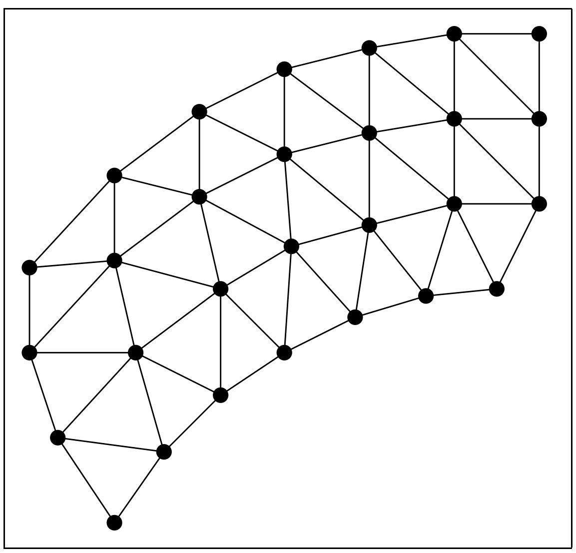

Figure 1. A typical Triangulation

Besides triangulations, we want to work with grids over $\mathcal{P}$, so we should mention our definition.

Definition 3 *A (polygonal)* **cell** C **over** $\mathcal{P}$ *is a polygon*

$$C = \{\sum_{j=1}^{L} b_j P_{i_j} | \sum_{j=1}^{L} b_j = 1\}$$

with

$$P_{i_j} \in \mathcal{P} \qquad C \subseteq D.$$

Definition 4 *A* **grid** $\mathcal{G}$ **over** $\mathcal{P}$ *is a set*

$$\mathcal{G} = \{C | C \text{ is cell over } \mathcal{P}\}$$

with the following properties :

(i) Every $P \in \mathcal{P}$ is a vertex of a cell $C \in \mathcal{G}$.

(ii) The interior of any two cells $C_1, C_2 \in \mathcal{G}$ do not intersect, i. e.

$$Int(C_1) \cap Int(C_2) = \emptyset.$$

(iii) The union of all cells is the whole domain, i. e.

$$\bigcup_{C \in \mathcal{G}} C = D$$

In figure 2 one can see a grid and one may recognize the differences to a triangulation.

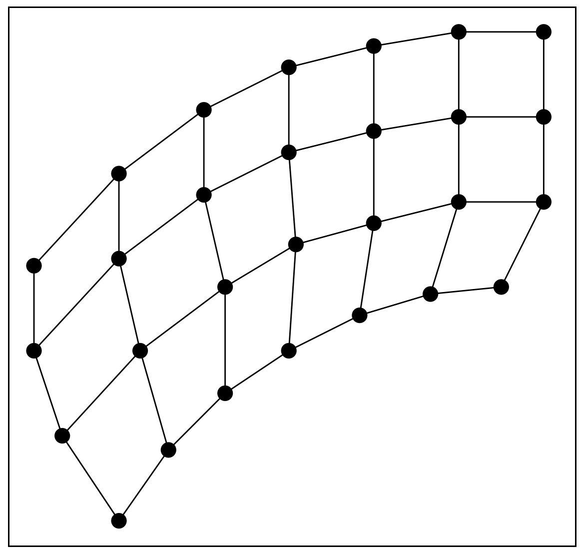

Figure 2. A curvilinear grid

We are aware of the fact that there are grids in the literature which do not fit to our definition, especially finite elements where the data position is sometimes different from the vertices. But for this article it is a good working definition.

3. Vector Fields and Triangulations

We want to inspect the topology of piecewise linear vector fields defined on triangulations $\mathcal{T}$ over a data point set $\mathcal{P}$. So let $D \subseteq R^2$ be a domain and $\mathcal{P}$ a data point set as in section 2. We denote the set of vector field values on $\mathcal{P}$ by

$$\mathcal{V} = \{V_i | i = 1, \ldots, N\} \subseteq R^2.$$

For each triangulation $\mathcal{T}$ over $\mathcal{P}$ we can define then a piecewise linear vector field

$$v_{\mathcal{T}} : D \rightarrow R^2$$

by

$$v_{\mathcal{T}}(X) := b_1 V_i + b_2 V_j + b_3 V_k$$

with

$$X = b_1 P_i + b_2 P_j + b_3 P_k \in T_{ijk}, \qquad T_{ijk} \in \mathcal{T}.$$

A critical point Q of $v_{\mathcal{T}}$ is a zero of the field, i. e.

$$v_{\mathcal{T}}(Q) = 0, \qquad Q \in D.$$

These points can be analysed by the winding number

$$n_Q = \frac{1}{2\pi} \int_{\gamma} \frac{v_1 dv_2 - v_2 dv_1}{v_1^2 + v_2^2}$$

counting the number of terms of the field in a small neighborhood bounded by γ containing no other zeros ([1], [8]). In our case we assume $Q \in Int(T_{ijk})$ and can take γ as the boundary of the triangle $T_{ijk} \in \mathcal{T}$. A simple example is given in the figures 3. There is a saddle point inside a triangulation. Several streamlines show the behavior of the field. A classification of the linear types can be found in [4].

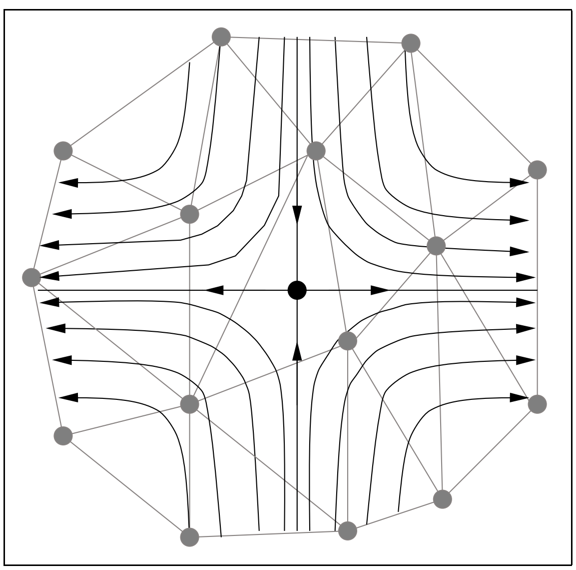

Figure 3. A simple critical point in a piecewise linear field

We want to look now in more detail at the appearance of two critical points in triangles having a common edge. If the quadrilateral constructed by the union of the two triangles is convex, we would have a second choice for the triangulation. The following theorem shows that one should always take this chance. To keep the notations simple, we assume that the domain D consists only of the convex quadrilateral but it should be seen as part of a bigger triangulation as can be seen in figure 4.

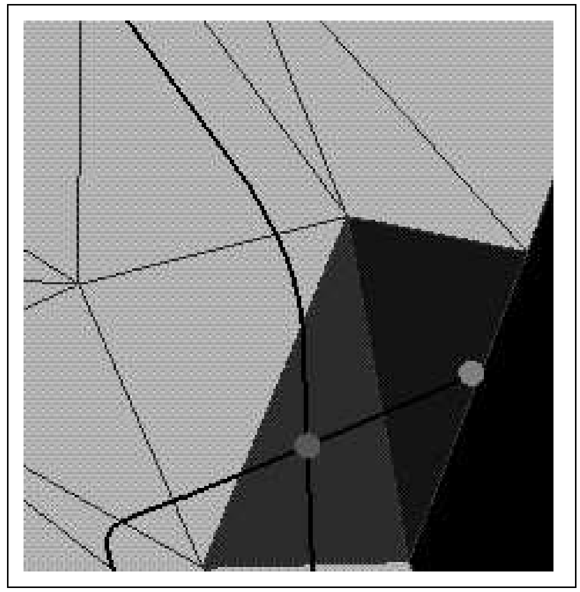

Figure 4. A saddle point and a source in neighboring triangles

Theorem 1 *Let $D \subseteq R^2$ be a convex quadrilateral,*

$$\mathcal{P} = \{P_1, \ldots, P_4\}$$

the vertices and

$$\mathcal{V} = \{V_1, \ldots, V_4\}$$

the data values. We assume that no triple (V_i, V_j, V_k) is collinear to garanty that all critical points are isolated.
If the vector field

$$v_{\mathcal{T}} : D \rightarrow R^2 \quad \textit{with} \quad \mathcal{T} = \{T_{123}, T_{134}\}$$

contains **two** *critical points*

$$Q_1 \in Int(T_{123}), \qquad Q_2 \in Int(T_{134}),$$

then has the vector field

$$v_{\mathcal{T}'} : D \rightarrow R^2 \quad \textit{with} \quad \mathcal{T}' = \{T_{124}, T_{234}\}$$

no *critical points.*

Proof : We show that $v_{\mathcal{T}'}$ contains no zero in T_{124}, the second part is analogous.
$v_{\mathcal{T}'}$ is defined on T_{124} by

$$v_{\mathcal{T}'}(P) = a_1 V_1 + a_2 V_2 + a_4 V_4,$$

with

$$a_1 + a_2 + a_4 = 1, \qquad P = a_1 P_1 + a_2 P_2 + a_3 P_3.$$

This is a linear field in the plane containing one critical point. We have to show that the zero is outside of T_{124}, i. e. in

$$0 = a_1 V_1 + a_2 V_2 + a_4 V_4$$

at least one a_i is negative.
We have

$$\begin{aligned} v_{\mathcal{T}}(Q_1) &= b_1 V_1 + b_2 V_2 + b_3 V_3 = 0, \\ & \quad b_1 + b_2 + b_3 = 1, b_i > 0 \\ v_{\mathcal{T}}(Q_2) &= c_1 V_1 + c_3 V_3 + c_4 V_4 = 0, \\ & \quad c_1 + c_3 + c_4 = 1, c_i > 0. \end{aligned}$$

This gives

$$V_3 = \frac{-b_1}{b_3} V_1 + \frac{-b_2}{b_3} V_2$$

and finally

$$c_1 V_1 + \frac{-b_1 c_3}{b_3} V_1 + \frac{-b_2 c_3}{b_3} V_2 + c_4 V_4 = 0,$$

so the zero is outside T_{124} because of $\frac{-b_2 c_3}{b_3} < 0$. QED.

This statement allows now to simplify the structure of vector fields.

4. Data Dependent Triangulation

We want to use now our considerations on the dependency between triangulation and vector field topology. We start here with a given triangulation. In the next step we compute the linear field inside each triangle and check for the position of the zero. For each critical point we determine the index and type as in the classification in [4, p.96]. If we have two critical points with different index in two triangles, we check if their union is convex. In this case we switch the diagonal and remove the two critical points by theorem 1. This simple local operation allows to simplify the structure.

Our algorithm for a data-dependent triangulation has now the following form :

```
FOR each triangle T in the triangulation
    compute linear interpolation
    IF critical point P in T
        determine the index and type
        put T in list L
FOR each unmarked triangle T1 in L
    FOR all neighbors T2 of T1
        IF (T2 in L) AND (T1 ∪ T2 convex)
            put pair (T1,T2) in list C
            mark T1,T2
FOR each pair (T1,T2) in list C
    switch diagonals
    compute linear interpolation in changed T1, T2
    remove (T1,T2) from C
```

5. Examples

We tested the algorithm with two randomly generated data sets. This garanties usually a lot of critical points and can be seen as purely turbulent motion.

The first set consists of the simple randomly distorted uniform grid with 36 vertices in figure 5. It was triangulated using a consistent choice for the diagonals. The data was randomly generated. The direct use of the piecewise linear approach shows 12 critical points as can be seen in figure 6. There are 20 separatrices so that one gets already a complicated topological structure for the resulting flow. Our data dependent triangulation in figure 7 reduces this to the 4 critical points in figure 8. It may be astonishing that both flows were constructed by piecewise linear interpolation of the same data if one only looks on the skeletons.

The second set contains 81 points on a similar grid. We start again with a consistent choice of the diagonals and obtain the triangulation in figure 9. The flow has 34 critical points with a total of 68 separatrices, showing a turbulent behavior over the whole area as can be seen in figure 10. The data dependent triangulation in figure 11 was constructed by switching 7 diagonals. The flow structure is shown in figure 12. There are now 20 critical points remaining with 40 separatrices.

6. Conclusion

We have shown that the structure of vector fields depends on the triangulation of the given data. If it is possible to choose a triangulation, one has some control over the topological skeleton by using a special grid structure. We have presented an algorithm that does some local edge switches to obtain a vector field with an easier structure. This was tested on two examples showing the effect of simplifying the topology. We think that this may help in finding ways to simplify the topological structures in other cases to get new insights in the analysis of vector fields.

References

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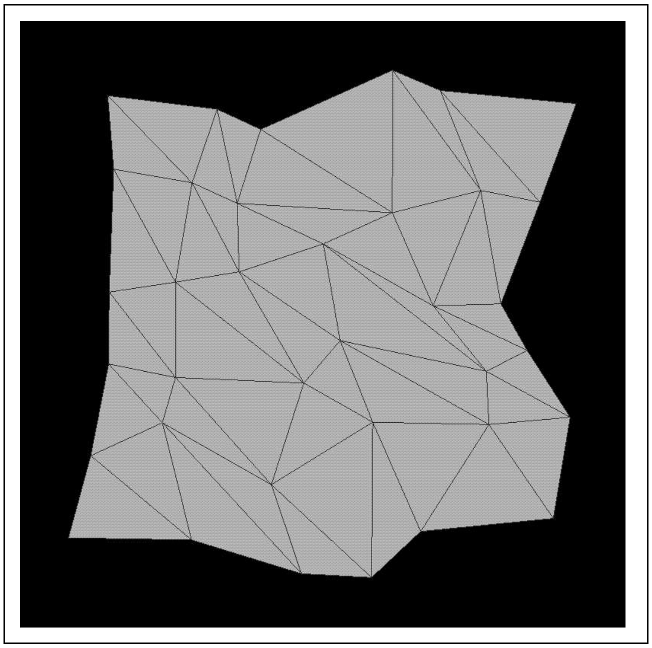

Figure 5. Consistent triangulation

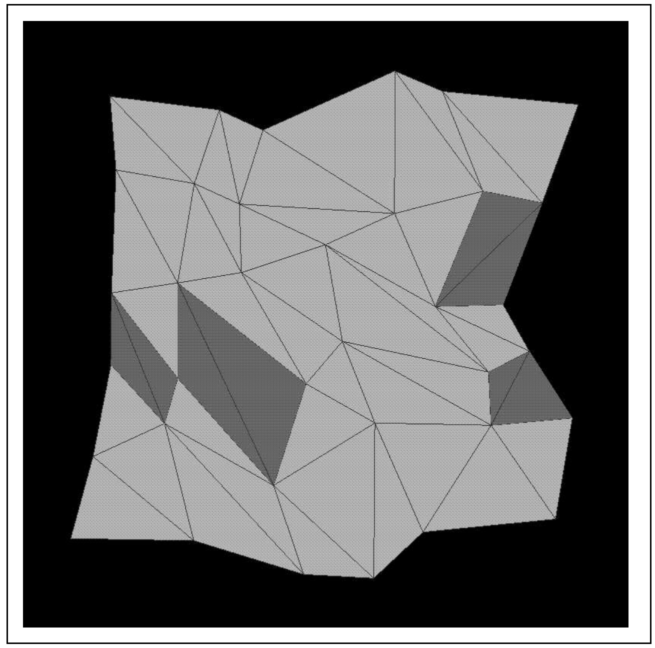

Figure 7. Data Dependent triangulation

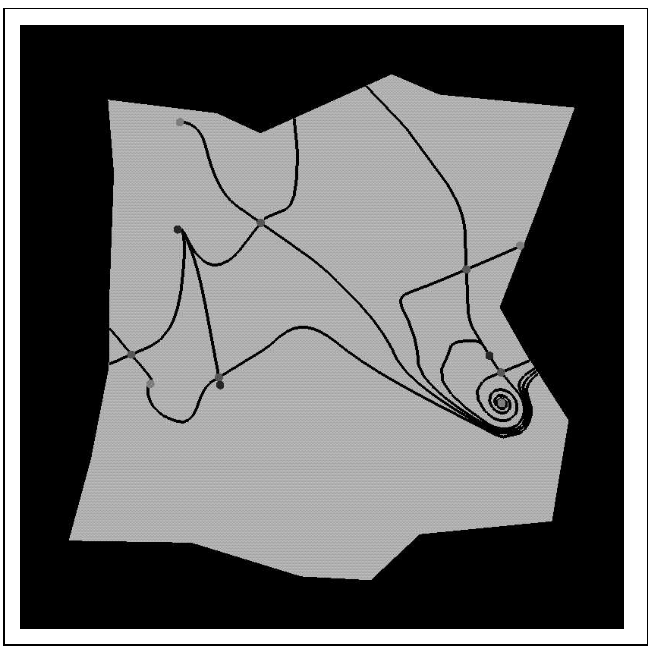

Figure 6. Structure with consistent triangulation

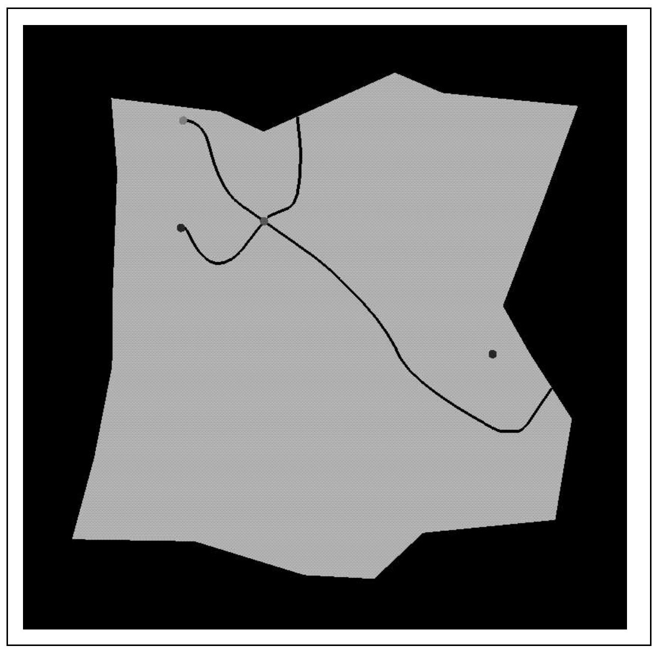

Figure 8. Struture with data dependent triangulation

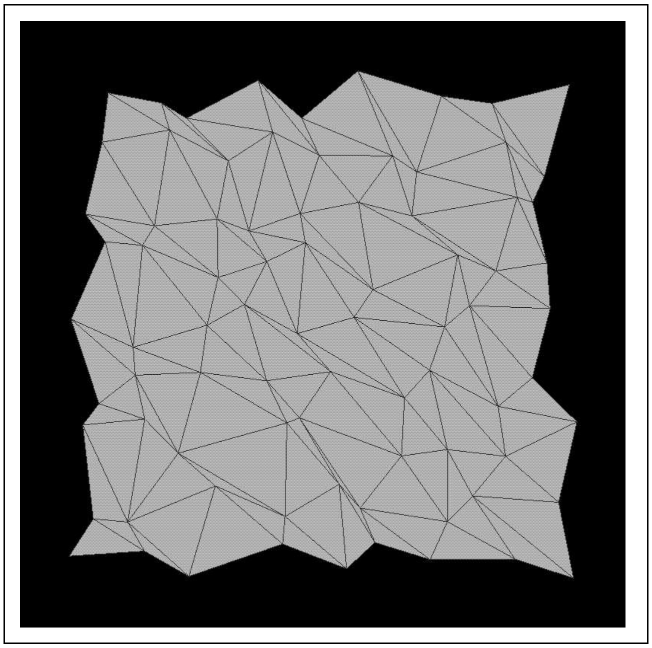

Figure 9. Consistent triangulation

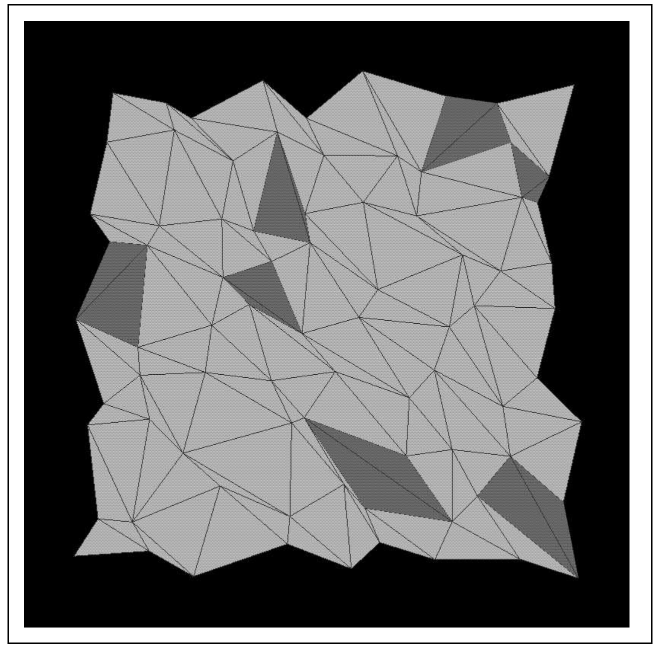

Figure 11. Data Dependent triangulation

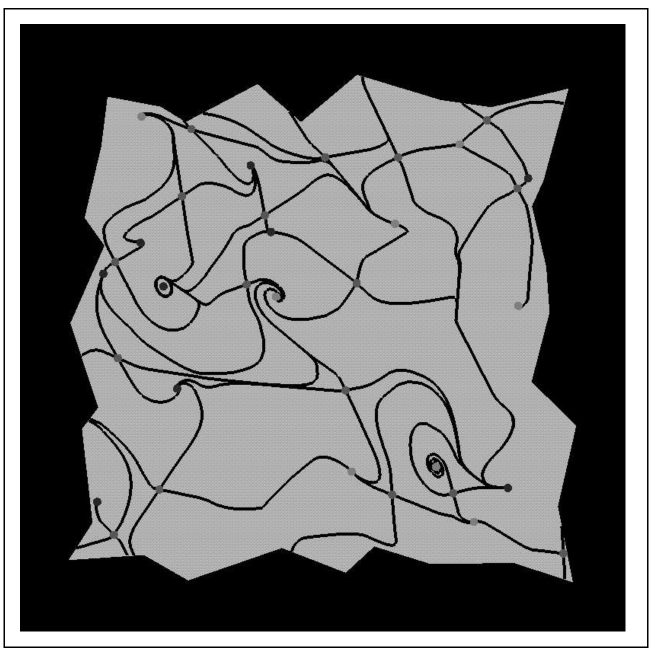

Figure 10. Structure with consistent triangulation

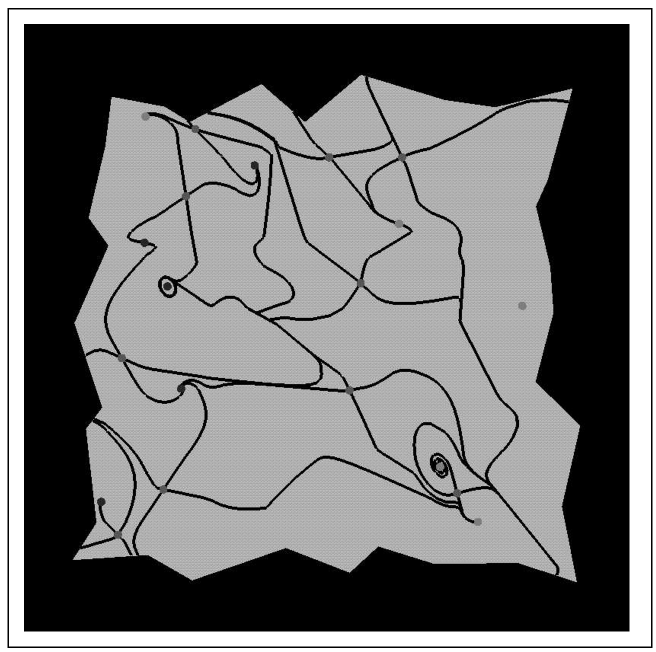

Figure 12. Struture with data dependent triangulation