A Backward and a Forward Simulation for Weighted Tree Automata*

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Abstract. Two types of simulations for weighted tree automata (wta) are considered. Wta process trees and assign a weight to each of them. The weights are taken from a semiring. The two types of simulations work for wta over additively idempotent, commutative semirings and can be used to reduce the size of wta while preserving their semantics. Such reductions are an important tool in automata toolkits.

1 Introduction

Automata minimization is an important and well-studied subject. Here we consider (finite-state) tree automata and weighted tree automata, which are used in applications such as model checking [1] and natural language processing [2]. Deterministic (bottom-up) tree automata can be minimized efficiently using, for example, an algorithm inspired by Hopcroft [3,4]. However, minimizing nondeterministic tree automata is PSPACE-complete [5] and cannot be approximated well [6,7,8] unless P = PSPACE. Consequently, alternative (efficient) methods to reduce the size of tree automata were explored [9,4,10,11]. An efficient minimization procedure for deterministic (bottom-up) weighted tree automata is presented in [12] and efficient reductions of nondeterministic weighted tree automata with the help of bisimulation relations are considered in [13].

Here we consider the simulation approach of [10] for weighted tree automata over additively idempotent, commutative semirings. A weighted tree automaton essentially is a tree automaton in which each transition carries a weight (an element of a semiring). Instead of accepting a certain set of trees, a weighted tree automaton assigns a weight to each tree. First, the automaton assigns a weight to each run, which is the same as a run of the corresponding unweighted automaton. The weight of the run is obtained by multiplying (in the semiring) the participating transition weights (each transition weight as often as it occurs in the run) and eventually the final weight associated to the state reached at the root. Should there be several runs on the same input tree, then the weights of those runs are summed up to obtain the weight assigned to this input tree.

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In [10] two types of simulation relations, called downward and upward simulations, are examined for tree automata. Roughly speaking, we generalize these notions to the setting of weighted tree automata. While there are several potential generalizations, our approach requires us to consider ordered semirings. Here we choose to work with additively idempotent (i.e., $a + a = a$ for all semiring elements a) semirings and their natural order. We define two types of simulation relations: backward and forward simulation. Intuitively, these notions correspond to backward and forward bisimulation of [13], but are unfortunately not generalizations of those concepts. Backward simulation generalizes downward simulation of [10] and our forward simulation generalizes upward simulation with respect to the identity as downward simulation [10]. We choose not to generalize upward simulations [10] with respect to arbitrary downward simulations since we believe that two completely separate notions are easier to handle and understand.

A simulation is a quasi-order (i.e., a reflexive and transitive relation) on the states of an input automaton M. A backward simulation is such that larger states dominate the smaller states; i.e., if the smaller state accepts a tree with weight a, then the larger state accepts the same tree with a weight that is larger than a (see Lemma 3). We take the equivalence induced by this quasi-order (i.e., two states are equivalent if they simulate each other) and reduce M with it (see Definition 6). This construction is simple for tree automata, however our reductions need to address the weights. This yields separate constructions for the backward (see Definition 6) and forward (see Definition 13) case. We show in Theorem 7 that the weighted tree automaton obtained with the help of a backward simulation, which never has more states than M, is equivalent to M.

In a forward simulation we do not consider the trees that a state can accept, but rather the contexts (i.e., trees over the input ranked alphabet with a unique occurrence of the extra symbol $\square$) that can be processed starting from that state. For those contexts a similar domination property as in the backward case must hold (see Lemma 12). Again, we use the induced equivalence to reduce the automaton. Theorem 15 shows that we obtain an equivalent weighted tree automaton.

Both types of simulations admit a greatest simulation that can be used for greatest gain in reduction (see Theorems 2 and 11). For deterministic weighted tree automata, we show that backward simulation is ineffective and forward simulation is only as effective as forward bisimulation [13]. This essentially means that our new tools do not surpass the existing tools in the deterministic case, but they can yield much greater reductions in the nondeterministic case. In summary, we add two more tools to the toolbox, which can be used to reduce nondeterministic weighted tree automata.

2 Preliminaries

We denote the nonnegative integers, which include 0, by $\mathbb{N}$. For every $l, u \in \mathbb{N}$, the subset $\{n \in \mathbb{N} \mid l \leqslant n \leqslant u\}$ is simply written as $[l, u]$. An alphabet is a nonempty and finite set. Its elements are called symbols. A ranked alphabet (Σ, rk) consists of an alphabet Σ and a mapping $\mathrm{rk}\colon \Sigma \to \mathbb{N}$, which associates to each symbol a rank. The set $\Sigma_k = \{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k\}$ contains the symbols of rank k. Henceforth, we will denote such a ranked alphabet by Σ alone and assume that the mapping rk is implicit. For a ranked alphabet Σ and a set T, we write $\Sigma(T)$ for $\{\sigma(t_1, \dots, t_k) \mid \sigma \in \Sigma_k, t_1, \dots, t_k \in T\}$. We generally write α instead of $\alpha()$ for $\alpha \in \Sigma_0$. The set $T_\Sigma(V)$ of Σ-trees indexed by a set V is the smallest set such that $V \subseteq T_\Sigma(V)$ and $\Sigma(T_\Sigma(V)) \subseteq T_\Sigma(V)$. We just write T_Σ for $T_\Sigma(\emptyset)$.

A relation ϱ on a set S is a subset of $S \times S$. The inverse ϱ^{-1} is the relation $\{(s', s) \mid s \; \varrho \; s'\}$ and the composition of two relations ϱ_1 and ϱ_2 on S is

$$\varrho_1 \,;\, \varrho_2 = \{(s, s'') \mid \exists s' \in S\colon s \; \varrho_1 \; s' \; \varrho_2 \; s''\} \enspace .$$

A quasi-order $\preceq$ on S is a reflexive, transitive relation on S. An up-set $A \subseteq S$ (with respect to $\preceq$) is such that for every $s \preceq s'$ with $s \in A$ also $s' \in A$. The smallest up-set containing $A \subseteq S$ is denoted by $\uparrow(A)$. If $A = \{s\}$, then we simply write $\uparrow(s)$. The quasi-order $\preceq$ is an equivalence relation if it is symmetric, and it is a partial order if it is anti-symmetric. A partial order $\leqslant$ on S is total if $s \leqslant s'$ or $s' \leqslant s$ for every $s, s' \in S$. Let $\equiv$ be an equivalence on S. We write $[s]_\equiv$ for the equivalence class of $s \in S$ and $(S/{\equiv})$ for the partition $\{[s]_\equiv \mid s \in S\}$. Whenever possible without confusion, we drop $\equiv$ from $[s]_\equiv$. Note that if $\preceq$ is a quasi-order on S, then $\simeq \,=\, \preceq \cap \preceq^{-1}$ is an equivalence relation on S and $\preceq$ induces a partial order on $S/{\simeq}$.

A commutative semiring is an algebraic structure $\mathcal{A} = (A, +, \cdot, 0, 1)$ comprising two commutative monoids $(A, +, 0)$ and $(A, \cdot, 1)$ such that $\cdot$ distributes over $+$ and 0 is absorbing for $\cdot$ (i.e., $0 \cdot a = 0$ for every $a \in A$). It is (additively) idempotent if $1 + 1 = 1$. Moreover, let $\leqslant$ be a partial order on A. It partially orders $\mathcal{A}$ if $a_1 + b_1 \leqslant a_2 + b_2$ and $a_1 \cdot b_1 \leqslant a_2 \cdot b_2$ for every $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$. Let $\sqsubseteq$ be the quasi-order on A such that $a \sqsubseteq b$ if there exists $c \in A$ with $a + c = b$. Whenever $\sqsubseteq$ is anti-symmetric, it is called the natural order. Note that for an idempotent semiring, the relation $\sqsubseteq$ is always a partial order. Morever, the natural order always (independent of idempotency) partially orders $\mathcal{A}$.

A tree series (over Σ and A) is a mapping $\varphi\colon T_\Sigma \to A$. The set of all such tree series is $A\langle\!\langle T_\Sigma \rangle\!\rangle$. We write (ψ, t) instead of $\psi(t)$ for every $t \in T_\Sigma$. A weighted tree automaton (wta) [14,15,16] is a tuple $M = (Q, \Sigma, \mathcal{A}, \mu, F)$ such that

- Q is a finite set of states,
- Σ is a ranked alphabet of input symbols,
- $\mathcal{A} = (A, +, \cdot, 0, 1)$ is a semiring,
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ is such that $\mu_k\colon \Sigma_k \to A^{Q \times Q^k}$, and
- $F\colon Q \to A$ is a final weight assignment.

The wta is deterministic if for every $\sigma \in \Sigma_k$ and $q_1, \dots, q_k \in Q$ there exists at most one $q \in Q$ such that $\mu_k(\sigma)_{q, q_1, \dots, q_k} \neq 0$. A wta computes a tree series as follows. Let $h_\mu\colon T_\Sigma(Q) \to A^Q$ be the mapping such that

- for every $p, q \in Q$
$$h_\mu(p)_q = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

– for every $\sigma \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma$, and $q \in Q$

$$h_\mu(\sigma(t_1, \ldots, t_k))_q = \sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,q_1,\ldots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \enspace .$$

The wta M recognizes the tree series $\varphi_M \in A\langle\langle T_\Sigma \rangle\rangle$, which is defined for every $t \in T_\Sigma$ by $(\varphi_M, t) = \sum_{q \in Q} F(q) \cdot h_\mu(t)_q$. Two wta are equivalent if they recognize the same tree series.

3 A Backward Simulation

In this section, we investigate backward simulation for wta [14,15,16]. Such simulations for unweighted tree automata were already considered in [10] and backward bisimulations, which are a related concept, for wta were considered in [13]. To avoid a very detailed discussion, we restrict ourselves to idempotent and commutative semirings and their natural order. With minor modifications, our arguments also work for other idempotent (even non-commutative) semirings that are partially ordered. In the following, we fix an idempotent semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$ and its natural order $\sqsubseteq$. In addition, let $M = (Q, \Sigma, \mathcal{A}, \mu, F)$ be a wta, and without loss of generality, suppose that Q is totally ordered. We will use $\min(P)$ with $P \subseteq Q$ for the minimal state of P with respect to that total order.

Let us start with the definition of a backward simulation. Note that our definition yields the definition of [10] when considered in the unweighted case. In that case, if a state q simulates a state p and there exists a transition $\sigma(p_1, \ldots, p_k) \to p$, then there also exists a transition $\sigma(q_1, \ldots, q_k) \to q$ such that the q_i simulate the corresponding p_i. Now, let us consider the weighted setting. In essence, for a state q to simulate a state p, written $p \preceq q$, we demand that for every transition weight $\mu_k(\sigma)_{p,p_1,\ldots,p_k}$ there exists a larger (with respect to the natural order $\sqsubseteq$) transition weight $\mu_k(\sigma)_{q,q_1,\ldots,q_k}$ such that, for every $i \in [1, k]$, the state q_i simulates p_i. Note that there is no condition on the final weights.

Definition 1 (cf. [10, Section 2]). *A quasi-order $\preceq$ on Q is a* backward simulation *for M if for every $p \preceq q$, $\sigma \in \Sigma_k$, and $p_1, \ldots, p_k \in Q$ there exist $q_1, \ldots, q_k \in Q$ such that $\mu_k(\sigma)_{p,p_1,\ldots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\ldots,q_k}$ and $p_i \preceq q_i$ for every $i \in [1, k]$.*

Let us discuss the definition. We already remarked that it coincides with the definition of a backward simulation [10] in the unweighted case [i.e., the case where $\mathcal{A} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ is the Boolean semiring]. However, the definition does not generalize the notion of backward bisimulation for wta of [13]. Next, let us establish some central properties of backward simulations. First, there is a greatest backward simulation for M. We prove this along the lines of [13, Theorem 22].

Theorem 2. *There exists a greatest (with respect to $\subseteq$) backward simulation for M.*

Proof. Let $\preceq$ and $\preceq'$ be backward simulations for M. We claim that $(\preceq \cup \preceq')^*$, the reflexive and transitive closure of $\preceq \cup \preceq'$, is again a backward simulation. Clearly, $(\preceq \cup \preceq')^*$ is a quasi-order. Now, let $(p, q) \in (\preceq \cup \preceq')^*$, $\sigma \in \Sigma$, and $p_1, \ldots, p_k \in Q$. Consequently, there exist $r_1, \ldots, r_n \in Q$ such that

$$p = r_0 \preceq r_1 \preceq' r_2 \preceq r_3 \preceq' \cdots \preceq' r_n = q \enspace .$$

By this chain of inequalities, there also exist $q_1, \ldots, q_k \in Q$ such that

$$\mu_k(\sigma)_{p,p_1,\ldots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\ldots,q_k} \quad \text{and} \quad p_i \; (\preceq ; \preceq' ; \preceq ; \preceq' ; \cdots ; \preceq') \; q_i$$

for every $i \in [1, k]$, which proves that $(\preceq \cup \preceq')^*$ is a backward simulation. $\square$

The main property of a state q that simulates a state p is that the state q accepts every input tree with a weight that is larger than the weight with which the same tree is accepted by p. In the unweighted case, this corresponds to the statement that the tree language accepted by p is a subset of the tree language accepted by q (see [10, Section 6.1]). In general, this immediately yields that any two states equivalent in $\preceq \cap \preceq^{-1}$, which is always an equivalence relation since $\preceq$ is a quasi-order, accept the same tree series.

Lemma 3. *Let $\preceq$ be a backward simulation for M. Then $h_\mu(t)_p \sqsubseteq h_\mu(t)_q$ for every $t \in T_\Sigma$ and $p \preceq q$.*

Proof. Let $t = \sigma(t_1, \ldots, t_k)$ for some $\sigma \in \Sigma_k$ and $t_1, \ldots, t_k \in T_\Sigma$. We compute as follows:

$$h_\mu(\sigma(t_1, \ldots, t_k))_p = \sum_{p_1,\ldots,p_k \in Q} \mu_k(\sigma)_{p,p_1,\ldots,p_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{p_i} \enspace .$$

For all $p_1, \ldots, p_k \in Q$ there exist $(q_1, \ldots, q_k) \in Q^k$ such that $p_i \preceq q_i$ for every $i \in [1, k]$ and $\mu_k(\sigma)_{p,p_1,\ldots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\ldots,q_k}$ because $p \preceq q$. Denote $(q_1, \ldots, q_k)$ by $f(p_1, \ldots, p_k)$ and q_i by $f(p_1, \ldots, p_k)_i$ for every $i \in [1, k]$. Then we continue with

$$h_\mu(\sigma(t_1, \ldots, t_k))_p \sqsubseteq \sum_{p_1,\ldots,p_k \in Q} \mu_k(\sigma)_{q,f(p_1,\ldots,p_k)} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{f(p_1,\ldots,p_k)_i}$$

by induction hypothesis and the fact that $\sqsubseteq$ partially orders $\mathcal{A}$ and

$$\begin{aligned} h_\mu(\sigma(t_1, \ldots, t_k))_p &\sqsubseteq \sum_{(q_1,\ldots,q_k) \in f(Q^k)} \mu_k(\sigma)_{q,q_1,\ldots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \\ &\sqsubseteq \sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,q_1,\ldots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \end{aligned}$$

$$= h_\mu(\sigma(t_1, \ldots, t_k))_q$$

by idempotency of $\mathcal{A}$ and the definition of the natural order. □

We already remarked that this yields that states $p, q \in Q$ such that $p \preceq q$ and $q \preceq p$, which we call equivalent, recognize the same tree series. Let us note another property of such states.

Note 4. Let $p \preceq q \preceq p$. For every $\sigma \in \Sigma_k$ and $p_1, \ldots, p_k \in Q$ there exist $q_1, \ldots, q_k \in Q$ and $r_1, \ldots, r_k \in Q$ such that $p_i \preceq q_i \preceq r_i \preceq q_i$ for every $i \in [1, k]$ and

$$\mu_k(\sigma)_{p,p_1,\ldots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\ldots,q_k} = \mu_k(\sigma)_{p,r_1,\ldots,r_k} \enspace .$$

So equivalent states enforce equally weighted transitions, but not necessarily within the same blocks (because, in general, we might have $q_i \not\preceq p_i$ for some $i \in [1, k]$ in Note 4). However, the property hints at an essential property of equivalent states p and q. If $p \neq q$, then there must be at least two transitions, one to p and one to q, with the same weight. Otherwise p and q cannot be equivalent.

Corollary 5 (of Lemma 3). *Let $\preceq$ be a backward simulation for M and $\simeq = (\preceq \cap \preceq^{-1})$. Then $h_\mu(t)_p = h_\mu(t)_q$ for every $p \simeq q$.*

This completes our investigation of the principal properties of backward simulation. Next, let us show how to reduce the size of a wta using a backward simulation. The main idea is, of course, to collapse equivalent states into just a single state. Recall, that we assumed a total order on Q and that $\min(P)$ with $P \subseteq Q$ denotes the smallest element in P with respect to that order. We use this order in our construction to obtain a unique wta. In contrast to [13, Definition 18], we thus need not discuss why the constructed wta is well-defined.

Definition 6. *Let $\preceq$ be a backward simulation for M and $\simeq = (\preceq \cap \preceq^{-1})$. The collapsed wta $(M/\simeq) = (Q', \Sigma, \mathcal{A}, \mu', F')$ is given by*

- *$Q' = (Q/\simeq)$,*
- *$F'(P) = \sum_{q \in P} F(q)$ for every $P \in Q'$, and*
- *for every $\sigma \in \Sigma_k$, states $P, P_1, \ldots, P_k \in Q'$*

$$\mu'_k(\sigma)_{P,P_1,\ldots,P_k} = \sum_{q_1 \in P_1, \ldots, q_k \in P_k} \mu_k(\sigma)_{\min(P),q_1,\ldots,q_k} \enspace .$$

Clearly, $(M/\simeq)$ never has strictly more states than M. Naturally, the best reduction is achieved by the greatest backward simulation. Next, let us show that $M/\simeq$ is equivalent to M, which proves that our construction preserves the semantics. Note that we make no assumptions on the total order on Q, so that the theorem will hold for any total order on Q.

Theorem 7 (cf. [10, Theorem 7]). *Let $\preceq$ be a backward simulation for M and $\simeq = (\preceq \cap \preceq^{-1})$. Then M and $M/\simeq$ are equivalent.*

Proof. Let $(M/{\simeq}) = (Q', \Sigma, \mathcal{A}, \mu', F')$ be the collapsed wta. We first prove that $h_{\mu'}(t)_P = h_\mu(t)_q$ for every $t \in T_\Sigma$, $P \in Q'$, and $q \in P$. Suppose that $t = \sigma(t_1, \ldots, t_k)$ for some $\sigma \in \Sigma_k$ and $t_1, \ldots, t_k \in T_\Sigma$. Then

$$
\begin{aligned}
& h_{\mu'}(\sigma(t_1, \ldots, t_k))_P \\
&= \sum_{P_1, \ldots, P_k \in Q'} \mu'_k(\sigma)_{P, P_1, \ldots, P_k} \cdot \prod_{i=1}^{k} h_{\mu'}(t_i)_{P_i} \\
&= \sum_{P_1, \ldots, P_k \in Q'} \Big(\sum_{q_1 \in P_1, \ldots, q_k \in P_k} \mu_k(\sigma)_{\min(P), q_1, \ldots, q_k} \Big) \cdot \prod_{i=1}^{k} h_{\mu'}(t_i)_{P_i} \\
&\overset{\dagger}{=} \sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{\min(P), q_1, \ldots, q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \\
&= h_\mu(\sigma(t_1, \ldots, t_k))_{\min(P)} \\
&= h_\mu(\sigma(t_1, \ldots, t_k))_q
\end{aligned}
$$

using the induction hypothesis at † and Corollary 5 in the last step where $q \simeq \min(P)$. With this auxiliary result

$$
\begin{aligned}
(\varphi_{(M/\simeq)}, t) &= \sum_{P \in Q'} F'(P) \cdot h_{\mu'}(t)_P = \sum_{P \in Q'} \Big(\sum_{q \in P} F(q) \Big) \cdot h_{\mu'}(t)_P \\
&= \sum_{q \in Q} F(q) \cdot h_\mu(t)_q = (\varphi_M, t) \enspace . \qquad \square
\end{aligned}
$$

An other negative property of our notion of backward simulation is that the result obtained by collapsing M with the greatest backward simulation is not minimal with respect to backward simulation. We call M *backward-simulation minimal* if every backward simulation for it is a partial order. If a backward simulation $\preceq$ is a partial order, then $\preceq \cap \preceq^{-1}$ is the identity, which yields no reduction in the number of states if used to collapse M. Thus, a backward-simulation minimal wta cannot be reduced any further using backward simulation, which justifies the name. Let us illustrate the definitions and the principal disadvantages of backward simulation on a very simplistic example. Similar examples can easily be constructed for more commonly used idempotent semirings such as the tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$.

Example 8. Let $S = \{1, 2\}$. We consider the semiring $\mathcal{P}(S) = (\mathcal{P}(S), \cup, \cap, \emptyset, S)$ where $\mathcal{P}(S)$ is the powerset of S. Moreover, consider the wta

$$M = ([1, 6], \Sigma, \mathcal{P}(S), \mu, F)$$

where

- $\Sigma = \{\alpha, \gamma\}$ contains the nullary symbol α and the unary symbol γ,
- $F(i) = \{1, 2\}$ for every $i \in [1, 6]$, and

– the following transitions

$$\begin{array}{lll} \mu_0(\alpha)_{1,\varepsilon} = \{1,2\} & \mu_0(\alpha)_{2,\varepsilon} = \{1,2\} & \mu_0(\alpha)_{3,\varepsilon} = \{1,2\} \\ \mu_1(\gamma)_{5,1} = \{1\} & \mu_1(\gamma)_{4,2} = \{1\} & \\ \mu_1(\gamma)_{5,2} = \{2\} & \mu_1(\gamma)_{4,1} = \{2\} & \\ \mu_1(\gamma)_{6,3} = \{1,2\} \ . & & \end{array}$$

Let $\preceq$ be the greatest backward simulation on M, and let $\simeq = (\preceq \cap \preceq^{-1})$. Then $1 \simeq 2 \simeq 3$ and $4 \simeq 5 \preceq 6$, but $6 \not\preceq 5$ and $6 \not\preceq 4$. Then the collapsed wta is $M' = (Q', \Sigma, \mathcal{P}(S), \mu', F')$ with $Q' = \{\{1,2,3\}, \{4,5\}, \{6\}\}$, $F'(P) = \{1,2\}$ for every $P \in Q'$, and

$$\mu'_0(\alpha)_{\{1,2,3\},\varepsilon} = \{1,2\} \quad \mu'_1(\gamma)_{\{4,5\},\{1,2,3\}} = \{1,2\} \quad \mu'_1(\gamma)_{\{6\},\{1,2,3\}} = \{1,2\} \ .$$

The wta M and M' are displayed in Figure 1. Now the states $\{4,5\}$ and $\{6\}$ are equivalent; i.e., $\{4,5\}$ simulates $\{6\}$ and vice versa. This demonstrates that the collapsed wta with respect to the greatest backward simulation need not be backward-simulation minimal. □

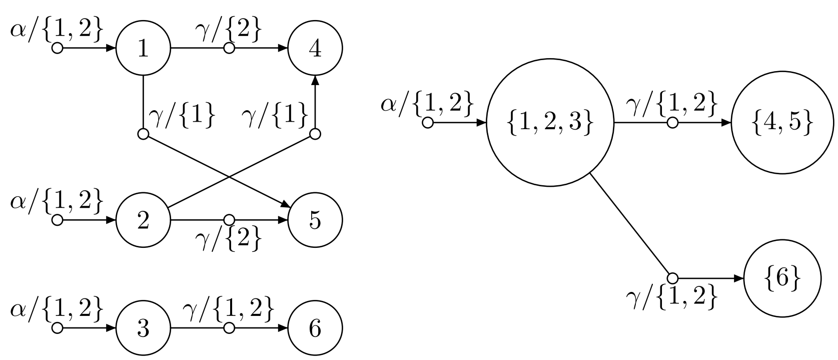

Fig. 1. The wta of Example 8 (without final weights).

Let us quickly consider deterministic wta. For every input tree $t \in T_\Sigma$, the vector $h_\mu(t)$ contains at most one nonzero entry if M is deterministic [17, Observation 4.1.6]. Thus, if M is deterministic and has no useless states (a state $q \in Q$ is useless if $h_\mu(t)_q = 0$ for every $t \in T_\Sigma$), then it is automatically backward-simulation minimal by Corollary 5. In other words, we cannot reduce a deterministic wta with the help of a backward simulation.

At the end of this section, let us develop a very simple algorithm to compute the greatest backward simulation. Our algorithm (Algorithm 1) starts with the optimistic assumption that all states simulate each other and the refines the relation as it finds evidence to the contrary (see [10,13]). To speed up the algorithm, we could also use the property mentioned in Note 4, but we present the simple, non-optimized version of the algorithm here for clarity.

Algorithm 1 Computing the greatest backward simulation for M.

$R_0 \leftarrow Q \times Q$
$i \leftarrow 0$
repeat
  $j \leftarrow i$
  for all $\sigma \in \Sigma_k$ and $p_1, \dots, p_k \in Q$ **do**
    $R_{i+1} \leftarrow \{(p, q) \in R_i \mid$
      $\exists (p_1, q_1), \dots, (p_k, q_k) \in R_i \colon \mu_k(\sigma)_{p,p_1,\dots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\dots,q_k}\}$
    $i \leftarrow i + 1$
until $R_i = R_j$

Theorem 9. *Algorithm 1 returns the greatest backward simulation for M.*

Proof. Let $\preceq$ be the greatest backward simulation for M. First, we prove that $\preceq\, \subseteq R_i$ for every i that is encountered during the run of the algorithm. Let us proceed by induction on i. Trivially $\preceq\, \subseteq R_0$ because $R_0 = Q \times Q$. Now, let us assume that $p \preceq q$. Consequently, $(p, q) \in R_i$ by the induction hypothesis. Let $\sigma \in \Sigma_k$ and $p_1, \dots, p_k \in Q$. Then by Definition 1 there exist $q_1, \dots, q_k \in Q$ such that $\mu_k(\sigma)_{p,p_1,\dots,p_k} \sqsubseteq \mu_k(\sigma)_{q,q_1,\dots,q_k}$ and $p_i \preceq q_i$ for every $i \in [1, k]$. By induction hypothesis, we also have $(p_i, q_i) \in R_i$ for every $i \in [1, k]$. Consequently, we obtain $(p, q) \in R_{i+1}$. This proves $\preceq\, \subseteq R_{i+1}$. At termination, R_i is a backward simulation (see Definition 1) and since $\preceq\, \subseteq R_i$ and $\preceq$ is the greatest backward simulation for M, we can conclude that $R_i = \preceq$. □

4 A Forward Simulation

Next, we consider a forward version of the simulation of Section 3. Similar simulations (called composed simulations) for the unweighted case are considered in [10] and forward bisimulation for wta is considered in [13]. Let us follow the structure of the previous section and start with the definition of a forward simulation. Note that we will use the same symbols here as in Section 3, but it should be clear that we exclusively speak about forward simulations here unless otherwise mentioned.

For state $q \in Q$ to (forward) simulate another state p, written $p \preceq q$, we demand that for every transition weight $\mu_k(\sigma)_{p',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k}$, there exists a larger transition weight $\mu_k(\sigma)_{q',q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_k}$ with the additional restriction that the state q' simulates p'. In addition, the final weight of q should be larger than the one of p. In the unweighted case, this coincides with the definition of an upward simulation [10, Section 2] with respect to the identity as a backward simulation. There it is demanded that q should be a final state if p is. Moreover, for every transition $\sigma(q_1, \dots, q_{i-1}, p, q_{i+1}, \dots, q_k) \to p'$ there should exist a transition $\sigma(q_1, \dots, q_{i-1}, q, q_{i+1}, \dots, q_k) \to q'$ such that q' simulates p'.

Definition 10. *A quasi-order $\preceq\, \subseteq Q \times Q$ is a* forward simulation *for M if for every $p \preceq q$ the following two conditions are satisfied:*

- *$F(p) \sqsubseteq F(q)$ and*
- *for every $\sigma \in \Sigma_k$, $i \in [1,k]$, and $p', q_1, \ldots, q_k \in Q$ there exist $q' \in Q$ such that $p' \preceq q'$ and*

$$\mu_k(\sigma)_{p',q_1,\ldots,q_{i-1},p,q_{i+1},\ldots,q_k} \sqsubseteq \mu_k(\sigma)_{q',q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_k} \enspace .$$

A forward simulation is also only a quasi-order and not an equivalence relation like every forward bisimulation. We do not consider upward simulations [10] here since we believe that two independent simulations are easier to understand and we can always first reduce with the help of a backward simulation and then with a forward simulation to achieve roughly the same as with an upward simulation of [10]. Let us proceed with the principal properties of forward simulations. As in the backward case, there exists a greatest forward simulation for M.

Theorem 11 (see [13, Theorem 7]). *There exists a greatest forward simulation for M.*

Proof. Let $\preceq$ and $\preceq'$ be forward simulations for M. Again, we claim that $(\preceq \cup \preceq')^*$ is a forward simulation. Clearly, $(\preceq \cup \preceq')^*$ is a quasi-order. Let $(p,q) \in (\preceq \cup \preceq')^*$, $\sigma \in \Sigma_k$, $i \in [1,k]$, and $p', q_1, \ldots, q_k \in Q$. Consequently, there exist $r_1, \ldots, r_n \in Q$ such that

$$p = r_0 \preceq r_1 \preceq' r_2 \preceq r_3 \preceq' \cdots \preceq' r_n = q \enspace .$$

By this chain of inequalities, there also exists $q' \in Q$ such that

$$\mu_k(\sigma)_{p',q_1,\ldots,q_{i-1},p,q_{i+1},\ldots,q_k} \sqsubseteq \mu_k(\sigma)_{q',q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_k}$$

and $p' \; (\preceq ; \preceq' ; \preceq ; \preceq' ; \cdots ; \preceq') \; q'$, which proves that $(\preceq \cup \preceq')^*$ is a forward simulation. ☐

To state the main property of similar states, we need some additional notions. A context is a tree of $T_\Sigma(\{\square\})$, where $\square$ is a distinguished (fixed) symbol, such that $\square$ occurs exactly once. The set of all contexts is denoted by C_Σ. The tree $c[t]$ is obtained by replacing the symbol $\square$ in the context $c \in C_\Sigma$ by the tree $t \in T_\Sigma$.

Lemma 12. *Let $\preceq$ be a forward simulation for M. Moreover, let $c \in C_\Sigma$ and $p \preceq q$. Then $\sum_{r \in B} h_\mu(c[p])_r \sqsubseteq \sum_{r \in B} h_\mu(c[q])_r$ for every up-set $B \subseteq Q$.*

Proof. We prove the statement by induction on $c \in C_\Sigma$. In the base case, let $c = \square$. Then

$$\sum_{r \in B} h_\mu(p)_r = \begin{cases} 1 & \text{if } p \in B \\ 0 & \text{otherwise.} \end{cases}$$

Since B is an up-set, $p \in B$ implies $q \in B$ and thus by $0 \sqsubseteq 1$

$$\sum_{r \in B} h_\mu(p)_r \sqsubseteq \begin{cases} 1 & \text{if } q \in B \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{r \in B} h_\mu(q)_r \enspace .$$

In the induction step, let $c = \sigma(t_1, \dots, t_{j-1}, c', t_{j+1}, \dots, t_k)$ for some $\sigma \in \Sigma_k$, $j \in [1, k]$, $c' \in C_\Sigma$, and $t_1, \dots, t_k \in T_\Sigma$. Then

$$\begin{aligned} & \sum_{r \in B} h_\mu(\sigma(t_1, \dots, t_{j-1}, c'[p], t_{j+1}, \dots, t_k))_r \\ &= \sum_{\substack{r \in B \\ p_1, \dots, p_k \in Q}} \mu_k(\sigma)_{r, p_1, \dots, p_k} \cdot h_\mu(c'[p])_{p_j} \cdot \prod_{i \in [1,k] \setminus \{j\}} h_\mu(t_i)_{p_i} \enspace . \end{aligned}$$

Then $h_\mu(c'[p])_{p_j} \sqsubseteq \sum_{p' \in \uparrow(p_j)} h_\mu(c'[q])_{p'}$ by the induction hypothesis, and thus

$$\begin{aligned} &\sqsubseteq \sum_{\substack{r \in B \\ p_1, \dots, p_k \in Q}} \mu_k(\sigma)_{r, p_1, \dots, p_k} \cdot \Big(\sum_{p' \in \uparrow(p_j)} h_\mu(c'[q])_{p'} \Big) \cdot \prod_{i \in [1,k] \setminus \{j\}} h_\mu(t_i)_{p_i} \\ &\sqsubseteq \sum_{\substack{r \in B \\ p_1, \dots, p_k \in Q \\ p' \in \uparrow(p_j)}} \Big(\sum_{r' \in \uparrow(r)} \mu_k(\sigma)_{r', p_1, \dots, p_{j-1}, p', p_{j+1}, \dots, p_k} \Big) \cdot h_\mu(c'[q])_{p'} \cdot \prod_{i \in [1,k] \setminus \{j\}} h_\mu(t_i)_{p_i} \end{aligned}$$

because $p_j \preceq p'$ and thus for every $r \in B$ there exists $r' \in Q$ such that $r \preceq r'$ and $\mu_k(\sigma)_{r, p_1, \dots, p_k} \sqsubseteq \mu_k(\sigma)_{r', p_1, \dots, p_{j-1}, p', p_{j+1}, \dots, p_k}$. Since B is an up-set and $\mathcal{A}$ idempotent, we continue with

$$\begin{aligned} &= \sum_{\substack{r \in B \\ p_1, \dots, p_k \in Q}} \mu_k(\sigma)_{r, p_1, \dots, p_k} \cdot h_\mu(c'[q])_{p_j} \cdot \prod_{i \in [1,k] \setminus \{j\}} h_\mu(t_i)_{p_i} \\ &= \sum_{r \in B} h_\mu(\sigma(t_1, \dots, t_{j-1}, c'[q], t_{j+1}, \dots, t_k))_r \enspace . \end{aligned}$$ □

Next, let us show how to reduce the size of a wta using a forward simulation. We again make use of the total order on Q to simplify the construction. In particular, the minimum operation in the construction refers to this total order and not to the forward simulation for M.

Definition 13 (cf. [13, Definition 3]). *Let $\preceq$ be a forward simulation for M and $\simeq = (\preceq \cap \preceq^{-1})$. The collapsed wta $(M/{\simeq}) = (Q', \Sigma, \mathcal{A}, \mu', F')$ is given by*

- *$Q' = (Q/{\simeq})$,*
- *$F'(P) = F(\min(P))$ for every $P \in Q'$, and*
- *for every $\sigma \in \Sigma_k$, states $P, P_1, \dots, P_k \in Q'$*

$$\mu'_k(\sigma)_{P, P_1, \dots, P_k} = \sum_{q \in P} \mu_k(\sigma)_{q, \min(P_1), \dots, \min(P_k)}$$

As before, the collapsed wta $M/\simeq$ never has more states than M itself and the best reduction is achieved by the greatest forward simulation. However, we first need to show that $M/\simeq$ is equivalent to M. Beforehand, let us note an important property of equivalent states (i.e., states that simulate each other) that follows immediately from Definition 10.

Note 14. Let $\preceq$ be a forward simulation for M, and let $\simeq = (\preceq \cap \preceq^{-1})$. Then for every $p \simeq q$, $\sigma \in \Sigma_k$, $i \in [1, k]$, and $p', q_1, \ldots, q_k \in Q$, there exist $q', r' \in Q$ such that $p' \preceq q' \simeq r'$ and

$$\begin{aligned}\mu_k(\sigma)_{p', q_1, \ldots, q_{i-1}, p, q_{i+1}, \ldots, q_k} &\sqsubseteq \mu_k(\sigma)_{q', q_1, \ldots, q_{i-1}, p, q_{i+1}, \ldots, q_k} \\ &= \mu_k(\sigma)_{r', q_1, \ldots, q_{i-1}, q, q_{i+1}, \ldots, q_k} \enspace .\end{aligned}$$

We will use this property in the proof of the next theorem, which will prove the correctness of our construction.

Theorem 15 (cf. [10, Theorem 7]). *Let $\preceq$ be a forward simulation for M and $\simeq = (\preceq \cap \preceq^{-1})$. Then M and $M/\simeq$ are equivalent.*

Proof. Let $(M/\simeq) = (Q', \Sigma, \mathcal{A}, \mu', F')$ be the collapsed wta. We first prove that $h_{\mu'}(t)_P = \sum_{q \in \uparrow(P)} h_\mu(t)_q$ for every $t \in T_\Sigma$, and $P \in Q'$. Suppose that $t = \sigma(t_1, \ldots, t_k)$ for some $\sigma \in \Sigma_k$ and $t_1, \ldots, t_k \in T_\Sigma$. Then we compute as follows where the equality marked † is explained below.

$$\begin{aligned}
& h_{\mu'}(\sigma(t_1, \ldots, t_k))_P \\
&= \sum_{P_1, \ldots, P_k \in Q'} \mu'_k(\sigma)_{P, P_1, \ldots, P_k} \cdot \prod_{i=1}^{k} h_{\mu'}(t_i)_{P_i} \\
&= \sum_{P_1, \ldots, P_k \in Q'} \mu'_k(\sigma)_{P, P_1, \ldots, P_k} \cdot \prod_{i=1}^{k} \Big(\sum_{q_i \in \uparrow(P_i)} h_\mu(t_i)_{q_i} \Big) \\
&= \sum_{\substack{P_1, \ldots, P_k \in Q' \\ q_1 \in \uparrow(P_1), \ldots, q_k \in \uparrow(P_k)}} \Big(\sum_{q \in \uparrow(P)} \mu_k(\sigma)_{q, \min(P_1), \ldots, \min(P_k)} \Big) \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \\
&= \sum_{q \in \uparrow(P)} \sum_{\substack{P_1, \ldots, P_k \in Q' \\ q_1 \in \uparrow(P_1), \ldots, q_k \in \uparrow(P_k)}} \mu_k(\sigma)_{q, \min(P_1), \ldots, \min(P_k)} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \\
&\stackrel{\dagger}{=} \sum_{q \in \uparrow(P)} \sum_{\substack{P_1, \ldots, P_k \in Q' \\ q_1 \in \uparrow(P_1), \ldots, q_k \in \uparrow(P_k)}} \mu_k(\sigma)_{q, q_1, \ldots, q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \\
&= \sum_{q \in \uparrow(P)} h_\mu(\sigma(t_1, \ldots, t_k))_q
\end{aligned}$$

Let us take a closer look at the equation marked †. We can show this equality by showing both inequalities. Let us consider the inequality $\sqsubseteq$ first. Clearly, it is sufficient to show that for each summand of the left-hand side there exists a larger summand in the right-hand side. For this we consider a summand $\mu_k(\sigma)_{p,\min(P_1),\dots,\min(P_k)} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i}$ of the left-hand side of † for some $P_1, \dots, P_k \in Q'$, $p \in \uparrow(P)$, and $q_1, \dots, q_k \in Q$ such that $q_i \in \uparrow(P_i)$ for every $i \in [1,k]$. Since $\min(P_i) \preceq q_i$ for every $i \in [1,k]$, there exists $q \in Q$ such that $\mu_k(\sigma)_{p,\min(P_1),\dots,\min(P_k)} \sqsubseteq \mu_k(\sigma)_{q,q_1\cdots q_k}$ by Definition 10. Consequently,

$$\mu_k(\sigma)_{p,\min(P_1),\dots,\min(P_k)} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \sqsubseteq \mu_k(\sigma)_{q,q_1,\dots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i}$$

and the latter is a summand on the right-hand side of †. For the converse inequality, let us consider a summand $\mu_k(\sigma)_{q,q_1,\dots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i}$ in the right-hand side where $q \in \uparrow(P)$ and $q_1, \dots, q_k \in Q$. For every $i \in [1,k]$ let $P_i = [q_i]$. Then $\min(P_i) \simeq q_i$ for every $i \in [1,k]$. Then by Definition 10 and the property remarked in Note 14, there exist $p, q' \in Q$ such that $q \preceq q' \simeq p$ and

$$\mu_k(\sigma)_{q,q_1,\dots,q_k} \sqsubseteq \mu_k(\sigma)_{q',q_1,\dots,q_k} = \mu_k(\sigma)_{p,\min(P_1),\dots,\min(P_k)} \enspace .$$

It follows that

$$\mu_k(\sigma)_{q,q_1,\dots,q_k} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \sqsubseteq \mu_k(\sigma)_{p,\min(P_1),\dots,\min(P_k)} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i} \enspace ,$$

which is a summand of the left-hand side because $q \preceq p$. This completes the proof of our auxiliary statement. For the statement of the theorem, we compute as follows:

$$\begin{aligned}(\varphi_{(M/\simeq)}, t) &= \sum_{P \in Q'} F'(P) \cdot h_{\mu'}(t)_P = \sum_{P \in Q'} F'(P) \cdot \Big(\sum_{q \in \uparrow(P)} h_\mu(t)_q \Big) \\ &= \sum_{P \in Q', q \in \uparrow(P)} F'(P) \cdot h_\mu(t)_q = \sum_{q \in Q} F(q) \cdot h_\mu(t)_q = (\varphi_M, t)\end{aligned}$$

because $F(p) \sqsubseteq F(q)$ if $p \preceq q$. This proves our theorem. □

Also the notion of forward simulation has the negative properties outlined in the section on backward simulation. For example, the result obtained by collapsing M with the greatest forward simulation is again not necessarily minimal with respect to forward simulation. Accordingly, we call M *forward-simulation minimal* if every forward simulation for it is a partial order. Let us also present a small example for forward simulation.

Example 16. Consider the wta M of Example 8. Let $\preceq$ be the coarsest forward simulation for it, and let $\simeq \; = \; (\preceq \cap \preceq^{-1})$. Then $4 \simeq 5 \simeq 6$ and $1 \simeq 2 \preceq 3$ but $3 \not\preceq 2$. Consequently, the collapsed wta is $M' = (Q', \Sigma, \mathcal{P}(S), \mu', F')$ where $Q' = \{\{1,2\}, \{3\}, \{4,5,6\}\}$, $F'(P) = \{1,2\}$ for every $P \in Q'$, and

$$\mu'_0(\alpha)_{\{1,2\},\varepsilon} = \{1,2\} \qquad \mu'_1(\gamma)_{\{4,5,6\},\{1,2\}} = \{1,2\}$$

$$\mu'_0(\alpha)_{\{3\},\varepsilon} = \{1,2\} \qquad \mu'_1(\gamma)_{\{4,5,6\},\{3\}} = \{1,2\} .$$

Figure 2 displays M and M'. In M' the states $\{1,2\}$ and $\{3\}$ are equivalent; i.e., $\{1,2\}$ simulates $\{3\}$ and vice versa. This again demonstrates that the collapsed wta with respect to the greatest forward simulation need not be forward-simulation minimal. □

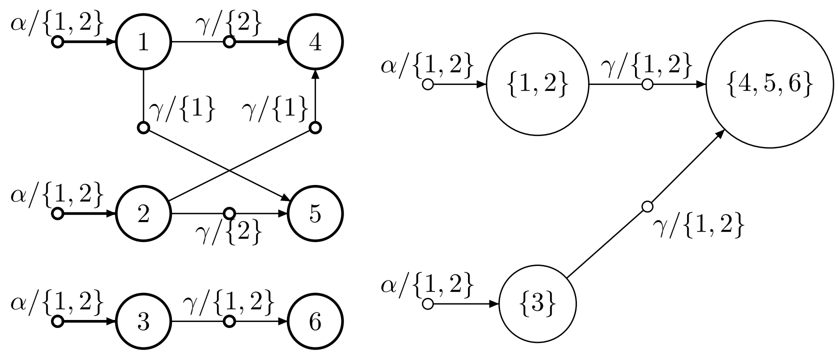

Fig. 2. The wta of Example 16 (without final weights).

Let us also discuss the deterministic case. Roughly speaking, we claim that reduction with the help of forward simulation is not more effective than reduction with the help of forward bisimulation on deterministic wta. Let us quickly recall the definition of a forward bisimulation [13] for M.

Definition 17 (see [13, Definition 1]). *An equivalence relation $\equiv$ on Q is a* forward bisimulation *for M if for every $p \equiv q$ the following two conditions hold:*

(i) $F(p) = F(q)$ *and*
(ii) *for every $\sigma \in \Sigma_k$, $i \in [1,k]$, $q_1, \ldots, q_k \in Q$, and $P \in (Q/\equiv)$*

$$\sum_{r \in P} \mu_k(\sigma)_{r,q_1,\ldots,q_{i-1},p,q_{i+1},\ldots,q_k} = \sum_{r \in P} \mu_k(\sigma)_{r,q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_k} .$$

Suppose that M is deterministic. To prove that reduction with the help of forward simulation is only as effective as reduction with the help of forward bisimulation, it is sufficient to show that any equivalence relation $\simeq$ obtained from a forward simulation $\preceq$ for M is indeed a forward bisimulation for M. The algorithm in [13] can then be used to compute an equivalent wta that has at most as many states as $(M/\simeq)$. Recall that a state $q \in Q$ is useless if $h_\mu(t)_q = 0$ for every $t \in T_\Sigma$.

Theorem 18. *Let M be deterministic and without useless states. Moreover, let $\preceq$ be a forward simulation for M, and $\simeq = (\preceq \cap \preceq^{-1})$. Then $\simeq$ is a forward bisimulation for M.*

Proof. Let $p \simeq q$, $\sigma \in \Sigma_k$, $i \in [1,k]$, and $p', q_1, \dots, q_k \in Q$ be such that $\mu_k(\sigma)_{p',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} \neq 0$. By determinism there exists at most one such p' and if M has no useless states, then there exists at least one such p'. Since $p \simeq q$, there exist $q', r' \in Q$ such that $p' \preceq q' \preceq r'$ and

$$\begin{aligned} \mu_k(\sigma)_{p',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} &\sqsubseteq \mu_k(\sigma)_{q',q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_k} \\ &\sqsubseteq \mu_k(\sigma)_{r',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} \enspace . \end{aligned}$$

By determinism, $r' = p'$ and thus $p' \simeq q'$ and

$$\mu_k(\sigma)_{p',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} = \mu_k(\sigma)_{q',q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_k} \enspace .$$

Consequently, for every $P \in (Q/\simeq)$

$$\begin{aligned} \sum_{r \in P} \mu_k(\sigma)_{r,q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} &= \begin{cases} \mu_k(\sigma)_{p',q_1,\dots,q_{i-1},p,q_{i+1},\dots,q_k} & \text{if } p' \in P \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_k(\sigma)_{q',q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_k} & \text{if } q' \in P \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{r \in P} \mu_k(\sigma)_{r,q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_k} \end{aligned}$$

because $p' \in P$ if and only if $q' \in P$. This proves condition (ii) of Definition 17. For condition (i) of the same definition, we simply observe that $F(p) \sqsubseteq F(q) \sqsubseteq F(p)$, which proves it and hence the statement that $\simeq$ is a forward bisimulation. □

Minimization (i.e., finding a minimal deterministic wta that is equivalent to M) of deterministic wta is discussed in [12]. Note that the previous theorem also proves that reduction with the help of the greatest forward simulation does not necessarily yield a minimal deterministic wta. This is due to the fact that forward bisimulation does not achieve that (cf. [18, Theorem 3.12]).

Algorithm 2 Computing the greatest forward simulation for M.

$R_0 \leftarrow \{(p,q) \in Q \times Q \mid F(p) \sqsubseteq F(q)\}$
$i \leftarrow 0$
repeat
  $j \leftarrow i$
  for all $\sigma \in \Sigma_k$, $n \in [1,k]$, and $p', q_1, \dots, q_k \in Q$ **do**
    $R_{i+1} \leftarrow \{(p,q) \in R_i \mid \exists (p',q') \in R_i \colon$
      $\mu_k(\sigma)_{p',q_1,\dots,q_{n-1},p,q_{n+1},\dots,q_k} \sqsubseteq \mu_k(\sigma)_{q',q_1,\dots,q_{n-1},q,q_{n+1},\dots,q_k}\}$
    $i \leftarrow i + 1$
until $R_i = R_j$

Finally, let us develop an algorithm for the greatest forward simulation. Our algorithm is displayed in Algorithm 2.

Theorem 19. *Algorithm 2 returns the greatest forward simulation for M.*

Proof. Let $\preceq$ be the greatest forward simulation for M. Again we prove that $\preceq \subseteq R_i$ for every relevant i as an auxiliary statement. Using the first condition of Definition 10, we have $\preceq \subseteq R_0$. Suppose that $p \preceq q$. Then $(p, q) \in R_i$ by the induction hypothesis. Moreover, let $\sigma \in \Sigma_k$, $n \in [1, k]$, and $p', q_1, \ldots, q_k \in Q$. Since $p \preceq q$, we can conclude that there exists $q' \in Q$ such that $p' \preceq q'$ and

$$\mu_k(\sigma)_{p',q_1,\ldots,q_{n-1},p,q_{n+1},\ldots,q_k} \sqsubseteq \mu_k(\sigma)_{q',q_1,\ldots,q_{n-1},q,q_{n+1},\ldots,q_k} \enspace .$$

Invoking the induction hypothesis, we obtain $(p', q') \in R_i$ and thus $(p, q) \in R_{i+1}$. Thus $\preceq \subseteq R_{i+1}$. Clearly, R_i is a forward simulation for M (see Definition 10) at termination. Since $\preceq \subseteq R_i$ and $\preceq$ is the greatest forward simulation for M, we can conclude that $R_i = \preceq$. ☐

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