The Power of Tree Series Transducers

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Diplom-Informatiker Andreas Maletti
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Gutachter:

Prof. Dr.-Ing. habil. *Heiko Vogler*
(Technische Universität Dresden)

Prof. Dr. rer. nat. habil. *Manfred Droste*
(Universität Leipzig)

O. Univ. Prof. Dr. phil. Dr. h.c. *Werner Kuich*
(Technische Universität Wien)

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Contents

CHAPTER 1

Introduction

There is nothing more difficult to take in hand,
more perilous to conduct or more uncertain in its success
than to take the lead in the introduction
of a new order of things.

Niccolò Machiavelli (1469–1527): "The Prince" [Il Principe]
Translation by William K. Marriott, 1513

1. Overview

This thesis scrutinizes the power of restricted tree series transducers that use either pure or o-substitution. We compare the classes of ε-tree-to-tree series and o-tree-to-tree-series transformations computed by several classes of tree series transducers. The analysis is very detailed (in the sense that we consider several syntactic restrictions on tree series transducers and several classes of semirings) for the classes of transformations computed by deterministic tree series transducers. We also consider classes of transformations computed by polynomial tree series transducers albeit in less detail. Finally, we also investigate compositions of transformations computed by bottom-up and top-down tree series transducers.

In this introduction we shortly recall the predecessors of tree series transducers and some of their applications. Then we present the IO tree series substitutions (namely pure and o-substitution) that are central in the definition of the semantics of tree series transducers. We shortly discuss these devices thereafter. Next we start the analysis of the power of deterministic tree series transducers and later also arbitrary polynomial tree series transducers. Finally, we consider compositions of transformations and investigate whether they can also be computed by a single tree series transducer.

Apart from this introduction, the thesis has six chapters. All chapters but the preliminaries are introduced here. The examples presented in the forthcoming chapters only serve to illustrate a certain construction or feature. Thus they are usually abstract in nature.

2. Historical notes and motivation

Tree series transducers [**79, 41, 55, 58**] were introduced as a joint generalization of tree transducers [**102, 106, 3, 35, 4**] and weighted tree automata [**9, 104, 77, 18, 17**]. They thereby serve as the transducing devices corresponding to weighted tree automata. Both historical predecessors of tree series transducers have successfully been motivated from and applied in practice. Specifically, tree transducers are motivated from syntax-directed translations in compilers [**68, 38, 57**], and they are applied in, *e. g.*, functional program analysis and transformation [**75, 62, 70, 109**], computational linguistics [**94, 73, 92, 72**], generation of pictures [**29, 30**], and query languages of XML databases [**11, 42**]. Weighted tree automata have been applied to code selection in compilers [**49, 14**] and tree pattern matching [**104**].

Weighted transducers on strings [**10**] are applied in image manipulation [**27**], where the images are coded as weighted string automata, and speech processing [**93**]. Since natural language processing features many transformations on parse trees, which come equipped with a degree of certainty, it seems natural to consider finite-state devices capable of transforming weighted trees. For natural language processing, the potential of tree series transducers over the semiring of the positive real numbers was recently discovered [**65**]. Moreover, a rich theory of tree transducers was developed (see [**35, 5, 39, 40**] as seminal papers and [**60, 96, 25, 61, 57**] as survey papers and monographs) during the seventies, whereas weighted tree automata just recently received some attention (*e. g.*, [**104, 77, 13, 17, 31, 32, 48, 82, 87, 33**]).

3. Tree series substitution

Tree substitution is at the core of the semantics of tree transducers, and tree series substitution fulfills this purpose for tree series transducers. In this thesis we discuss two tree series substitutions, namely pure [**20, 41**] and o-substitution [**58**]. A tree series is a mapping from a set of trees [**60, 61**] into a semiring [**66, 64**]. For the illustration here, we use the natural numbers with minimum and addition as the underlying semiring (a tropical semiring). A tree series can then be seen as a multiset [**12**] (sometimes also called bag) of trees. Let us suppose that we want to substitute k tree series $\psi_1, \dots, \psi_k$ into a tree series ψ. On the tree level the tree series substitutions just perform tree substitution. But how do we obtain the natural number that is associated with a tree u? For this we consider all decompositions $u'[u_1, \dots, u_k]$ of the tree u into a tree u', which may contain the variables $\{\mathrm{z}_1, \dots, \mathrm{z}_k\}$, and trees $u_1, \dots, u_k$. The natural number that is associated with u in the resulting tree series is computed as follows:

- for pure substitution (also called ε-substitution)

$$\min_{\substack{u', u_1, \dots, u_k \text{ trees,} \\ u = u'[u_1, \dots, u_k]}} \Big(\psi(u') + \psi_1(u_1) + \dots + \psi_k(u_k) \Big)$$

- for o-substitution

$$\min_{\substack{u',u_1,\ldots,u_k \text{ trees,}\\ u=u'[u_1,\ldots,u_k]}} \Big(\psi(u') + n_1 \cdot \psi_1(u_1) + \cdots + n_k \cdot \psi_k(u_k)\Big)$$

  where $n_1, \ldots, n_k$ hold the number of occurrences of the variables $\mathrm{z}_1, \ldots, \mathrm{z}_k$ in u', respectively.

We deal with these two modes of tree series substitution. The first is called pure tree series substitution [**20, 41**] (for short: pure substitution) and represents a computational approach; *i. e.*, the output trees represent values of computations, and the natural number that is associated to an output tree can be viewed as the cost of computing this value. When we combine the results of smaller problems (output trees), then their costs are simply added to obtain the cost of the combined result (output tree). This happens irrespective of the number of uses of a computed result; *i. e.*, a result may be copied without penalty, which represents the computational approach in the sense that a value is available and can be reused without recomputation. Finally, we take the minimum of the costs over all possible combinations that yield the same final result.

On the other hand, we also investigate a tree series substitution that respects occurrences [**58**] (for short: o-substitution), which represents a more material approach. There the natural number, that is associated with an output tree, is taken n times, if the output tree is used in n copies (the corresponding variable occurs n times). In this approach, an output tree stands for a composite, and the natural number associated with the output tree reflects the (monetary) cost of creating or obtaining this particular composite. When we combine composites into a new composite, then we obtain the cost of the composite by a simple addition of the costs of its components; each component taken as often as needed to assemble the composite.

Tree series substitutions have also been studied in relation with recognizable tree series [**9, 16**]. Substitution is a standard operation on tree series, and in particular, OI-substitution [**18, 79**] was studied with respect to preservation of recognizability [**78, 80**]. A tree series is called recognizable, if there exists a finite state automaton (more specifically, a bottom-up weighted tree automaton [**16**]) that computes this tree series. Recognizable tree series are of particular interest, because they are finitely representable.

In Chapter 3 on tree series substitutions we introduce the mentioned substitutions formally and study pure and o-substitution with respect to the fundamental properties of distributivity, linearity, and associativity. Finally, we consider preservation of recognizability for pure and o-substitution. The main result of this chapter, apart from the lemmata for distributivity, linearity, and associativity, is Theorem 3.28. It states that o-substitution preserves recognizable tree series in the

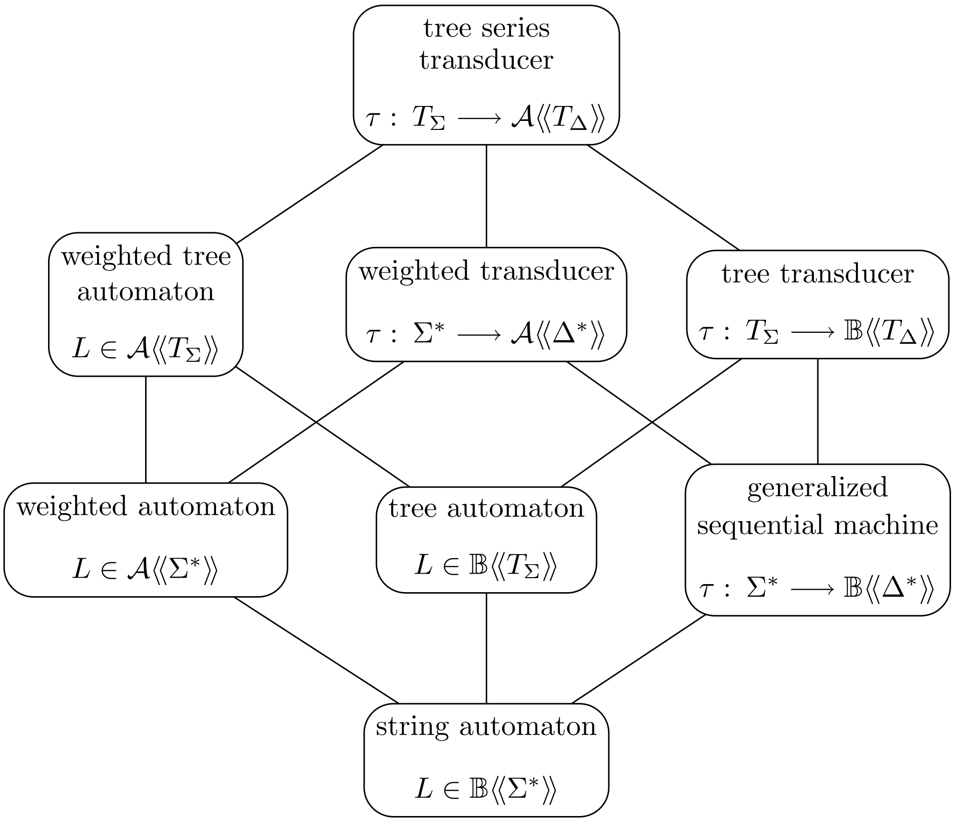

FIGURE 1. Generalization hierarchy.

tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$, whenever the tree series are linear (*i. e.*, each variable may occur at most once in the trees). In general, the result holds for all semirings $(A, +, \cdot)$ that are continuous [**48**] and additively idempotent (*i. e.*, $a + a = a$ for every $a \in A$).

4. Tree series transducers

Figure 1 attempts to display the automata and transducer concepts subsumed by tree series transducers. Roughly speaking, moving upwards-left in this figure adds weights (costs or multiplicity), moving upwards performs the generalization from strings to trees, and finally, moving upwards-right adds an output component.

Intuitively, a (bottom-up or top-down) tree series transducer is a (bottom-up or top-down) tree transducer [**106, 102, 107**] in which the transitions carry a weight; a weight is an element of some semiring [**66, 64**]. The rewrite semantics works as follows. Along a successful computation on some input tree, the weights of the involved transitions are combined by means of the semiring multiplication; if there is more than one successful computation for some pair of input and output trees, then the weights of these computations are combined by means of the semiring addition.

Tree series transducers, henceforth abbreviated to tst, capture both (a) the way of translating input trees into output trees, which they inherit from bottom-up and top-down tree transducers, and (b) the computation of a weight (or cost) in a semiring, which they inherit from weighted tree automata. More formally, a (bottom-up or top-down) tst is a tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, where

- Q is a finite set of states;
- Σ and Δ are ranked alphabets of input and output symbols, respectively;
- $\mathcal{A} = (A, +, \cdot)$ is a semiring;
- $F\colon Q \longrightarrow A^{C_\Delta(\mathrm{Z}_1)}$ assigns top-most output to each state (with $C_\Delta(\mathrm{Z}_1)$ being the set of all Δ-trees that contain the variable z_1 exactly once); and
- $\mu = (\mu_k)_{k\in\mathbb{N}}$ is a (bottom-up or top-down) tree representation.

The tree representation consists of mappings μ_k that map the set of k-ary symbols of Σ into $(Q \times Q(\mathrm{X}_k)^*)$-matrices over $A^{T_\Delta(\mathrm{Z})}$, where $T_\Delta(\mathrm{Z})$ denotes the set of Δ-trees indexed by variables of $\mathrm{Z} = \{\mathrm{z}_1, \mathrm{z}_2, \dots\}$ and $Q(\mathrm{X}_k) = \{q(\mathrm{x}_i) \mid q \in Q, 1 \leqslant i \leqslant k\}$. The entries are mappings $\varphi\colon T_\Delta(\mathrm{Z}) \longrightarrow A$, and such a mapping is called a tree series.

We use η-substitution of tree series (with $\eta \in \{\varepsilon, \mathrm{o}\}$; see [**58**]) to substitute tree series into tree series. This way we can impose a Σ-algebraic structure on the set of all mappings $V\colon Q \longrightarrow A^{T_\Delta}$ and thereby obtain the unique Σ-homomorphism h^η_μ from T_Σ to $\{V \mid V\colon Q \longrightarrow A^{T_\Delta}\}$. Then the η-tree-to-tree-series (for short: η-t-ts) transformation computed by M is the mapping $\|M\|_\eta\colon T_\Sigma \longrightarrow A^{T_\Delta}$ defined by

$$\|M\|_\eta(t) = \sum_{q\in Q} F(q) \xleftarrow[\eta]{} (h^\eta_\mu(t))(q)$$

where $\xleftarrow[\eta]{}$ denotes η-substitution. Thus, for a given input tree $t \in T_\Sigma$, the tst M computes a (potentially infinite) set

$$\mathrm{supp}(\|M\|_\eta(t)) = \{u \in T_\Delta \mid (\|M\|_\eta(t))(u) \neq 0\} \enspace ,$$

where 0 is the additively neutral element of $\mathcal{A}$, of output trees and associates a coefficient $(\|M\|_\eta(t))(u) \in A$ to every output tree $u \in T_\Delta$.

In the same way as tree transducers, also tst can have particular properties. For every so-called polynomial tst M and input tree $t \in T_\Sigma$, the set $\mathrm{supp}(\|M\|_\eta(t))$ of computed and relevant output trees is finite. Polynomial bottom-up and top-down tst over the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge)$ essentially are bottom-up and top-down tree transducers, respectively (see [**41**, Section 4]). Moreover, a tst can be deterministic or a homomorphism (see, *e. g.*, [**35**]). Note that homomorphism tst are deterministic, and deterministic tst are polynomial. The classes of η-t-ts transformations computed by bottom-up and top-down tst that have the properties x (*e. g.*, that are deterministic) over $\mathcal{A}$ are denoted by x–$\mathrm{BOT}_\eta(\mathcal{A})$ and x–$\mathrm{TOP}_\eta(\mathcal{A})$, respectively. For brevity, we usually use only the first letter of the property.

In [**41, 58, 59, 55**] several generalizations of well-known theorems of the theory of tree transducers have been proved for bottom-up tst, *e. g.*,

- a generalization of the decomposition of the class of bottom-up tree transformations (see [**41**, Theorem 5.7] and [**35**, p. 220]); in its turn the result of [**35**] generalizes the decomposition of gsm-mappings as proved in [**95**];
- a generalization of (some) composition hierarchy results for bottom-up and top-down tree transformation classes (see [**55**, Theorem 6.24] and [**60**, Corollary 8.13(iii)]);
- a generalization of the equivalence of a rewrite semantics and the initial algebra semantics for bottom-up and top-down tree transducers (see [**59**, Theorems 5.10 and 6.9] and [**35**, Lemmata 5.6 and 5.5]).

Chapter 4 formally recalls tree series transducers from [**41**] along with some syntactic properties. We also present examples and some simple statements on tree series transducers.

5. Deterministic tree series transducers

We start our investigation of the power of tree series transducers with deterministic devices. The generating power of deterministic tree transducers was already studied in [**91**].

We concentrate on deterministic bottom-up tst (for short: bu-tst) and deterministic top-down tst (for short: td-tst). For such tst there is at most one successful computation (see [**41**, Proposition 3.12] and Proposition 5.1) for every input tree; *i. e.*, at most one computed output tree and its associated weight. Deterministic and total top-down tree transducers formalize a restricted class of functional programs [**75**]. Consequently, they and their generalizations (*e. g.*, macro tree transducers [**37, 26, 45**] and modular tree transducers [**46**]) were intensively studied (*e. g.*, [**90, 74, 61**] and references provided therein), in particular in the area of functional programming [**75, 76, 67, 81, 71, 109, 108**]. Deterministic top-down tree transducers are also applied in syntax-directed semantics [**68, 38, 57**]. All these applications could potentially benefit from the additional information (the weight or the cost) that is attached to the output tree of a deterministic td-tst. In the functional programming application, we could count reduction steps using the weights, and this could enable us to study efficiency effects of constructions (as, *e. g.*, in [**108**]) in a uniform setting. Moreover, we can perform a multitude of statistical computations while processing the input tree. Such features are relevant in natural language and speech processing [**93, 65**], for example.

Clearly the classes of ε-t-ts and o-t-ts transformations computed by (restricted) deterministic bu-tst and (restricted) deterministic td-tst can be ordered by inclusion. For tree transducers (*i. e.*, polynomial bu-tst and td-tst over the boolean semiring) there exist several results relating classes of transformations computed by deterministic top-down and bottom-up tree transducers [**35**]. In [**56**] a HASSE diagram [**28**] shows the order relation between several classes of translations induced by deterministic top-down tree transducers.

Our goal is to convey the order relation between classes of η-t-ts transformations computed by restricted deterministic bottom-up and top-down tst over "most" commutative semirings by means of HASSE diagrams. In order to explain "most" in the previous sentence we need two simple concepts. A multiplicatively nonperiodic semiring is a semiring in which there exists an element such that all of its (multiplicative) powers are different. Moreover, a semiring has zero-divisors, if there exist nonzero elements whose product is zero. In fact, we cannot present a HASSE diagram for those commutative semirings that are multiplicatively nonperiodic and have zero-divisors. For all other commutative semirings we present a HASSE diagram. For our investigation we restrict ourselves to the properties of nondeletion, linearity, totality, and homomorphism [**41, 58**] and their combinations. The boolean property is not considered because boolean deterministic bu-tst and td-tst compute essentially the same class of transformations as deterministic bottom-up and top-down tree transducers (see [**41**, Theorem 4.6] and [**58**, Theorem 5.8], where in both results the additive idempotency is not needed for the results on deterministic devices), respectively.

For the discussion of the main results of this investigation we need the residue semirings $\mathbb{Z}_1$ and $\mathbb{Z}_2$. Generally, for every natural number n the residue semiring $\mathbb{Z}_n$ is $\mathbb{Z}_n = (\{0, \ldots, n-1\}, +, \cdot)$ with the usual operations of addition and multiplication modulo n. The main results of this investigation are presented in the HASSE diagrams contained in Chapter 5 (see Theorems 5.5, 5.19, 5.28, and 5.32). Specifically, we conclude that:

- the semirings $\mathbb{Z}_1$, $\mathbb{Z}_2$, and $\mathbb{B}$ are (up to isomorphism) the only semirings $\mathcal{A}$ such that $x\text{–BOT}_\varepsilon(\mathcal{A}) = x\text{–BOT}_\text{o}(\mathcal{A})$ holds for every combination x of restrictions (see Corollary 5.17); and
- only in multiplicatively idempotent semirings $\mathcal{A}$, which are semirings in which the square of every element coincides with the element itself, the equality $\text{hn–BOT}_\varepsilon(\mathcal{A}) = \text{hn–BOT}_\text{o}(\mathcal{A})$ holds, where hn abbreviates homomorphism and nondeletion (see Corollary 5.31).

Let us discuss the first item in some detail. It is rather clear that, for $\mathbb{Z}_1$, $\mathbb{Z}_2$, and $\mathbb{B}$, pure and o-substitution coincide, and for all other semirings $\mathcal{A} = (A, +, \cdot)$ with additive neutral element 0 and multiplicative neutral element 1 there is at least one element a different from both 0 and 1. Consider an output tree weighted a and another one weighted 1. The property, which separates pure and o-substitution in this case, is that pure substitution may tell those two different output trees apart even when deleting them. This is due to the fact that, when we use pure substitution, the weight of the deleted output tree is still accounted for, which is not the case for o-substitution.

Considering the second item, it is again straightforward to observe the equality, because $a^n = a$ for all elements a of the multiplicatively idempotent semiring and $n \geqslant 1$. In a semiring, that is not multiplicatively idempotent, the property $a \neq a^2$ can be used to separate pure and o-substitution with the help of a copying (*i. e.*, nonlinear) homomorphism bu-tst. For this, imagine an output tree with weight a. If this output is used in a transition that copies it, then pure substitution records a just once while o-substitution records a twice.

A multiplicatively periodic semiring is one in which, for every element a, all powers of a form a finite set. In the following let us consider combinations x of properties which do not contain the homomorphism property. Moreover, let us consider only commutative semirings $\mathcal{A}$. It turns out that:

- x–$\mathrm{BOT}_\varepsilon(\mathcal{A}) = x$–$\mathrm{BOT}_\mathrm{o}(\mathcal{A})$ if $\mathcal{A}$ is multiplicatively periodic (see Corollary 5.23 and Lemma 5.24);
- h–$\mathrm{BOT}_\mathrm{o}(\mathcal{A}) =$ h–$\mathrm{TOP}_\varepsilon(\mathcal{A})$ if and only if $\mathcal{A}$ is zero-divisor free (see Proposition 5.13);
- hl–$\mathrm{BOT}_\varepsilon(\mathcal{A}) \nsubseteq$ d–$\mathrm{BOT}_\mathrm{o}(\mathcal{A})$ and hl–$\mathrm{BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq$ d–$\mathrm{BOT}_\varepsilon(\mathcal{A})$ if $\mathcal{A}$ is multiplicatively nonperiodic (see Lemma 5.18); and
- hn–$\mathrm{BOT}_\varepsilon(\mathcal{A}) \nsubseteq$ h–$\mathrm{TOP}_\varepsilon(\mathcal{A})$ and hn–$\mathrm{BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq$ h–$\mathrm{BOT}_\varepsilon(\mathcal{A})$ if $\mathcal{A}$ is multiplicatively non-idempotent (see Lemma 5.27).

The first result builds on the properties of periodicity and commutativity, of which the former allows us to keep track of the weights in the states (because there are only finitely many different powers of any element), and the latter allows us to reorder the factors. Thus, we can keep the current weight in the state and apply the weight only in the very last step (to the top-most output). This result essentially shows that the states can perform the duty of the weight computation in this setting. The result does not hold for x containing the homomorphism property because of the additional states required for the book-keeping.

The second result strengthens [**58**, Theorem 5.12]. There sufficiency of the above statement is shown. Necessity is shown by an exploit of the different deletion behavior of bu-tst and td-tst. A bu-tst deletes output trees (or better yet: tree series), whereas a td-tst deletes input trees. This means that a td-tst never inspects the deleted input subtree, whereas a bu-tst first translates the input tree and then deletes the computed output. Now let us imagine that the translation of the input subtree fails (the input subtree is transformed into the zero tree series $\widetilde{0}$; *i. e.*, there is no output tree into which the input tree can be translated). The td-tst is not affected by this failure, because it does not inspect the offending input subtree. The bu-tst, however, also fails to translate the whole input tree, because the failure is propagated (see Observation 4.9). This feature is called "checking followed by deletion" [**35**, Section 4.3] and is not implementable in td-tst. A homomorphism bu-tst or td-tst can only reject input trees (*i. e.*, offer no translation for them), if the semiring has a zero-divisor. We use exactly this property to separate the class of o-t-ts transformations computed by homomorphism bu-tst from the class of ε-t-ts transformations computed by homomorphism td-tst in the second result.

We observe in Lemma 5.18 that, for a given input tree, pure and o-substitution can realize different powers of a as the weight of an output tree. Pure substitution takes the weights of deleted subtrees into account, so that with respect to deletion pure substitution can realize larger exponents of a. However, bu-tst using o-substitution enjoy the property that they can reset the computed weight to a predetermined weight with the help of deletion. Essentially, these two properties separate the classes in the third item. Actually, the statement in the third item also holds for the restriction hn instead of hl (see Lemma 5.18). This strategy reappears in Chapter 6, where we essentially also use the powers of an element a to separate classes of η-t-ts transformations.

Finally, for every semiring $\mathcal{A}$ we have $x\text{–BOT}_\varepsilon(\mathcal{A}) = x\text{–BOT}_\text{o}(\mathcal{A})$, if both the nondeletion and linearity restriction are present in x (see Theorem 5.5 of [**58**] and Proposition 4.21). Several small results complement the presented results; all of them are required to show the validity of the presented Hasse diagrams.

In summary, we compare the transformational power of deterministic bu-tst and td-tst in this chapter. Moreover, we also study the effect of the two types of substitution (pure and o-substitution). The investigation is detailed in the sense that we not only consider the classes of η-t-ts transformations computed by deterministic bu-tst and td-tst, but also the classes of η-t-ts transformations computed by restricted deterministic bu-tst and td-tst where the restriction is any combination of nondeletion, linearity, totality, and homomorphism. Moreover, we present results for "most" commutative semirings.

6. Polynomial tree series transducers

In Chapter 6 we continue the investigations of Chapter 5. However, we consider polynomial (not necessarily deterministic) bu-tst and td-tst. Since equality and inclusion results already exist in sizable number [**41, 58**], we concentrate on incomparability results.

Nondeterministic tst are interesting, because they can capture several (combinatorial) possibilities that yield the same final output tree. For example, in natural language processing the parse tree of an input sentence is usually not uniquely determined because natural language is inherently ambiguous. Thus the output consists of several parse trees each annotated with its likelihood. The author believes that nondeterministic tst provide a suitable formal model in this scenario [**65**].

In [**58**, Section 5] several classes of the form x–$\mathrm{BOT}_\eta(\mathcal{A})$ and the form x–$\mathrm{TOP}_\eta(\mathcal{A})$ have been compared with respect to inclusion. Generally speaking, [**58**] introduces o-substitution and investigates the relation to pure substitution. For instance, it is proved there that:

- x–$\mathrm{TOP}_\varepsilon(\mathcal{A}) = x$–$\mathrm{TOP}_\mathrm{o}(\mathcal{A})$ for every $x \in \{\mathrm{p}, \mathrm{d}, \mathrm{h}\}$ (see [**58**, Theorem 5.2]);
- p–$\mathrm{BOT}_\varepsilon(\mathbb{N}_\infty) \bowtie$ p–$\mathrm{BOT}_\mathrm{o}(\mathbb{N}_\infty)$ where the semiring $\mathbb{N}_\infty$ of nonnegative integers (with infinity) is $(\mathbb{N} \cup \{\infty\}, +, \cdot)$, and $\bowtie$ denotes incomparability with respect to inclusion (see [**58**, Corollary 5.18]); and
- p–$\mathrm{BOT}_\varepsilon(\mathbb{T}) \bowtie$ p–$\mathrm{BOT}_\mathrm{o}(\mathbb{T})$ where $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +)$ is the tropical semiring on the natural numbers (see [**58**, Corollary 5.23]).

The latter two incomparability results motivate us to investigate the question whether this incomparability also holds for semirings different from $\mathbb{N}_\infty$ and $\mathbb{T}$. In Chapter 6 we answer this question in the affirmative. Additionally, we compare classes of ε-t-ts transformations that are computed by different types of tst; *i. e.*, bu-tst and td-tst. Our main result here is Theorem 6.30, which states that p–$\mathrm{BOT}_\varepsilon(\mathcal{A}) \bowtie$ p–$\mathrm{BOT}_\mathrm{o}(\mathcal{A})$ and p–$\mathrm{BOT}_\varepsilon(\mathcal{A}) \bowtie$ p–$\mathrm{TOP}_\varepsilon(\mathcal{A})$ in every weakly growing and additively idempotent semiring $\mathcal{A}$.

Let us add some details and then briefly discuss the way how to prove this theorem. A partially ordered semiring $\mathcal{A} = (A, +, \cdot)$ is a semiring equipped with a partial order $\leqslant$ on A such that the order is preserved by both semiring operations. A semiring that is partially ordered by $\leqslant$ is called weakly growing, if there exists an element a such that:

(1) $a^i < a^j$ for all nonnegative integers $i < j$; and
(2) for every $a_1, a_2, b \in A \setminus \{0\}$ and $d \in A$ and $n \in \mathbb{N}$, if we have $a^n = a_1 \cdot b \cdot a_2 + d$, then there exists an $m \in \mathbb{N}$ such that $b \leqslant a^m$.

Roughly speaking, Condition (2) requires that every element b that occurs in a decomposition of a power of a can be bounded (from above) by another power of a. In particular, the following semirings are weakly growing:

- $\mathbb{N}_\infty$ with the total order $\leqslant$, $a = 2$, and $m = n$;

- $\mathbb{T}$ with the total order $\leqslant$, $a = 1$, and $m = \max(n, d)$;
- the arctic semiring $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +)$ with the total order $\leqslant$, $a = 1$, and $m = n$; and
- the formal language semiring $(\mathfrak{P}(S^*), \cup, \circ)$ over the nonempty and finite set S with concatenation $\circ$ of languages [**111**], partial order $\subseteq$, $a = \{\varepsilon, s\}$ for some $s \in S$, and $m = n$.

In order to prove the non-inclusion results of the main theorem, we use the partial order on the semiring and establish a framework of mappings called coefficient majorizations. For a given η-t-ts transformation $\tau \colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta \rangle\rangle$, a coefficient majorization $f \colon \mathbb{N} \longrightarrow A$ is a mapping such that $f(n)$ is an upper bound of the set $\mathcal{C}^\eta_\tau(n)$, which is the set of all nonzero coefficients of output trees generated from input trees of height at most n (in the formal development in Chapter 6 a coefficient majorization is defined with respect to a tst M). Given two classes $\mathfrak{T}_1$ and $\mathfrak{T}_2$ of transformations, we can prove $\mathfrak{T}_1 \not\subseteq \mathfrak{T}_2$ by exhibiting (i) a mapping f that is a coefficient majorization for the class $\mathfrak{T}_2$ (*i. e.*, a coefficient majorization for every $\tau \in \mathfrak{T}_2$) and (ii) a transformation $\tau \in \mathfrak{T}_1$ for which f is no coefficient majorization. For particular classes, this is achieved in Lemma 6.29.

The idea of coefficient majorizations is not new. Coefficient majorizations have been investigated for the specific case in which the coefficient is the height or size of the output tree, *e. g.*, for top-down tree transducers (see [**57**, Lemma 3.27]), for attributed tree transducers (see [**38**, Lemma 3.3] and [**57**, Lemma 5.40]), for macro tree transducers (see [**38**, Lemma 3.3] and [**57**, Lemma 4.22]), and for bottom-up tree transducers (which have the same coefficient majorization as top-down tree transducers, which follows in a straightforward manner from the decomposition in [**35**, Theorem 3.15]). As a byproduct of our investigation we obtain coefficient majorizations for polynomial bu-tst (see Theorem 6.8) and polynomial tst (see Theorem 6.15) over partially ordered semirings.

7. Composition of tree series transducers

In the final chapter, we consider compositions of η-t-ts transformations. Such compositions arise naturally, because the strategy of breaking down a transformation into several stages is well-known and well-established in software development, for example [**6, 99**]. The implementation of a single stage is often easier to understand and easier to validate (and thus less prone to errors). The final result is then obtained by running the stages one after the other where the output of one stage becomes the input of the next stage. This strategy introduces an overhead of communication between the stages (called intermediate result [**99**]), which might deteriorate efficiency. Strategies that avoid this intermediate result exist in abundance [**99, 76, 7, 24, 97**]. In those approaches the transformations are composed by composing the specifications of the stages at compile time. The external communication overhead is avoided because the system now uses internal communication; however, this need not yield an efficiency gain [**108**].

In our setting, the specification of the stage is a tst with some properties (*e. g.*, deterministic or linear). Let us explain the scenario of natural language processing [**93, 112**] in some more detail. A tree bank is a collection of parse trees (of natural language sentences) each annotated with a weight (usually the relative frequency). When we translate a natural language sentence from one language into another, we first parse the original sentence in order to obtain a parse tree. Since natural language is usually highly ambiguous we obtain a collection of parse trees each annotated with a probability. The probability is derived from the evidence found in the tree bank. The transformation stage translates the annotated parse trees into parse trees of the output language. Again there may be more than one possible translation for one parse tree, so that for each input parse tree we again obtain a collection of annotated output parse trees. A tree bank containing parse trees of sentences in the output languages delivers the coefficients required to compute the probability.

Such collections of annotated parse trees are formal tree series. The translation stage can thus be seen as a transformation which transforms tree series into tree series and tst are finite-state devices that compute such transformations.

The complexity of the transformations involved in the translation stage is usually high (automata requiring several million states), so that modularity is of utmost importance. One designs small transducers that only deal with one phenomenon at a time and then composes the transformations (*i. e.*, uses the output of the first transformation as the input of a second transformation) to obtain the final result. However, this approach is usually inefficient because many intermediate results are computed. By composing the transducers we can avoid these intermediate results. Moreover, the analysis of a single transducer is usually simpler than the analysis of a series of transducers. An important problem in natural language processing is to find the most likely path (*i. e.*, the path with the highest probability) that yields a given parse tree. This problem is very difficult for compositions of transformations, so that the composition of the transducers helps to reduce the complexity.

We call the class of all tree series transformations, that are computable by bu-tst (respectively, td-tst), simply bottom-up (respectively, top-down) tree series transformations. In the same manner we deal with other restrictions. In the unweighted case, bottom-up tree transformations are closed under left-composition with linear bottom-up tree transformations [**35**, Theorem 4.5] and right-composition with deterministic bottom-up tree transformations [**35**, Theorem 4.6] (see also [**5**, Theorem 6]). In this chapter we try to extend these results to bottom-up tree series transformations. The first result was already generalized to bottom-up tree series transformations [**79, 41**]. Essentially the authors obtain that, for arbitrary commutative and complete semirings [**66**], bottom-up tree series transformations are closed under left-composition with nondeleting, linear bottom-up tree series transformations. We generalize this further by showing that the mentioned class of bottom-up tree series transformations is even closed under left-composition with linear bottom-up tree series transformations (see Theorem 7.13).

Roughly speaking, the construction that is required to show this statement is as follows. Let

$$M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu') \quad \text{and} \quad M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$$

be bu-tst over the commutative and complete [**66, 64, 58**] semiring $\mathcal{A}$. We construct a bu-tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ that computes the composition of the transformations computed by M' and M''. We set $Q = Q' \times Q''$. If we consider a transition that reads a k-ary symbol σ in the input, changes into the state (p, q), and supposes that the subtrees $t_1, \dots, t_k$ have respectively been processed in states $(p_1, q_1), \dots, (p_k, q_k)$, then we first consult the tree representation entry $\mu'_k(\sigma)_{p,p_1\cdots p_k}$, which represents a transition of M'. Each output tree present in this entry is processed using the tree representation μ'' such that the computation (of M'') ends in state q. Such an output tree may contain variables from $\{\mathrm{z}_1, \dots, \mathrm{z}_k\}$. At a variable z_i we start the computation of M'' in state q_i. The such processed output trees constitute the entry $\mu_k(\sigma)_{(p,q),(p_1,q_1)\cdots(p_k,q_k)}$. It shows however that some preprocessing of M'' is necessary, otherwise the construction may return a tst that does not compute the composition of the transformations computed by M' and M''.

For the next result, the stated construction works without modification. Let $\mathcal{A}$ be a commutative and continuous semiring. It is shown in [**41**, Corollary 5.5] that the class of bottom-up tree series transformations over $\mathcal{A}$ is closed under right-composition with boolean homomorphism bottom-up tree series transformations over $\mathcal{A}$. Using our construction, we also show that this class of bottom-up tree series transformations is actually closed under right-composition with boolean, deterministic bottom-up tree series transformations (see Theorem 7.18).

In the top-down case, we have that the class of top-down tree transformations is closed under right-composition with nondeleting, linear top-down tree transformations [**5**, Theorem 1]. Moreover, it is closed under left-composition with deterministic, total tree transformations [**106, 102**] (see also [**5**, Theorem 1]). These results were generalized for deterministic tst by [**41**, Theorem 5.18]. They showed that, for every commutative and complete semiring, the class of deterministic top-down tree series transformations is closed under right-composition with nondeleting, linear, and deterministic tree series transformations and under left-composition with boolean, deterministic, total tree series transformations. We present a generalization of the former statement and a statement similar to the latter. More precisely, we show that the class of top-down tree series transformations is closed under right-composition with nondeleting, linear top-down tree series transformations. Secondly, we show that the composition of a boolean, deterministic, total top-down tree series transformation with a linear top-down tree series transformation is a top-down tree series transformation.

CHAPTER 2

Preliminaries

Do not worry if you have built your castles in the air.
They are where they should be.
Now put the foundations under them.
Henry David Thoreau (1817-1862)

1. Sets, relations, and mappings

A naive treatment of set theory [**51**] is sufficient for our purposes. We assume that the reader is acquainted with knowledge of the empty set ($\emptyset$), the relations of membership ($\in$), subset ($\subseteq$), and strict subset ($\subset$), and the operations of union ($\cup$), intersection ($\cap$), cartesian product ($\times$), and set difference ($\setminus$).

From a given set A we can construct the set of all subsets of A, which is called the *power set of* A and is denoted by $\mathfrak{P}(A)$, and the set of all elements of A that fulfill a property p, which is called *set comprehension* and is written as $\{a \in A \mid p(a)\}$. Whenever it is obvious from which set A the element a is chosen, we simply write a instead of $a \in A$. An excellent introduction into set theory can be found in [**100**]; more historically inclined readers may consult [**50, 8**]. Classes are treated similarly [**69**].

Moreover, we assume that the reader is familiar with the notion of cardinality. The denotation $\operatorname{card}(A)$ is used for the cardinality of the set A. Sets with cardinality 1 are called *singletons.* The set of nonnegative integers is denoted by $\mathbb{N}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. The cardinality of $\mathbb{N}$ is $\aleph_0$ and any set that has a cardinality of at most $\aleph_0$ is called *countable.* A set whose cardinality is smaller than $\aleph_0$ is said to be *finite.* For every $k, n \in \mathbb{N}$ we use the shorthand $[k, n]$ for the set $\{i \in \mathbb{N} \mid k \leqslant i \leqslant n\}$ and $[n]$ for $[1, n]$.

Let A, B, and C be sets. A *relation ϱ from A to B* is a subset of $A \times B$. Instead of the cumbersome $(a, b) \in \varrho$ we usually write $a \mathrel{\varrho} b$. The relation ϱ^{-1} from B to A is then defined by $\{(b, a) \mid a \mathrel{\varrho} b\}$. Moreover, we write $\varrho(a)$ instead of $\{b \in B \mid a \mathrel{\varrho} b\}$ for every $a \in A$, and $\varrho(A')$ instead of $\bigcup_{a' \in A'} \varrho(a')$ for every $A' \subseteq A$. Given a relation ϱ_1 from A to B and a relation ϱ_2 from B to C, the *composition of ϱ_1 with ϱ_2*, denoted by $\varrho_1 \circ \varrho_2$, is defined by $\varrho_1 \circ \varrho_2 = \{(a, c) \mid \varrho_1(a) \cap \varrho_2^{-1}(c) \neq \emptyset\}$.

A *relation ϱ on A* is a subset of $A \times A$. In particular, the *identity relation* id_A on A is defined by $\mathrm{id}_A = \{(a, a) \mid a \in A\}$ for every set A. Let ϱ be a relation on A. We say that ϱ is:

- *reflexive,* if $\mathrm{id}_A \subseteq \varrho$;
- *symmetric,* if $\varrho \subseteq \varrho^{-1}$;
- *anti-symmetric,* if $\varrho \cap \varrho^{-1} \subseteq \mathrm{id}_A$; and
- *transitive,* if $\varrho \circ \varrho \subseteq \varrho$.

A reflexive, anti-symmetric, and transitive relation is also called a *partial order (relation),* and a reflexive, symmetric, and transitive relation is also called an *equivalence relation.* A thorough introduction into relations can be found, *e. g.*, in [**103**].

Let $\leqslant$ be a partial order on A. If $\leqslant \cup \leqslant^{-1} = A \times A$, then $\leqslant$ is called *total order.* Let $a, b \in A$. The elements a and b are said to be *incomparable (with respect to $\leqslant$),* if neither $a \leqslant b$ nor $b \leqslant a$. We write $a \bowtie b$ to denote incomparability of a and b. The *strict order* $<$ on A is derived from $\leqslant$ by setting $a < b$, if and only if $a \leqslant b$ and $a \neq b$. Moreover, the *covering relation* $\lessdot$ on A is defined by $a \lessdot b$, if (i) $a < b$ and (ii) $a \leqslant c < b$ implies that $a = c$ for every $c \in A$.

Let $B \subseteq A$. An element $a \in A$ is called an *upper bound* (with respect to $\leqslant$) of B, if $b \leqslant a$ for every $b \in B$. The set of all upper bounds of B is denoted by $\uparrow B$. If $\uparrow B$ has a *smallest element* (*i. e.*, an element $c \in \uparrow B$ such that $c \leqslant b$ for every $b \in \uparrow B$), then this element is called the *supremum of* B and denoted by $\sup B$. For more details on partial orders we refer the reader to [**28**].

Finite partial orders (*i. e.*, partial orders on finite sets) can be visualized by means of Hasse diagrams [**28**, p. 11]. The Hasse diagram of a partial order $\leqslant$ on A is the (directed, acyclic, and unlabeled) graph [**103**] $(A, \lessdot)$ with the set A of vertices and the set $\lessdot$ of edges; *i. e.*, for every $a, b \in A$ there is a directed edge from the vertex a to the vertex b, if and only if $a \lessdot b$. In pictorial expressions the vertices are displayed by naming the element of A, and the edges are drawn as line segments connecting vertices, where we assume that all edges are directed upwards (unless otherwise indicated by an arrow) and a line segment is only supposed to intersect with a vertex, if the vertex is either its starting or ending point.

A *partial function f (from A to B),* denoted by $f\colon A \dashrightarrow B$, is a relation from A to B such that $\mathrm{card}(f(a)) \leqslant 1$ for every $a \in A$. The set $\mathrm{ran}(f) \subseteq B$, called *range of* f, is given by $\mathrm{ran}(f) = f(A)$, and the set $\mathrm{dom}(f) \subseteq A$, called *domain of* f, is given by $\mathrm{dom}(f) = f^{-1}(B)$. For partial functions f we write $f(a) = b$, whenever $b \in f(a)$. Given $f, g\colon A \dashrightarrow B$ and $a \in A$ we may write $f(a) = g(a)$ to express that either $f(a) = \emptyset = g(a)$ [*i. e.*, f and g are undefined on a] or $f(a) = b = g(a)$ for some $b \in B$ [*i. e.*, f and g are both defined on a and both return b]. We say that f is a *mapping,* denoted by $f\colon A \longrightarrow B$, whenever $\mathrm{dom}(f) = A$. For example, id_A is a mapping from A to A. Further details on partial functions and mappings can be found in [**105**, Section 1.3].

Let $k \in \mathbb{N}$ and A be a set. We use A^k to stand for the k-fold cartesian product of A with itself; *e. g.*, $A^3 = A \times A \times A$. Note that $A^0 = \{()\}$. We often write the empty tuple $()$ as ε. Finally, a *k-ary operation $*$ (on A)* is a mapping $*\colon A^k \longrightarrow A$. With binary (2-ary) operations we prefer the notation $a * b$ to $*(a, b)$ for every $a, b \in A$.

Let A and I and J be sets. An *(I-indexed) family f (over A)* is a mapping $f\colon I \longrightarrow A$. Such a family is also called an *I-vector (over A)*. The set of all mappings from I to A is denoted by A^I. Let $f \in A^I$. We occasionally abbreviate $f(i)$ with $i \in I$ to just f_i. When possible, we commonly write f as $(f_i)_{i\in I}$. Let $(A_i)_{i\in I}$ be a family over $\mathfrak{P}(A)$. We call $(A_i)_{i\in I}$ a *partition of* A, if $\bigcup_{i\in I} A_i = A$ and $A_i \cap A_j = \emptyset$ for every $i, j \in I$ with $i \neq j$. Note that in comparison with definitions found in standard textbooks (*e. g.*, [**23**, Definition 4.10]), we do not require that $A_i \neq \emptyset$ for every $i \in I$. Finally, an *$(I \times J)$-matrix M (over A)* is a mapping $M\colon I \times J \longrightarrow A$. The element $M(i, j)$, usually written $M_{i,j}$, is called the *(i, j)-entry* of M.

2. Words and trees

This section draws heavily on [**60, 61**]. Finite, nonempty sets are also called *alphabets* and their elements are called *symbols.* Let A be a set. The set of all *words (over A),* denoted by A^*, is $\bigcup_{n\in\mathbb{N}} A^n$. Note that $\emptyset^* = \{\varepsilon\}$. We usually write $(a_1, \ldots, a_n)$ as $a_1 \cdots a_n$ and define the operation of *concatenation* (on words) as follows. Let $k, n \in \mathbb{N}$ and $a_i, b_j \in A$ for every $i \in [n]$ and $j \in [k]$. The concatenation of $v = a_1 \cdots a_n$ and $w = b_1 \cdots b_k$, denoted by vw, is $a_1 \cdots a_n b_1 \cdots b_k$. We denote by $|w|$ the *length* of $w \in A^*$ (*i. e.*, the unique $n \in \mathbb{N}$ such that $w \in A^n$). Moreover, we use $|w|_a$ to denote the number of occurrences of an $a \in A$ in w. Finally, given $w \in A^n$ and $i \in [n]$ we write w_i to refer to the i-th symbol in w.

Let Σ be a set and $(\Sigma_k)_{k\in\mathbb{N}}$ be a partition of Σ. Then $(\Sigma_k)_{k\in\mathbb{N}}$ is called *ranked set,* though we usually just say that Σ is a ranked set and implicitly assume the partition. A *ranked alphabet* is a ranked set Σ in which Σ is an alphabet. We use mx_Σ to denote the maximal rank of the symbols in the ranked alphabet Σ; *i. e.*, $\mathrm{mx}_\Sigma = \max\{k \in \mathbb{N} \mid \Sigma_k \neq \emptyset\}$.

In the sequel we often specify a ranked alphabet by a set of symbols with their rank put in parentheses as superscript as in $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Let Σ be a ranked set and V be a set. The set of *Σ-trees (indexed by V),* denoted by $T_\Sigma(V)$, is the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T$ also $\sigma(t_1, \ldots, t_k) \in T$. For $T_\Sigma(\emptyset)$ we simply write T_Σ. We generally assume that Σ and V are disjoint, and thus we may write α instead of $\alpha()$ for every $\alpha \in \Sigma_0$. Moreover, for any ranked set Σ that we consider we presuppose that $\Sigma_0 \neq \emptyset$. Thus, also $T_\Sigma \neq \emptyset$. Finally, for every $n \in \mathbb{N}$, $t \in T_\Sigma(V)$, and $\gamma \in \Sigma_1$ we occasionally abbreviate $\underbrace{\gamma(\cdots(\gamma(}_{n \text{ times } \gamma}t))\cdots)$ to just $\gamma^n(t)$.

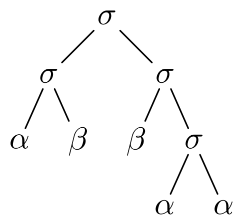

FIGURE 1. Graphical representation of $\sigma\big(\sigma(\alpha,\beta),\sigma(\beta,\sigma(\alpha,\alpha))\big)$, which is a Σ-tree for the ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$.

In universal algebra [**110**], a tree is usually called a *term* [**23**, Definition 10.1]; we use "tree" because it is an established notion in formal language theory [**61**]. Our pictorial representation of trees is straightforward and an example is presented in Figure 1.

For every $t \in T_\Sigma(V)$ we define the following mappings [**16**]:

$$\begin{aligned} \mathrm{pos}\colon T_\Sigma(V) &\longrightarrow \mathfrak{P}(\mathbb{N}_+^*) & \mathrm{size}\colon T_\Sigma(V) &\longrightarrow \mathbb{N}_+ \\ \mathrm{height}\colon T_\Sigma(V) &\longrightarrow \mathbb{N} & \mathrm{sub}\colon T_\Sigma(V) &\longrightarrow \mathfrak{P}(T_\Sigma(V)) \\ \mathrm{lab}_t\colon \mathrm{pos}(t) &\longrightarrow \Sigma \cup V \end{aligned}$$

For every $v \in V$

$$\begin{aligned} \mathrm{pos}(v) &= \{\varepsilon\} & \mathrm{size}(v) &= 1 & \mathrm{height}(v) &= 0 \\ \mathrm{sub}(v) &= \{v\} & \mathrm{lab}_v(\varepsilon) &= v \end{aligned}$$

and for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma(V)$

$$\begin{aligned} \mathrm{pos}\big(\sigma(t_1,\ldots,t_k)\big) &= \{\varepsilon\} \cup \{iw \mid i \in [k], w \in \mathrm{pos}(t_i)\} \\ \mathrm{size}\big(\sigma(t_1,\ldots,t_k)\big) &= 1 + \sum_{i \in [k]} \mathrm{size}(t_i) \\ \mathrm{height}\big(\sigma(t_1,\ldots,t_k)\big) &= \max\{1 + \mathrm{height}(t_i) \mid i \in [k]\} \\ \mathrm{sub}\big(\sigma(t_1,\ldots,t_k)\big) &= \{\sigma(t_1,\ldots,t_k)\} \cup \bigcup_{i \in [k]} \mathrm{sub}(t_i) \end{aligned}$$

and for every $p \in \mathrm{pos}(\sigma(t_1,\ldots,t_k))$

$$\mathrm{lab}_{\sigma(t_1,\ldots,t_k)}(p) = \begin{cases} \sigma & \text{if } p = \varepsilon, \\ \mathrm{lab}_{t_i}(p') & \text{if } p = ip' \text{ with } i \in [k], p' \in \mathrm{pos}(t_i). \end{cases}$$

We note that for every $\alpha \in \Sigma_0$

$$\mathrm{pos}(\alpha) = \{\varepsilon\} \;\; \mathrm{size}(\alpha) = 1 \;\; \mathrm{height}(\alpha) = 0 \;\; \mathrm{sub}(\alpha) = \{\alpha\} \;\; \mathrm{lab}_\alpha(\varepsilon) = \alpha \;\; .$$

A subset of $T_\Sigma(V)$ is also called a *tree language.* Let $t \in T_\Sigma(V)$ and $s \in \Sigma \cup V$ and $V' \subseteq V$. We denote by $|t|_s$ the number of occurrences of s in t; *i. e.*,

$$|t|_s = \mathrm{card}(\{p \in \mathrm{pos}(t) \mid \mathrm{lab}_t(p) = s\}) \;\; .$$

Moreover, we say that t is *linear* (respectively, *nondeleting) in* V', if $|t|_v \leqslant 1$ (respectively, $|t|_v \geqslant 1$) for every $v \in V'$. The set of Σ-trees that are linear and nondeleting in V is denoted by $C_\Sigma(V)$. Let $L \subseteq T_\Sigma(V)$. We say that L is *linear* (respectively, *nondeleting) in* V', if every $t \in L$ is linear (respectively, nondeleting) in V'.

In the sequel we use the sets $\mathrm{X} = \{\mathrm{x}_i \mid i \in \mathbb{N}_+\}$ and $\mathrm{Z} = \{\mathrm{z}_i \mid i \in \mathbb{N}_+\}$ of (formal) variables. Moreover, we use $\mathrm{X}_k = \{\mathrm{x}_i \mid i \in [k]\}$ and $\mathrm{Z}_k = \{\mathrm{z}_i \mid i \in [k]\}$ for every $k \in \mathbb{N}$ (note that $\mathrm{X}_0 = \mathrm{Z}_0 = \emptyset$). The elements of X and Z are used as variables in trees, however such that for all considered trees $t \in T_\Sigma(\mathrm{X} \cup \mathrm{Z})$ it holds that $t \in T_\Sigma(\mathrm{X})$ or $t \in T_\Sigma(\mathrm{Z})$; *i. e.*, we do not mix variables from X and Z.

The following notions and notations deal with variables and are defined for $t \in T_\Sigma(\mathrm{X})$ and $t \in T_\Sigma(\mathrm{Z})$. Since $T_\Sigma(\mathrm{X}) \cap T_\Sigma(\mathrm{Z}) = T_\Sigma$, this should not lead to confusion. Let (i) $V = \mathrm{X}$ and $v = \mathrm{x}$ or (ii) $V = \mathrm{Z}$ and $v = \mathrm{z}$. The set $\mathrm{var}(t)$ is defined by $\mathrm{var}(t) = \{i \in \mathbb{N}_+ \mid 1 \leqslant |t|_{v_i}\}$ for every $t \in T_\Sigma(V)$. Moreover, for every $L \subseteq T_\Sigma(V)$ we let $\mathrm{var}(L)$ stand for $\bigcup_{t \in L} \mathrm{var}(t)$. Tree languages $L_1, L_2 \subseteq T_\Sigma(V)$ are called *variable-disjoint,* if $\mathrm{var}(L_1) \cap \mathrm{var}(L_2) = \emptyset$. We use $V_I = \{v_i \mid i \in I\}$ for every $I \subseteq \mathbb{N}_+$. Let $I \subseteq \mathbb{N}_+$ be finite, $t \in T_\Sigma(V)$, and $u_i \in T_\Sigma(V)$ for every $i \in I$. We denote by $t[u_i]_{i \in I}$ the result obtained from t by replacing, for every $j \in I$, each occurrence of v_j by u_j; *i. e.*:

- $v_j[u_i]_{i \in I} = u_j$ for every $j \in I$;
- $v_j[u_i]_{i \in I} = v_j$ for every $j \in \mathbb{N}_+ \setminus I$; and
- $\sigma(t_1, \ldots, t_k)[u_i]_{i \in I} = \sigma(t_1[u_i]_{i \in I}, \ldots, t_k[u_i]_{i \in I})$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma(V)$.

For every $n \in \mathbb{N}$ we write $t[u_1, \ldots, u_n]$ instead of $t[u_i]_{i \in [n]}$. Substitution is generalized to tree languages as follows:

$$L[L_i]_{i \in I} = \{t[u_i]_{i \in I} \mid t \in L, (\forall i \in I) \colon u_i \in L_i\}$$

for every finite $I \subseteq \mathbb{N}_+$, $L \subseteq T_\Sigma(V)$, and family $(L_i)_{i \in I} \in \mathfrak{P}(T_\Sigma(V))^I$. This notion of substitution is called *IO substitution* [**43, 44**].

Let Σ be a ranked set. A Σ-*algebra* is a pair $(A, (f_\sigma)_{\sigma \in \Sigma})$ where A is an arbitrary set, which is called the *carrier,* and $f_\sigma \colon A^k \longrightarrow A$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$. Let $\mathcal{A} = (A, (f_\sigma)_{\sigma \in \Sigma})$ and $\mathcal{B} = (B, (g_\sigma)_{\sigma \in \Sigma})$ be two Σ-algebras. A *homomorphism (of Σ-algebras) from $\mathcal{A}$ to $\mathcal{B}$* is a mapping $h \colon A \longrightarrow B$ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $a_1, \ldots, a_k \in A$

$$h\big(f_\sigma(a_1, \ldots, a_k)\big) = g_\sigma\big(h(a_1), \ldots, h(a_k)\big) \ .$$

We express that h is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ by $h \colon \mathcal{A} \longrightarrow \mathcal{B}$. A homomorphism $h \colon \mathcal{A} \longrightarrow \mathcal{B}$ such that $h^{-1} \colon B \longrightarrow A$ is called an *isomorphism* and the algebras $\mathcal{A}$ and $\mathcal{B}$ are called *isomorphic.* We denote this fact by $\mathcal{A} \cong \mathcal{B}$.

Clearly, T_Σ can easily be turned into a Σ-algebra $(T_\Sigma, (\mathrm{top}_\sigma)_{\sigma\in\Sigma})$ where for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$

$$\mathrm{top}_\sigma(t_1, \ldots, t_k) = \sigma(t_1, \ldots, t_k) \ .$$

The Σ-algebra $(T_\Sigma, (\mathrm{top}_\sigma)_{\sigma\in\Sigma})$ is called the *Σ-term algebra* and denoted by $\mathcal{T}_\Sigma$. It is known that $\mathcal{T}_\Sigma$ is the *initial* Σ-algebra; *i. e.*, for every Σ-algebra $\mathcal{A} = (A, (f_\sigma)_{\sigma\in\Sigma})$ there exists a unique homomorphism from $\mathcal{T}_\Sigma$ to $\mathcal{A}$ [**23**, Lemma 10.6].

Let Σ be a ranked alphabet. The set of *fully balanced trees (over Σ)* [or *mirror trees*] is the smallest set $T \subseteq T_\Sigma$ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t \in T$ also

$$\sigma(\underbrace{t, \ldots, t}_{k \text{ times}}) \in T \ .$$

Consequently, α is a fully balanced tree for every $a \in \Sigma_0$.

3. Tree automata and tree transducers

Let us recall the notions of bottom-up and top-down tree transducers [**102, 106, 107**] from [**36**]. Let Σ and Δ be ranked alphabets and $Q(V) \subseteq (Q \cup V \cup \{(,)\})^*$ be given by

$$Q(V) = \{q(v) \mid q \in Q, v \in V\}$$

for all sets Q and V. Let $n \in \mathbb{N}$, $q \in Q$, $t \in T_\Sigma(\mathrm{X})$ a tree, and $t_1, \ldots, t_n \in T_\Sigma(\mathrm{X})$. Then $q(t)[t_1, \ldots, t_n] = q(t[t_1, \ldots, t_n])$; *i. e.*, we treat $q(t)$ as the tree that is build from the unary symbol q and t. Analogously, substitution for variables of Z shall be defined.

A *bottom-up tree transducer* [**107**] *(over Σ and Δ)* is a tuple

$$(Q, \Sigma, \Delta, F, R)$$

where:

- Q is a finite set of *states;*
- $F \subseteq Q$ is a set of *final states;* and
- R is a finite set of *transitions* (or *rules*) of the form:

$$\sigma(q_1(\mathrm{z}_1), \ldots, q_k(\mathrm{z}_k)) \to q(u)$$

  where $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q, q_1, \ldots, q_k \in Q$, and $u \in T_\Delta(\mathrm{Z}_k)$.

Let $M = (Q, \Sigma, \Delta, F, R)$ be a bottom-up tree transducer. We follow the presentation of [**36**] and present the *rewrite semantics.* The relation $\Rightarrow_M$ on $T_\Sigma(Q(T_\Delta))$ is defined for every $s, s' \in T_\Sigma(Q(T_\Delta))$ by $s \Rightarrow_M s'$ if and only if:

- there exist $C \in T_\Sigma(\mathrm{X}_1 \cup Q(T_\Delta))$, that is nondeleting and linear in X_1, and $t \in T_\Sigma(Q(T_\Delta))$ such that $C[t] = s$;
- there exist $u_1, \ldots, u_k \in T_\Delta$ and $(l \to r) \in R$ such that $l[u_1, \ldots, u_k] = t$; and
- $s' = C[r[u_1, \ldots, u_k]\,]$.

Let $\Rightarrow_M^*$ denote the reflexive, transitive closure [**110**] of $\Rightarrow_M$. The *tree transformation computed by* M is denoted by τ_M and defined by

$$\tau_M = \{(t, u) \in T_\Sigma \times T_\Delta \mid (\exists q \in F)\colon t \Rightarrow_M^* q(u)\} \ .$$

A (bottom-up) *tree automaton* $M = (Q, \Sigma, F, R)$ *[over* Σ*]* is a bottom-up tree transducer $M = (Q, \Sigma, \Sigma, F, R)$ whose rules are of the form

$$\sigma(q_1(\mathrm{z}_1), \ldots, q_k(\mathrm{z}_k)) \to q(\sigma(\mathrm{z}_1, \ldots, \mathrm{z}_k))$$

where $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q, q_1, \ldots, q_k \in Q$. The tree language *recognized by* M is denoted by $L(M)$ and defined by $L(M) = \tau_M(T_\Sigma)$. A tree language $L \subseteq T_\Sigma$ is called *recognizable*, if there exists a tree automaton M over Σ such that $L(M) = L$. The set of all recognizable tree languages over Σ is denoted by $\mathrm{RECOG}(\Sigma)$.

On the other hand, a *top-down tree transducer* [**102, 106**] *(over* Σ *and* Δ) is a tuple $M = (Q, \Sigma, \Delta, F, R)$ in which:

- Q is a finite set of *states;*
- $F \subseteq Q$ is a set of *initial states;* and
- R is a finite set of *transitions* (or *rules*) of the form:

$$q(\sigma(\mathrm{x}_1, \ldots, \mathrm{x}_k)) \to u$$

  where $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $u \in T_\Delta(Q(\mathrm{X}_k))$.

We define the relation $\Rightarrow_M$ on $T_\Delta(Q(T_\Sigma))$ for every $s, s' \in T_\Delta(Q(T_\Sigma))$ by $s \Rightarrow_M s'$ if and only if:

- there exist $C \in T_\Delta(\mathrm{Z}_1 \cup Q(T_\Sigma))$, that is nondeleting and linear in Z_1, and $t \in Q(T_\Sigma)$ such that $C[t] = s$;
- there exist trees $t_1, \ldots, t_k \in T_\Sigma$ and a rule $(l \to r) \in R$ such that $l[t_1, \ldots, t_k] = t$; and
- $s' = C[r[t_1, \ldots, t_k]\,]$.

By $\Rightarrow_M^*$ we denote the reflexive, transitive closure [**110**] of $\Rightarrow_M$. The *tree transformation computed by* M, denoted by τ_M, is defined by

$$\tau_M = \{(t, u) \in T_\Sigma \times T_\Delta \mid (\exists q \in F)\colon q(t) \Rightarrow_M^* u\} \ .$$

In Chapter 4 we present a generalization of bottom-up and top-down tree transducers, so we refrain from introducing a multitude of properties for tree transducers and refer the reader to Chapter 4. Excellent, detailed introductions into the theory of tree automata and tree transducers can be found in [**36, 35, 37, 60, 61**].

4. Monoids and semirings

Let A be a set and $\cdot\colon A^2 \longrightarrow A$. We say that $\cdot$:

- is *associative,* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in A$;
- is *commutative,* if $a \cdot b = b \cdot a$ for every $a, b \in A$;
- is *extremal,* if $a \cdot b \in \{a, b\}$ for every $a, b \in A$;
- admits a *neutral element,* if there exists an $e \in A$ such that $e \cdot a = a = a \cdot e$ for every $a \in A$; and

- admits an *absorbing element,* if there exists an $o \in A$ such that for every $a \in A$ we have $a \cdot o = o = o \cdot a$.

It is easy to see, that if $\cdot$ admits a neutral (respectively, absorbing) element, then this element is unique. We usually denote it by 1 (respectively, 0). For brevity we denote by A_+ the set A, if $\cdot$ does not admit an absorbing element, and the set $A \setminus \{0\}$, when $\cdot$ admits the absorbing element 0.

The pair $(A, \cdot)$ is termed a *monoid,* if $\cdot$ is associative and admits a neutral element 1. We usually abbreviate the n-fold product $a \cdot \ldots \cdot a$ to a^n and set $a^0 = 1$. Moreover, for every finite $I \subseteq \mathbb{N}_+$ we use $\prod_{i \in I} a_i$ as an abbreviation for $a_{i_1} \cdot \ldots \cdot a_{i_n}$ where $I = \{i_1, \ldots, i_n\}$ with $i_1 < \cdots < i_n$.

A *submonoid* of a monoid $\mathcal{A} = (A, \cdot_A)$ is a monoid $\mathcal{B} = (B, \cdot_B)$ such that $B \subseteq A$ and $a \cdot_B b = a \cdot_A b$ for every $a, b \in B$. Note that with this definition, the monoids $\mathcal{A}$ and $\mathcal{B}$ may have different neutral elements. Let $C \subseteq A$. The *closure of C (with respect to $\cdot_A$)* is the smallest set D such that $1 \in D$ and $C \subseteq D \subseteq A$ and $a \cdot_A b \in D$ for every $a, b \in D$. We denote the closure of C by $\langle C \rangle$. The submonoid generated by C is the monoid $(\langle C \rangle, \cdot_C)$ where $a \cdot_C b = a \cdot_A b$ for every $a, b \in \langle C \rangle$.

Let $\mathcal{A} = (A, \cdot)$ be a monoid with neutral element 1. An element $a \in A$ is called:

- *idempotent,* if $a^2 = a$;
- *periodic,* if there exist $i, j \in \mathbb{N}$ such that $i \neq j$ and $a^i = a^j$;
- *(left-) cancellative,* if $a \cdot b = a \cdot c$ implies $b = c$ for every $b, c \in A$; and
- *invertible,* if there exists a $b \in A$, called *the inverse of a* and denoted by a^{-1}, such that $a \cdot b = 1 = b \cdot a$.

It follows from the definitions that the neutral element 1 is invertible, cancellative, idempotent and hence periodic. Additionally, the inverse of an element is unique as the following observation shows.

OBSERVATION 2.1 (see [**2**, Chapter 1]). *Let $\mathcal{A} = (A, \cdot)$ be a monoid with neutral element 1. Moreover, let $a, b, c \in A$ be such that*

$$b \cdot a = 1 = a \cdot c \ .$$

Then $b = c$.

PROOF. We derive $b = b \cdot 1 = b \cdot (a \cdot c) = (b \cdot a) \cdot c = 1 \cdot c = c$. □

Should $\cdot$ admit an absorbing element 0, then it is idempotent and periodic, but neither cancellative nor invertible (unless $A_+ = \emptyset$). This motivates the following definition. The monoid $\mathcal{A}$ is called *idempotent* (respectively, *periodic, cancellative,* and *a group),* if every $a \in A_+$ is idempotent (respectively, periodic, cancellative, and invertible). Moreover, we say that $\mathcal{A}$ is *finite,* whenever A is finite. Finally, a *commutative* (respectively, *extremal)* monoid is a monoid $(A, \cdot)$ such that $\cdot$ is commutative (respectively, extremal). We note the following trivial interrelations between the aforementioned properties. Every finite monoid is periodic, every extremal monoid is idempotent, and every idempotent monoid is periodic.

Let $\mathcal{A} = (A, \cdot)$ be a monoid such that $\cdot$ admits the absorbing element 0. An element $a \in A_+$ is called a *(left) zero-divisor,* if there exists a $b \in A_+$ such that $a \cdot b = 0$. If $\mathcal{A}$ does not possess any zero-divisors, then $\mathcal{A}$ is called *zero-divisor free.*

OBSERVATION 2.2 (see [**64**, p. 54]). *Let $\mathcal{A} = (A, \cdot)$ be a monoid with an absorbing element* 0*, and let $a \in A$ be an invertible element. Then a is not a zero-divisor.*

PROOF. Assume the contrary; *i. e.*, there exists a $b \in A_+$ such that $a \cdot b = 0$. Then $b = 1 \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$, which is a contradiction. □

Finally, given $+\colon A^2 \longrightarrow A$ and $\cdot\colon A^2 \longrightarrow A$, the operation $\cdot$ *distributes over* $+$, if for every $a, b, c \in A$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (a + b) \cdot c = (a \cdot c) + (b \cdot c) \ .$$

For succinctness we agree on the following conventions. The (binding) priority of multiplicative operation symbols is assumed to be higher than the one of additive operation symbols. This means, *e. g.*, that $a \cdot b + a \cdot c$ is read as $(a \cdot b) + (a \cdot c)$. Occasionally, we omit multiplicative operation symbols altogether and just use juxtaposition (as in $ab + ac$).

A *semiring* [**66, 64**] *(with neutral one and absorbing zero)* is a tuple $(A, +, \cdot)$ that consists of a multiplicative monoid $(A, \cdot)$ and an additive monoid $(A, +)$ subject to the following restrictions:

- the neutral element of $(A, +)$ acts as the absorbing element with respect to $\cdot$;
- $(A, +)$ is commutative; and
- $\cdot$ distributes over $+$.

Now let us present some well-known semirings. Numerous examples of additional semirings can be found in [**66, 64**].

- the *boolean semiring* $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$ with disjunction $\vee$ and conjunction $\wedge$;
- the *natural numbers* $\mathbb{N} = (\mathbb{N}, +, \cdot)$ with addition and multiplication;
- the *natural numbers extended by infinity* $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot)$ with addition and multiplication extended such that ∞ is the absorbing element of $+$ and $n \cdot \infty = \infty = \infty \cdot n$ for every $n \in \mathbb{N}_+ \cup \{\infty\}$;
- the *nonnegative reals* $\mathbb{R}_+ = (R, +, \cdot)$ with $R = \{r \in \mathbb{R} \mid r \geqslant 0\}$ and the usual operations of addition and multiplication;
- the *tropical semiring (on the natural numbers)*

$$\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +)$$

TABLE 1. Properties of semirings.

	$\mathbb{B}$	$\mathbb{N}$	$\mathbb{N}_\infty$	$\mathbb{R}_+$	$\mathbb{T}$	$\mathbb{T}_{\mathrm{sf}}$	$\mathbb{A}$	$\mathbb{A}_\infty$	$\mathbb{Z}_n$
commutative	yes	yes	yes	yes	yes	yes	yes	yes	yes
cancellative	yes	yes	no	yes	yes	yes	yes	no	yes, if n prime no, otherwise
mult. idemp.	yes	no	no	no	no	no	no	no	no, if $n > 2$ yes, otherwise
mult. periodic	yes	no	no	no	no	no	no	no	yes
zero-div. free	yes	yes	yes	yes	yes	yes	yes	yes	yes, if n prime no, otherwise
semifield	yes	no	no	yes	no	yes	no	no	yes, if n prime no, otherwise
ring	no	no	no	no	no	no	no	no	yes
subtractive	no	yes	no	yes	no	no	no	no	yes
add. extremal	yes	no	no	no	yes	yes	yes	yes	no, if $n > 1$ yes, otherwise
add. idemp.	yes	no	no	no	yes	yes	yes	yes	no, if $n > 1$ yes, otherwise
zero-sum free	yes	yes	yes	yes	yes	yes	yes	yes	no, if $n > 1$ yes, otherwise
naturally ord.	yes	yes	yes	yes	yes	yes	yes	yes	no, if $n > 1$ yes, otherwise

with minimum and addition such that ∞ is the neutral element of min and the absorbing element of $+$;

- the *tropical semifield (on the integers)* $\mathbb{T}_{\mathrm{sf}} = (\mathbb{Z} \cup \{\infty\}, \min, +)$ with minimum and addition such that ∞ is the neutral element of min and the absorbing element of $+$;
- the *arctic semiring (on the natural numbers)*
$$\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +)$$
with maximum and addition such that $-\infty$ is the neutral element of max and the absorbing element of $+$;
- the *arctic semiring (on the natural numbers) extended by infinity*
$$\mathbb{A}_\infty = (\mathbb{N} \cup \{\infty, -\infty\}, \max, +)$$
with the operations of $\mathbb{A}$ extended to ∞ such that ∞ is the absorbing element of max and $n + \infty = \infty = \infty + n$ for every $n \in \mathbb{N} \cup \{\infty\}$; and
- for every $n \in \mathbb{N}_+$ the *residue semiring* $\mathbb{Z}_n = ([0, n-1], +, \cdot)$ with addition and multiplication modulo n.

Let $\mathcal{A} = (A, +, \cdot)$ be a semiring. We again use A_+ to denote the set $A \setminus \{0\}$ where 0 is the multiplicatively absorbing element. We say that $\mathcal{A}$ is:

- *(multiplicatively) commutative,* if $(A, \cdot)$ is commutative;
- *(multiplicatively) cancellative,* if $(A, \cdot)$ is cancellative;
- *multiplicatively idempotent,* if $(A, \cdot)$ is idempotent;
- *multiplicatively periodic,* if $(A, \cdot)$ is periodic;
- *zero-divisor free,* if $(A, \cdot)$ is zero-divisor free;
- *a semifield,* if $(A, \cdot)$ is a group;
- *a ring,* if every $a \in A$ is invertible in $(A, +)$;
- *subtractive,* if every $a \in A$ is cancellative in $(A, +)$;
- *additively extremal,* if $(A, +)$ is extremal;
- *additively idempotent,* if $(A, +)$ is idempotent;
- *zero-sum free,* if $a + b = 0$ implies that $a = 0 = b$ for every $a, b \in A$; and
- *naturally ordered,* if the condition $a + b + c = a$ implies that $a + b = a$ for every $a, b, c \in A$.

The properties of the introduced semirings are displayed in Table 1. The relation $\sqsubseteq$ on A is defined for every $a, b \in A$ by

$$a \sqsubseteq b \quad \iff \quad (\exists c \in A)\colon a + c = b \ .$$

If $\mathcal{A}$ is naturally ordered, then $\sqsubseteq$ is a partial order [**66**, Theorem III.1.8]. Clearly, additively idempotent semirings are naturally ordered [**66**, Corollary III.1.12], and naturally ordered semirings in turn are zero-sum free [**66**, Corollary III.1.11].

Let $\mathcal{A} = (A, +, \cdot)$ and $\mathcal{B} = (B, \oplus, \odot)$ be semirings. A mapping $f\colon A \longrightarrow B$ is a *homomorphism (of semirings) from $\mathcal{A}$ to $\mathcal{B}$*, denoted by $f\colon \mathcal{A} \longrightarrow \mathcal{B}$, if for every $a, b \in A$ we have $f(a + b) = f(a) \oplus f(b)$ and $f(a \cdot b) = f(a) \odot f(b)$.

Let $\mathcal{A} = (A, +, \cdot)$ be a semiring and $\sum_I\colon A^I \dashrightarrow A$ for every countable set I. Instead of the cumbersome $\sum_I (a_i)_{i \in I}$ with $(a_i)_{i \in I} \in A^I$ we generally write $\sum_{i \in I} a_i$. We say that the class $\sum = (\sum_I)_{I \text{ countable set}}$ constitutes an *infinitary summation* (for $\mathcal{A}$), if the following five axioms [**66**, Section IV.1] hold.

(U) $\sum_{i \in \{x\}} a_i = a_x$ for every $a_x \in A$;

(E) $\sum_{i \in \{x,y\}} a_i = a_x + a_y$ for every $x \neq y$ and $a_x, a_y \in A$.

(GP) For all countable sets I and J, every family $(a_i)_{i \in I} \in A^I$, and every partition $(I_j)_{j \in J}$ of I we have that if $\sum_{i \in I} a_i$ is defined, then

$$\sum_{i \in I} a_i = \sum_{j \in J} \Big(\sum_{i \in I_j} a_i \Big) \ .$$

($\mathrm{GP}'_{\mathrm{F}}$) For every finite set J, countable set I, family $(a_i)_{i \in I} \in A^I$, and partition $(I_j)_{j \in J}$ of I we have that if $\sum_{j \in J} (\sum_{i \in I_j} a_i)$ is defined, then

$$\sum_{i \in I} a_i = \sum_{j \in J} \Big(\sum_{i \in I_j} a_i \Big) \ .$$

(D) For all countable sets I, J and $(a_i)_{i\in I} \in A^I$ and $(b_j)_{j\in J} \in A^J$ we have that if $\sum_{i\in I} a_i$ and $\sum_{j\in J} b_j$ are defined, then

$$\sum_{(i,j)\in I\times J} (a_i \cdot b_j) = \Big(\sum_{i\in I} a_i\Big) \cdot \Big(\sum_{j\in J} b_j\Big) \ .$$

The foundational problems of this definition are discussed in [**66**, Remark IV.1.19].

The *infinitary summation induced by* $+$ is the class

$$\bigoplus = (\bigoplus_I)_{I \text{ countable set}}$$

where $\bigoplus_I \colon A^I \dashrightarrow A$ for every countable set I, and we define $\bigoplus_I$ for every family $(a_i)_{i\in I} \in A^I$ by

$$\bigoplus_{i\in I} a_i = \begin{cases} \sum_{i\in I, a_i\neq 0} a_i & \text{if } \{i\in I \mid a_i \neq 0\} \text{ is finite,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It can readily be checked that $\bigoplus$ fulfills axioms (U), (E), (GP), ($\mathrm{GP}'_{\mathrm{F}}$), and (D) [**66**, Theorem IV.1.14]. Thus for each semiring, there exists an infinitary summation. A semiring is called $\aleph_0$-*complete,* if there exists an infinitary summation $\sum = (\sum_I)_{I \text{ countable set}}$ with $\sum_I \colon A^I \longrightarrow A$ for every countable set I. Note that in $\aleph_0$-complete semirings the axiom ($\mathrm{GP}'_{\mathrm{F}}$) trivially holds even for countable sets J. Moreover, an $\aleph_0$-complete semiring is zero-sum free [**64**, Proposition 22.28].

Let $\mathcal{A} = (A, +, \cdot)$ be a semiring and $\sum$ be an infinitary summation for $\mathcal{A}$. Then by (GP) and ($\mathrm{GP}'_{\mathrm{F}}$) the following law [**41**, Observation 2.2] holds whenever the left hand side is well-defined (*i. e.*, whenever the left hand side is not undefined, then the right hand side is also not undefined and both sides are equal).

$$\sum_{j\in J}\Big(\sum_{i\in I} a_{ij}\Big) = \sum_{i\in I}\Big(\sum_{j\in J} a_{ij}\Big) \tag{1}$$

for every finite index set J, countable index set I, and $(I \times J)$-matrix $(a_{ij})_{(i,j)\in I\times J} \in A^{I\times J}$.

Let $\mathcal{A} = (A, +, \cdot)$ be a naturally ordered semiring that is $\aleph_0$-complete with respect to $\sum$. We say that $\mathcal{A}$ is *continuous,* if for every countable set I and family $(a_i)_{i\in I} \in A^I$ the following supremum exists and

$$\sum_{i\in I} a_i = \sup\{\sum_{i\in F} a_i \mid F \subseteq I, F \text{ finite}\} \ ,$$

where the supremum is taken with respect to the natural order $\sqsubseteq$. Note that the particular form of the above equation is suitable to define an infinitary summation, so that if a semiring is continuous then the infinitary summation is uniquely determined. It is easily checked that $\mathbb{B}$, $\mathbb{N}_\infty$, $\mathbb{T}$, $\mathbb{A}_\infty$, and $\mathbb{Z}_1$ are continuous, whereas $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{T}_{\mathrm{sf}}$, $\mathbb{A}$, and $\mathbb{Z}_n$

for every $n \in \mathbb{N}$ with $n \geqslant 2$ are not continuous. In fact, the semirings $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{T}_{\mathrm{sf}}$, $\mathbb{A}$, and $\mathbb{Z}_n$ are not even $\aleph_0$-complete [**64**, Proposition 22.27].

Excellent introductions into abstract algebra can be found in [**52, 2**], and we recommend [**66, 64**] as lucid expositions into semiring theory.

5. Tree series

Our notions and notations for tree series are heavily influenced by the presentation of tree series in [**41**, Section 2.5]. Let $\mathcal{A} = (A, +, \cdot)$ be a semiring, Δ be a ranked alphabet, and S and V be sets. A mapping $\psi\colon S \longrightarrow A$ is also called a *(formal) power series (over S and $\mathcal{A}$)*. If $S \subseteq T_\Delta(V)$, then such a power series is also called a *(formal) tree series*. Let $\psi\colon S \longrightarrow A$. For every $s \in S$ the semiring element $\psi(s)$ is called the *coefficient of s in ψ*. According to the conventions, which are established in the literature [**79**], we write (ψ, s) instead of $\psi(s)$ for every $s \in S$. Moreover, we occasionally write the power series ψ as $\sum_{s\in S}(\psi, s)\, s$ where the summation is merely formal (*i. e.*, this representation resembles a table with entries from $A \times S$; the entries are separated by $+$). Commonly we omit those summands where the coefficient is 0, so that we write $1\,\alpha + 1\,\gamma(\alpha) + 1\,\gamma^2(\alpha) + \cdots$ instead of $\sum_{t\in T_\Sigma}|t|_\alpha\, t$ where we use the semiring $\mathbb{N}$ and the ranked alphabet $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$.

The set of all power series over S and $\mathcal{A}$ is denoted by $\mathcal{A}\langle\!\langle S\rangle\!\rangle$. This set can be turned into a semiring $\mathcal{B} = (\mathcal{A}\langle\!\langle S\rangle\!\rangle, +, \cdot)$ where for every $\psi, \varphi \in \mathcal{A}\langle\!\langle S\rangle\!\rangle$ and $s \in S$ we define $(\psi + \varphi, s) = (\psi, s) + (\varphi, s)$ and $(\psi \cdot \varphi, s) = (\psi, s) \cdot (\varphi, s)$ and the neutral elements of $+$ and $\cdot$ are, respectively, $\widetilde{0}$ and $\widetilde{1}$, where, for every $a \in A$, $\widetilde{a}$ is defined by $(\widetilde{a}, s) = a$ for every $s \in S$. Provided that $\mathcal{A}$ is equipped with an infinitary summation $\sum$, then $\sum$ can be lifted to an infinitary summation for $\mathcal{B}$ (in a manner analogous to $+$). Given $a \in A$ and $\psi \in \mathcal{A}\langle\!\langle S\rangle\!\rangle$, we write $a\cdot\psi$ to denote the pointwise multiplication of ψ with a; *i. e.*, $(a\cdot\psi, s) = a\cdot(\psi, s)$ for every $s \in S$.

Power series in which all coefficients are 0 or 1 are called *boolean*. The *support* of $\psi \in \mathcal{A}\langle\!\langle S\rangle\!\rangle$, denoted by $\mathrm{supp}(\psi)$, is defined by

$$\mathrm{supp}(\psi) = \{s \in S \mid (\psi, s) \neq 0\}\ .$$

Clearly, $\mathrm{supp}(\widetilde{0}) = \emptyset$. Conversely, the *characteristic power series* corresponding to a given set $L \subseteq S$, denoted by $\chi(L)$, is defined for every $s \in S$ by

$$(\chi(L), s) = \begin{cases} 1 & \text{if } s \in L, \\ 0 & \text{otherwise.} \end{cases}$$

OBSERVATION 2.3. *Let S be a set.*

$$(\mathfrak{P}(S), \cup, \cap) \cong (\mathbb{B}\langle\!\langle S\rangle\!\rangle, \vee, \wedge)$$

PROOF. The homomorphism h from $(\mathfrak{P}(S), \cup, \cap)$ to $(\mathbb{B}\langle\langle S\rangle\rangle, \vee, \wedge)$ is χ and the inverse homomorphism is supp. $\square$

A power series with finite support is called *polynomial,* and a power series with at most one support element is called *monomial.* The set of all polynomial power series over S and $\mathcal{A}$ is denoted by $\mathcal{A}\langle S\rangle$, and the set of all monomial tree series over S and $\mathcal{A}$ is denoted by $\mathcal{A}[S]$.

Let $S \subseteq T_\Delta(V)$ and $\psi \in \mathcal{A}\langle\langle S\rangle\rangle$ and $V' \subseteq V$. We say that ψ is *linear* (respectively, *nondeleting)* in V', if u is linear (respectively, nondeleting) in V' for every $u \in \mathrm{supp}(\psi)$. The set $\mathrm{var}(\psi)$ is defined by $\mathrm{var}(\psi) = \bigcup_{u \in \mathrm{supp}(\psi)} \mathrm{var}(u)$.

CHAPTER 3

Tree Series Substitution

I soon realized, however,
that I could generalize this capability [...].
I could also greatly increase its usefulness
by not limiting it to a simple, single substitution.

Chris Mair: "Enabling Constant Substitution in Property Values"
Java Developer's Journal **6(12)**, 2001

1. Bibliographic information

In this chapter we present the definition of pure and o-tree-series-substitution (for short: pure and o-substitution, respectively). Moreover, we investigate the basic properties of distributivity, linearity, and associativity. Most of the results for pure substitution are known [**41, 58, 55, 84**]. We add corresponding results for o-substitution. Finally, we also consider preservation of recognizability [**9**]. OI tree series substitution [**18, 77**] has already been studied with respect to this property [**79**], and we present results for pure and o-substitution.

2. Definition and simple properties

Several notions of substitution on tree series have been defined in the literature. Basically we distinguish between IO and OI tree series substitutions. Each type is a generalization of the corresponding type of substitution on tree languages [**43, 44**]. Roughly speaking, an IO substitution replaces all occurrences of a variable in a tree with the same tree, which is chosen out of a set of trees. For example, let $L = \{\delta(\mathrm{z}_1, \mathrm{z}_1)\}$ and $L' = \{\alpha, \beta\}$ where δ is binary and both α and β are nullary. Then IO substitution of L' (for z_1) in L yields the tree language $\{\delta(\alpha, \alpha), \delta(\beta, \beta)\}$, whereas OI substitution yields $\{\delta(\alpha, \alpha), \delta(\alpha, \beta), \delta(\beta, \alpha), \delta(\beta, \beta)\}$.

OI tree series substitution is introduced in [**18**, p. 7] and [**77**, p. 14]. Several authors define IO tree series substitutions; *e. g.*, [**20**, Section 3.3] and [**41**, Definition 2.5] introduce pure substitution, [**58**, Definition 3.2] introduces o-substitution, and [**22**] introduces [IO] substitution.

Here we concern ourselves with IO tree series substitutions, and in particular, with pure and o-substitution. We have chosen these substitutions because bottom-up tree series transducers that use either pure or o-substitution are genuine generalizations of the well-known bottom-up tree transducers (see [**41**, Section 4] and [**58**, Corollary 5.9]). Essentially, this is true because the substitutions enjoy a property (see Observation 3.4) that allows "checking followed by deletion" (see [**35**, Section 2]). Next we recall those central notions of substitution. For the rest of this chapter, let Δ be a ranked alphabet and $\mathcal{A} = (A, +, \cdot)$ be a semiring with infinitary summation $\sum$ such that either (i) $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$ or (ii) $\sum$ is the infinitary summation induced by $+$.

Definition 3.1 (see [**58**, Definitions 3.1 and 3.2]). *Let $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$, and $\psi_i \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$ for every $i \in I$. The* pure substitution of $(\psi_i)_{i\in I}$ into ψ, *denoted by $\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$, is defined by*

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I} = \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall i\in I)\colon u_i\in\mathrm{supp}(\psi_i)}} \Big((\psi, u) \cdot \prod_{i\in I} (\psi_i, u_i)\Big)\, u[u_i]_{i\in I} \ . \tag{2}$$

For brevity we shortened the index of the sum. The line

$$(\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i)$$

deserves explanation. Formally, the index should read

$$\substack{u\in\mathrm{supp}(\psi),\\ (u_i)_{i\in I}\in T_\Delta(\mathrm{Z})^I,\\ (\forall i\in I)\colon u_i\in\mathrm{supp}(\psi_i)} \ .$$

Since the quantification is usually obvious, we simply omit the second line.

The o-substitution of $(\psi_i)_{i\in I}$ into ψ, *denoted by $\psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I}$, is defined by*

$$\psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I} = \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall i\in I)\colon u_i\in\mathrm{supp}(\psi_i)}} \Big((\psi, u) \cdot \prod_{i\in I} (\psi_i, u_i)^{|u|_{z_i}}\Big)\, u[u_i]_{i\in I} \ . \tag{3}$$

The binding priority of $\xleftarrow[\varepsilon]{}$ and $\xleftarrow[\mathrm{o}]{}$ is assumed to be higher than the priority of every additive symbol (like $+$ or $\sum$), but lower than the priority of every multiplicative symbol (like $\cdot$ or $\prod$). Thus, the term $\sum_{j\in J} a_j \cdot \psi_j \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$ reads as

$$\sum_{j\in J} \big((a_j \cdot \psi_j) \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}\big) \ .$$

Note that compared to [**58**] we have defined pure and o-substitution also for non-contiguous blocks of variables. Let us illustrate on an example, how the original notion can be obtained. Let $I = \{2, 4\}$. Then $\psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I}$ in our notation is equal to $\psi \overset{o}{\leftarrow} (1\, \mathrm{z}_1, \psi_2, 1\, \mathrm{z}_3, \psi_4)$ in the notation of [**58**, Definition 3.2]. We do not explicitly mention this little difference when we reference results of [**41, 58, 55**]. The advantage of this additional freedom is that it simplifies some statements considerably (*e. g.*, compare [**41**, Proposition 2.10] with Proposition 3.19).

The following mapping unifies some of the discussions in the sequel. Let $\mathrm{sel}_\Delta \colon T_\Delta(\mathrm{Z}) \times \mathbb{N}_+ \times \{\varepsilon, \mathrm{o}\} \longrightarrow \mathbb{N}$ be defined for every $u \in T_\Delta(\mathrm{Z})$, $n \in \mathbb{N}_+$, and $\eta \in \{\varepsilon, \mathrm{o}\}$ by

$$\mathrm{sel}_\Delta(u, n, \eta) = \begin{cases} 1 & \text{if } \eta = \varepsilon, \\ |u|_{\mathrm{z}_n} & \text{if } \eta = \mathrm{o}. \end{cases} \tag{4}$$

Since Δ is usually obvious from the context, we regularly omit it and just write sel.

Pure substitution represents a computational approach; *i. e.*, the output trees represent values of computations, and the coefficient associated to an output tree can be viewed as the cost of computing this value. When we combine output trees, we simply multiply their coefficients to obtain the coefficient of the combined output tree. This is done irrespective of the number of uses of an output tree; *i. e.*, an output tree may be copied without penalty, which represents the computational approach in the sense that a value is available and can be reused without recomputation (call-by-value or eager evaluation [**1**]).

On the other hand, o-substitution represents a more material approach. There the coefficient of an output tree is taken to the n-th power, if the tree is used in n copies. In this approach, an output tree stands for a composite, and the coefficient of an output tree reflects the (monetary) cost of creating or obtaining this particular composite. When we combine composites into a new composite, we obtain the cost by multiplying the costs of the components; each component taken as often as needed to assemble the composite.

Certainly, those analogies reach their limits when stressed. It may be argued that a computation is successful only if all of its subcomputations are successful (*cf.* eager evaluation [**1**]); *i. e.*, if $\psi_i = \widetilde{0}$ for some $i \in I$, then $\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i \in I} = \widetilde{0}$ (even if $i \notin \mathrm{var}(\psi)$; see Observation 3.4). However, the analogy breaks down for o-substitution, which enjoys the same property. If we follow the terminology of our analogy, then this means that we cannot assemble a composite whenever one type of component is not available. It holds true even for composites that do not require the missing component.

Subsequently, we use ε-substitution as a synonym for pure substitution. Let $\eta \in \{\varepsilon, \mathrm{o}\}$. In an expression $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ we call ψ the *target* and each ψ_i a *source*. If $I = [n]$ for some $n \in \mathbb{N}$, we occasionally write $\psi \xleftarrow[\eta]{} (\psi_1, \dots, \psi_n)$ instead of $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in [n]}$. Now let us illustrate pure and o-substitution on examples.

EXAMPLE 3.2. *Let* $\Delta = \{\delta^{(2)}, \alpha^{(0)}\}$.

(1) *Consider the semiring* $\mathbb{N}$ *and the three monomial tree series* $\psi_1 = 2\,\delta(\alpha, \alpha)$ *and* $\psi_1' = 2\,\delta(\mathrm{z}_1, \alpha)$ *and* $\psi_1'' = 2\,\delta(\mathrm{z}_1, \mathrm{z}_1)$. *The results of several substitutions are displayed in Table 1.*

TABLE 1. Pure and o-substitution compared; see Example 3.2(1).

	$\psi_1 \xleftarrow[\eta]{} (\psi_1)$	$\psi_1' \xleftarrow[\eta]{} (\psi_1')$	$\psi_1'' \xleftarrow[\eta]{} (\psi_1'')$
$\eta = \varepsilon$	$4\,\delta(\alpha,\alpha)$	$4\,\delta(\delta(\mathrm{z}_1,\alpha),\alpha)$	$4\,\delta(\delta(\mathrm{z}_1,\mathrm{z}_1),\delta(\mathrm{z}_1,\mathrm{z}_1))$
$\eta = \mathrm{o}$	$2\,\delta(\alpha,\alpha)$	$4\,\delta(\delta(\mathrm{z}_1,\alpha),\alpha)$	$8\,\delta(\delta(\mathrm{z}_1,\mathrm{z}_1),\delta(\mathrm{z}_1,\mathrm{z}_1))$

Decomposition along some path: *Decomposition along maximal path:*

FIGURE 1. Decompositions of $\delta(\delta(\alpha,\alpha),\delta(\alpha,\alpha))$; see Example 3.2(2).

(2) *Consider the tropical semiring* $\mathbb{T}$ *and the tree series*

$$\psi_2 = \min_{u \in C_\Delta(\mathrm{Z}_1)} \mathrm{height}(u)\, u \quad \textit{and} \quad \psi_2' = \min_{u \in T_\Delta} \mathrm{height}(u)\, u \enspace .$$

Then $\psi_2' = \psi_2 \xleftarrow[\varepsilon]{} (\psi_2') = \psi_2 \xleftarrow[\mathrm{o}]{} (\psi_2')$ *[see Figure 1].*

(3) *Consider the arctic semiring* $\mathbb{A}$ *and the tree series*

$$\psi_3 = \max_{u \in T_\Delta(\mathrm{Z}_1)} |u|_\delta\, u \enspace .$$

Then $\psi_3 \xleftarrow[\varepsilon]{} (\psi_3)$ *and* $\psi_3 \xleftarrow[\mathrm{o}]{} (\psi_3)$ *are not well-defined. If we reconsider the above substitutions in the continuous semiring* $\mathbb{A}_\infty$ *with infinitary summation* sup, *then* $\psi_3 = \psi_3 \xleftarrow[\mathrm{o}]{} (\psi_3)$, *but* $\psi_3 \neq \psi_3 \xleftarrow[\varepsilon]{} (\psi_3)$. *The latter can be seen on the example* $u = \delta(\alpha,\alpha)$. *Clearly,* $u = \delta(\alpha,\alpha)[\delta(\alpha,\alpha)]$ *and* $|\delta(\alpha,\alpha)|_\delta = 1$. *Consequently,* $(\psi_3 \xleftarrow[\varepsilon]{} (\psi_3), u) \geqslant 2$ *but* $|u|_\delta = 1$. *In fact, for every* $u \in T_\Delta(\mathrm{Z}_1)$

$$\big(\psi_3 \xleftarrow[\varepsilon]{} (\psi_3), u\big) = \begin{cases} \infty & \textit{if } u \in T_\Delta, \\ |u|_\delta & \textit{otherwise.} \end{cases}$$

The last two examples raise the question of well-definedness of η-substitutions. Recall that we consider a countable sum $\sum_{i \in I} \psi_i$ well-defined, if and only if $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$ or for every $u \in T_\Delta(\mathrm{Z})$ there exist only finitely many $i \in I$ such that $(\psi_i, u) \neq 0$. In the latter case, the coefficient of every u in $\sum_{i \in I} \psi_i$ is given by an essentially finite sum. When one uses this notion of well-definedness, it is evident that, in general, an η-substitution might be undefined. However, the substitutions in Example 3.2(2) are well-defined because ψ_2 is nondeleting in Z_1. Clearly, all η-substitutions are well-defined in $\aleph_0$-complete semirings. The next observation presents sufficient conditions for the well-definedness of $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ in not necessarily $\aleph_0$-complete semirings. For the rest of this section, let $\eta \in \{\varepsilon, \mathrm{o}\}$, $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$, and $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ for every $i \in I$. Some of the following results are not needed in their full generality in this thesis. For the sake of completeness, we nevertheless present the general statements unless this would be overly cumbersome.

Observation 3.3. *The η-substitution $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is well-defined, if there exists a $J \subseteq I$ such that:*

- *ψ is nondeleting in Z_J; and*
- *ψ_i is polynomial for every $i \in I \setminus J$.*

Proof. The η-substitution is well-defined, if for every $u \in T_\Delta(\mathrm{Z})$ there exist only finitely many $u' \in \mathrm{supp}(\psi)$ and $u_i \in \mathrm{supp}(\psi_i)$ for every $i \in I$ such that $u = u'[u_i]_{i \in I}$. Clearly, $\mathrm{supp}(\psi_i)$ is finite for every $i \in I \setminus J$. Moreover, since u' in nondeleting in Z_J we immediately obtain $u_j \in \mathrm{sub}(u)$ for every $j \in J$ as well as $\mathrm{size}(u') \leqslant \mathrm{size}(u)$ and $\mathrm{var}(u') \subseteq I \cup \mathrm{var}(u)$. Obviously, there are only finitely many such trees u' and u_j, which proves the statement. $\square$

The above observation is most useful in the cases $J = \emptyset$ and $J = I$. We obtain that $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is well-defined, if ψ_i is polynomial for every $i \in I$ (see the discussion following [**55**, Definition 2.1]), or ψ is nondeleting in Z_I. Next, we recall three properties of paramount importance from [**58**, Proposition 3.4]. In the sequel we use these properties without explicit mention.

Observation 3.4 (see [**58**, Proposition 3.4]).

(1) $\psi \xleftarrow[\eta]{} () = \psi$.
(2) $\widetilde{0} \xleftarrow[\eta]{} (\psi_i)_{i \in I} = \widetilde{0}$.
(3) *If $\psi_i = \widetilde{0}$ for some $i \in I$, then $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I} = \widetilde{0}$.*

In essence it is the third property that ensures that bottom-up tree series transducers [**41**] have the "checking followed by deletion"-property. Note that in all cases η-substitution is well-defined. Observation 3.3 shows this for the first two items but not for the third. We present another example that shows that Observation 3.3 does not characterize well-defined η-substitutions in semirings that are not $\aleph_0$-complete. We consider the semiring $\mathbb{Z}_4$ and the tree series $\psi = \sum_{u \in T_\Delta(\mathrm{Z})} 2u$. Then $\psi \xleftarrow[\varepsilon]{} (\psi)$ is well-defined and equal to $\widetilde{0}$ (because $\sum_{i \in I} 0 = 0$ by (D) whenever the left hand side is well-defined), whereas $\psi \xleftarrow[\mathrm{o}]{} (\psi)$ is not well-defined. The next observation, however, shows that a straightforward characterization can be obtained for zero-divisor free semirings. Clearly, if $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$, then all substitutions are well-defined. Thus we explicitly consider the infinitary summation induced by $+$ in the next observation.

Observation 3.5. *Let $\mathcal{A}$ be zero-divisor free and $\sum$ be the infinitary summation induced by $+$. Then $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is well-defined, if and only if:*

(1) *there exists $i \in I$ such that $\psi_i = \widetilde{0}$; or*

(2) *there exists* $J \subseteq I$ *such that* ψ *is nondeleting in* Z_J *and* ψ_i *is polynomial for every* $i \in I \setminus J$.

PROOF. Observations 3.4(3) and 3.3 show that $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is well-defined in Cases (1) and (2), respectively. For the converse, assume that there exists a $j \in I$ such that ψ_j is not polynomial, ψ is not nondeleting in $\{\mathrm{z}_j\}$, and $\psi_i \neq \widetilde{0}$ for every $i \in I$. Thus there exist $u_i \in \mathrm{supp}(\psi_i)$ for all $i \in I$, and moreover, $u' \in \mathrm{supp}(\psi)$ such that $j \notin \mathrm{var}(u')$. Clearly, u_j does not influence $u = u'[u_i]_{i \in I}$ and since $\mathrm{supp}(\psi_j)$ is infinite, there exist infinitely many decompositions of u. Moreover, none of the coefficients is zero, because $(\psi, u') \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u', i, \eta)}$ is not zero by zero-divisor freeness of $\mathcal{A}$. Thus we obtain an infinite summation of nonzero coefficients, which is not well-defined (because $\sum$ is the infinitary summation induced by $+$). $\square$

Well-definedness of summations plays a major role in the next section, but let us now investigate which properties are preserved under substitution. The next observation shows that substitution of polynomial (respectively, monomial) tree series into a polynomial (respectively, monomial) tree series yields a polynomial (respectively, monomial) tree series. This was already observed in [**58**, Proposition 3.11]. For our discussion of deterministic tree series transducers, we also need that the substitution of boolean monomial tree series into a boolean monomial tree series yields a boolean monomial tree series. A similar statement for polynomial tree series holds in additively idempotent semirings [**55**, Lemma 6.1]. However, in $\mathbb{N}$ we have

$$(1\, \delta(\mathrm{z}_1, \alpha) + 1\, \delta(\alpha, \mathrm{z}_1)) \xleftarrow[\eta]{} (1\, \alpha) = 2\, \delta(\alpha, \alpha) \enspace ,$$

which is not boolean.

OBSERVATION 3.6 (see [**58**, Proposition 3.11]).

(1) *If* ψ *is polynomial and* ψ_i *is polynomial for every* $i \in I$, *then* $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ *is polynomial.*
(2) *If* ψ *is monomial and* ψ_i *is monomial for every* $i \in I$, *then* $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ *is monomial.*
(3) *If* ψ *is boolean and monomial and* ψ_i *is boolean and monomial for every* $i \in I$, *then* $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ *is boolean.*

PROOF. In [**58**, Proposition 3.11] one finds the proof of the first two statements. Since $(\{0, 1\}, \cdot)$ is a submonoid of $(A, \cdot)$, it is straightforward to prove that $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I}$ is boolean whenever the preconditions of the last statement are met. $\square$

Table 2 presents some more results, which additionally assume that the substitution is well-defined. The conditions stated there are very restrictive and can easily be relaxed, but the statements shall serve as examples here. The proofs of the last two statements in Table 2 are straightforward and omitted. We consider the preservation of recognizability in Section 5.

TABLE 2. Preservation of properties under substitution.

$\psi \xleftarrow[\eta]{} (\psi_i)_{i\in I}$ is ...	whenever ψ is ...	and every ψ_i is ...
polynomial	polynomial	polynomial
monomial	monomial	monomial
boolean	boolean and monomial	boolean and monomial
linear in Z_J	linear in Z_I and $\mathrm{var}(\psi) \cap J = \emptyset$	linear in Z_J and $\mathrm{var}(\psi_i) \cap \mathrm{var}(\psi_j) \cap J = \emptyset$ $(i \neq j)$
nondeleting in Z_J	nondeleting in $Z_J \cup \mathrm{Z}_I$	nondeleting in Z_J

Finally, Example 3.2(2) hints at some property relating pure and o-substitution. Whenever the target tree series is nondeleting and linear in Z_I, pure and o-substitution coincide (see [**58**, Proposition 3.10]).

OBSERVATION 3.7 (*cf.* [**58**, Proposition 3.10]). *We have*

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I} = \psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I} \ , \tag{5}$$

whenever there exists $J \subseteq I$ such that ψ is nondeleting and linear in Z_J, and ψ_i is boolean for every $i \in I \setminus J$.

We obtain [**58**, Proposition 3.10], which claims (5) provided that (i) ψ is nondeleting and linear in Z_I or (ii) ψ_i is boolean for every $i \in I$, as a corollary.

3. Distributivity and linearity

Distributivity and linearity are key properties for the composition results of Chapter 7 and several other results (*e. g.*, associativity in Section 4 and preservation of recognizability in Section 5). In this section we investigate pure and o-substitution with respect to distributivity and linearity.

3.1. Pure substitution. The first central result, which we recall from [**41**], is that pure substitution is distributive and linear. For the rest of this section, let $I \subseteq \mathbb{N}_+$ be a finite set, J a finite set, and J_i a finite set for every $i \in I$.

PROPOSITION 3.8 (see [**41**, Proposition 2.9]). *Let $\psi_j \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$ for every $j \in J$, and for every $i \in I$ and $j_i \in J_i$ let $\psi_{j_i} \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$.*

$$\sum_{\substack{j\in J,\\ (\forall i\in I)\colon j_i\in J_i}} \psi_j \xleftarrow[\varepsilon]{} (\psi_{j_i})_{i\in I} = \Big(\sum_{j\in J} \psi_j\Big) \xleftarrow[\varepsilon]{} \Big(\sum_{j_i\in J_i} \psi_{j_i}\Big)_{i\in I} \ , \tag{6}$$

provided that the left hand side is well-defined.

PROOF. This result is stated for continuous semirings and proved for $\aleph_0$-complete semirings in [**41**, Proposition 2.9]. The index sets J and J_i may be countable provided that $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$. A similar statement for semirings that are not $\aleph_0$-complete with respect to $\sum$ is claimed in [**55**, Proposition 2.3], but unfortunately it is wrong.

$$\begin{aligned}
&\sum_{\substack{j\in J,\\ (\forall i\in I)\colon j_i\in J_i}} \psi_j \underset{\varepsilon}{\longleftarrow} (\psi_{j_i})_{i\in I} \\
= \quad & \textit{(by definition of } \underset{\varepsilon}{\longleftarrow} \textit{ and Observation 3.4(3))} \\
&\sum_{\substack{j\in J,\\ (\forall i\in I)\colon j_i\in J_i}} \Bigl(\sum_{\substack{u\in T_\Delta(\mathrm{Z}),\\ (\forall i\in I)\colon u_i\in T_\Delta(\mathrm{Z})}} \bigl((\psi_j, u) \cdot \prod_{i\in I} (\psi_{j_i}, u_i)\bigr)\, u[u_i]_{i\in I} \Bigr) \\
= \quad & \textit{(by (1) because } J \textit{ and all } J_i \textit{ are finite)} \\
&\sum_{\substack{u\in T_\Delta(\mathrm{Z}),\\ (\forall i\in I)\colon u_i\in T_\Delta(\mathrm{Z})}} \Biggl(\Bigl(\sum_{j\in J} \psi_j, u\Bigr) \cdot \prod_{i\in I} \Bigl(\sum_{j_i\in J_i} \psi_{j_i}, u_i\Bigr) \Biggr) u[u_i]_{i\in I} \\
= \quad & \textit{(by definition of } \underset{\varepsilon}{\longleftarrow} \textit{ and Observation 3.4(3))} \\
&\Bigl(\sum_{j\in J} \psi_j\Bigr) \underset{\varepsilon}{\longleftarrow} \Bigl(\sum_{j_i\in J_i} \psi_{j_i} \Bigr)_{i\in I}
\end{aligned}$$

By assumption the left hand side of (6) is well-defined. We conclude that $\psi_j \underset{\varepsilon}{\longleftarrow} (\psi_{j_i})_{i\in I}$ is well-defined for every $j \in J$, $i \in I$, and $j_i \in J_i$. We leave it to the reader to check that well-definedness is preserved. ☐

Proposition 3.9 (see [**41**, Proposition 2.8]). *Let $\mathcal{A}$ be commutative, $a \in A$, and $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$. Moreover, let $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ and $a_i \in A$ for every $i \in I$.*

$$\Bigl(a \cdot \prod_{i\in I} a_i\Bigr) \cdot \bigl(\psi \underset{\varepsilon}{\longleftarrow} (\psi_i)_{i\in I}\bigr) = a \cdot \psi \underset{\varepsilon}{\longleftarrow} (a_i \cdot \psi_i)_{i\in I} \ , \tag{7}$$

provided that the left hand side is well-defined.

The previous propositions essentially state that pure substitution is distributive and linear in the target tree series as well as in every source tree series. Let $\Omega \subseteq \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ be a ranked set with $\Omega_k \subseteq \mathcal{A}\langle T_\Delta(\mathrm{Z}_k)\rangle$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ and $\omega \in \Omega_k$ we define the operation $\overline{\omega}^\varepsilon_k \colon \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle^k \longrightarrow \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ by $\overline{\omega}^\varepsilon_k(\psi_1, \dots, \psi_k) = \omega \underset{\varepsilon}{\longleftarrow} (\psi_1, \dots, \psi_k)$ for every $\psi_1, \dots, \psi_k \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$. Using linearity and distributivity we can conclude that $(\mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle, +, \cdot, (\overline{\omega}^\varepsilon_k)_{k\in\mathbb{N},\omega\in\Omega})$ is an $\mathcal{A}$-Ω-algebra [**18**, Section 3].

3.2. The case of o-substitution. Next we study distributivity and linearity of o-substitution. First we recall the full linearity and distributivity in the target tree series.

Proposition 3.10 (see [**58**, Propositions 3.12 and 3.14]). *Let $\mathcal{A}$ be commutative, $a_j \in A$ and $\psi_j \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ for every $j \in J$, and for every $i \in I$ let $b_i \in \{0, 1\}$ and $\psi_i \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$.*

$$\sum_{j\in J} \bigl(a_j \cdot \prod_{i\in I} b_i\bigr) \cdot \bigl(\psi_j \underset{\mathrm{o}}{\longleftarrow} (\psi_i)_{i\in I}\bigr) = \Bigl(\sum_{j\in J} a_j \cdot \psi_j\Bigr) \underset{\mathrm{o}}{\longleftarrow} (b_i \cdot \psi_i)_{i\in I} \ , \tag{8}$$

provided that the left hand side is well-defined.

Clearly, pure and o-substitution coincide on boolean tree series (see [**58**, Proposition 3.10] and Observation 3.7), so that we can derive linearity from pure substitution in this setting. If we recall the definition of o-substitution from Definition 3.1, we see that for the distributivity of o-substitution in source tree series we obviously need a law like $(\sum_{i\in I} a_i)^n = \sum_{i\in I} a_i^n$ for every nonempty and finite I, family $(a_i)_{i\in I} \in A^I$, and $n \in \mathbb{N}$.

DEFINITION 3.11. *Let $\mathcal{A} = (A, +, \cdot)$ be a semiring and $N \subseteq \mathbb{N}$. The semiring $\mathcal{A}$ is called N-*FROBENIUS*, if for every $n \in N$, nonempty and finite I, and family $(a_i)_{i\in I} \in A^I$ the equality $(\sum_{i\in I} a_i)^n = \sum_{i\in I} a_i^n$ holds. Semirings that are $\mathbb{N}$-*FROBENIUS *are also called* FROBENIUS *semirings.*

Note that we require I to be nonempty in the previous definition because $\sum_{i\in\emptyset} a_i^n = 0$ for every $n \in \mathbb{N}$ but $(\sum_{i\in\emptyset} a_i)^n = 1$ if $n = 0$. Thus only the trivial semiring $\mathbb{Z}_1$, where $0 = 1$, would be $\{0, 1\}$-FROBENIUS, if we would omit the nonemptiness condition for I in Definition 3.11. Moreover, note that a semiring that is N-FROBENIUS is also N'-FROBENIUS for every $N' \subseteq N$. Let $\mathcal{A}$ be N-FROBENIUS for some $N \subseteq \mathbb{N}$. It immediately follows that for every $n \in N$ the n-th power FROBENIUS mapping $f_n \colon A \longrightarrow A$ defined for every $a \in A$ by $f(a) = a^n$ is a semiring homomorphism.

EXAMPLE 3.12. *Let us show examples of N-*FROBENIUS *semirings.*

- *Every semiring is $\{1\}$-*FROBENIUS.
- *Every additively idempotent semiring is $\{0, 1\}$-*FROBENIUS.
- *Every additively extremal semiring is* FROBENIUS.
- *Every additively idempotent, multiplicatively cancellative, and commutative semiring is* FROBENIUS [**64**, Proposition 4.43].

PROOF. We proof the third statement. Let $\mathcal{A}$ be an additively extremal semiring, $k, n \in \mathbb{N}$ with $k \geqslant 1$, and $a_i \in A$ for every $i \in [k]$. By [**64**, p. 228] we have that $\mathcal{A}$ is naturally ordered. Clearly, there exists a $j \in [k]$ such that $\sum_{i\in[k]} a_i = a_j$ and hence $a_i \sqsubseteq a_j$ for every $i \in [k]$. Let $B = \{a_i \mid i \in [k]\}$. We show that for every element $b \in B$ we have $b^n \sqsubseteq a_j^n$. Clearly, for every $c, c', d, d' \in A$ we have that $c \sqsubseteq d$ and $c' \sqsubseteq d'$ imply that $c \cdot c' \sqsubseteq d \cdot d'$. With this, $b^n \sqsubseteq a_j^n$ for every $b \in B$. Since for every $c, d \in A$ we have $c + d = d$ if and only if $c \sqsubseteq d$, we obtain $\sum_{i\in[k]} a_i^n = a_j^n = (\sum_{i\in[k]} a_i)^n$. □

In fact, a semiring is $\{0, 1\}$-FROBENIUS if and only if it is additively idempotent. Using the notion "N-FROBENIUS" we can prove distributivity of o-substitution (*cf.* [**58**, Proposition 3.14]), *e. g.*, for linear tree series over additively idempotent semirings. Note that in this scenario pure and o-substitution do not coincide (consider, *e. g.*, the tropical semiring $\mathbb{T}$, the ranked alphabet $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$, and the tree series

$\psi = 5\,\sigma(\alpha,\alpha)$, which is linear in Z_1. Then $\psi \xleftarrow[\varepsilon]{} (\psi) = 10\,\sigma(\alpha,\alpha)$ and $\psi \xleftarrow[\mathrm{o}]{} (\psi) = \psi)$, so the statement is not trivial.

LEMMA 3.13. *Let $\mathcal{A}$ be N-FROBENIUS for some $N \subseteq \mathbb{N}$. Moreover, for every $j \in J$ let $\psi_j \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$, and for every $i \in I$ and $j_i \in J_i$ let $\psi_{j_i} \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$. Then*

$$\sum_{\substack{j\in J,\\ (\forall i\in I):\, j_i\in J_i}} \psi_j \xleftarrow[\mathrm{o}]{} (\psi_{j_i})_{i\in I} = \Big(\sum_{j\in J}\psi_j\Big) \xleftarrow[\mathrm{o}]{} \Big(\sum_{j_i\in J_i}\psi_{j_i}\Big)_{i\in I} \tag{9}$$

holds provided that the left hand side is well-defined and $|u|_{z_i} \in N$ for every $i \in I$, $j \in J$, and $u \in \mathrm{supp}(\psi_j)$.

PROOF. Clearly, by Observation 3.4(3) the statement holds if $J_i = \emptyset$ for some $i \in I$. Thus we assume that $J_i \neq \emptyset$ for every $i \in I$. Moreover, we use the fact that

$$\mathrm{supp}\Big(\sum_{i\in I'}\psi_i\Big) \subseteq \bigcup_{i\in I'}\mathrm{supp}(\psi_i)$$

for all finite sets I' and $(\psi_i)_{i\in I'} \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle^{I'}$.

$$\begin{aligned}
&\sum_{\substack{j\in J,\\ (\forall i\in I):\, j_i\in J_i}} \psi_j \xleftarrow[\mathrm{o}]{} (\psi_{j_i})_{i\in I}\\
=\;& \textit{(by definition of } \xleftarrow[\mathrm{o}]{}\textit{)}\\
&\sum_{\substack{j\in J,\\ (\forall i\in I):\, j_i\in J_i}} \Big(\sum_{\substack{u\in\mathrm{supp}(\psi_j),\\ (\forall i\in I):\, u_i\in\mathrm{supp}(\psi_{j_i})}} \Big((\psi_j,u)\cdot\prod_{i\in I}(\psi_{j_i},u_i)^{|u|_{z_i}}\Big)\, u[u_i]_{i\in I}\Big)\\
=\;& \textit{(because } J \textit{ and } J_i \textit{ are finite, } J_i \textit{ is nonempty, and } |u|_{z_i}\in N\textit{)}\\
&\sum_{\substack{u\in\bigcup_{j\in J}\mathrm{supp}(\psi_j),\\ (\forall i\in I):\, u_i\in\bigcup_{j_i\in J_i}\mathrm{supp}(\psi_{j_i})}} \bigg(\Big(\sum_{j\in J}\psi_j,u\Big)\cdot\prod_{i\in I}\Big(\sum_{j_i\in J_i}\psi_{j_i},u_i\Big)^{|u|_{z_i}}\bigg)\, u[u_i]_{i\in I}\\
=\;& \textit{(see below)}\\
&\sum_{\substack{u\in\mathrm{supp}(\sum_{j\in J}\psi_j),\\ (\forall i\in I):\, u_i\in\mathrm{supp}(\sum_{j_i\in J_i}\psi_{j_i})}} \bigg(\Big(\sum_{j\in J}\psi_j,u\Big)\cdot\prod_{i\in I}\Big(\sum_{j_i\in J_i}\psi_{j_i},u_i\Big)^{|u|_{z_i}}\bigg)\, u[u_i]_{i\in I}\\
=\;& \textit{(by definition of } \xleftarrow[\mathrm{o}]{}\textit{)}\\
&\Big(\sum_{j\in J}\psi_j\Big) \xleftarrow[\mathrm{o}]{} \Big(\sum_{j_i\in J_i}\psi_{j_i}\Big)_{i\in I}
\end{aligned}$$

In the second-to-last step we used the property that either (i) $|u|_{z_i} \geqslant 1$ for every $i \in I$ and $u \in \mathrm{supp}\big(\sum_{j\in J}\psi_j\big)$ or (ii) $0 \in N$. In the former case, validity of this step is immediate. Now consider the latter case. Since $0 \in N$, the semiring $\mathcal{A}$ is additively idempotent and thus zero-sum free [**64**, p. 4]. In zero-sum free semirings the property $\mathrm{supp}\big(\sum_{i\in I'}\psi_i\big) = \bigcup_{i\in I'}\mathrm{supp}(\psi_i)$ holds [**34**, Section VI.3], which proves that the second-to-last step is valid. □

As a corollary we obtain the already stated distributivity result for linear target tree series in additively idempotent semirings.

COROLLARY 3.14 (of Lemma 3.13). *Let $\mathcal{A}$ be additively idempotent. Moreover, for every $j \in J$ let $\psi_j \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ be linear in Z_I, and for every $i \in I$ and $j_i \in J_i$ let $\psi_{j_i} \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$.*

$$\sum_{\substack{j \in J, \\ (\forall i \in I)\colon j_i \in J_i}} \psi_j \underset{\mathrm{o}}{\longleftarrow} (\psi_{j_i})_{i \in I} = \Big(\sum_{j \in J} \psi_j\Big) \underset{\mathrm{o}}{\longleftarrow} \Big(\sum_{j_i \in J_i} \psi_{j_i}\Big)_{i \in I} , \tag{10}$$

provided that the left hand side is well-defined.

PROOF. Additively idempotent semirings are $\{0,1\}$-FROBENIUS. Thus the statement follows from Lemma 3.13. $\square$

Let $\mathcal{A}$ be an additively idempotent semiring. For every $k \in \mathbb{N}$ and $\omega \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_k)\rangle$ we define the mapping $\overline{\omega}^{\mathrm{o}}_k\colon \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle^k \longrightarrow \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ by $\overline{\omega}^{\mathrm{o}}_k(\psi_1, \dots, \psi_k) = \omega \underset{\mathrm{o}}{\longleftarrow} (\psi_1, \dots, \psi_k)$ for all $\psi_1, \dots, \psi_k \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$. Let $\Omega \subseteq \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ be a ranked set of linear (in Z) tree series such that $\Omega_k \subseteq \mathcal{A}\langle T_\Delta(\mathrm{Z}_k)\rangle$ for every $k \in \mathbb{N}$. The algebraic structure $(\mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle, +, (\overline{\omega}^{\mathrm{o}}_k)_{k \in \mathbb{N}, \omega \in \Omega})$ is a distributive Ω-algebra [**48**, p. 222]. We can derive an analogous result (without the linearity restriction) for additively extremal semirings.

4. Associativity

This section is devoted to the study of associativity-like laws for pure and o-substitution. These laws are of paramount importance for compositions of tree series transformations. The main problem is that our IO tree series substitutions generalize IO-substitution on tree languages, which is not associative. Thus we cannot establish associativity in general (neither for pure nor for o-substitution). However, in [**43**, Lemma 2.4.3] it is shown that for every $k, n \in \mathbb{N}$ with $k \geqslant 1$ and $L \subseteq T_\Delta(\mathrm{Z}_k)$, $L_1, \dots, L_k \subseteq T_\Delta(\mathrm{Z}_n)$, and $L'_1, \dots, L'_n \subseteq T_\Delta(\mathrm{Z})$

$$\big(L[L_1, \dots, L_k]\big)[L'_1, \dots, L'_n] = L\big[L_1[L'_1, \dots, L'_n], \dots, L_k[L'_1, \dots, L'_n]\big]$$

holds, whenever all $L'_1, \dots, L'_n$ are singletons or $L_1, \dots, L_k$ are pairwise variable-disjoint. When we want $k = 0$ to be eligible, we have to demand that $L'_i \neq \emptyset$ for every $i \in [n]$. Now we formalize this condition (including the case $k = 0$) on tree series.

DEFINITION 3.15. *Let $I, J \subseteq \mathbb{N}_+$ be finite and $\psi_j \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle$ for every $j \in J$. Finally, let $\mathcal{I} = (I_j)_{j \in J}$ be a partition of I. The partition $\mathcal{I}$ is said to* conform to *$(\psi_j)_{j \in J}$, if for every $j \in J$ the condition $\mathrm{var}(\psi_j) \subseteq I_j$ holds.*

Note that for every family $\Psi = (\psi_j)_{j \in J}$ with $J \neq \emptyset$ of pairwise variable-disjoint tree series a partition of I conforming to Ψ exists. Further, if $J = \emptyset$ then such a partition only exists when $I = \emptyset$.

Before we can prove an associativity result comparable to the one presented above, we need to generalize a preliminary statement. For this, let $u \in T_\Delta(\mathrm{Z})$ and $u_i \in T_\Delta(\mathrm{Z})$ for every $i \in I$. We present a proposition, which generalizes the result that $u[u_i]_{i\in I} = u[u_j]_{j\in J}$ for every $J \subseteq I$ such that $J \cap \mathrm{var}(u) = I \cap \mathrm{var}(u)$. Intuitively speaking, this asserts that u_i is irrelevant in $u[u_i]_{i\in I}$, if $i \notin \mathrm{var}(u)$. This generalizes nicely to tree languages $L, L_i \subseteq T_\Delta(\mathrm{Z})$; *i. e.*, $L[L_i]_{i\in I} = L[L_j]_{j\in J}$ for every $J \subseteq I$ such that $J \cap \mathrm{var}(L) = I \cap \mathrm{var}(L)$ and $L_i \neq \emptyset$ for every $i \in I \setminus J$. The additional restriction is derived from the fact that $L[L_i]_{i\in I} = \emptyset$, whenever $L_i = \emptyset$ for some $i \in I$. In order to generalize the statement to tree series we need the notion of a necessary summation from [**64**, p. 251].

DEFINITION 3.16. *We say that* $\sum$ *is* necessary [**64**, p. 251], *whenever* $\sum_{i\in I} a_i = \sum_{i\in I} b_i$ *for all countable sets* I *and* $(a_i)_{i\in I}, (b_i)_{i\in I} \in A^I$ *such that:*

(1) $\sum_{i\in I} a_i$ *and* $\sum_{i\in I} b_i$ *are well-defined; and*
(2) *for each finite subset* $I' \subseteq I$ *there exists a finite set* I'' *with* $I' \subseteq I'' \subseteq I$ *and* $\sum_{i\in I''} a_i = \sum_{i\in I''} b_i$.

Note that the infinitary summation induced by $+$ is clearly necessary [**64**], but not every infinitary summation of an $\aleph_0$-complete semiring is necessary. The infinitary summations introduced for the semirings $\mathbb{B}$, $\mathbb{N}_\infty$, $\mathbb{T}$, and $\mathbb{A}_\infty$ are all necessary. For examples of infinitary summations that are not necessary, we refer the reader to [**64**, Example 22.18].

OBSERVATION 3.17. *Let* $\mathcal{A} = (A, +, \cdot)$ *be a continuous semiring with respect to* $\sum$. *Then* $\sum$ *is necessary.*

PROOF. Clearly, $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$, so all sums with countably many summands are defined. Let I be a countable index set and $(a_i)_{i\in I} \in A^I$ and $(b_i)_{i\in I} \in A^I$ be families. By definition of $\sqsubseteq$ we have

$$\sum_{i\in J} a_i \sqsubseteq \sum_{i\in I} a_i \tag{11}$$

for every $J \subseteq I$. Let $\mathcal{F} = \{F \subseteq I \mid F \text{ finite}, \sum_{i\in F} a_i = \sum_{i\in F} b_i\}$. Moreover, suppose that for every finite $J \subseteq I$ there exists a $J' \in \mathcal{F}$ such that $J \subseteq J'$. With (11) we obtain

$$\begin{aligned}
\sum_{i\in I} a_i &= \sup\{\sum_{i\in F} a_i \mid F \subseteq I, F \text{ finite}\} &&= \sup\{\sum_{i\in F} a_i \mid F \in \mathcal{F}\} \\
&= \sup\{\sum_{i\in F} b_i \mid F \in \mathcal{F}\} &&= \sup\{\sum_{i\in F} b_i \mid F \subseteq I, F \text{ finite}\} \\
&= \sum_{i\in I} b_i \enspace .
\end{aligned}$$

Thus $\sum_{i\in I} a_i = \sum_{i\in I} b_i$ and hence $\sum$ is necessary. $\square$

Note that every semiring that is $\aleph_0$-complete with respect to a necessary summation is naturally ordered [**64**, Proposition 22.29]. Finally, in an additively idempotent semiring with necessary summation we have $\sum_{i \in I} a = a$ for every $a \in A$ and countable index set I such that $\sum_{i \in I} a$ is well-defined.

PROPOSITION 3.18. *Let $\eta \in \{\varepsilon, \mathrm{o}\}$ be a modifier, $J \subseteq I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$ such that $J \cap \mathrm{var}(\psi) = I \cap \mathrm{var}(\psi)$, and for every $i \in I$ let $\psi_i \in \mathcal{A}\langle\!\langle T_\Delta(\mathrm{Z})\rangle\!\rangle$ such that $\psi_i \neq \widetilde{0}$ for every $i \in I \setminus J$. Then*

$$\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I} = \psi \xleftarrow[\eta]{} (\psi_j)_{j \in J} \quad , \tag{12}$$

provided that:

(1) *the left hand side is well-defined;*
(2) *if $\eta = \varepsilon$, then ψ_i is boolean for every $i \in I \setminus J$; and*
(3) *$\mathcal{A}$ is additively idempotent and $\sum$ necessary, or ψ_i is monomial for every $i \in I \setminus J$.*

PROOF.

$$\begin{aligned}
& \psi \xleftarrow[\eta]{} (\psi_i)_{i \in I} \\
= {} & \quad \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i)}} \Big((\psi, u) \cdot \prod_{i \in I} (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} \Big)\, u[u_i]_{i \in I} \\
= {} & \quad \textit{(because } i \notin \mathrm{var}(u) \subseteq \mathrm{var}(\psi) \textit{ for every } i \in I \setminus J \textit{ and:} \\
& \quad \bullet\ \textit{if } \eta = \varepsilon\textit{, then } \psi_i \textit{ is boolean and hence } (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} = 1\textit{; or} \\
& \quad \bullet\ \textit{if } \eta = \mathrm{o}\textit{, then } \mathrm{sel}(u,i,\eta) = 0 \textit{ and hence } (\psi_i, u_i)^{\mathrm{sel}(u,i,\eta)} = 1\textit{)} \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall i \in I)\colon u_i \in \mathrm{supp}(\psi_i)}} \Big((\psi, u) \cdot \prod_{j \in J} (\psi_j, u_j)^{\mathrm{sel}(u,j,\eta)} \Big)\, u[u_j]_{j \in J} \\
= {} & \quad \textit{(because } \mathrm{supp}(\psi_i) \neq \emptyset \textit{ for every } i \in I \setminus J \textit{ and:} \\
& \quad \bullet\ \mathcal{A} \textit{ is additively idempotent and } \textstyle\sum \textit{ is necessary; or} \\
& \quad \bullet\ \psi_i \textit{ is monomial for every } i \in I \setminus J\textit{)} \\
& \sum_{\substack{u \in \mathrm{supp}(\psi), \\ (\forall j \in J)\colon u_j \in \mathrm{supp}(\psi_j)}} \Big((\psi, u) \cdot \prod_{j \in J} (\psi_j, u_j)^{\mathrm{sel}(u,j,\eta)} \Big)\, u[u_j]_{j \in J} \\
= {} & \quad \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& \psi \xleftarrow[\eta]{} (\psi_j)_{j \in J}
\end{aligned}$$

□

The condition $J \cap \mathrm{var}(\psi) = I \cap \mathrm{var}(\psi)$ just asserts that J covers all those variables of ψ which are also covered by I; *i. e.*, $(I \setminus J) \cap \mathrm{var}(\psi) = \emptyset$. We have already seen that this restriction is necessary even for substitution on trees. Moreover, we have seen that for the corresponding statement on tree languages the condition $L_i \neq \emptyset$ is necessary for every $i \in I \setminus J$.

4.1. Pure substitution. In [**41**, Proposition 2.10] an associativity law for monomial tree series is proved and [**55**, Proposition 2.5] presents a generalized version. We present yet another straightforward generalization for pairwise variable-distinct polynomial tree series. The restriction to polynomial tree series avoids the problem of well-definedness. Similar results can usually be obtained for non-polynomial tree series, if we require that the semiring $\mathcal{A}$ is additionally $\aleph_0$-complete. We often mention the exact requirements after the statement for polynomial tree series.

PROPOSITION 3.19 (*cf.* [**55**, Proposition 2.5]). *Let $\mathcal{A}$ be commutative, $I, J \subseteq \mathbb{N}_+$ be finite, and $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_J)\rangle$. Let $\psi_j \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $j \in J$ and $(I_j)_{j\in J}$ be a partition of I conforming to $(\psi_j)_{j\in J}$. Finally, let $\tau_i \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $i \in I$.*

$$\big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} = \psi \xleftarrow[\varepsilon]{} \big(\, \psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\big)_{j\in J} \tag{13}$$

PROOF. Note that $J = \emptyset$ implies that $I = \emptyset$.

$$\begin{aligned}
& \big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} \\
= \quad & \textit{(by Proposition 3.8)} \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\mathrm{supp}(\psi_j)}} \Big((\psi, u)\, u \xleftarrow[\varepsilon]{} \big((\psi_j, u_j)\, u_j\big)_{j\in J}\Big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} \\
= \quad & \textit{(by [\textbf{55}, \text{Proposition 2.4}] and conformance of } (I_j)_{j\in J} \textit{ to } (\psi_j)_{j\in J}\textit{)} \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\mathrm{supp}(\psi_j)}} (\psi, u)\, u \xleftarrow[\varepsilon]{} \Big((\psi_j, u_j)\, u_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\Big)_{j\in J} \\
= \quad & \textit{(by Proposition 3.8)} \\
& \psi \xleftarrow[\varepsilon]{} \big(\psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\big)_{j\in J}
\end{aligned}$$

□

The previous statement also holds for non-polynomial tree series ψ, ψ_j, and τ_i provided that $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$. We presented an associativity-like law where the inner substitutions are $\psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}$. Proposition 3.18 tells us when we can equivalently replace such a substitution by $\psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I}$ where $I_j \subseteq I$. Hence we can easily derive our first associativity result from the previous two statements.

COROLLARY 3.20 (of Propositions 3.18 and 3.19). *Let $\mathcal{A}$ be commutative and additively idempotent, $I, J \subseteq \mathbb{N}_+$ be finite sets, and let $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_J)\rangle$. Moreover, let $\psi_j \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $j \in J$ and $(I_j)_{j\in J}$ be a partition of I conforming to $(\psi_j)_{j\in J}$. Finally, let $\tau_i \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ be boolean for every $i \in I$.*

$$\big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} = \psi \xleftarrow[\varepsilon]{} \big(\, \psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I}\big)_{j\in J} \tag{14}$$

PROOF. Note that $J = \emptyset$ implies that $I = \emptyset$. By Proposition 3.19 we have

$$\big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} = \psi \xleftarrow[\varepsilon]{} \big(\, \psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\big)_{j\in J}$$

and we can furthermore assume that $\operatorname{supp}(\tau_i) \neq \emptyset$ for every $i \in I$; otherwise both sides of (14) are $\widetilde{0}$ by Observation 3.4(3). Hence we have $\psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j} = \psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I}$ for every $j \in J$ by Proposition 3.18. Thus the statement follows. $\square$

This corollary can be generalized to non-polynomial tree series by adding the restriction that $\mathcal{A}$ is $\aleph_0$-complete with respect to a necessary $\sum$. A more detailed analysis of the proof shows that we can actually drop the commutativity condition, because only products of the form $(\psi_j, u_j) \cdot (\tau_i, v_i)$ need to commute. Since the τ_i are boolean, those products commute automatically. Next we consider the other case hinted already in Proposition 3.18, namely that the τ_i are monomial.

LEMMA 3.21. *Let $\mathcal{A}$ be commutative, $I, J \subseteq \mathbb{N}_+$ be finite sets, and $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_J)\rangle$. Moreover, let $(I_j)_{j\in J}$ be a family of $I_j \subseteq I$ such that $\bigcup_{j\in J} I_j = I$, $\psi_j \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ such that $\operatorname{var}(\psi_j) \subseteq I_j$ for every $j \in J$, and $\tau_i \in \mathcal{A}[T_\Delta(\mathrm{Z})]$ for every $i \in I$. If (τ_i, v_i) is multiplicatively idempotent for every $v_i \in T_\Delta(\mathrm{Z})$ and $i \in I$, then*

$$\big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I} = \psi \xleftarrow[\varepsilon]{} \big(\, \psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\big)_{j\in J} \ . \tag{15}$$

PROOF. Firstly, let $J = \emptyset$. Then also $I = \emptyset$ and both sides of (15) are ψ by Observation 3.4(1). Secondly, let $\operatorname{supp}(\tau_i) = \emptyset$ for some $i \in I$. It follows that $J \neq \emptyset$ and hence both sides of (15) are $\widetilde{0}$ again by Observation 3.4(3). Finally, we assume that $J \neq \emptyset$, and for every $i \in I$ let $v_i \in T_\Delta(\mathrm{Z})$ be such that $\operatorname{supp}(\tau_i) = \{v_i\}$.

$$\begin{aligned}
&\big(\psi \xleftarrow[\varepsilon]{} (\psi_j)_{j\in J}\big) \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I}\\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)}\\
& \sum_{\substack{u\in\operatorname{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\operatorname{supp}(\psi_j)}} \Big((\psi, u) \cdot \big(\prod_{j\in J}(\psi_j, u_j)\big) \cdot \prod_{i\in I}(\tau_i, v_i)\Big)\, u[u_j]_{j\in J}[v_i]_{i\in I}\\
= \quad & \textit{(because } \mathcal{A} \textit{ is commutative, } J \neq \emptyset,\ \operatorname{var}(u_j) \subseteq \operatorname{var}(\psi_j) \subseteq I_j,\\
& \quad \textit{and } (\tau_i, v_i) \textit{ is multiplicatively idempotent for every } i \in I\textit{)}\\
& \sum_{\substack{u\in\operatorname{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\operatorname{supp}(\psi_j)}} \left((\psi, u) \cdot \prod_{j\in J}\Big((\psi_j, u_j) \cdot \prod_{i\in I_j}(\tau_i, v_i)\Big)\right) u\big[u_j[v_i]_{i\in I_j}\big]_{j\in J}\\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)}\\
& \psi \xleftarrow[\varepsilon]{} \big(\psi_j \xleftarrow[\varepsilon]{} (\tau_i)_{i\in I_j}\big)_{j\in J}
\end{aligned}$$

$\square$

Note that if we set $I_j = I$ for every $j \in J$, then we obtain associativity. Moreover, if we restrict ourselves to boolean tree series τ_i, then every (τ_i, v_i) is automatically multiplicatively idempotent, and we can furthermore drop the commutativity restriction. This immediately yields that $\mathbb{B}[T_\Delta(\mathrm{Z})]$ forms an n-polypode [**19, 21**] under pure substitution for every $n \in \mathbb{N}$.

4.2. The case of o-substitution. Surprisingly, [**58**] does not investigate associativity of o-substitution, which can be established using the same approach. In particular, we need that $\mathcal{A}$ is N-Frobenius for some $N \subseteq \mathbb{N}$. Since o-substitution coincides with pure substitution on boolean tree series (*i. e.*, tree languages), we again distinguish two cases—there exists a conforming partition or the tree series τ_i are monomial.

Lemma 3.22. *Let $\mathcal{A}$ be commutative and N-Frobenius for some $N \subseteq \mathbb{N}$, $I, J \subseteq \mathbb{N}_+$ be finite, and $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_J)\rangle$. Moreover, let $(I_j)_{j\in J}$ be a family of $I_j \subseteq I$ such that $\bigcup_{j\in J} I_j = I$, $\psi_j \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ such that $\mathrm{var}(\psi_j) \subseteq I_j$ for every $j \in J$, and $\tau_i \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $i \in I$. Finally, let $\mathcal{A}$ be zero-divisor free if $0 \in N$, and let $|u|_{\mathrm{z}_j} \in N$ for every $u \in \mathrm{supp}(\psi)$ and $j \in J$.*

$$\big(\psi \xleftarrow[\mathrm{o}]{} (\psi_j)_{j\in J}\big) \xleftarrow[\mathrm{o}]{} (\tau_i)_{i\in I} = \psi \xleftarrow[\mathrm{o}]{} \Big(\, \psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i\in I_j} \Big)_{j\in J} \;, \qquad (16)$$

provided that:

- *$(I_j)_{j\in J}$ is a partition of I; or*
- *τ_i is monomial for every $i \in I$.*

Proof. Firstly, let $J = \emptyset$. Then $I = \emptyset$ and both sides of (16) are ψ by Observation 3.4(1). Secondly, let $J \neq \emptyset$. Moreover, suppose that $\tau_i = \widetilde{0}$ for some $i \in I$ or $\psi_j = \widetilde{0}$ for some $j \in J$. Then the both sides of (16) are $\widetilde{0}$ by Observation 3.4(3) because $J \neq \emptyset$.

Finally, let us assume that $J \neq \emptyset$ and $\tau_i \neq \widetilde{0} \neq \psi_j$ for every $i \in I$ and $j \in J$.

$$\begin{aligned}
&\big(\psi \xleftarrow[\mathrm{o}]{} (\psi_j)_{j\in J}\big) \xleftarrow[\mathrm{o}]{} (\tau_i)_{i\in I} \\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{o}]{}\textit{)} \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\mathrm{supp}(\psi_j),\\ (\forall i\in I)\colon v_i\in\mathrm{supp}(\tau_i)}} \Bigg((\psi, u) \cdot \Big(\prod_{j\in J} (\psi_j, u_j)^{|u|_{\mathrm{z}_j}} \Big) \cdot \prod_{i\in I} (\tau_i, v_i)^{|u[u_j]_{j\in J}|_{\mathrm{z}_i}} \Bigg) \; u[u_j]_{j\in J}[v_i]_{i\in I} \\
= \quad & \textit{(by commutativity and } |u[u_j]_{j\in J}|_{\mathrm{z}_i} = \textstyle\sum_{j\in J} |u|_{\mathrm{z}_j} \cdot |u_j|_{\mathrm{z}_i}\textit{)} \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\mathrm{supp}(\psi_j),\\ (\forall i\in I)\colon v_i\in\mathrm{supp}(\tau_i)}} \Bigg((\psi, u) \cdot \prod_{j\in J} \Big((\psi_j, u_j) \cdot \prod_{i\in I} (\tau_i, v_i)^{|u_j|_{\mathrm{z}_i}} \Big)^{|u|_{\mathrm{z}_j}} \Bigg) \; u[u_j[v_i]_{i\in I}]_{j\in J} \\
= \quad & \textit{(because } \tau_i \textit{ is monomial or } (I_j)_{j\in J} \textit{ is a partition of } I\textit{)} \\
& \sum_{\substack{u\in\mathrm{supp}(\psi),\\ (\forall j\in J)\colon u_j\in\mathrm{supp}(\psi_j),\\ (\forall i\in I_j)\colon v_{ij}\in\mathrm{supp}(\tau_i)}} \Bigg((\psi, u) \cdot \prod_{j\in J} \Big((\psi_j, u_j) \cdot \prod_{i\in I_j} (\tau_i, v_{ij})^{|u_j|_{\mathrm{z}_i}} \Big)^{|u|_{\mathrm{z}_j}} \Bigg) \; u[u_j[v_{ij}]_{i\in I_j}]_{j\in J}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \textit{(since } |u|_{z_j} \in N\text{, } \mathcal{A} \textit{ is } N\text{-}\textsc{Frobenius},\\
& \quad \textit{and } \mathcal{A} \textit{ is zero-divisor free if } |u|_{z_j} = 0\textit{)}\\
& \sum_{\substack{u \in \mathrm{supp}(\psi),\\ (\forall j \in J)\colon u'_j \in \mathrm{supp}(\psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I_j})}} (\psi, u) \cdot \\
& \cdot \prod_{j \in J} \Big(\sum_{\substack{u_j \in \mathrm{supp}(\psi_j),\\ (\forall i \in I_j)\colon v_{ij} \in \mathrm{supp}(\tau_i)}} \Big((\psi_j, u_j) \cdot \prod_{i \in I_j} (\tau_i, v_{ij})^{|u_j|_{z_i}} \Big) \, u_j[v_{ij}]_{i \in I_j}, u'_j \Big)^{|u|_{z_j}} \, u[u'_j]_{j \in J}\\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{o}]{} \textit{)}\\
& \sum_{\substack{u \in \mathrm{supp}(\psi),\\ (\forall j \in J)\colon u'_j \in \mathrm{supp}(\psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I_j})}} \Big((\psi, u) \cdot \prod_{j \in J} (\psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I_j}, u'_j)^{|u|_{z_j}} \Big) \, u[u'_j]_{j \in J}\\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{o}]{} \textit{)}\\
& \psi \xleftarrow[\mathrm{o}]{} \big(\psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I_j} \big)_{j \in J}
\end{aligned}
$$

□

Note that compared to Lemma 3.21 the condition that requires the τ_i to be boolean vanished, but the additional requirement of zero-divisor freeness emerged. However, if we again restrict the τ_i to be boolean, then we only have to consider products of a nonzero element with several factors being 1, hence we could drop the zero-divisor freeness condition in this case.

Corollary 3.23 (of Lemma 3.22 and Proposition 3.18). *Let $\mathcal{A}$ be commutative, additively idempotent, and N-*Frobenius *for some $N \subseteq \mathbb{N}$. Let $I, J \subseteq \mathbb{N}_+$ be finite and $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{Z}_J)\rangle$. Moreover, let $\psi_j \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $j \in J$, and let $(I_j)_{j \in J}$ be a partition of I conforming to $(\psi_j)_{j \in J}$. Let $\tau_i \in \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle$ for every $i \in I$. Finally, let $\mathcal{A}$ be zero-divisor free if $0 \in N$, and let $|u|_{z_j} \in N$ for every $u \in \mathrm{supp}(\psi)$ and $j \in J$.*

$$\big(\psi \xleftarrow[\mathrm{o}]{} (\psi_j)_{j \in J}\big) \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I} = \psi \xleftarrow[\mathrm{o}]{} \big(\psi_j \xleftarrow[\mathrm{o}]{} (\tau_i)_{i \in I} \big)_{j \in J} \tag{17}$$

Proof. The statement follows from Lemma 3.22 and Proposition 3.18. □

5. Preservation of recognizability

In this section we consider the question whether tree series substitution preserves recognizability. Let Σ be a ranked alphabet. It is known that substitution of the same tree t for two occurrences of a variable x, in general, does not preserve recognizability; *i. e.*, already for $n \in \mathbb{N}_+$, recognizable tree languages $L_1, \ldots, L_n \in \mathrm{RECOG}(\Sigma)$ and $L = \{u\}$ with $u \in T_\Sigma(\mathrm{X}_n)$ we have that $L[L_1, \ldots, L_n]$ is not necessarily recognizable (although $L_1, \ldots, L_n$ and $\{u\}$ are recognizable). However, IO-substitution on tree languages preserves recognizable tree languages, if the target tree language is linear (see [**36**, Theorem 3.65] or [**60**, Theorem 4.16]); *i. e.*, for every $n \in \mathbb{N}$ and $L, L_1, \dots, L_n \subseteq T_\Sigma(\mathrm{X})$ such that L is linear in X_n and $L, L_1, \dots, L_n$ are recognizable also $L[L_1, \dots, L_n]$ is recognizable.

First, let us clarify the notion of recognizable tree series [**9, 77, 18**]. We refer the reader to [**16**] for a detailed introduction and references to further models and results. We have chosen the automaton model called bu-w-fta-f (bottom-up finite-state weighted tree automaton with final weights) in [**16**, Section 4.1.3].

Definition 3.24 (see [**16**, Chapter 4]). *A* bottom-up weighted tree automaton (over Σ and $\mathcal{A}$) [**16**] *is a tuple $M = (Q, \Sigma, \mathcal{A}, F, \mu)$ where:*

- *Q is a nonempty, finite set of* states;
- *Σ is a ranked alphabet of* input symbols;
- *$\mathcal{A} = (A, +, \cdot)$ is a semiring;*
- *$F \colon Q \longrightarrow A$ is a* final weight distribution; *and*
- *$\mu = (\mu_k)_{k \in \mathbb{N}}$ with $\mu_k \colon \Sigma_k \longrightarrow A^{Q \times Q^k}$ is a* tree representation.

The initial algebra semantics *of M* [**16**, Section 4.1] *is determined by the Σ-algebra*

$$\mathcal{D} = (A^Q, (\overline{\mu_k(\sigma)})_{k \in \mathbb{N}, \sigma \in \Sigma_k})$$

where for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $V_1, \dots, V_k \in A^Q$

$$\overline{\mu_k(\sigma)}(V_1, \dots, V_k)_q = \sum_{q_1, \dots, q_k \in Q} \mu_k(\sigma)_{q, q_1 \cdots q_k} \cdot \prod_{i \in [k]} (V_i)_{q_i} \enspace .$$

Let $h_\mu \colon T_\Sigma \longrightarrow A^Q$ be the unique homomorphism from $\mathcal{T}_\Sigma$ to $\mathcal{D}$. The tree series recognized by *M, denoted by $\|M\|$, is defined for every $t \in T_\Sigma$ by*

$$(\|M\|, t) = \sum_{q \in Q} F_q \cdot h_\mu(t)_q \enspace .$$

We use the method of [**98**] and [**16**, Example 3.1.2] to graphically represent weighted tree automata. It is similar to the method used in Chapter 4 to represent tree series transducers.

Note that we write $\mu_0(\alpha)_q$ instead of $\mu_0(\alpha)_{q,\varepsilon}$ for every $\alpha \in \Sigma_0$ and $q \in Q$. Let Σ be a ranked alphabet and $\mathcal{A}$ be a semiring. A tree series $\psi \in \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$ is termed *recognizable,* if there exists a bottom-up weighted tree automaton M over Σ and $\mathcal{A}$ such that $\psi = \|M\|$. The class of all recognizable tree series over Σ and $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma \rangle\!\rangle$. For every finite $I \subseteq \mathbb{N}_+$, we say that a tree series $\psi \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X}_I) \rangle\!\rangle$ is *recognizable,* if $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta \rangle\!\rangle$ where $\Delta_k = \Sigma_k$ for every $k \in \mathbb{N}_+$ and $\Delta_0 = \Sigma_0 \cup \mathrm{X}_I$; *i. e.*, the elements of X_I are treated as new nullary symbols. Consequently, $\mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma(\mathrm{X}_I) \rangle\!\rangle = \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Delta \rangle\!\rangle$ denotes the class of all recognizable tree series over Σ, $\mathcal{A}$, and I.

Let us illustrate the previous definition. Let $\Sigma = \{\delta^{(2)}, \alpha^{(0)}\}$. We show that $\psi = \max_{u \in T_\Sigma(\mathrm{X}_1)} \mathrm{height}(u)\ u$ is a recognizable tree series (using the arctic semiring $\mathbb{A}_\infty$) by presenting a bottom-up weighted tree automaton that recognizes ψ.

EXAMPLE 3.25. *Let* $M_{3.25} = (Q, \Delta, \mathbb{A}_\infty, F, \mu)$ *be the bottom-up weighted tree automaton specified by*

- $Q = \{1, 2\}$;
- $\Delta = \{\delta^{(2)}, \alpha^{(0)}, \mathrm{x}_1^{(0)}\}$;
- $F_1 = 0$ *and* $F_2 = -\infty$; *and*
- $\mu_0(\alpha)_1 = \mu_0(\alpha)_2 = \mu_0(\mathrm{x}_1)_1 = \mu_0(\mathrm{x}_1)_2 = 0$ *and*

$$\mu_2(\delta)_{1,12} = \mu_2(\delta)_{1,21} = 1 \quad \textit{and} \quad \mu_2(\delta)_{2,22} = 0 \ .$$

(All remaining entries of $\mu_2(\delta)$ *are assumed to be* $-\infty$*.)*

The automaton is illustrated in Figure 2. We claim that

$$(\|M_{3.25}\|, u) = \mathrm{height}(u)$$

for every $u \in T_\Delta$*. This claim can be proved by a straightforward induction. We just demonstrate the previous definition on a particular example. Let* $u = \delta(\alpha, \delta(\alpha, \mathrm{x}_1))$*.*

$$\begin{aligned}
& h_\mu\big(\delta(\alpha, \delta(\alpha, \mathrm{x}_1))\big)_1 \\
&= \max_{p,q \in Q} \{\mu_2(\delta)_{1,pq} + h_\mu(\alpha)_p + h_\mu(\delta(\alpha, \mathrm{x}_1))_q\} \\
&= \max\Big(1 + h_\mu(\alpha)_1 + h_\mu(\delta(\alpha, \mathrm{x}_1))_2, 1 + h_\mu(\alpha)_2 + h_\mu(\delta(\alpha, \mathrm{x}_1))_1\Big) \\
&= \max\Big(1 + \mu_0(\alpha)_1 + h_\mu(\delta(\alpha, \mathrm{x}_1))_2, 1 + \mu_0(\alpha)_2 + h_\mu(\delta(\alpha, \mathrm{x}_1))_1\Big) \\
&= \max\Big(1 + \max_{p,q \in Q} \{\mu_2(\delta)_{2,pq} + h_\mu(\alpha)_p + h_\mu(\mathrm{x}_1)_q\}, \\
&\qquad\quad 1 + \max_{p,q \in Q} \{\mu_2(\delta)_{1,pq} + h_\mu(\alpha)_p + h_\mu(\mathrm{x}_1)_q\}\Big) \\
&= \max\Big(1 + h_\mu(\alpha)_2 + h_\mu(\mathrm{x}_1)_2, \\
&\qquad\quad 1 + \max\big(1 + h_\mu(\alpha)_1 + h_\mu(\mathrm{x}_1)_2, 1 + h_\mu(\alpha)_2 + h_\mu(\mathrm{x}_1)_1\big)\Big) \\
&= \max\Big(1 + \mu_0(\alpha)_2 + \mu_0(\mathrm{x}_1)_2, \\
&\qquad\quad 1 + \max\big(1 + \mu_0(\alpha)_1 + \mu_0(\mathrm{x}_1)_2, 1 + \mu_0(\alpha)_2 + \mu_0(\mathrm{x}_1)_1\big)\Big) \\
&= \max\big(1, 1 + \max(1, 1)\big) \\
&= \max(1, 2) \\
&= 2 = \mathrm{height}\big(\delta(\alpha, \delta(\alpha, \mathrm{x}_1))\big)
\end{aligned}$$

So $(\|M_{3.25}\|, u) = \max(F_1 + h_\mu(u)_1, F_2 + h_\mu(u)_2) = h_\mu(u)_1 = \mathrm{height}(u)$*.*

In fact, for every ranked alphabet Δ we can give a bottom-up weighted tree automaton (over the arctic semiring $\mathbb{A}_\infty$) recognizing $\max_{u \in T_\Delta} \mathrm{height}(u)\, u$.

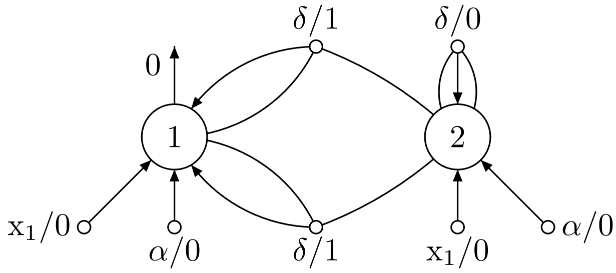

FIGURE 2. Bottom-up weighted tree automaton $M_{3.25}$ over $\mathbb{A}_\infty$ (see Example 3.25).

Now let us return to the question of preservation of recognizability. In [**79**, Corollary 14] it is proved that recognizability is preserved whenever the target tree series is nondeleting and linear. Since this statement is proved for OI-substitution in [**79**], we first relate OI-substitution to pure substitution. Therefore we present the definition of OI-substitution of [**79**]. Recall that $\mathcal{A}$ is a semiring that is (i) $\aleph_0$-complete with respect to $\sum$ or (ii) $\sum$ is the infinitary summation induced by $+$. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $\psi_1, \dots, \psi_k \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$, we define

$$\sigma(\psi_1, \dots, \psi_k) = \sum_{t_1,\dots,t_k \in T_\Sigma(\mathrm{X})} \big((\psi_1, t_1) \cdot \ldots \cdot (\psi_k, t_k)\big)\, \sigma(t_1, \dots, t_k) \enspace .$$

Note that this sum is always well-defined. Let $t \in T_\Sigma(\mathrm{X})$, $I \subseteq \mathbb{N}_+$ be finite, and $\psi_i \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$ for every $i \in I$. For every $j \in I$, $\ell \in \mathbb{N}_+ \setminus I$, $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma(\mathrm{X})$ let

$$\begin{aligned}
\mathrm{x}_j \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I} &= \psi_j \\
\mathrm{x}_\ell \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I} &= 1\, \mathrm{x}_\ell \\
\sigma(t_1, \dots, t_k) \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I} &= \sigma(t_1 \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I}, \dots, t_k \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I}) \enspace .
\end{aligned}$$

Finally, we define $\psi \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I}$ for every $\psi \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$ by

$$\psi \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I} = \sum_{t \in T_\Sigma(\mathrm{X})} (\psi, t) \cdot \big(t \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I}\big) \enspace .$$

Note that also this sum is always well-defined. With the help of [**79**, Theorem 6] we can easily relate pure and OI-substitution.

OBSERVATION 3.26. *Let $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X}_I)\rangle\!\rangle$ be nondeleting and linear in X_I, and $\psi_i \in \mathcal{A}\langle\!\langle T_\Sigma(\mathrm{X})\rangle\!\rangle$ for every $i \in I$.*

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I} = \psi \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i\in I}$$

PROOF. Clearly, $t \xleftarrow[\mathrm{OI}]{} (1\, t_i)_{i\in I} = 1\, t[t_i]_{i\in I}$ for every $t \in T_\Sigma(\mathrm{X}_I)$ and $(t_i)_{i\in I} \in T_\Sigma(\mathrm{X})^I$.

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I}$$

$$
\begin{aligned}
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& \sum_{\substack{t \in T_\Sigma(\mathrm{X}_I), \\ (\forall i \in I)\colon t_i \in T_\Sigma(\mathrm{X})}} \Big((\psi, t) \cdot \prod_{i \in I} (\psi_i, t_i) \Big) \cdot (1\, t[t_i]_{i \in I}) \\
= \quad & \textit{(by } t \xleftarrow[\mathrm{OI}]{} (1\, t_i)_{i \in I} = 1\, t[t_i]_{i \in I}\textit{)} \\
& \sum_{\substack{t \in T_\Sigma(\mathrm{X}_I), \\ (\forall i \in I)\colon t_i \in T_\Sigma(\mathrm{X})}} \Big((\psi, t) \cdot \prod_{i \in I} (\psi_i, t_i) \Big) \cdot \big(t \xleftarrow[\mathrm{OI}]{} (1\, t_i)_{i \in I}\big) \\
= \quad & \textit{(by } [\mathbf{79}, \text{Theorem 6}]\textit{)} \\
& \sum_{t \in T_\Sigma(\mathrm{X}_I)} (\psi, t) \cdot \big(t \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i \in I}\big) \\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{OI}]{}\textit{)} \\
& \psi \xleftarrow[\mathrm{OI}]{} (\psi_i)_{i \in I}
\end{aligned}
$$

□

THEOREM 3.27 (see [**79**, Corollary 14]). *For every $\eta \in \{\varepsilon, \mathrm{o}\}$, finite $I \subseteq \mathbb{N}_+$, family $(\psi_i)_{i \in I} \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma \rangle\!\rangle^I$, and $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma(\mathrm{X}_I) \rangle\!\rangle$ such that ψ is nondeleting and linear in X_I, we have $\psi \xleftarrow[\eta]{} (\psi_i)_{i \in I} \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma \rangle\!\rangle$.*

PROOF. The statement is proved for OI-substitution in [**79**, Corollary 14]. Since OI-substitution coincides with η-substitution on nondeleting and linear target tree series (see Observations 3.26 and 3.7), we obtain the statement. □

5.1. The case of o-substitution. Let us consider the preservation of recognizability for o-substitution first. Since o-substitution is an IO-type of substitution we immediately restrict the target tree series to be linear. Let us illustrate the main idea in a simplified setting. Let $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma(\mathrm{X}_1) \rangle\!\rangle$ be linear in X_1 and $\psi_1 \in \mathcal{A}^{\mathrm{rec}}\langle\!\langle T_\Sigma \rangle\!\rangle$. We want to show that $\psi \xleftarrow[\mathrm{o}]{} (\psi_1)$ is recognizable, thus we need to present a bottom-up weighted tree automaton $M' = (Q', \Sigma, \mathcal{A}, F', \mu')$ that recognizes $\psi \xleftarrow[\mathrm{o}]{} (\psi_1)$. Let $M = (Q, \Delta, \mathcal{A}, F, \mu)$ be a bottom-up weighted tree automaton recognizing ψ and $M_1 = (Q_1, \Sigma, \mathcal{A}, F_1, \mu_1)$ be a bottom-up weighted tree automaton recognizing ψ_1. We employ a standard idea for the construction of M'. Roughly speaking, we take the disjoint union of M and M_1 and add transitions that nondeterministically change from M_1 to M. More precisely, for every $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, $q \in Q$, and $q_1, \ldots, q_k \in Q_1$ we set

$$\mu'_k(\sigma)_{q, q_1 \cdots q_k} = \sum_{p \in Q_1} \mu_0(\mathrm{x}_1)_q \cdot (F_1)_p \cdot (\mu_1)_k(\sigma)_{p, q_1 \cdots q_k} \ .$$

Informally speaking, for each state p of M_1 we take the weight of the standard transition $(\mu_1)_k(\sigma)_{p, q_1 \cdots q_k}$ of M_1, multiply the corresponding entry $(F_1)_p$ in the final distribution, and multiply the weight $\mu_0(\mathrm{x}_1)_q$ for entering M (via x_1) in state q. Nullary symbols σ are treated similarly. We employ a proof method, which requires us to make the input alphabets Σ and Δ disjoint. This simplifies the proof because each tree then admits a unique decomposition into (at most one) part that needs to be processed by M_1 and a part that needs to be processed by M.

THEOREM 3.28. *Let $\mathcal{A}$ be a commutative and additively idempotent semiring, which is $\aleph_0$-complete with respect to the necessary $\sum$. Moreover, let $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma(\mathrm{X}_I)\rangle\rangle$ be linear in X_I, and $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle$ for every $i \in I$. Then*

$$\psi \underset{\mathrm{o}}{\longleftarrow} (\psi_i)_{i\in I} \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle \quad . \tag{18}$$

PROOF. Let $\psi_i = \widetilde{0}$ for some $i \in I$. Then $\psi \underset{\mathrm{o}}{\longleftarrow} (\psi_i)_{i\in I} = \widetilde{0}$ by Observation 3.4(3) which is clearly recognizable. Hence for the remainder of the proof we assume that $\psi_i \neq \widetilde{0}$ for all $i \in I$. For every $k \in \mathbb{N}_+$ let $\Delta_k = \Sigma_k$ and $\Delta_0 = \Sigma_0 \cup \mathrm{X}_I$. Since $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma(\mathrm{X}_I)\rangle\rangle$ and $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle$ for every $i \in I$, there exist bottom-up weighted tree automata $M = (Q, \Delta, \mathcal{A}, F, \mu)$ and $M_i = (Q_i, \Sigma, \mathcal{A}, F_i, \mu_i)$ such that $\|M\| = \psi$ and $\|M_i\| = \psi_i$ for every $i \in I$.

For every $i \in I$ let $\overline{\Sigma}^i$ be the ranked alphabet given by $\overline{\Sigma}^i_k = \{\overline{\sigma}^i \mid \sigma \in \Sigma_k\}$ for every $k \in \mathbb{N}$. For every $i \in I$ we define the mapping $\mathrm{bar}_i \colon T_\Sigma \longrightarrow T_{\overline{\Sigma}^i}$ by

$$\mathrm{bar}_i(\sigma(t_1, \dots, t_k)) = \overline{\sigma}^i(\mathrm{bar}_i(t_1), \dots, \mathrm{bar}_i(t_k))$$

for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$. Moreover, we extend bar_i to tree series as follows. We define the mapping $\mathrm{bar}_i \colon \mathcal{A}\langle\langle T_\Sigma\rangle\rangle \longrightarrow \mathcal{A}\langle\langle T_{\overline{\Sigma}^i}\rangle\rangle$, which relabels all σ-nodes by their corresponding i-overlined version, for every $\varphi \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle$ by

$$\mathrm{bar}_i(\varphi) = \sum_{t\in T_\Sigma} (\varphi, t)\ \mathrm{bar}_i(t) \quad .$$

Without loss of generality, we assume that for every $i \in I$ we have that (i) Σ and $\overline{\Sigma}^i$ are disjoint and (ii) Q and Q_i are disjoint. We let $\Sigma'_k = \Sigma_k \cup \bigcup_{i\in I} \overline{\Sigma}^i_k$ for every $k \in \mathbb{N}$, and $Q' = Q \cup \bigcup_{i\in I} Q_i$. We construct a bottom-up weighted tree automaton M' recognizing $\psi \underset{\mathrm{o}}{\longleftarrow} (\mathrm{bar}_i(\psi_i))_{i\in I}$ as follows. Let $M' = (Q', \Sigma', \mathcal{A}, F', \mu')$ where for every $i \in I$, $k \in \mathbb{N}$, $\sigma \in \Sigma_k$:

- $F'_q = F_q$ for every $q \in Q$ and $F'_p = 0$ for every $p \in \bigcup_{i\in I} Q_i$;
- $\mu'_k(\overline{\sigma}^i)_{p,w} = (\mu_i)_k(\sigma)_{p,w}$ for every $p \in Q_i$ and $w \in (Q_i)^k$;
- $\mu'_k(\sigma)_{q,w} = \mu_k(\sigma)_{q,w}$ for every $q \in Q$ and $w \in Q^k$; and
- $\mu'_k(\overline{\sigma}^i)_{q,w} = \sum_{p\in Q_i} \mu_0(\mathrm{x}_i)_q \cdot (F_i)_p \cdot (\mu_i)_k(\sigma)_{p,w}$ for every $q \in Q$ and $w \in (Q_i)^k$.

All the remaining entries in μ' are set to 0.

Clearly, $h_{\mu'}(\mathrm{bar}_i(t))_p = h_{\mu_i}(t)_p$ for every $i \in I$, $t \in T_\Sigma$, and $p \in Q_i$. Next we prove that for every $q \in Q$ and $t \in T_\Sigma(\mathrm{X}_I)$, which is linear in X_I, and family $(u_i)_{i\in\mathrm{var}(t)} \in T_\Sigma^{\mathrm{var}(t)}$ we have

$$h_{\mu'}(t[\mathrm{bar}_i(u_i)]_{i\in\mathrm{var}(t)})_q = h_\mu(t)_q \cdot \prod_{i\in\mathrm{var}(t)} (\|M_i\|, u_i) \quad .$$

We prove this statement inductively, so let $t = \mathrm{x}_j$ for some $j \in I$. Moreover, let $u_j = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$
\begin{aligned}
& h_{\mu'}(\mathrm{x}_j[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(\mathrm{x}_j)})_q \\
= \quad & \textit{(by substitution and definition of } \mathrm{bar}_j\textit{)} \\
& h_{\mu'}\big(\overline{\sigma}^j(\mathrm{bar}_j(t_1), \ldots, \mathrm{bar}_j(t_k))\big)_q \\
= \quad & \textit{(by Definition 3.24)} \\
& \sum_{q_1,\ldots,q_k \in Q'} \mu'_k(\overline{\sigma}^j)_{q,q_1\cdots q_k} \cdot \prod_{i \in [k]} h_{\mu'}(\mathrm{bar}_j(t_i))_{q_i} \\
= \quad & \textit{(by definition of } \mu' \textit{ and } h_{\mu'}(\mathrm{bar}_j(t_i))_{q_i} = h_{\mu_j}(t_i)_{q_i}\textit{)} \\
& \sum_{q_1,\ldots,q_k \in Q_j} \sum_{p \in Q_j} \mu_0(\mathrm{x}_j)_q \cdot (F_j)_p \cdot (\mu_j)_k(\sigma)_{p,q_1\cdots q_k} \cdot \prod_{i \in [k]} h_{\mu_j}(t_i)_{q_i} \\
= \quad & \textit{(by Definition 3.24)} \\
& \sum_{p \in Q_j} \mu_0(\mathrm{x}_j)_q \cdot (F_j)_p \cdot h_{\mu_j}(\sigma(t_1, \ldots, t_k))_p \\
= \quad & \textit{(by Definition 3.24)} \\
& \mu_0(\mathrm{x}_j)_q \cdot \big(\|M_j\|, \sigma(t_1, \ldots, t_k)\big) \\
= \quad & \textit{(by Definition 3.24)} \\
& h_\mu(\mathrm{x}_j)_q \cdot \prod_{i \in \mathrm{var}(\mathrm{x}_j)} \big(\|M_i\|, u_i\big)
\end{aligned}
$$

Now, let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma(\mathrm{X}_I)$.

$$
\begin{aligned}
& h_{\mu'}(\sigma(t_1, \ldots, t_k)[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)})_q \\
= \quad & \textit{(by substitution)} \\
& h_{\mu'}(\sigma(t_1[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_1)}, \ldots, t_k[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_k)}))_q \\
= \quad & \textit{(by Definition 3.24)} \\
& \sum_{q_1,\ldots,q_k \in Q'} \mu'_k(\sigma)_{q,q_1\cdots q_k} \cdot \prod_{j \in [k]} h_{\mu'}(t_j[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t_j)})_{q_j} \\
= \quad & \textit{(by induction hypothesis and definition of } \mu'\textit{)} \\
& \sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,q_1\cdots q_k} \cdot \prod_{j \in [k]} \Big(h_\mu(t_j)_{q_j} \cdot \prod_{i \in \mathrm{var}(t_j)} \big(\|M_i\|, u_i\big)\Big) \\
= \quad & \textit{(by Definition 3.24)} \\
& h_\mu(\sigma(t_1, \ldots, t_k))_q \cdot \prod_{j \in [k], i \in \mathrm{var}(t_j)} \big(\|M_i\|, u_i\big) \\
= \quad & \textit{(because } t \textit{ is linear in } \mathrm{X}_I\textit{)} \\
& h_\mu(\sigma(t_1, \ldots, t_k))_q \cdot \prod_{i \in \mathrm{var}(t)} \big(\|M_i\|, u_i\big)
\end{aligned}
$$

This completes the proof of the auxiliary statement. Consequently,

$$(\|M'\|, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}) = (\|M\|, t) \cdot \prod_{i \in \mathrm{var}(t)} (\|M_i\|, u_i)$$

$$= (\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\psi_i, u_i) \ . \tag{19}$$

Using this result, we can show that $\psi' = \psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_i(\psi_i))_{i \in I}$ is recognizable. In fact, this is the tree series that is recognized by M'.

$$\begin{aligned}
&\psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_i(\psi_i))_{i \in I} \\
= &\quad \textit{(by Proposition 3.10)} \\
&\sum_{t \in \mathrm{supp}(\psi)} (\psi, t) \cdot \Big((1\, t) \xleftarrow[\mathrm{o}]{} \big(\mathrm{bar}_i(\psi_i)\big)_{i \in I} \Big) \\
= &\quad \textit{(by Proposition 3.18)} \\
&\sum_{t \in \mathrm{supp}(\psi)} (\psi, t) \cdot \Big((1\, t) \xleftarrow[\mathrm{o}]{} \big(\mathrm{bar}_i(\psi_i)\big)_{i \in \mathrm{var}(t)} \Big) \\
= &\quad \textit{(by definition of } \xleftarrow[\mathrm{o}]{} \textit{ because } t \textit{ is linear)} \\
&\sum_{\substack{t \in \mathrm{supp}(\psi), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in \mathrm{supp}(\mathrm{bar}_i(\psi_i))}} \Big((\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\mathrm{bar}_i(\psi_i), u_i) \Big)\ t[u_i]_{i \in \mathrm{var}(t)} \\
= &\quad \textit{(by definition of } \mathrm{bar}_i\textit{)} \\
&\sum_{\substack{t \in T_\Sigma(\mathrm{X}_I), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in T_\Sigma}} \Big((\psi, t) \cdot \prod_{i \in \mathrm{var}(t)} (\psi_i, u_i) \Big)\ t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)} \\
= &\quad \textit{(by (19))} \\
&\sum_{\substack{t \in T_\Sigma(\mathrm{X}_I), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in T_\Sigma}} \big(\|M'\|, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)} \big)\ t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)} \\
= &\sum_{u \in T_{\Sigma'}} \Big(\sum_{\substack{t \in T_\Sigma(\mathrm{X}_I), \\ (\forall i \in \mathrm{var}(t))\colon u_i \in T_\Sigma}} \big(\|M'\|, t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)} \big)\ t[\mathrm{bar}_i(u_i)]_{i \in \mathrm{var}(t)}\ ,\ u \Big)\ u \\
= &\quad \textit{(because } t \textit{ and } u_i \textit{ are uniquely determined by } u\textit{)} \\
&\sum_{u \in T_{\Sigma'}} \big(\|M'\|, u \big)\ u \ = \ \|M'\|
\end{aligned}$$

Finally, we need to remove the annotation. To this end we define the mapping unbar: $T_{\Sigma'}(\mathrm{X}) \longrightarrow T_\Sigma(\mathrm{X})$ for every $x \in \mathrm{X}$, $k \in \mathbb{N}$, $i \in I$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_{\Sigma'}(\mathrm{X})$ by

$$\begin{aligned}
\mathrm{unbar}(x) &= x \\
\mathrm{unbar}(\sigma(t_1, \ldots, t_k)) &= \sigma(\mathrm{unbar}(t_1), \ldots, \mathrm{unbar}(t_k)) \\
\mathrm{unbar}(\overline{\sigma}^i(t_1, \ldots, t_k)) &= \sigma(\mathrm{unbar}(t_1), \ldots, \mathrm{unbar}(t_k)) \ .
\end{aligned}$$

Note that unbar is a homomorphism. We lift unbar to tree series as follows. Let unbar: $\mathcal{A}\langle\langle T_{\Sigma'}(\mathrm{X})\rangle\rangle \longrightarrow \mathcal{A}\langle\langle T_\Sigma(\mathrm{X})\rangle\rangle$ be the mapping defined for every $\varphi \in \mathcal{A}\langle\langle T_{\Sigma'}(\mathrm{X})\rangle\rangle$ by

$$\mathrm{unbar}(\varphi) = \sum_{t \in T_{\Sigma'}(\mathrm{X})} (\varphi, t)\ \mathrm{unbar}(t) \ .$$

Clearly, $\mathrm{unbar}(\psi') = \psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i \in I}$. Moreover, unbar can be realized by a nondeleting, linear tree transducer (with one state and with OI-substitution) of [**79**] (because it is a relabeling homomorphism). Since ψ' is a recognizable tree series and nondeleting, linear tree transducers of [**79**] preserve recognizability [**79**, Corollary 14], also $\mathrm{unbar}(\psi')$ is recognizable, which proves the statement. The proof of [**79**, Corollary 14] additionally assumes a continuous semiring, but this assumption is not needed for the special case considered here. Alternatively the last step can be shown by referring to the closure of the class of recognizable tree series under linear and nondeleting tree homomorphisms [**15**]. □

Let us illustrate the previous proposition on two examples. The first example shows that the linearity restriction is necessary and the second example demonstrates a successful application of Theorem 3.28.

EXAMPLE 3.29. *Let* $\Sigma = \{\delta^{(2)}, \alpha^{(0)}\}$. *Let* $\psi_1 = \max_{u \in T_\Sigma} \mathrm{height}(u)\, u$ *and* $\psi = \max_{u \in T_\Sigma(\mathrm{X}_1)} \mathrm{height}(u)\, u$ *over the semiring* $\mathbb{A}_\infty$. *Note that* ψ *is not linear in* X_1. *Nevertheless we apply the construction found in the proof of Theorem 3.28 (see Example 3.25 for the bottom-up weighted tree automaton recognizing* ψ*) and obtain the bottom-up weighted tree automaton* $M_{3.29} = (Q, \Gamma, \mathbb{A}_\infty, F, \mu)$ *with*

- $Q = \{1, 2, 3, 4\}$;
- $\Gamma_2 = \{\delta, \overline{\delta}\}$ *and* $\Gamma_0 = \{\alpha, \overline{\alpha}\}$ *(for the sake of readability we omit the* 1 *at the overlining);*
- $F_1 = 0$ *and* $F_2 = F_3 = F_4 = -\infty$; *and*
- $\mu_0(\alpha)_1 = \mu_0(\alpha)_2 = \mu_0(\overline{\alpha})_1 = \mu_0(\overline{\alpha})_2 = \mu_0(\overline{\alpha})_3 = \mu_0(\overline{\alpha})_4 = 0$ *and*

$$\mu_2(\delta)_{1,12} = \mu_2(\delta)_{1,21} = \mu_2(\overline{\delta})_{3,34} = \mu_2(\overline{\delta})_{3,43} = 1$$
$$\mu_2(\delta)_{2,22} = \mu_2(\overline{\delta})_{4,44} = 0$$
$$\mu_2(\overline{\delta})_{1,34} = \mu_2(\overline{\delta})_{1,43} = 1 \ .$$

(All remaining entries of $\mu_2(\delta)$ *and* $\mu_2(\overline{\delta})$ *are assumed to be* $-\infty$*.)*

The automaton $M_{3.29}$ *is displayed in Figure 3. However,* $M_{3.29}$ *does not recognize* $\psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_1(\psi_1))$. *To demonstrate, let* $u = \delta(\overline{\delta}(\overline{\alpha}, \overline{\alpha}), \overline{\delta}(\overline{\alpha}, \overline{\alpha}))$. *Clearly,* $(\|M_{3.29}\|, u) = -\infty$, *but* $(\psi \xleftarrow[\mathrm{o}]{} (\mathrm{bar}_1(\psi_1)), u) = 3$. *The latter statement can be seen using the decomposition* $u = \delta(\mathrm{x}_1, \mathrm{x}_1)[\overline{\delta}(\overline{\alpha}, \overline{\alpha})]$.

EXAMPLE 3.30. *Let* $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$. *Let* $\psi_1 = \max_{u \in T_\Sigma} \mathrm{height}(u)\, u$ *and* $\psi = \max_{u \in T_\Sigma(\mathrm{X}_1)} \mathrm{height}(u)\, u$ *be tree series over the arctic semiring* $\mathbb{A}_\infty$. *Then* $\psi \xleftarrow[\mathrm{o}]{} (\psi_1)$ *is recognizable. In fact,* $\psi \xleftarrow[\mathrm{o}]{} (\psi_1) = \psi_1$. *We show the bottom-up weighted tree automata that recognize* ψ *and* $\psi \xleftarrow[\mathrm{o}]{} (\psi_1)$ *[the automaton that is constructed in Theorem 3.28] in Figure 4.*

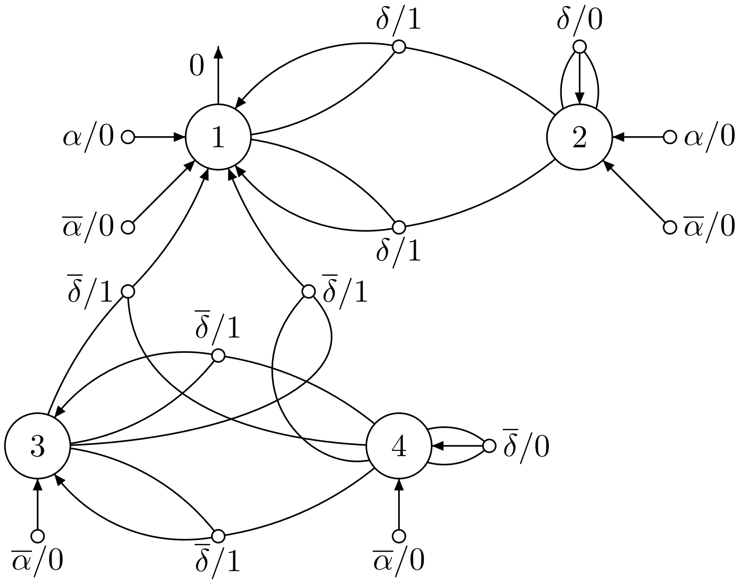

FIGURE 3. Bottom-up weighted tree automaton $M_{3.29}$ over $\mathbb{A}_\infty$ (see Example 3.29).

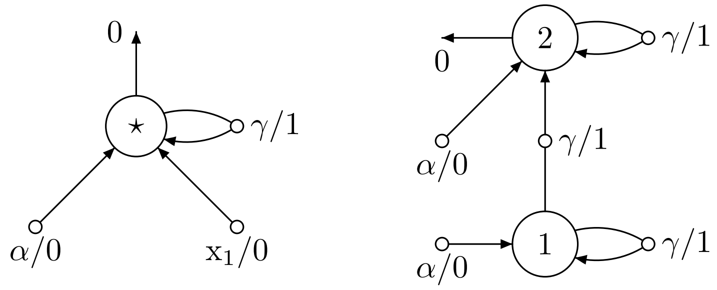

FIGURE 4. Bottom-up weighted tree automata recognizing ψ [left] and $\psi \xleftarrow[\mathrm{o}]{} (\psi_1)$ [right] over $\mathbb{A}_\infty$ of Example 3.30.

5.2. Pure substitution. Finally, this section is concluded by a result which shows preservation of recognizability for pure substitution.

COROLLARY 3.31 (of Theorem 3.28). *Let $\mathcal{A}$ be commutative, additively idempotent, and $\aleph_0$-complete with respect to a necessary $\sum$. Moreover, let $I \subseteq \mathbb{N}_+$ be finite, $\psi \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma(\mathrm{X}_I)\rangle\rangle$ be linear in X_I, and $\psi_i \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle$ be boolean for every $i \in I$.*

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I} \in \mathcal{A}^{\mathrm{rec}}\langle\langle T_\Sigma\rangle\rangle \qquad (20)$$

PROOF. Actually, under the given conditions

$$\psi \xleftarrow[\varepsilon]{} (\psi_i)_{i\in I} = \psi \xleftarrow[\mathrm{o}]{} (\psi_i)_{i\in I}$$

by Observation 3.7 and o-substitution preserves recognizability by Theorem 3.28. □

6. Open problems and future work

A detailed analysis of [IO] and OI tree series substitution with respect to the properties of distributivity, linearity, and associativity is still missing. Some results are already present in the literature (*e. g.*, [**77, 18**]), but a systematic study would, *e. g.*, help to unravel more composition results for tree series transducers using those types of substitution. Another interesting line of research would be to investigate preservation of recognizability (as in [**79**]) for [IO] and OI substitution. This together with composition results could potentially benefit the theory of abstract families of tree series.

CHAPTER 4

Tree Series Transducers

To see distinctly the machinery—the wheels and pinions—
of any work of Art is, unquestionably, of itself, a pleasure,
but one which we are able to enjoy only just in proportion as
we do not enjoy the legitimate effect designed by the artist.

Edgar Allan Poe (1809–1849): "Marginalia 266"
Southern Literary Messenger, 1849

1. Bibliographic information

In this chapter we present the definition of tree series transducers, the device which we investigate in the rest of the thesis. The definition we present is taken from [**41**] with one minor extension. We illustrate the definition using two examples and study the ramifications of our slight extension. The core definition is complemented with several properties of tree series transducers mostly taken from [**41**] and [**83**]. Finally, this chapter is concluded by the presentation of some simple statements about tree series transducers.

2. Core definition

In this section we recall from [**41**] the notion of tree series transducers, abbreviated to tst henceforth. In particular, we present the definition of bottom-up and top-down tst, which are subsequently abbreviated to bu-tst and td-tst, respectively.

2.1. Syntax and graphical representation. We have seen in Figure 1 of Chapter 1 that tst generalize tree transducers [**102, 106, 107**]. So let us start from those devices. A bottom-up tree transducer $M = (Q, \Sigma, \Delta, F, R)$ has rules of the form

$$\sigma\big(q_1(\mathrm{z}_1), \dots, q_k(\mathrm{z}_k)\big) \to q(u)$$

where $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q, q_1, \dots, q_k \in Q$, and $u \in T_\Delta(\mathrm{Z}_k)$. Several such rules with different u can be grouped together with the help of a mapping

$$\nu_k \colon \Sigma_k \times Q^k \times Q \longrightarrow \mathfrak{P}(T_\Delta(\mathrm{Z}_k)) \enspace .$$

More precisely,

$$\Big(\sigma\big(q_1(\mathrm{z}_1), \dots, q_k(\mathrm{z}_k)\big) \to q(u)\Big) \in R \qquad \Longleftrightarrow \qquad u \in \nu_k(\sigma, q_1, \dots, q_k, q) \enspace .$$

We have seen in Observation 2.3 that

$$(\mathfrak{P}(T_\Delta(\mathrm{Z}_k)), \cup, \cap) \cong (\mathbb{B}\langle\langle T_\Delta(\mathrm{Z}_k)\rangle\rangle, \vee, \wedge) \enspace .$$

Using a matrix representation we can refine the mapping to

$$\mu_k \colon \Sigma_k \longrightarrow \mathbb{B}\langle\langle T_\Delta(\mathrm{Z}_k)\rangle\rangle^{Q \times Q^k}$$

with the correspondence

$$u \in \nu_k(\sigma, q_1, \dots, q_k, q) \qquad \Longleftrightarrow \qquad (\mu_k(\sigma)_{q, q_1 \cdots q_k}, u) = 1 \enspace .$$

This representation of the rules can now be easily generalized to arbitrary semirings. For this we simply replace the semiring $\mathbb{B}$ by an arbitrary semiring $\mathcal{A}$. Essentially such a family $(\mu_k)_{k \in \mathbb{N}}$ of mappings μ_k is called a (bottom-up) tree representation [**79**].

Next we present the definition of tree representations, which encode the rules, of arbitrary tst from [**41**]. Let Q be a finite set, which represents the state set of a tst. Recall that for every $V \subseteq \mathrm{X}$ we abbreviate $\{q(v) \mid q \in Q, v \in V\}$ simply by $Q(V)$. Roughly speaking, a tree representation is a family of mappings, each of which maps an input symbol to a matrix indexed by sequences of (annotated) states (more formally, by a state and an element of $Q(\mathrm{X})^*$). The entries of those matrices are tree series over Δ, Z, and $\mathcal{A}$, where Δ is an output ranked alphabet and $\mathcal{A}$ is a semiring. Note that in contrast to [**41**] we use elements of X to refer to input trees and elements of Z to refer to output trees.

Definition 4.1 (see [**41**, Definition 3.1]). *Given a finite set Q, ranked alphabets Σ and Δ, and a semiring $\mathcal{A} = (A, +, \cdot)$, a* tree representation (over Q, Σ, Δ, and $\mathcal{A}$) *is a family $(\mu_k)_{k \in \mathbb{N}}$ of mappings $\mu_k \colon \Sigma_k \longrightarrow \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle^{Q \times Q(\mathrm{X}_k)^*}$ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q \in Q$:*

- *there exist only finitely many $w \in Q(\mathrm{X}_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$; and*
- $\mathrm{var}(\mu_k(\sigma)_{q,w}) \subseteq \mathrm{Z}_{|w|}$ *for every $w \in Q(\mathrm{X}_k)^*$.*

Let us explain this notion in more detail. Let Σ and Δ be ranked alphabets, $\mathcal{A}$ a semiring, and Q a set of states. Further, let μ be a tree representation over Q, Σ, Δ, and $\mathcal{A}$. Roughly speaking, for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ encodes output trees (which may contain variables of Z_n) together with their weight. If for some $t_1, \dots, t_k \in T_\Sigma$ we would like to process the input tree $\sigma(t_1, \dots, t_k)$ in state q, then according to this entry we have to process the i_j-th direct subtree t_{i_j} in state q_j for every $j \in [n]$. This way we obtain, for every $j \in [n]$, the tree series ψ_j. Those tree series are combined with $\mu_k(\sigma)_{q,w}$ by means of tree series substitution, in which $\mu_k(\sigma)_{q,w}$ is the target and $\psi_1, \dots, \psi_n$ are the sources. Thus the variables $\mathrm{z}_1, \dots, \mathrm{z}_n$ in the output trees of $\mathrm{supp}(\mu_k(\sigma)_{q,w})$ represent placeholders for the transformations of the subtrees. With this

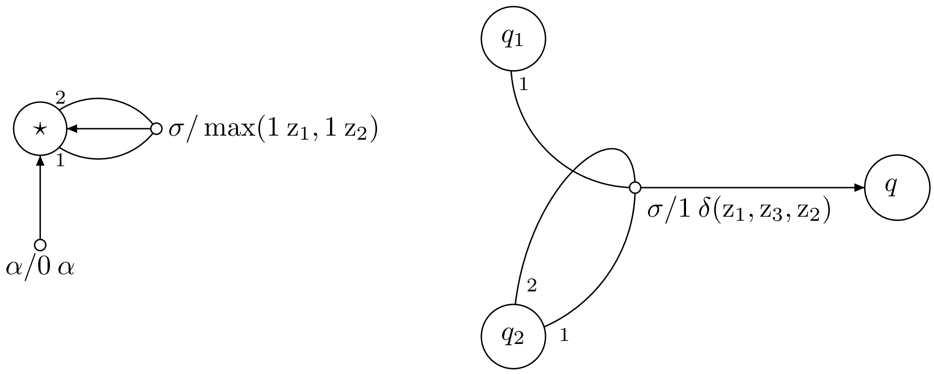

FIGURE 1. Examples of tree representations.

intuition the second item in Definition 4.1 is a simple sanity condition. If the considered transition calls for n translations of input trees (*i. e.*, $|w| = n$), then $\mathrm{var}(\mu_k(\sigma)_{q,w}) \subseteq \mathrm{Z}_n$. The first item in Definition 4.1 ensures that all matrices in the range of μ_k are essentially finite; *i. e.*, almost all entries of the matrices are $\widetilde{0}$. Let us present an example.

EXAMPLE 4.2. *Let* $Q = \{\star\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, *and* $\Delta = \{\alpha^{(0)}\}$. *Let* μ *be the tree representation over* Q, Σ, Δ, *and* $\mathbb{A}_\infty$ *with*

$$\mu_2(\sigma)_{\star,\star(\mathrm{x}_1)\star(\mathrm{x}_2)} = \max(1\,\mathrm{z}_1,\ 1\,\mathrm{z}_2) \quad \textit{and} \quad \mu_0(\alpha)_{\star,\varepsilon} = 0\,\alpha\ .$$

All remaining entries of $\mu_2(\sigma)$ *are supposed to be* $\widetilde{-\infty}$. *Clearly, the conditions imposed in the items of Definition 4.1 are satisfied.*

Let us now show how we illustrate tree representations. In Figure 1(left) we display the tree representation μ of Example 4.2. Figure 1(right) attempts to illustrate the relation between the pictorial display and the formal tree representation entry. The large circles represent states; the name of the state is inscribed. Smaller circles (without inscription) denote transitions (entries in the tree representation). Let us explain how to construct the tree representation entry from a transition given graphically. First we note that a transition is connected to a number of states by means of edges. In particular, there is exactly one edge marked with an arrow (leading away from the transition toward a state). This edge is called *outgoing edge;* all the remaining edges (without arrows) that connect to the given transition are called *incoming edges.* In Figure 1(right) the outgoing edge leads to the state q and the incoming edges connect the states q_1 and q_2 to the transition. Let us consider the circumference of the circle representing the transition. All incoming and outgoing edges intersect the circumference, and we implicitly order the incoming edges by starting at the intersection of the outgoing edge and the circumference and then going anti-clockwise along the circumference. Let us number the first incoming edge crossed by 1 (do not confuse this number with the annotation;

2 in the example), the second is numbered 2, and so on. Eventually, we return to the outgoing edge and have thus numbered all incoming edges. Now we are ready to construct a word $w \in Q(\mathrm{X})^*$ for the transition. The length of the word is determined by the number of incoming edges, thus in our example of Figure 1(right) we have $|w| = 3$. The i-th symbol w_i in w is determined by the incoming edge numbered i in the following way. The symbol is supposed to be an element of $Q(\mathrm{X})$; *i. e.*, for some $p \in Q$ and $j \in \mathbb{N}_+$ we have $w_i = p(\mathrm{x}_j)$. The state p is the state to which the incoming edge with number i connects, and j is annotated at the incoming edge with number i (close to the state). In the running example, we have $w = q_2(\mathrm{x}_2)q_1(\mathrm{x}_1)q_2(\mathrm{x}_1)$.

Finally, at the circle representing the transition there is an annotation of the form σ/ψ where $\sigma \in \Sigma$ is an input symbol and ψ is a tree series. Let k be the rank of σ. Such a transition determines the tree representation entry $\mu_k(\sigma)_{q,w}$, namely $\mu_k(\sigma)_{q,w} = \psi$. In the example, we have $\mu_2(\sigma)_{q,q_2(\mathrm{x}_2)q_1(\mathrm{x}_1)q_2(\mathrm{x}_1)} = 1\ \delta(\mathrm{z}_1, \mathrm{z}_3, \mathrm{z}_2)$, if we suppose that σ is binary and δ is ternary. We leave it to the reader to check that Figure 1(left) really corresponds to the tree representation given in Example 4.2 (only the nonzero entries are represented in the graphical representation).

Two restricted tree representations are of paramount importance in the sequel of the thesis. These are top-down and bottom-up tree representations. We recall the definition from [**41**].

Definition 4.3 (see [**41**, Definition 3.1]). *We say that a tree representation μ over Q, Σ, Δ, and $\mathcal{A}$ is:*

- top-down, *if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ we have* $\mathrm{supp}(\mu_k(\sigma)_{q,w}) \subseteq C_\Delta(\mathrm{Z}_{|w|})$; *and*
- bottom-up, *if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$ and $w \in Q(\mathrm{X}_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ there exist states $q_1, \dots, q_k \in Q$ such that $w = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$.*

This definition reflects the main difference between top-down and bottom-up devices. The top-down device decides which subtrees to process in which states (possibly even one subtree with several—not necessarily different—states), and then inserts the result obtained by the translation. It may neither delete nor copy the obtained output and hence each output tree in $\mathrm{supp}(\mu_k(\sigma)_{q,w})$ should be nondeleting and linear in $\mathrm{Z}_{|w|}$. A bottom-up device, however, has already processed each subtree in a certain state and decides how to arrange the obtained outputs. It may delete and copy the obtained outputs, but has only restricted influence on how the input trees are processed. In particular, it may not request that a subtree is not processed at all or processed twice. Thus every nonzero entry $\mu_k(\sigma)_{q,w}$ should obey that $w = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$ for some $q_1, \dots, q_k \in Q$.

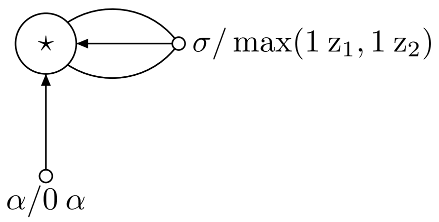

FIGURE 2. Tree representation μ of Example 4.4.

The tree representation given in Example 4.2 is bottom-up, but not top-down. We agree on the following conventions that simplify the presentation of tree representations:

- With any tree representation μ we just write $\mu_0(\alpha)_q$ instead of $\mu_0(\alpha)_{q,\varepsilon}$ for every $\alpha \in \Sigma_0$ and $q \in Q$.
- For bottom-up tree representations μ we write $\mu_k(\sigma)_{q,q_1\cdots q_k}$ instead of $\mu_k(\sigma)_{q,q_1(\mathrm{x}_1)\cdots q_k(\mathrm{x}_k)}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q, q_1, \ldots, q_k \in Q$.
- When we define a tree representation we usually only specify values for some entries; the remaining entries are silently assumed to be $\widetilde{0}$.
- In graphical representations we only illustrate the nonzero entries of a tree representation.
- In graphical representations of bottom-up tree representations the number annotated to an incoming edge of a transition is omitted because it always coincides with the number that is implicitly assigned to that edge.

With the previous conventions in mind we restate Example 4.2.

EXAMPLE 4.4. *Let $Q = \{\star\}$, $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, and $\Delta = \{\alpha^{(0)}\}$. Let μ be the bottom-up tree representation over Q, Σ, Δ, and $\mathbb{A}_\infty$ with*

$$\mu_2(\sigma)_{\star,\star\star} = \max(1\,\mathrm{z}_1,\ 1\,\mathrm{z}_2) \quad \textit{and} \quad \mu_0(\alpha)_\star = 0\,\alpha \ .$$

The tree representation μ is illustrated in Figure 2.

Now we are ready to recall the definition of tst from [**41**]. Note that we introduce one small extension, which we discuss in Section 4. A tst is basically a tree representation together with supportive information about the state set, the input and output ranked alphabets, and the semiring. Additionally, we adjoin a mapping which associates top-most output to each state.

DEFINITION 4.5. *A* tree series transducer, *abbreviated to tst, is a sixtuple*

$$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu) \ ,$$

where:

- *Q is an alphabet of* states;

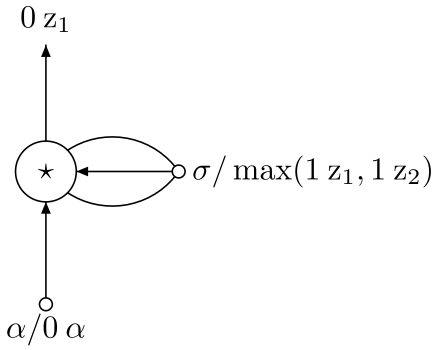

FIGURE 3. Bu-tst $M_{4.6}$ over $\mathbb{A}_\infty$ of Example 4.6.

- *Σ and Δ are ranked alphabets (both disjoint to Q) of input and output symbols, respectively;*
- *$\mathcal{A} = (A, +, \cdot)$ is a semiring;*
- *$F \in \mathcal{A}\langle\!\langle C_\Delta(\mathrm{Z}_1)\rangle\!\rangle^Q$ assigns* top-most output *to each state; and*
- *μ is a tree representation over Q, Σ, Δ, and A.*

The tst M inherits the properties bottom-up and top-down from its tree representation μ; i. e., M is called bottom-up tree series transducer, *abbreviated to bu-tst (respectively,* top-down tree series transducer, *abbreviated to td-tst), if μ is bottom-up (respectively, top-down). Moreover, for every $q \in Q$ we also say that F_q is* final output *(respectively,* initial output), *if M is bottom-up (respectively, top-down). Accordingly, we call a state $q \in Q$ such that $F_q \neq \widetilde{0}$ a* final state, *if M is bottom-up, and an* initial state, *if M is top-down.*

For the rest of the thesis we mostly consider tst that are bottom-up or top-down. For an investigation of general tst, we refer the reader to [**41**, Section 4]. Now we complete our bottom-up tree representation from Example 4.4 to a bu-tst.

EXAMPLE 4.6. *Let Q, Σ, Δ, and μ be as in Example 4.4. Moreover, let $F \in \mathcal{A}\langle\!\langle C_\Delta(\mathrm{Z}_1)\rangle\!\rangle^Q$ be such that $F_\star = 0\,\mathrm{z}_1$. Then*

$$M_{4.6} = (Q, \Sigma, \Delta, \mathbb{A}_\infty, F, \mu)$$

is a bu-tst with final state $\star$.

We display a tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ graphically by depicting its tree representation μ together with information about F. The top-most output of a state $q \in Q$ is annotated at a special edge leading away (as indicated by an arrow) from the circle representing q but pointing nowhere. We again adopt the convention that only nonzero top-most output will be depicted. The bu-tst $M_{4.6}$ of Example 4.6 is displayed in Figure 3.

2.2. Semantics. Let us show the behavior of a tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$. Provided that the behavior of M is well-defined, M computes a transformation that maps a tree (over the input ranked alphabet Σ) to a tree series (over the semiring $\mathcal{A}$ and the output ranked alphabet Δ). Depending on the type of tree series substitution used, the computed transformation is either called ε-tree-to-tree-series or o-tree-to-tree-series transformation. It can easily be extended to a transformation that maps tree series to tree series. The such obtained transformations are called ε-tree-series-to-tree-series and o-tree-series-to-tree-series transformations. The problem of well-definedness occurs because tree series substitution is not always well-defined (see Section 2 in Chapter 3). We discuss several restrictions, which imply that the behavior is always well-defined, of the general model of tst in Section 3.

As in Chapter 3 on tree series substitutions we now fix a semiring $\mathcal{A} = (A, +, \cdot)$ and an infinitary summation $\sum$ such that either (i) $\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$ or (ii) $\sum$ is the infinitary summation induced by $+$.

Definition 4.7. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tstand $\eta \in \{\varepsilon, \mathrm{o}\}$.*

(1) *For every $k \in \mathbb{N}$, and $\sigma \in \Sigma_k$ the tree representation μ induces a partial function $\overline{\mu_k(\sigma)}^{\eta} \colon \big(\mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q\big)^k \dashrightarrow \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q$ defined for every $q \in Q$ and $V_1, \dots, V_k \in \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q$ by*

$$\overline{\mu_k(\sigma)}^{\eta}(V_1, \dots, V_k)_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} \big((V_{i_j})_{q_j}\big)_{j \in [n]} \enspace .$$

Note that

$$\mathcal{D} = \Big(\mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q, (\overline{\mu_k(\sigma)}^{\eta})_{k \in \mathbb{N}, \sigma \in \Sigma_k}\Big)$$

defines a partial Σ-algebra [**101**]. *The partial function*

$$h_\mu^\eta \colon T_\Sigma \dashrightarrow \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q$$

is inductively defined for every $k \in \mathbb{N}$, symbol $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$ by

$$h_\mu^\eta(\sigma(t_1, \dots, t_k)) = \overline{\mu_k(\sigma)}^{\eta}(h_\mu^\eta(t_1), \dots, h_\mu^\eta(t_k)) \enspace .$$

It is lifted to a partial function $h_\mu^\eta \colon \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle \dashrightarrow \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle^Q$ by

$$h_\mu^\eta(\psi)_q = \sum_{t \in T_\Sigma} (\psi, t) \cdot h_\mu^\eta(t)_q$$

for every $\psi \in \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$ and $q \in Q$.

(2) *The* η-tree-to-tree-series transformation, *abbreviated to η-t-ts transformation, computed by M, denoted by $\|M\|_\eta$, is the partial function*

$$\|M\|_\eta \colon T_\Sigma \dashrightarrow \mathcal{A}\langle\!\langle T_\Delta \rangle\!\rangle$$

given, for every $t \in T_\Sigma$*, by* $\|M\|_\eta(t) = \sum_{q \in Q} F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q)$.

(3) *The* η-tree-series-to-tree-series, *abbreviated to* η*-ts-ts,* transformation *computed by* M*, which is denoted by* $\|M\|^{\mathrm{ts}}_\eta$*, is the partial function*

$$\|M\|^{\mathrm{ts}}_\eta \colon \mathcal{A}\langle\langle T_\Sigma\rangle\rangle \dashrightarrow \mathcal{A}\langle\langle T_\Delta\rangle\rangle$$

given, for every $\psi \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle$*, by*

$$\|M\|^{\mathrm{ts}}_\eta(\psi) = \sum_{t \in T_\Sigma} (\psi, t) \cdot \|M\|_\eta(t) \enspace .$$

Two tst M' and M'' are called *(semantically)* η*-equivalent,* whenever $\|M'\|_\eta = \|M''\|_\eta$. If η is obvious from the context, then we simply say that M' and M'' are equivalent.

We discuss well-definedness in more detail in Section 3. At this point we note that whenever the semiring is $\aleph_0$-complete with respect to $\sum$, then the obtained partial Σ-algebra $\mathcal{D}$ is indeed a Σ-algebra and the partial function h^η_μ is in fact the unique homomorphism [**23**, Lemma II.10.6] from the initial term algebra $\mathcal{T}_\Sigma$ to $\mathcal{D}$ (see [**41**, Observation 3.3]). Due to this fact, this type of semantics is usually called *initial algebra semantics* [**63**]. It immediately follows that in this setting both $\|M\|_\eta$ and $\|M\|^{\mathrm{ts}}_\eta$ are defined on their whole domain and thus are in fact mappings. Let us demonstrate the previous definitions on our running example.

Example 4.8. *Let* $M = M_{4.6} = (Q, \Sigma, \Delta, \mathbb{A}_\infty, F, \mu)$ *be the bu-tst from Example 4.6. We claim that* $\|M\|_{\mathrm{o}}(t) = \mathrm{height}(t)\,\alpha$ *for every* $t \in T_\Sigma$.

Proof. To illustrate Definition 4.7, we prove this property by structural induction on t.

Induction base: Let $t = \alpha$.

$$\begin{aligned}
& \|M\|_{\mathrm{o}}(\alpha) \\
= \quad & \text{(by Definition 4.7(2))} \\
& \max_{q \in \{\star\}} \Bigl(F_q \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(\alpha)_q)\Bigr) \;=\; (0\,\mathrm{z}_1) \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(\alpha)_\star) \\
= \quad & \text{(by definition of } \xleftarrow[\mathrm{o}]{} \text{ and Definition 4.7(1))} \\
& h^{\mathrm{o}}_\mu(\alpha)_\star \;=\; \overline{\mu_0(\alpha)}^{\mathrm{o}}()_\star \\
= \quad & \text{(by Definition 4.7(1) and Observation 3.4(1))} \\
& \mu_0(\alpha)_\star \xleftarrow[\mathrm{o}]{} () \;=\; \mu_0(\alpha)_\star \\
= \quad & \text{(by definition of } \mu \text{ and height)} \\
& 0\,\alpha \;=\; \mathrm{height}(\alpha)\,\alpha
\end{aligned}$$

Induction step: Let $t = \sigma(t_1, t_2)$ for some $t_1, t_2 \in T_\Sigma$. Note that in the arctic semiring $\mathbb{A}_\infty$ we have $n^0 = 0$ for every $n \in \mathbb{N} \cup \{\infty, -\infty\}$.

$$
\begin{aligned}
& \|M\|_{\mathrm{o}}\big(\sigma(t_1,t_2)\big) \\
= \quad & \textit{(by Definition 4.7(2) and definition of } \xleftarrow[\mathrm{o}]{}\textit{)} \\
& \max_{q\in\{\star\}}\Big(F_q \xleftarrow[\mathrm{o}]{} \big(h^{\mathrm{o}}_\mu\big(\sigma(t_1,t_2)\big)_q\big)\Big) \;=\; h^{\mathrm{o}}_\mu\big(\sigma(t_1,t_2)\big)_\star \\
= \quad & \textit{(by Definition 4.7(1) and the fact that } M \textit{ is bottom-up)} \\
& \overline{\mu_2(\sigma)}^{\mathrm{o}}\big(h^{\mathrm{o}}_\mu(t_1), h^{\mathrm{o}}_\mu(t_2)\big)_\star \;=\; \max_{q_1,q_2\in\{\star\}}\Big(\mu_2(\sigma)_{\star,q_1q_2} \xleftarrow[\mathrm{o}]{} \big(h^{\mathrm{o}}_\mu(t_1)_{q_1}, h^{\mathrm{o}}_\mu(t_2)_{q_2}\big)\Big) \\
= {} & \mu_2(\sigma)_{\star,\star\star} \xleftarrow[\mathrm{o}]{} \big(h^{\mathrm{o}}_\mu(t_1)_\star, h^{\mathrm{o}}_\mu(t_2)_\star\big) \\
= \quad & \textit{(by definition of } \mu \textit{ and definition of } \xleftarrow[\mathrm{o}]{}\textit{)} \\
& \max(1\,\mathrm{z}_1, 1\,\mathrm{z}_2) \xleftarrow[\mathrm{o}]{} \Big(\max_{q\in\{\star\}}\big(F_q \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t_1)_q)\big), \max_{q\in\{\star\}}\big(F_q \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t_2)_q)\big)\Big) \\
= \quad & \textit{(by Definition 4.7(2))} \\
& \max(1\,\mathrm{z}_1, 1\,\mathrm{z}_2) \xleftarrow[\mathrm{o}]{} \big(\|M\|_{\mathrm{o}}(t_1), \|M\|_{\mathrm{o}}(t_2)\big) \\
= \quad & \textit{(by induction hypothesis)} \\
& \max(1\,\mathrm{z}_1, 1\,\mathrm{z}_2) \xleftarrow[\mathrm{o}]{} \big(\mathrm{height}(t_1)\,\alpha,\ \mathrm{height}(t_2)\,\alpha\big) \\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{o}]{} \textit{ because } \mathrm{supp}(\max(1\,\mathrm{z}_1,\ 1\,\mathrm{z}_2)) = \{\mathrm{z}_1, \mathrm{z}_2\} \\
& \quad \textit{and } \mathrm{supp}(\mathrm{height}(t_1)\,\alpha) = \{\alpha\} = \mathrm{supp}(\mathrm{height}(t_2)\,\alpha)\textit{)} \\
& \max\big(1+\mathrm{height}(t_1), 1+\mathrm{height}(t_2)\big)\,\alpha \\
= \quad & \textit{(by definition of } \mathrm{height}\textit{)} \\
& \mathrm{height}\big(\sigma(t_1,t_2)\big)\,\alpha
\end{aligned}
$$

We have proved that $\|M\|_{\mathrm{o}}(t) = \mathrm{height}(t)\,\alpha$ for every $t \in T_\Sigma$. □

Let $t \in T_\Sigma$. Finally, we verify that bu-tst can implement "checking followed by deletion" [**35**, Section 2]. To this end, we show an important observation that states: if there is a subtree $t' \in \mathrm{sub}(t)$ which cannot be translated successfully by any state (*i. e.*, $h^\eta_\mu(t')_q = \widetilde{0}$ for every state q), then the tree t cannot be translated successfully by any state.

Observation 4.9. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a bu-tst, and let $t \in T_\Sigma$. If for some $t' \in \mathrm{sub}(t)$ we have $h^\eta_\mu(t')_q = \widetilde{0}$ for every $q \in Q$, then also $h^\eta_\mu(t)_q = \widetilde{0}$ for every $q \in Q$. Moreover, $\|M\|_\eta(t) = \widetilde{0}$ in this situation.*

Proof. Let $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ be such that for some $i \in [k]$ we have $h^\eta_\mu(t_i)_q = \widetilde{0}$ for every $q \in Q$. We prove $h^\eta_\mu(\sigma(t_1, \ldots, t_k))_q = \widetilde{0}$ for every $q \in Q$.

$$
\begin{aligned}
& h^\eta_\mu(\sigma(t_1,\ldots,t_k))_q \\
= \quad & \textit{(by Definition 4.7(1) and the fact that } M \textit{ is bottom-up)} \\
& \sum_{q_1,\ldots,q_k\in Q} \mu_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\eta]{} \big(h^\eta_\mu(t_i)_{q_i}\big)_{i\in[k]}
\end{aligned}
$$

$$= \quad \textit{(by Observation 3.4(3))}$$
$$\sum_{q_1,\dots,q_k \in Q} \widetilde{0} = \widetilde{0}$$

Note that $h_\mu^\eta(\sigma(t_1, \dots, t_k))_q$ is indeed well-defined. With the help of the previous statement we can now prove that $h_\mu^\eta(t)_q = \widetilde{0}$ for every $q \in Q$ by a straightforward induction. Finally, with the help of Observation 3.4(3) we thus obtain

$$\|M\|_\eta(t) = \sum_{q \in Q} F_q \xleftarrow[\eta]{} (h_\mu^\eta(t)_q) = \sum_{q \in Q} \widetilde{0} = \widetilde{0} \ . \qquad \square$$

Note that $\|M\|_\eta(t') = \widetilde{0}$ would not be a sufficient precondition for the above observation. Furthermore, we note that Observation 4.9 does not hold for td-tst.

3. Syntactic restrictions

In this section we introduce the various syntactic restrictions on tst that we use and explore in the forthcoming material. Many of the restrictions already exist in the literature [**41, 58**], so that we illustrate only few restrictions by examples. Some properties carry the prefix "bu" or "td" to show that they are intended for bu-tst or td-tst, respectively. First we define the properties only for tree representations.

Definition 4.10 (see [**41**, Definitions 3.9 and 3.10]). *We say that a tree representation μ over Q, Σ, Δ, and $\mathcal{A}$ is:*

- polynomial *(respectively,* monomial *and* boolean), *if $\mu_k(\sigma)_{q,w}$ is polynomial (respectively, monomial and boolean) for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$;*
- bu-deterministic *(respectively,* bu-total), *if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q_1, \dots, q_k \in Q$ there exists at most (respectively, at least) one pair $(q, u) \in Q \times T_\Delta(\mathrm{Z})$ such that*
$$u \in \operatorname{supp}(\mu_k(\sigma)_{q,q_1(\mathrm{x}_1)\cdots q_k(\mathrm{x}_k)}) \ ;$$
- td-deterministic *(respectively,* td-total), *if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q \in Q$ there exists at most (respectively, at least) one $(w, u) \in Q(\mathrm{X}_k)^* \times T_\Delta(\mathrm{Z})$ such that $u \in \operatorname{supp}(\mu_k(\sigma)_{q,w})$;*
- input-linear *(respectively,* input-nondeleting), *if for every integer $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ the word w contains each $x \in \mathrm{X}_k$ at most (respectively, at least) once;*
- output-linear *(respectively,* output-nondeleting), *if $\mu_k(\sigma)_{q,w}$ is linear (respectively, nondeleting) in $\mathrm{Z}_{|w|}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$; and*
- linear *(respectively,* nondeleting), *if μ is input- and output-linear (respectively, input- and output-nondeleting).*

Note that polynomial tree representations are finitely representable due to the finiteness of the input ranked alphabet Σ, the fact that for every $k \in \mathbb{N}$ almost all entries in the matrices in the range of μ_k are $\widetilde{0}$, and the property that each entry is finitely representable. The properties monomial and boolean are straightforward. A classic property is determinism. Let $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$. For bottom-up tree representations determinism amounts to the restriction that for each given combination of k states, in which the k subtrees are processed, there is at most one state and output tree, which allow the input symbol σ to be processed. Similarly, for totality there should be at least one state and output tree.

Let us note some trivial relations between the introduced properties. Bu-deterministic bottom-up as well as td-deterministic tree representations are monomial, and monomial tree representations are polynomial. Moreover, a bottom-up tree representation is necessarily input-linear and input-nondeleting, whereas a top-down tree representation is output-linear and output-nondeleting by definition. We agree upon the following convention. Whenever we explicitly mention the property "bottom-up" (respectively, "top-down"), then we drop the prefix "bu" (respectively, "td") from the remaining properties. Thus, instead of the cumbersome "bu-total and bu-deterministic bottom-up tree representation" we only write "total and deterministic bottom-up tree representation". Let us examine our example bottom-up tree representation μ from Example 4.4 and determine the properties it has.

EXAMPLE 4.11. *Let μ be the bottom-up tree representation of Example 4.4. We easily observe that μ is polynomial, bu-total, td-total, input-nondeleting, and linear. Clearly, μ is not monomial and thereby also neither bu-deterministic nor td-deterministic. Recall that μ uses the arctic semiring $\mathbb{A}_\infty$. With this in mind it turns out that μ is not boolean, because $(\mu_2(\sigma)_{\star,\star\star}, \mathrm{z}_1) = 1 \notin \{-\infty, 0\}$.*

Now we are ready to define the properties for tst. Most of the properties are simply lifted from the tree representation but occasionally restrictions are added. So, *e. g.*, for td-determinism we additionally require that there is at most one initial state. Finally, the homomorphism property is a combination of determinism, totality, and single-state (*i. e.*, $\mathrm{card}(Q) = 1$).

DEFINITION 4.12. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tst. We say that M is:*

- linear *(respectively,* nondeleting, input-linear, input-nondeleting, output-linear, *and* output-nondeleting), *if μ is linear (respectively, nondeleting, input-linear, input-nondeleting, output-linear, and output-nondeleting);*

- polynomial *(respectively,* monomial *and* boolean), *if μ and F_q are polynomial (respectively, monomial and boolean) for every $q \in Q$;*
- bu-deterministic *(respectively,* bu-total), *if μ is bu-deterministic (respectively, bu-total) and for every $q \in Q$ there exists at most (respectively, at least) one $u \in C_\Delta(\mathrm{Z}_1)$ such that $u \in \mathrm{supp}(F_q)$;*
- td-deterministic *(respectively,* td-total), *if μ is td-deterministic (respectively, td-total) and there exists at most (respectively, at least) one pair $(q, u) \in Q \times C_\Delta(\mathrm{Z}_1)$ such that $u \in \mathrm{supp}(F_q)$; and*
- *a* bu-homomorphism *(respectively,* td-homomorphism), *if M is bu-total and bu-deterministic (respectively, td-total and td-deterministic), Q is a singleton, and $F_q = 1\mathrm{z}_1$ for every $q \in Q$.*

Note that polynomial tst are finitely representable. The properties monomial, boolean, nondeleting, and linear are straightforward. Determinism is special, because in addition to a deterministic tree representation, also their top-most output must be uniquely determined. This means that F_q is monomial for every $q \in Q$. Moreover, for td-tst there exists at most one $q \in Q$ such that $F_q \neq \widetilde{0}$, thus there is at most one initial state. We again note that every deterministic bu-tst is monomial and every td-deterministic tst is monomial (we study deterministic tst intensively in Chapter 5). Monomial tst are clearly polynomial.

We also observe that bu-tst are necessarily input-linear and input-nondeleting, whereas td-tst are by definition output-linear and output-nondeleting. We agree upon the following convention. Whenever we explicitly mention the property "bottom-up" (respectively, "top-down"), then we drop the prefix "bu" (respectively, "td") from the remaining properties. Thus, instead of the cumbersome "bu-total and bu-deterministic bu-tst" we only write "total and deterministic bu-tst". Let us examine our example bu-tst $M_{4.6}$ from Example 4.6 and determine the properties it has.

EXAMPLE 4.13. *Let $M_{4.6}$ be the bu-tst of Example 4.6. It is straightforward to show that $M_{4.6}$ is polynomial, bu-total, td-total, input-nondeleting, and linear. It is not monomial and thereby also neither bu-deterministic nor td-deterministic. From this it follows that $M_{4.6}$ is not a bu-homomorphism and not a td-homomorphism. Finally, $M_{4.6}$ is not boolean because its tree representation is not boolean.*

Let us return to the issue of well-definedness of the computed η-t-ts transformation. With the help of Observation 3.3 we can conclude the following observation.

OBSERVATION 4.14. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tst, and let $\eta \in \{\varepsilon, \mathrm{o}\}$. The η-t-ts transformation $\|M\|_\eta$ is well-defined for every $t \in T_\Sigma$, whenever:*

(1) *$\mathcal{A}$ is $\aleph_0$-complete with respect to $\sum$;*
(2) *M is a td-tst; or*
(3) *M is polynomial.*

PROOF. The first two cases are straightforward. For the second case, we observe that

$$\overline{\mu_k(\sigma)}^{\eta}(V_1, \ldots, V_k)_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} \big((V_{i_j})_{q_j}\big)_{j \in [n]} \tag{21}$$

is well-defined by Observation 3.3 because $\mu_k(\sigma)_{q,w}$ is nondeleting and linear in $\mathrm{Z}_{|w|}$ for every $w \in Q(\mathrm{X}_k)^*$ and there are only finitely many $w \in Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ by Definition 4.1.

In the third case, we have that (21) is well-defined by Observation 3.3, if the tree series $V_1, \ldots, V_k$ are polynomial. It can easily be shown by structural induction that $h_\mu^\eta(t)_q$ is well-defined and polynomial for every $t \in T_\Sigma$ and $q \in Q$ [**55**, Proposition 3.4]. Thus the statement follows. □

In the sequel we use tables such as Table 2 to present the properties that are preserved by constructions. The abbreviations used in those tables are presented in Table 1. A "✓"-symbol states that the property is preserved by the construction (*i. e.*, given that the input tst have the considered property, also the constructed tst has this property), whereas a "✗"-symbol states that the property is not necessarily preserved.

Since we are mostly interested in bu-tst and td-tst, we introduce the set

$$\mathrm{Pref} = \{\mathrm{p}, \mathrm{d}, \mathrm{t}, \mathrm{l}, \mathrm{n}, \mathrm{b}, \mathrm{h}\}$$

where the letters stand for polynomial, deterministic, total, linear, nondeleting, boolean, and homomorphism, respectively. We introduce the set $\Pi = \mathfrak{P}(\mathrm{Pref})$, and for every $x \in \Pi$ we also set $\Pi_x = \{\pi \in \Pi \mid x \subseteq \pi\}$. Henceforth, we sometimes say that a tst M, which is bottom-up or top-down, has properties x where $x \in \Pi$ to mean that M has all the properties whose abbreviation belongs to x. For brevity, we usually omit set braces and commata for elements of Π and just write ptl instead of $\{\mathrm{p}, \mathrm{t}, \mathrm{l}\}$. So we would say that the bu-tst $M_{4.6}$ from Example 4.6 has properties ptl.

In the forthcoming chapters we are interested in the computational power of certain restricted bu-tst and td-tst. More precisely, to every class of restricted bu-tst or td-tst (see the properties in Definition 4.12) we associate the class of all η-t-ts transformations computed by them. Then we compare such classes of η-t-ts transformations by means of inclusion. The next definition establishes shorthands for classes of η-t-ts transformations.

DEFINITION 4.15. *Let $\eta \in \{\varepsilon, \mathrm{o}\}$ and $x \in \Pi$. The class of all η-t-ts transformations computable by bu-tst with properties x over the*

TABLE 1. Abbreviations of properties.

bu	bottom-up	td	top-down
p	polynomial	m	monomial
bu-d	bu-deterministic	bu-t	bu-total
td-d	td-deterministic	td-t	td-total
i-l	input-linear	i-n	input-nondeleting
o-l	output-linear	o-n	output-nondeleting
bu-h	bu-homomorphism	b	boolean
td-h	td-homomorphism		

semiring $\mathcal{A}$ is denoted by x–$\mathrm{BOT}_\eta(\mathcal{A})$, and the class of all η-t-ts transformations computable by td-tst with properties x over $\mathcal{A}$ is denoted by x–$\mathrm{TOP}_\eta(\mathcal{A})$.

We just write $\mathrm{BOT}_\eta(\mathcal{A})$ and $\mathrm{TOP}_\eta(\mathcal{A})$ instead of $\emptyset$–$\mathrm{BOT}_\eta(\mathcal{A})$ and $\emptyset$–$\mathrm{TOP}_\eta(\mathcal{A})$, respectively.

4. Relating top-most output and designated states

In [**41**, Definition 3.4] tree series transducers are introduced with a set $D \subseteq Q$ of so-called *designated states* instead of the top-most output F in our Definition 4.5. Our notion is obviously slightly stronger because we can simulate designated states. Given a set $D \subseteq Q$ of designated states we construct F by

$$F_q = \begin{cases} 1\,\mathrm{z}_1 & \text{if } q \in D, \\ \widetilde{0} & \text{otherwise;} \end{cases}$$

for every $q \in Q$. Consequently, we call a tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ a *tst with designated states* whenever $F_q \in \{\widetilde{0}, 1\,\mathrm{z}_1\}$ for every $q \in Q$. Next we show that for every tst we can construct a semantically equivalent tst with designated states. However, the involved construction does not preserve bu-determinism, and we show in Chapter 5 that deterministic bu-tst are indeed more powerful than the deterministic bottom-up tree series transducers of [**41**].

LEMMA 4.16. *Let M be a polynomial tst, and let $\eta \in \{\varepsilon, \mathrm{o}\}$. There exists a tst M' with designated states such that $\|M'\|_\eta = \|M\|_\eta$.*

PROOF. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ and let $\overline{Q} = \{\overline{q} \mid q \in Q\}$ be disjoint with Q. We construct $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows:

- $Q' = Q \cup \overline{Q}$;
- for every $q \in Q$ let $F'_q = \widetilde{0}$ and

$$F'_{\overline{q}} = \begin{cases} 1\,\mathrm{z}_1 & \text{if } F_q \neq \widetilde{0}, \\ \widetilde{0} & \text{otherwise;} \end{cases}$$

- for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ let

$$\mu'_k(\sigma)_{q,w} = \mu_k(\sigma)_{q,w} \quad \text{and} \quad \mu'_k(\sigma)_{\overline{q},w} = F_q \xleftarrow[\eta]{} (\mu_k(\sigma)_{q,w}) \ .$$

It remains to prove that $\|M'\|_\eta = \|M\|_\eta$. It is obvious that $h^\eta_{\mu'}(t)_q = h^\eta_\mu(t)_q$ for every $t \in T_\Sigma$ and $q \in Q$. Using this auxiliary statement we prove the main statement. Let $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$
\begin{aligned}
& \|M'\|_\eta(\sigma(t_1, \ldots, t_k)) \\
= \quad & \textit{(by Definition 4.7(2))} \\
& \sum_{q \in Q'} F'_q \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(\sigma(t_1, \ldots, t_k))_q\big) \\
= \quad & \textit{(by definition of } F' \textit{ and Observation 3.4(2))} \\
& \sum_{q \in Q} F'_{\overline{q}} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(\sigma(t_1, \ldots, t_k))_{\overline{q}}\big) \\
= \quad & \textit{(by definition of } F' \textit{ and } \xleftarrow[\eta]{}\textit{)} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} h^\eta_{\mu'}(\sigma(t_1, \ldots, t_k))_{\overline{q}} \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} \Big(\sum_{\substack{w \in Q'(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu'_k(\sigma)_{\overline{q},w} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_{i_j})_{q_j}\big)_{j \in [n]} \Big) \\
= \quad & \textit{(by definition of } \mu'\textit{, Observation 3.4(2), and } h^\eta_{\mu'}(t)_q = h^\eta_\mu(t)_q\textit{)} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} \Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu'_k(\sigma)_{\overline{q},w} \xleftarrow[\eta]{} \big(h^\eta_{\mu}(t_{i_j})_{q_j}\big)_{j \in [n]} \Big) \\
= \quad & \textit{(by definition of } \mu'_k(\sigma)_{\overline{q},w}\textit{)} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} \Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \big(F_q \xleftarrow[\eta]{} (\mu_k(\sigma)_{q,w})\big) \xleftarrow[\eta]{} \big(h^\eta_{\mu}(t_{i_j})_{q_j}\big)_{j \in [n]} \Big) \\
= \quad & \textit{(by Observation 3.7 and Proposition 3.19;} \\
& \quad \textit{because of the special shape of } F_q\textit{, commutativity is not necessary)} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} \Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} F_q \xleftarrow[\eta]{} \big(\mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h^\eta_{\mu}(t_{i_j})_{q_j})_{j \in [n]}\big) \Big) \\
= \quad & \textit{(by Observation 3.7 and Proposition 3.8)} \\
& \sum_{q \in Q, F_q \neq \widetilde{0}} F_q \xleftarrow[\eta]{} \Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} \big(h^\eta_{\mu}(t_{i_j})_{q_j}\big)_{j \in [n]} \Big) \\
= \quad & \textit{(by Observation 3.4(2) and Definition 4.7(1))} \\
& \sum_{q \in Q} F_q \xleftarrow[\eta]{} \big(h^\eta_\mu(\sigma(t_1, \ldots, t_k))_q\big)
\end{aligned}
$$

TABLE 2. Preservation of properties for the construction of Lemma 4.16.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✓	✗	✗	✓	✓	✓	✓	✓	✓	✗	✗	✗

$=$ *(by Definition 4.7(2))*

$\|M\|_\eta(\sigma(t_1, \ldots, t_k))$ □

The preserved properties are displayed in Table 2. Note that bu-homomorphism and td-homomorphism are not preserved, but homomorphism bu-tst and homomorphism td-tst have designated states by definition.

5. Principal properties

In this section we collect some essential statements about tst, that (in most cases) are known from the literature. First we show that the choice of pure or o-substitution is irrelevant for td-tst.

THEOREM 4.17 (see [**58**, Lemma 5.1]). *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a td-tst.*

(1) $h^\varepsilon_\mu = h^{\mathrm{o}}_\mu$ *and* $\|M\|_\varepsilon = \|M\|_{\mathrm{o}}$.
(2) x–$\mathrm{TOP}_\varepsilon(\mathcal{A}) = x$–$\mathrm{TOP}_{\mathrm{o}}(\mathcal{A})$ *for every* $x \in \Pi$.

PROOF. Essentially, the two statements are proved in [**58**, Lemma 5.1 and Theorem 5.2]. The main argument required for the proof is delivered by Observation 3.7. □

Similarly, there is no difference in terms of transformational power between bu-tst using pure and o-substitution in the boolean semiring. In fact, this is proved for boolean polynomial bu-tst over additively idempotent semirings in [**58**, Theorem 5.8] and can easily be extended to boolean bu-tst over additively idempotent semirings with necessary $\sum$ [**58**, Observation 5.10].

OBSERVATION 4.18 (see [**58**, Corollary 5.9]). *For every* $x \in \Pi$:

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) = x\text{–}\mathrm{BOT}_{\mathrm{o}}(\mathbb{B}) \ . \tag{22}$$

OBSERVATION 4.19. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tst with boolean tree representation μ. Then* $h^\varepsilon_\mu = h^{\mathrm{o}}_\mu$ *and* $\|M\|_\varepsilon = \|M\|_{\mathrm{o}}$.

The next proposition relates homomorphism bu-tst and homomorphism td-tst over the boolean semiring. In fact, over the boolean semiring the transformational power of homomorphism bu-tst and homomorphism td-tst coincides [**35**, Lemma 3.2].

PROPOSITION 4.20 (see [**35**, Lemma 3.2]). *For every* $x \in \Pi_{\mathrm{h}}$:

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) = x\text{–}\mathrm{TOP}_\varepsilon(\mathbb{B}) \ . \tag{23}$$

Next, we restate the equality of the classes of ε-t-ts and o-t-ts transformations computed by bu-tst for all properties that contain both the nondeletion as well as the linearity property. This equality is shown in [**58**, Theorem 5.5], but can also be seen from the definition of pure and o-substitution, because both notions coincide whenever the target tree series is nondeleting and linear (see Observation 3.7). If $\mathcal{A}$ is commutative, then we even have that the classes of ε-t-ts transformations computed by bu-tst and td-tst coincide [**41**, Theorem 5.24].

PROPOSITION 4.21 (see [**58**, Theorem 5.5] and [**41**, Theorem 5.24]). *For every* $x \in \Pi_{\mathrm{nl}}$:

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{BOT}_{\mathrm{o}}(\mathcal{A}) \ . \tag{24}$$

If $\mathcal{A}$ *is commutative, then for every* $x \in \Pi_{\mathrm{nl}} \setminus (\Pi_{\mathrm{d}} \cup \Pi_{\mathrm{t}})$:

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{TOP}_\varepsilon(\mathcal{A}) \ . \tag{25}$$

In Theorem 3.28 we showed that o-substitution preserves recognizability. We now lift this statement to the level of tst. A tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ is called *recognizable,* if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ and F_q are recognizable.

THEOREM 4.22. *Let* $\mathcal{A}$ *be commutative, additively idempotent, and* $\aleph_0$*-complete with respect to a necessary* $\sum$. *Let* $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ *be an output-linear recognizable tst. Then for every* $t \in T_\Sigma$ *the tree series* $\|M\|_{\mathrm{o}}(t)$ *is recognizable.*

PROOF. We prove the auxiliary statement that $h^{\mathrm{o}}_\mu(t)_q$ is recognizable for every $t \in T_\Sigma$ and $q \in Q$ by induction on t. So let $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$. By Definition 4.7(1)

$$h^{\mathrm{o}}_\mu(\sigma(t_1, \dots, t_k))_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*,\\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t_{i_j})_{q_j})_{j \in [n]} \ .$$

By induction hypothesis $h^{\mathrm{o}}_\mu(t_{i_j})_{q_j}$ is recognizable for every $j \in [n]$. Since M is recognizable, $\mu_k(\sigma)_{q,w}$ is recognizable. By Theorem 3.28 also

$$\mu_k(\sigma)_{q,w} \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t_{i_j})_{q_j})_{j \in [n]}$$

is recognizable because $\mu_k(\sigma)_{q,w}$ is linear in Z_n. Since recognizable tree series are closed under finite sums [**9**, Proposition 3.1] (see also [**31**, Lemma 6.4]) we obtain that $h^{\mathrm{o}}_\mu(t)_q$ is recognizable.

For every $t \in T_\Sigma$ we have

$$\|M\|_{\mathrm{o}}(t) = \sum_{q \in Q} F_q \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t)_q)$$

by Definition 4.7(2). In the auxiliary statement we showed that $h^{\mathrm{o}}_\mu(t)_q$ is recognizable. Moreover, $F_q \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t)_q)$ is recognizable due to Theorem 3.28. Thus, also $\|M\|_{\mathrm{o}}(t)$ is recognizable. □

Finally, we present an easy observation that simplifies some of the discussions in the sequel.

OBSERVATION 4.23. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a homomorphism bu-tst or a homomorphism td-tst. Moreover, let $\eta \in \{\varepsilon, \mathrm{o}\}$ and $Q = \{\star\}$. Then for every $t \in T_\Sigma$ we have $\|M\|_\eta(t) = h^\eta_\mu(t)_\star$. In particular, $\|M\|_\eta(\alpha) = \mu_0(\alpha)_\star$ for every $\alpha \in \Sigma_0$.*

PROOF. We can give a straightforward direct proof.

$$\|M\|_\eta(t) = \sum_{q \in Q} F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q) = (1\, \mathrm{z}_1) \xleftarrow[\eta]{} (h^\eta_\mu(t)_\star) = h^\eta_\mu(t)_\star$$

Note that the restriction to designated states is essential here. □

Several more statements on bu-tst and td-tst can be found in the next chapters.

6. Two final examples

Finally, let us present two examples, which are relevant in Chapter 6. Moreover, we chose to present a deterministic bu-tst and a deterministic td-tst, because deterministic tst play the central role in Chapter 5. We demonstrate how to achieve exponentiation of a coefficient using a deterministic bu-tst with just a single state in Example 4.24 and a deterministic td-tst with just a single state in Example 4.26.

EXAMPLE 4.24. *Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $a \in A$. We consider the bu-tst $M^a_{4.24} = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ with topmost output $F_\star = a\, \mathrm{z}_1$ and*

$$\mu_0(\alpha)_\star = a\, \alpha \quad \text{and} \quad \mu_1(\gamma)_{\star,\star} = a\, \sigma(\mathrm{z}_1, \mathrm{z}_1) \ .$$

Clearly, $M^a_{4.24}$ is a nondeleting deterministic bu-tst, which is not a homomorphism. We illustrate $M^a_{4.24}$ in Figure 4(left).

Let us state that $M^a_{4.24}$ indeed realizes the exponentiation of a, when we employ o-substitution. Furthermore, it is noteworthy that $M^a_{4.24}$ outputs fully balanced trees. Since we have seen several examples of applications of Definitions 4.7 and 3.1 and Observation 3.4, we only occasionally refer to those basic statements in the sequel.

LEMMA 4.25. *Let $a \in A$ and $M = M^a_{4.24} = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be the bu-tst of Example 4.24. For every $t \in T_\Sigma$ we have the equality $\|M\|_{\mathrm{o}}(t) = a^{2^{n+1}}\, u_n$ where $n = \mathrm{height}(t)$ and u_n is the fully balanced tree over Δ of height n.*

PROOF. We first prove that $h^{\mathrm{o}}_\mu(t)_\star = a^{2^{n+1}-1}\, u_n$ for every $t \in T_\Sigma$.

Induction base: Let $t = \alpha$ and hence $n = 0$.

$$h^{\mathrm{o}}_\mu(\alpha)_\star = \overline{\mu_0(\alpha)}^{\mathrm{o}}()_\star = \mu_0(\alpha)_\star \xleftarrow[\mathrm{o}]{} () = \mu_0(\alpha)_\star = a\, \alpha = a^{2^{n+1}-1}\, u_n$$

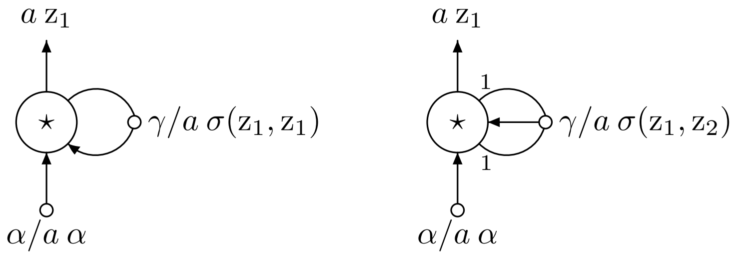

FIGURE 4. Bu-tst $M^a_{4.24}$ (left) and td-tst $M^a_{4.26}$ (right) over $\mathcal{A}$ (see Examples 4.24 and 4.26).

Induction step: Let $t = \gamma(t')$ for some $t' \in T_\Sigma$. Recall that $n = \mathrm{height}(t)$.

$$
\begin{aligned}
& h^{\mathrm{o}}_\mu\bigl(\gamma(t')\bigr)_\star \;=\; \overline{\mu_1(\gamma)}^{\mathrm{o}}\bigl(h^{\mathrm{o}}_\mu(t')\bigr)_\star \\
= & \sum_{q \in \{\star\}} \mu_1(\gamma)_{\star,q} \xleftarrow[\mathrm{o}]{} \bigl(h^{\mathrm{o}}_\mu(t')_q\bigr) \;=\; \mu_1(\gamma)_{\star,\star} \xleftarrow[\mathrm{o}]{} \bigl(h^{\mathrm{o}}_\mu(t')_\star\bigr) \\
= & \qquad \textit{(by induction hypothesis and definition of } \mu\textit{)} \\
& a\,\sigma(\mathrm{z}_1,\mathrm{z}_1) \xleftarrow[\mathrm{o}]{} (a^{2^n-1}\,u_{n-1}) \;=\; \bigl(a \cdot (a^{2^n-1})^2\bigr)\,\sigma(u_{n-1},u_{n-1}) \\
= & \; a^{2^{n+1}-1}\,u_n
\end{aligned}
$$

We have

$$\|M\|_{\mathrm{o}}(t) = F_\star \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t)_\star) = a \cdot h^{\mathrm{o}}_\mu(t)_\star = a^{2^{n+1}}\,u_n$$

for every $t \in T_\Sigma$ with $\mathrm{height}(t) = n$, which proves the statement. □

Note that $\|M^a_{4.24}\|_\varepsilon(t) = a^{n+2}\,u_n$ for every $t \in T_\Sigma$ and $n = \mathrm{height}(t)$. Now let us show that the transformation $\|M^a_{4.24}\|_{\mathrm{o}}$ can also be realized by a deterministic td-tst. In the next example we present the td-tst, and in the following lemma we state the equivalence of the computed transformations.

EXAMPLE 4.26. *Let* $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ *and* $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$ *and* $a \in A$. *We consider the td-tst* $M^a_{4.26} = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ *with topmost output* $F_\star = a\,\mathrm{z}_1$ *and*

$$\mu_0(\alpha)_\star = a\,\alpha \quad \textit{and} \quad \mu_1(\gamma)_{\star,\star(\mathrm{x}_1)\star(\mathrm{x}_1)} = a\,\sigma(\mathrm{z}_1,\mathrm{z}_2) \;\; .$$

Clearly, $M^a_{4.26}$ *is a nondeleting deterministic td-tst, which is not a homomorphism. We illustrate* $M^a_{4.26}$ *in Figure 4(right).*

LEMMA 4.27. *Let* $a \in A$. *Moreover, let*

$$M = M^a_{4.26} = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$$

be the td-tst of Example 4.26. For every $t \in T_\Sigma$ *we have the equality* $\|M\|_\varepsilon(t) = a^{2^{n+1}}\,u_n$ *where* $n = \mathrm{height}(t)$ *and* u_n *is the fully balanced tree over* Δ *of height* n.

PROOF. The proof is analogous to the one of Lemma 4.25. □

7. Open problems and future work

In the literature [**79, 41, 58**] well-definedness is usually enforced by considering only $\aleph_0$-complete semirings. We presented three simple conditions in Observation 4.14, each of which implies that the computed η-t-ts transformation is indeed a mapping (and thus defined for every input tree). It would be worthwhile to investigate more elaborate conditions that enforce well-definedness for bu-tst. Along this line of research, the question of well-definedness of the computed η-ts-ts transformation should be investigated for bu-tst as well as td-tst.

Another potential research direction concerns more powerful devices. Bottom-up and top-down tree transducers were generalized to attributed tree transducers [**53**], macro tree transducers [**37, 26, 45**], modular tree transducers [**46**], tree-to-graph-transducers [**47**], and many more devices. If we presuppose that there are strong enough (practical or theoretical) applications for those devices, then we could also generalize them to their corresponding weighted versions and expose the weighted devices to meticulous study.

CHAPTER 5

Deterministic Tree Series Transducers

With Earth's first Clay They did the Last Man's knead,
And then of the Last Harvest sow'd the Seed:
Yea, the first Morning of Creation wrote
What the Last Dawn of Reckoning shall read.

Omar Khayyam (1048–1123): "Rubaiyee LIII"
Rubaiyat, rendered into English verse by Edward J. Fitzgerald, 1120

1. Bibliographic information

In this chapter, we investigate the inclusion relation between classes of ε-t-ts and o-t-ts transformations computed by deterministic bottom-up and top-down tst. We derive several HASSE diagrams displaying the relationships given certain properties of the underlying semiring.

This chapter is a heavily revised and extended version of [**85**]. Therein the author studies deterministic bu-tst with designated states; *i. e.*, deterministic bu-tst whose final output is a vector of $1\,\mathrm{z}_1$ and $\widetilde{0}$. Here we consider deterministic bu-tst with final outputs. Surprisingly, the results obtained in this chapter are essentially different from those of [**85**], so we only present the results for bu-tst with final outputs and refer the reader to [**85**] for the results on bu-tst with designated states. Moreover, we additionally investigate classes of transformations computed by deterministic td-tst with initial outputs.

2. Properties of deterministic tree series transducers

In this section we present properties of deterministic bu-tst and deterministic td-tst, which we use in the forthcoming sections. In the sequel, we often write deterministic bu-tst or td-tst to mean bu-deterministic bu-tst or td-deterministic td-tst. Most of the material in this section is known, so that we refrain from giving plenty of examples. Moreover, we recall that deterministic bu-tst and td-tst are polynomial, so that their η-t-ts transformation is well-defined by Observation 4.14.

First we recall a central property of deterministic bu-tst and td-tst. Roughly speaking, the additive operation of the underlying semiring is irrelevant concerning computations of a deterministic bu-tst or td-tst; *i. e.*, all computations are performed in the multiplicative monoid of the semiring. If $\eta = \varepsilon$ then the proof of this statement is in [**41**, Proposition 3.12]. The proof of the statement with $\eta = \mathrm{o}$ uses exactly the same argumentation.

PROPOSITION 5.1. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a deterministic bu-tst or td-tst, and let $\eta \in \{\varepsilon, \mathrm{o}\}$.*

(1) *If M is bottom-up then for every $t \in T_\Sigma$ there exists at most one $q \in Q$ such that $h_\mu^\eta(t)_q \neq \widetilde{0}$.*
(2) *For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma$, and $q \in Q$ there exists a $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$ such that*

$$h_\mu^\eta(\sigma(t_1, \ldots, t_k))_q = \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h_\mu^\eta(t_{i_j})_{q_j})_{j \in [n]} \enspace .$$

(3) *There exists at most one $q \in Q$ such that $F_q \xleftarrow[\eta]{} (h_\mu^\eta(t)_q) \neq \widetilde{0}$.*
(4) *For every $t \in T_\Sigma$ there exists a $q \in Q$ such that*

$$\|M\|_\eta(t) = F_q \xleftarrow[\eta]{} (h_\mu^\eta(t)_q) \enspace .$$

(5) *For every $t \in T_\Sigma$ and $q \in Q$ we have that $h_\mu^\eta(t)_q$ and $\|M\|_\eta(t)$ are monomial.*

PROOF. Let us prove the statements individually.

(1) This statement is essentially proved in [**58**, Proposition 4.11], but we repeat the proof for clarity. We prove the statement inductively, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$. Moreover, let $q \in Q$ be arbitrary.

$$\begin{aligned}
& h_\mu^\eta(\sigma(t_1, \ldots, t_k))_q \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\eta]{} (h_\mu^\eta(t_i)_{q_i})_{i \in [k]} \\
= \quad & \textit{(for } i \in [k] \textit{ there exists at most one } p_i \in Q \textit{ such that } h_\mu^\eta(t_i)_{p_i} \neq \widetilde{0} \\
& \quad \textit{by induction hypothesis, and } \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\eta]{} (h_\mu^\eta(t_i)_{q_i})_{i \in [k]} = \widetilde{0} \\
& \quad \textit{if there exists an } i \in [k] \textit{ such that } q_i \neq p_i \textit{ by Observation 3.4(3);} \\
& \quad \textit{so for every } i \in [k] \textit{ let } p_i \in Q \textit{ be such that } h_\mu^\eta(t_i)_{p_i} \neq \widetilde{0} \textit{ and} \\
& \quad \textit{if no such } p_i \textit{ exists then let } p_i \in Q \textit{ be arbitrary)} \\
& \mu_k(\sigma)_{q, p_1 \cdots p_k} \xleftarrow[\eta]{} (h_\mu^\eta(t_i)_{p_i})_{i \in [k]}
\end{aligned}$$

There exists at most one $p \in Q$ such that $\mu_k(\sigma)_{p, p_1 \cdots p_k} \neq \widetilde{0}$ because M is bu-deterministic. So, if $q \neq p$ then $h_\mu^\eta(t)_q = \widetilde{0}$ by Observation 3.4. This proves the statement and note that we also proved Statement (2) for deterministic bu-tst.

(2) For deterministic bu-tst, this statement is already proved in the proof of Statement (1). So let M be a deterministic td-tst.

$$\begin{aligned}
& h_\mu^\eta(\sigma(t_1, \ldots, t_k))_q \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h_\mu^\eta(t_{i_j})_{q_j})_{j \in [n]}
\end{aligned}$$

$=$ *(by td-determinism there exists at most one*
$w' = p_1(\mathrm{x}_{i'_1}) \cdots p_l(\mathrm{x}_{i'_l}) \in Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w'} \neq \widetilde{0}$,
and for every $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$ with $w \neq w'$:
$\mu_k(\sigma)_{q,w} \underset{\eta}{\longleftarrow} (h^\eta_\mu(t_{i_j})_{q_j})_{j\in[n]} = \widetilde{0}$ by Observation 3.4(2);
so let $w' = p_1(\mathrm{x}_{i'_1}) \cdots p_l(\mathrm{x}_{i'_l}) \in Q(\mathrm{X}_k)^*$ be such that $\mu_k(\sigma)_{q,w'} \neq \widetilde{0}$
and if no such w' exists then let $w' = p_1(\mathrm{x}_{i'_1}) \cdots p_l(\mathrm{x}_{i'_l}) \in Q(\mathrm{X}_k)^*$
be arbitrary)

$\mu_k(\sigma)_{q,w'} \underset{\eta}{\longleftarrow} (h^\eta_\mu(t_{i'_j})_{p_j})_{j\in[l]}$

(3) Assume that M is a deterministic bu-tst. Then by Statement (1) there exists at most one $p \in Q$ such that $h^\eta_\mu(t)_p \neq \widetilde{0}$. In particular, $h^\eta_\mu(t)_q = \widetilde{0}$ for every $q \in Q$ with $q \neq p$. Thus $F_q \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_q) = \widetilde{0}$ for every such q by Observation 3.4(3), which proves the statement.

Now let M be a deterministic td-tst. By definition there exists at most one $p \in Q$ such that $F_p \neq \widetilde{0}$. Thus $F_q = \widetilde{0}$ and thereby $F_q \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_q) = \widetilde{0}$ by Observation 3.4(2) for every $q \in Q$ with $q \neq p$.

(4) We recall that

$$\|M\|_\eta(t) = \sum_{q\in Q} F_q \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_q) \enspace .$$

By Statement (3) there exists at most one state $p \in Q$ such that $F_p \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_p) \neq \widetilde{0}$. If no such $p \in Q$ exists then let $p \in Q$ be arbitrary. It follows that

$$\|M\|_\eta(t) = F_p \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_p) \enspace .$$

(5) This statement is proved in [**58**, Proposition 4.11] for deterministic bu-tst and in [**58**, Proposition 4.12] for deterministic td-tst. Note that those propositions additionally assume that M is total and $\mathcal{A}$ zero-divisor free, but these properties are only required to show that $h^\eta_\mu(t)_q \neq \widetilde{0}$ and $\|M\|_\eta(t) \neq \widetilde{0}$. We resupply the proof. The statement is proved by induction on t, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$. By Statement (2) we have

$$h^\eta_\mu(\sigma(t_1, \ldots, t_k))_q = \mu_k(\sigma)_{q,w} \underset{\eta}{\longleftarrow} (h^\eta_\mu(t_{i_j})_{q_j})_{j\in[n]}$$

for some $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$. By induction hypothesis $h^\eta_\mu(t_{i_j})_{q_j}$ is monomial for every $j \in [n]$ and further $\mu_k(\sigma)_{q,w}$ is monomial by definition. It follows from Observation 3.6(2) that $h^\eta_\mu(\sigma(t_1, \ldots, t_k))_q$ is monomial. This proves the first part of the statement.

For the second part we observe that

$$\|M\|_\eta(t) = F_q \underset{\eta}{\longleftarrow} (h^\eta_\mu(t)_q)$$

for some $q \in Q$ by Statement (4). By the first part of the statement, $h^\eta_\mu(t)_q$ is monomial and F_q is monomial because M is deterministic. It follows from Observation 3.6(2) that $\|M\|_\eta(t)$ is monomial. $\square$

Now let us show that boolean deterministic bu-tst and td-tst compute transformations that map each input tree to a boolean tree series. The determinism restriction is necessary because boolean tree series are not closed under η-substitution (see Table 2 in Chapter 3). Even the restriction to monomial tst is too weak because the η-t-ts transformation computed by a monomial tst not necessarily maps each input tree to a monomial tree series.

OBSERVATION 5.2. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a boolean and deterministic bu-tst or td-tst, and let $\eta \in \{\varepsilon, \mathrm{o}\}$. Then $h_\mu^\eta(t)_q$ and $\|M\|_\eta(t)$ are boolean for every $t \in T_\Sigma$ and $q \in Q$.*

PROOF. We have already remarked that deterministic bu-tst and td-tst compute using the multiplicative monoid of $\mathcal{A}$ only. Thus, if M is boolean, then all tree series in the range of the tree representation μ are boolean. Since $\{0, 1\}$ is closed under $\cdot$, we obtain the stated. □

In the preliminaries we recalled the notions of homomorphism and isomorphism for semirings. Here we present the version for monoids. Let $\mathcal{B} = (B, \cdot)$ and $\mathcal{D} = (D, \circ)$ be monoids. A *homomorphism (of monoids) from $\mathcal{B}$ to $\mathcal{D}$* is a mapping $h\colon B \longrightarrow D$ such that

$$h(b_1 \cdot b_2) = h(b_1) \circ h(b_2)$$

for every $b_1, b_2 \in B$. The homomorphism $h\colon B \longrightarrow D$ is called an *isomorphism,* if $h^{-1}\colon D \longrightarrow B$. The monoids $\mathcal{B}$ and $\mathcal{D}$ are said to be *isomorphic,* if there exists an isomorphism from $\mathcal{B}$ to $\mathcal{D}$. We denote by $\mathcal{B} \cong \mathcal{D}$ the fact that $\mathcal{B}$ and $\mathcal{D}$ are isomorphic.

Let $\mathcal{A} = (A, +, \cdot)$ and $\mathcal{B} = (B, \oplus, \odot)$ be semirings and $\mathcal{A}' = (A', \cdot)$ be a submonoid of $(A, \cdot)$. In addition, let $\tau_1\colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta\rangle\rangle$ and $\tau_2\colon T_\Sigma \longrightarrow \mathcal{B}\langle\langle T_\Delta\rangle\rangle$ be such that $(\tau_1(t), u) \in A'$ for every $t \in T_\Sigma$ and $u \in T_\Delta$. We write $\tau_1 \cong_{\mathrm{D}} \tau_2$, if there exists an isomorphism $h\colon A' \longrightarrow B$ from $(A', \cdot)$ to $(B, \odot)$ such that $h((\tau_1(t), u)) = (\tau_2(t), u)$ for every $t \in T_\Sigma$ and $u \in T_\Delta$. The capital "D" at $\cong_{\mathrm{D}}$ reminds us that the isomorphism only works for deterministic devices because it is only an isomorphism between the multiplicative monoids of $\mathcal{A}'$ and $\mathcal{B}$. The relation $\cong_{\mathrm{D}}$ is lifted to classes of transformations in the usual manner.

Since deterministic bu-tst and td-tst compute in the multiplicative monoid only, we observe that boolean deterministic bu-tst and td-tst only use the coefficients 0 and 1. Since the submonoid $(\{0, 1\}, \cdot)$ of $(A, \cdot)$ is isomorphic to the multiplicative monoid $(\{0, 1\}, \wedge)$ of $\mathbb{B}$, we obtain the following statement.

OBSERVATION 5.3. *Let $\mathcal{A}$ be a semiring. For every $x \in \Pi_{\mathrm{db}}$ and $\eta \in \{\varepsilon, \mathrm{o}\}$*

$$x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \cong_{\mathrm{D}} x\text{–}\mathrm{BOT}_\eta(\mathbb{B}) \quad \textit{and} \quad x\text{–}\mathrm{TOP}_\eta(\mathcal{A}) \cong_{\mathrm{D}} x\text{–}\mathrm{TOP}_\eta(\mathbb{B}) \ .$$

PROOF. The proof is straightforward and omitted. □

Finally we recall an important observation from [**58**]. We have seen in Observation 5.2 that for boolean and deterministic bu-tst and td-tst, we have that $\|M\|_\eta(t)$ is boolean for every input tree t. Moreover, by Proposition 5.1(5) $\|M\|_\eta(t)$ is monomial. If we add the totality restriction we obtain $\|M\|_\eta(t) \neq \widetilde{0}$ for every input tree t. This essentially means that such tst (at the level of h^η_μ) cannot implement "checking"; *i. e.*, selective rejection of some input trees. They may still reject input trees by entering a state whose top-most output is $\widetilde{0}$.

OBSERVATION 5.4 (see [**58**, Propositions 4.11 and 4.12]). *Suppose $\eta \in \{\varepsilon, \mathrm{o}\}$, and let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a boolean, total, and deterministic bu-tst or td-tst. For every $t \in T_\Sigma$ we have $\|M\|_\eta(t) = 1\, u$ for some $u \in T_\Delta$.*

3. The boolean semiring

As a starting point, we state the HASSE diagram (see Figure 1) for deterministic bu-tst and td-tst over the boolean semiring $\mathbb{B}$ (*i. e.*, for deterministic bottom-up and top-down tree transducers with top-most outputs). Clearly, the classes are ordered by inclusion, so the order in all our HASSE diagrams is inclusion. The diagram is well-known for deterministic bottom-up and top-down tree transducers (with designated states), and the adaption to devices with top-most outputs is straightforward. In order to present concise diagrams, we shorten the denotation of the classes from $x\text{–BOT}_\eta(\mathcal{A})$ to just $x^\perp_\eta$ and $x\text{–TOP}_\eta(\mathcal{A})$ to $x^\top_\eta$ for every $x \in \Pi$ and $\eta \in \{\varepsilon, \mathrm{o}\}$. Moreover, we use $x^=_\eta$ to express that $x^\perp_\eta = x^\top_\eta$ and $x^\ell_=$ to express that $x^\ell_\varepsilon = x^\ell_\mathrm{o}$ for every $\ell \in \{\perp, \top, =\}$. Thus $x^=_=$ stands for $x^\perp_\varepsilon = x^\perp_\mathrm{o} = x^\top_\varepsilon = x^\top_\mathrm{o}$.

THEOREM 5.5. *Figure 1 is the* HASSE *diagram for $\mathcal{A} = \mathbb{B}$.*

PROOF. The equalities are concluded from Observation 4.18 and Theorem 4.17 and Proposition 4.20, and all the inclusions hold by definition. Finally, the following eight statements are sufficient to prove strictness and incomparability (this statement was checked by an algorithm, which computes the minimal set of statements given the supposed HASSE diagram).

$$\text{dnlt–BOT}_\varepsilon(\mathbb{B}) \not\subseteq \text{d–TOP}_\varepsilon(\mathbb{B}) \tag{26}$$
$$\text{dnlt–TOP}_\varepsilon(\mathbb{B}) \not\subseteq \text{d–BOT}_\varepsilon(\mathbb{B}) \tag{27}$$
$$\text{dnl–BOT}_\varepsilon(\mathbb{B}) \not\subseteq \text{dt–BOT}_\varepsilon(\mathbb{B}) \tag{28}$$
$$\text{dnl–TOP}_\varepsilon(\mathbb{B}) \not\subseteq \text{dt–TOP}_\varepsilon(\mathbb{B}) \tag{29}$$
$$\text{hn–BOT}_\varepsilon(\mathbb{B}) \not\subseteq \text{dl–BOT}_\varepsilon(\mathbb{B}) \tag{30}$$
$$\text{hn–TOP}_\varepsilon(\mathbb{B}) \not\subseteq \text{dl–TOP}_\varepsilon(\mathbb{B}) \tag{31}$$
$$\text{hl–BOT}_\varepsilon(\mathbb{B}) \not\subseteq \text{dn–BOT}_\varepsilon(\mathbb{B}) \tag{32}$$
$$\text{hl–TOP}_\varepsilon(\mathbb{B}) \not\subseteq \text{dn–TOP}_\varepsilon(\mathbb{B}) \tag{33}$$

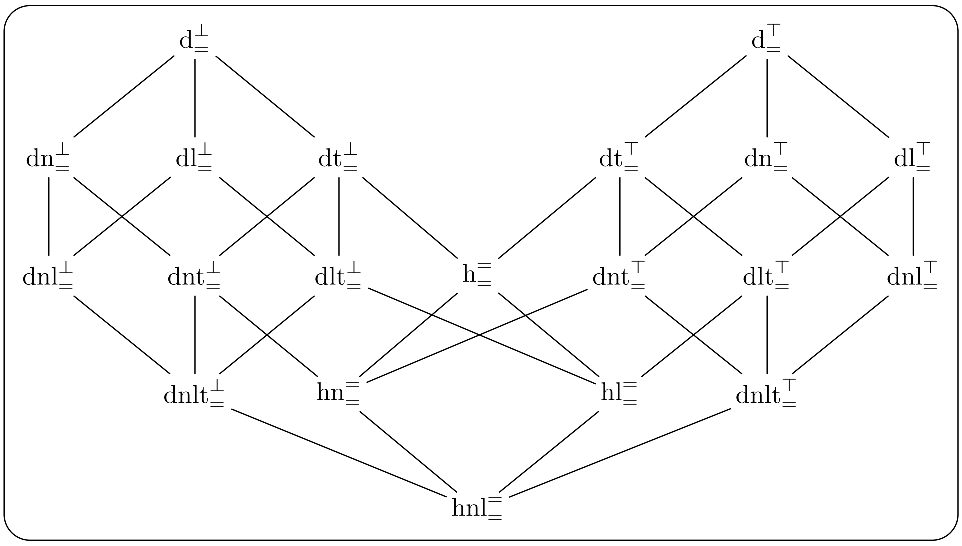

FIGURE 1. HASSE diagram for the semifield $\mathbb{B}$ and the field $\mathbb{Z}_2$.

The inequalities (28) and (29) are trivial. For this let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\tau\colon T_\Sigma \longrightarrow \mathbb{B}\langle\!\langle T_\Delta \rangle\!\rangle$ be such that $\tau(t) = \widetilde{0}$ for every $t \in T_\Sigma$. In a straightforward manner we can show that

$$\tau \in \mathrm{dnl}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) \cap \mathrm{dnl}\text{–}\mathrm{TOP}_\varepsilon(\mathbb{B})$$

but $\tau \notin \mathrm{dt}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) \cup \mathrm{dt}\text{–}\mathrm{TOP}_\varepsilon(\mathbb{B})$, which follows from [**58**, Propositions 4.11 and 4.12]. Inequalities (30)–(33) are also easily seen (*cf.* [**54**, Theorem 3.3]). So, *e. g.*, let $\tau\colon T_\Sigma \longrightarrow \mathbb{B}\langle\!\langle T_\Delta \rangle\!\rangle$ be such that for every $t \in T_\Sigma$ we have $\tau(t) = 1\, \gamma^n(\alpha)$ where $n = \mathrm{height}(t)$. It can easily be seen that $\tau \in \mathrm{hl}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B})$, but $\tau \notin \mathrm{dn}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) \cup \mathrm{dn}\text{–}\mathrm{TOP}_\varepsilon(\mathbb{B})$. Moreover, let $\tau'\colon T_\Delta \longrightarrow \mathbb{B}\langle\!\langle T_\Sigma \rangle\!\rangle$ be such that for every $t \in T_\Delta$ we have $\tau'(t) = 1\, u_t$ where $u_t \in T_\Sigma$ is the fully balanced tree (over Σ) such that $\mathrm{height}(u_t) = \mathrm{height}(t)$. Again, an easy proof shows that $\tau' \in \mathrm{hn}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B})$ and

$$\tau' \notin \mathrm{dl}\text{–}\mathrm{BOT}_\varepsilon(\mathbb{B}) \cup \mathrm{dl}\text{–}\mathrm{TOP}_\varepsilon(\mathbb{B}) \enspace .$$

Inequality (26) is easily proved using the fact that deterministic top-down tree automata are less powerful than deterministic bottom-up tree automata (see, *e. g.*, [**60**, Example II.2.11]). We construct a nondeleting, linear, total, and deterministic bu-tst M following the spirit of [**60**, Example II.2.11]. Since all states of M are final (because M is total), we use the output tree to signal acceptance. So let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Delta = \Sigma \cup \{\delta^{(2)}\}$ and $Q = \{\star, \alpha, \beta\}$. Intuitively, a δ-symbol in the output marks acceptance. Moreover, let $\mu_0(\alpha)_\alpha = 1\, \alpha$ and $\mu_0(\beta)_\beta = 1\, \beta$ and

$$\mu_2(\sigma)_{\star,\alpha\beta} = \mu_2(\sigma)_{\star,\beta\alpha} = 1\, \delta(\mathrm{z}_1, \mathrm{z}_2)$$
$$\mu_2(\sigma)_{\star,\alpha\alpha} = \mu_2(\sigma)_{\star,\beta\beta} = \mu_2(\sigma)_{\star,\star q} = \mu_2(\sigma)_{\star,q\star} = 1\, \sigma(\mathrm{z}_1, \mathrm{z}_2)$$

for every $q \in Q$. As usual all remaining entries in the tree representation μ are assumed to be $\widetilde{0}$. Then $M = (Q, \Sigma, \Delta, \mathbb{B}, F, \mu)$ with $F_q = 1\, \mathrm{z}_1$ for every $q \in Q$ (see Figure 2) computes the ε-t-ts transformation $\tau = \|M\|_\varepsilon$, which maps every $t \in T_\Sigma$ to $1\,u$ where u is obtained from t by replacing subtrees $\sigma(\alpha, \beta)$ and $\sigma(\beta, \alpha)$ in t by $\delta(\alpha, \beta)$ and $\delta(\beta, \alpha)$, respectively. It is straightforward to show that no deterministic td-tst over $\mathbb{B}$ can compute τ.

Informally speaking, Inequality (27) can be shown by exploiting the fact that a deterministic td-tst can count (modulo some $n \geqslant 2$) the number of symbols on a path from the root of the input tree to some other node. According to its current count, it can then perform output. A deterministic bu-tst cannot emulate this behavior because at each node it is unaware of the distance (modulo n) of the current node to the root node. Thus it could only delay the correct output at this step, which is impossible in the example we present. More formally, let $\Sigma = \{\gamma_1^{(1)}, \gamma_2^{(1)}, \alpha^{(0)}\}$ and $Q = \{0, 1\}$. Moreover, let $\mu_0(\alpha)_q = 1\,\alpha$ for every $q \in Q$ and

$$\mu_1(\gamma_i)_{0,1(\mathrm{x}_1)} = 1\,\mathrm{z}_1 \quad \text{and} \quad \mu_1(\gamma_i)_{1,0(\mathrm{x}_1)} = 1\,\gamma_i(\mathrm{z}_1)$$

for every $i \in [2]$. The td-tst $M = (Q, \Sigma, \Sigma, \mathbb{B}, F, \mu)$ with $F_0 = \widetilde{0}$ and $F_1 = 1\mathrm{z}_1$ (see Figure 3) is clearly nondeleting, linear, total, and deterministic and computes the ε-t-ts transformation $\tau = \|M\|_\varepsilon$, which maps every $t \in T_\Sigma$ to $1\,u$ where u is obtained from t by deleting every second position in the input string; *i. e.*, $\tau(\gamma_1(\gamma_2(\gamma_1(\alpha)))) = 1\,\gamma_1(\gamma_1(\alpha))$. It is easily seen that no deterministic bu-tst can compute τ. □

Let $\mathcal{A}$ be a commutative semiring with at least three elements. In Section 4, we derive some statements which hold for every such semiring $\mathcal{A}$. Moreover, we completely describe the situation for ε-t-ts transformations computed by deterministic td-tst provided that $\mathcal{A}$ is zero-divisor free. We continue in Section 5 with semirings $\mathcal{A}$ that are not multiplicatively periodic. Section 6 is dedicated to multiplicatively periodic, but not multiplicatively idempotent semirings $\mathcal{A}$. The final case, which is handled in Section 7, assumes that $\mathcal{A}$ is multiplicatively idempotent.

4. Arbitrary semirings

In this section, we derive some statements that hold for every commutative semiring $\mathcal{A}$ that has at least three elements (*i. e.*, $0 \neq 1$ and $\mathcal{A}$ is not isomorphic to $\mathbb{B}$ or $\mathbb{Z}_2$). Then we consider ε-t-ts transformations that are computed by deterministic td-tst. Foremost important, we show how to use the results of the HASSE diagram in Figure 1 in order to obtain incomparability results for classes of ε-t-ts and o-t-ts transformations over semirings $\mathcal{A}$ different from $\mathbb{B}$ and $\mathbb{Z}_2$. Roughly speaking, we show that all inequalities present in Figure 1 are preserved in the transition from $\mathbb{B}$ to $\mathcal{A}$. This is mainly due to the fact that $(\{0, 1\}, \cdot)$ is a submonoid (with absorbing 0) of $(A, \cdot)$, which is isomorphic to $(\{0, 1\}, \wedge)$. For the rest of this section, let $\mathcal{A} = (A, +, \cdot)$ be a nontrivial (*i. e.*, $0 \neq 1$) and commutative semiring and $\eta, \kappa \in \{\varepsilon, \mathrm{o}\}$. Moreover, let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tst. Note

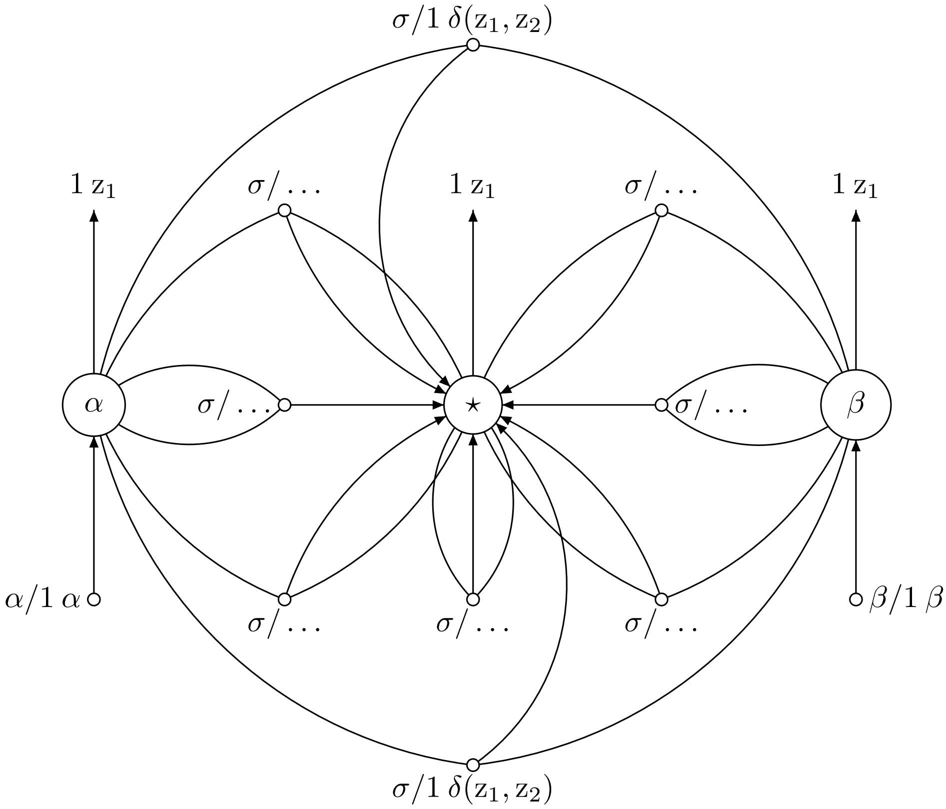

FIGURE 2. Bu-tst over $\mathbb{B}$ that is used to show Inequality (26) of Theorem 5.5 where $\sigma/\ldots$ stands for $\sigma/1\,\sigma(\mathrm{z}_1, \mathrm{z}_2)$.

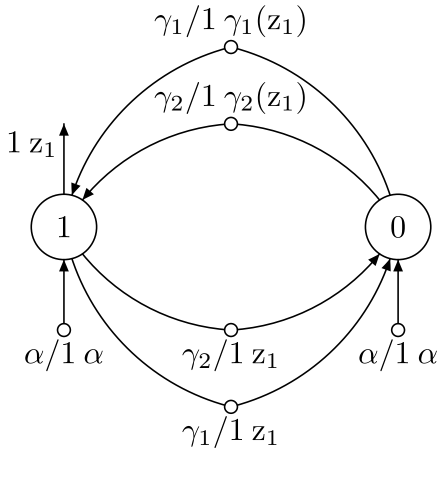

FIGURE 3. Td-tst over $\mathbb{B}$ that is used to show Inequality (27) of Theorem 5.5.

that well-definedness of η-t-ts transformations is no issue, because deterministic bu-tst and td-tst are polynomial (see Observation 4.14).

The result, that all inequalities are preserved, is achieved by two preparatory propositions and one lemma. First we present a proposition that shows that top-most outputs F can equivalently be replaced by boolean top-most outputs, if F is a vector of monomial tree series and the nonzero coefficients in F are (multiplicatively) invertible. Note that the first condition is automatically fulfilled, if M is td-deterministic or bu-deterministic. In [**16**, Lemma 6.1.4] a similar construction is stated for weighted tree automata. There it is shown that given a commutative semifield and a weighted tree automaton with final weights, there exists a semantically equivalent weighted tree automaton with final states (*i. e.*, boolean final weights). Our statement generalizes this statement to tst and only assumes that the nonzero coefficients in the top-most output tree series are invertible as opposed to all nonzero semiring elements are invertible. This slight change is necessitated by a forthcoming proposition (see Proposition 5.9), where we only show this weaker condition for particular tst.

The main idea (see [**16**, Lemma 6.1.4]) of the construction (explained below for bu-tst) is to move the coefficient from F into the tree representation μ. Roughly speaking, the top-most output coefficient corresponding to a state q (the coefficient is unique because F_q is monomial) is applied, whenever M changes into state q. Of course, it may happen that the computation does not terminate in q, but rather proceeds by changing into another state. Thus, we multiply the inverses of the top-most output coefficients corresponding to the states q_i of subcomputations to all tree representation entries $\mu_k(\sigma)_{q,q_1(\mathrm{x}_{i_1})\cdots q_n(\mathrm{x}_{i_n})}$. Since these factors are separated from their inverses by other coefficients, we assume that $\mathcal{A}$ is commutative. Finally, a technical problem (division by zero) arises, whenever the top-most output coefficient corresponding to a state is 0. Such a coefficient may not be moved into the tree representation μ, but should remain in F.

PROPOSITION 5.6 (*cf.* [**16**, Lemma 6.1.4]). *If F_q is monomial and (F_q, u) is invertible for every $q \in Q$ and $u \in \mathrm{supp}(F_q)$, then there exists a tst M' with top-most outputs F' such that $\|M'\|_\eta = \|M\|_\eta$ and every entry in F' is boolean and monomial.*

PROOF. Since F_q is monomial for every $q \in Q$, we let $a_q \in A$ and $u_q \in C_\Delta(\mathrm{Z}_1)$ be such that $F_q = a_q\, u_q$. Moreover, let $a'_q = a_q$ if $a_q \neq 0$, and $a'_q = 1$ otherwise. Note that a'_q is invertible for every $q \in Q$. We construct $M' = (Q, \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows. For every $q \in Q$ we let $F'_q = \chi(\mathrm{supp}(F_q))$, and for every $k \in \mathbb{N}$, symbol $\sigma \in \Sigma_k$, state $q \in Q$,

$w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$, and $u \in T_\Delta(\mathrm{Z}_n)$ let

$$(\mu'_k(\sigma)_{q,w}, u) = a'_q \cdot (\mu_k(\sigma)_{q,w}, u) \cdot \prod_{j \in [n]} (a'_{q_j})^{-\operatorname{sel}(u,j,\eta)} \enspace .$$

(Recall that (4) defines sel.) Clearly, a'_q and $(a'_{q_j})^{-\operatorname{sel}(u,j,\eta)}$ are nonzero and invertible for every $j \in [n]$. Thus, it follows from Observation 2.2 that $(\mu'_k(\sigma)_{q,w}, u) = 0$ if and only if $(\mu_k(\sigma)_{q,w}, u) = 0$. Hence

$$\operatorname{supp}(\mu'_k(\sigma)_{q,w}) = \operatorname{supp}(\mu_k(\sigma)_{q,w}) \enspace .$$

By definition, F'_q is boolean and monomial for every $q \in Q$. It remains to prove $\|M'\|_\eta = \|M\|_\eta$. For this we first prove

$$h^\eta_{\mu'}(t)_q = a'_q \cdot h^\eta_\mu(t)_q \tag{34}$$

for every $q \in Q$ and $t \in T_\Sigma$ by induction on t, so let $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$.

$$\begin{aligned}
& h^\eta_{\mu'}\big(\sigma(t_1, \dots, t_k)\big)_q \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu'_k(\sigma)_{q,w} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_{i_j})_{q_j}\big)_{j \in [n]} \\
= \quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{ and induction hypothesis)} \\
& \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \Bigg(\sum_{\substack{u \in \operatorname{supp}(\mu'_k(\sigma)_{q,w}), \\ (\forall j \in [n]) \colon u_j \in \operatorname{supp}(a'_{q_j} \cdot h^\eta_\mu(t_{i_j})_{q_j})}} \\
& \quad \Big((\mu'_k(\sigma)_{q,w}, u) \cdot \prod_{j \in [n]} (a'_{q_j} \cdot h^\eta_\mu(t_{i_j})_{q_j}, u_j)^{\operatorname{sel}(u,j,\eta)} \Big) \, u[u_j]_{j \in [n]} \Bigg) \\
= \quad & \textit{(by definition of } \mu'_k(\sigma)_{q,w} \textit{ and } \operatorname{supp}(\mu'_k(\sigma)_{q,w}) = \operatorname{supp}(\mu_k(\sigma)_{q,w}) \\
& \quad \textit{and } \operatorname{supp}(a'_{q_j} \cdot h^\eta_\mu(t_{i_j})_{q_j}) = \operatorname{supp}(h^\eta_\mu(t_{i_j})_{q_j}) \textit{ since } a'_{q_j} \textit{ is invertible)} \\
& \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \Bigg(\sum_{\substack{u \in \operatorname{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall j \in [n]) \colon u_j \in \operatorname{supp}(h^\eta_\mu(t_{i_j})_{q_j})}} \Big(a'_q \cdot (\mu_k(\sigma)_{q,w}, u) \cdot \\
& \quad \cdot \Big(\prod_{j \in [n]} (a'_{q_j})^{-\operatorname{sel}(u,j,\eta)} \Big) \cdot \Big(\prod_{j \in [n]} (a'_{q_j})^{\operatorname{sel}(u,j,\eta)} \Big) \cdot \\
& \quad \cdot \prod_{j \in [n]} (h^\eta_\mu(t_{i_j})_{q_j}, u_j)^{\operatorname{sel}(u,j,\eta)} \Big) \, u[u_j]_{j \in [n]} \Bigg) \\
= a'_q \cdot & \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \Bigg(\sum_{\substack{u \in \operatorname{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall j \in [n]) \colon u_j \in \operatorname{supp}(h^\eta_\mu(t_{i_j})_{q_j})}} \Big((\mu_k(\sigma)_{q,w}, u) \cdot \\
& \quad \cdot \prod_{j \in [n]} (h^\eta_\mu(t_{i_j})_{q_j}, u_j)^{\operatorname{sel}(u,j,\eta)} \Big) \, u[u_j]_{j \in [n]} \Bigg)
\end{aligned}$$

TABLE 1. Preservation of properties for the construction of Proposition 5.6.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

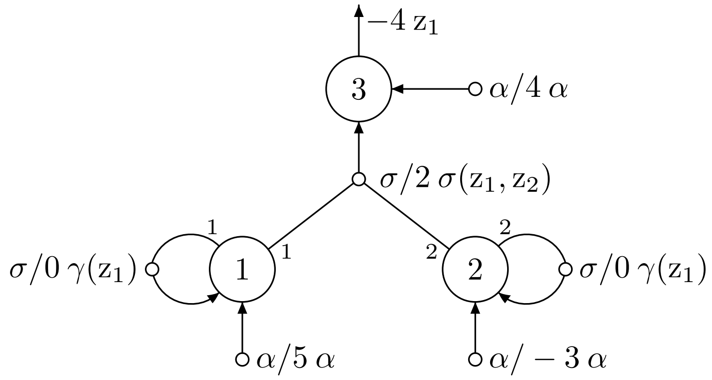

FIGURE 4. Td-tst $M_{5.7}$ over $\mathbb{T}_{\text{sf}}$ (see Example 5.7).

$$= \quad \textit{(by definition of } \xleftarrow[\eta]{}\textit{)}$$
$$a'_q \cdot \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1})\cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} \big(h^\eta_\mu(t_{i_j})_{q_j}\big)_{j \in [n]}$$
$$= \quad \textit{(by Definition 4.7(1))}$$
$$a'_q \cdot h^\eta_\mu\big(\sigma(t_1, \dots, t_k)\big)_q$$

Now we can prove $\|M'\|_\eta = \|M\|_\eta$ as follows. For every $q \in Q$, let $b_q = 0$ if $a_q = 0$, and $b_q = 1$ otherwise. We observe that according to these definitions we have $F'_q = b_q\, u_q$ and $a_q = a'_q \cdot b_q$.

$$\|M'\|_\eta(t)$$
$$= \quad \textit{(by Definition 4.7(2), Equation (34), and definition of } F'_q\textit{)}$$
$$\sum_{q \in Q} F'_q \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t)_q\big) \;=\; \sum_{q \in Q} b_q\, u_q \xleftarrow[\eta]{} \big(a'_q \cdot h^\eta_\mu(t)_q\big)$$
$$= \quad \textit{(by Proposition 3.9 and Definition 4.7(2))}$$
$$\sum_{q \in Q} (a'_q \cdot b_q)\, u_q \xleftarrow[\eta]{} \big(h^\eta_\mu(t)_q\big) \;=\; \sum_{q \in Q} F_q \xleftarrow[\eta]{} \big(h^\eta_\mu(t)_q\big) \;=\; \|M\|_\eta(t) \qquad \square$$

Note that the previous construction preserves all introduced properties (see Table 1); *i. e.*, if M has a property $x \in \text{Pref}$, then also M' has property x. Let us first illustrate the construction on a deterministic td-tst.

EXAMPLE 5.7. *Let* $M_{5.7} = (Q, \Sigma, \Delta, \mathbb{T}_{\text{sf}}, F, \mu)$ *be the deterministic td-tst with:*

- $Q = \{1, 2, 3\}$;

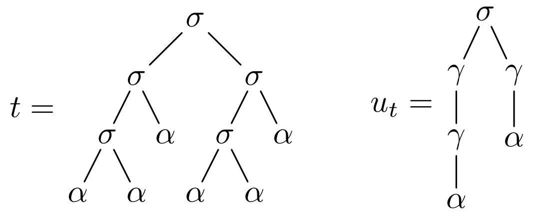

FIGURE 5. Illustrating the relation between t and u_t (see Example 5.7).

- $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ *and* $\Delta = \Sigma \cup \{\gamma^{(1)}\}$;
- $F_1 = F_2 = \widetilde{\infty}$ *and* $F_3 = -4\,\mathrm{z}_1$; *and*
- *all entries in* μ *are* $\widetilde{\infty}$ *except*

$$\mu_2(\sigma)_{1,1(\mathrm{x}_1)} = 0\,\gamma(\mathrm{z}_1) \quad \mu_2(\sigma)_{3,1(\mathrm{x}_1)2(\mathrm{x}_2)} = 2\,\sigma(\mathrm{z}_1,\mathrm{z}_2) \quad \mu_0(\alpha)_2 = -3\,\alpha$$
$$\mu_2(\sigma)_{2,2(\mathrm{x}_2)} = 0\,\gamma(\mathrm{z}_1) \qquad \mu_0(\alpha)_1 = 5\,\alpha \qquad \mu_0(\alpha)_3 = 4\,\alpha\ .$$

The td-tst $M_{5.7}$ is displayed in Figure 4. Clearly, $M_{5.7}$ computes the ε-t-ts transformation $\|M_{5.7}\|_\varepsilon(t) = 0\,u_t$ for every $t \in T_\Sigma$, where $u_t = \alpha$ if $t = \alpha$, and $u_t = \sigma(\gamma^{n_1}(\alpha), \gamma^{n_2}(\alpha))$ if $t = \sigma(t_1, t_2)$ and n_1 (respectively, n_2) is the number of σ-symbols on the left (respectively, right) spine of t_1 (respectively, t_2). Thus, for

$$t = \sigma(\sigma(\sigma(\alpha,\alpha),\alpha),\sigma(\sigma(\alpha,\alpha),\alpha))$$

we have $u_t = \sigma(\gamma^2(\alpha), \gamma(\alpha))$ [see Figure 5]. Moreover, F is a vector of monomial tree series and -4 is invertible in $\mathbb{T}_{\mathrm{sf}}$. Thus we can apply Proposition 5.6 and obtain the deterministic td-tst

$$M'_{5.7} = (Q, \Sigma, \Delta, \mathbb{T}_{\mathrm{sf}}, F', \mu')$$

with:

- $F'_1 = F'_2 = \widetilde{\infty}$ *and* $F'_3 = 0\,\mathrm{z}_1$; *and*
- *all entries in* μ' *are* $\widetilde{\infty}$ *except*

$$\mu'_2(\sigma)_{1,1(\mathrm{x}_1)} = 0\,\gamma(\mathrm{z}_1) \quad \mu'_2(\sigma)_{3,1(\mathrm{x}_1)2(\mathrm{x}_2)} = -2\,\sigma(\mathrm{z}_1,\mathrm{z}_2) \quad \mu'_0(\alpha)_2 = -3\,\alpha$$
$$\mu'_2(\sigma)_{2,2(\mathrm{x}_2)} = 0\,\gamma(\mathrm{z}_1) \qquad \mu'_0(\alpha)_1 = 5\,\alpha \qquad \mu'_0(\alpha)_3 = 0\,\alpha\ .$$

We display $M'_{5.7}$ in Figure 6. It is easily seen that $\|M'_{5.7}\|_\varepsilon = \|M_{5.7}\|_\varepsilon$.

If $\eta = \varepsilon$ then Proposition 5.6 is a straightforward adaption of [**16**, Lemma 6.1.4]. To illustrate Proposition 5.6 in case $\eta = \mathrm{o}$, let us consider a nonlinear bu-tst using o-substitution.

EXAMPLE 5.8 (*cf.* [**58**, Example 4.8]). *Let*

$$M_{5.8} = (\{\star\}, \Sigma, \Delta, \mathbb{R}_+, F, \mu)$$

be the deterministic bu-tst with:

- $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ *and* $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$;
- $F_\star = 2\,\mathrm{z}_1$; *and*
- $\mu_0(\alpha)_\star = 2\,\alpha$ *and* $\mu_1(\gamma)_{\star,\star} = 2\,\sigma(\mathrm{z}_1,\mathrm{z}_1)$.

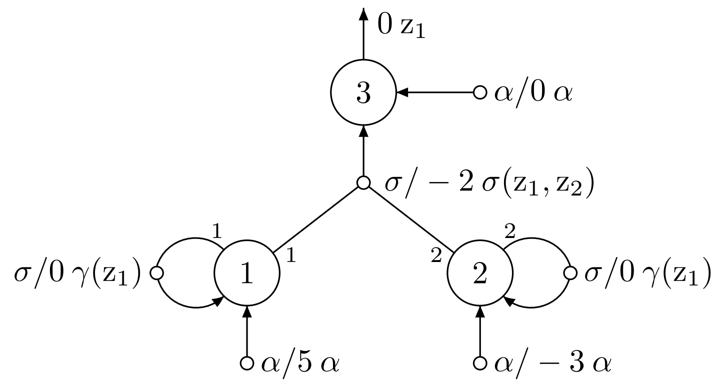

FIGURE 6. Td-tst $M'_{5.7}$ over $\mathbb{T}_{\mathrm{sf}}$ (see Example 5.7).

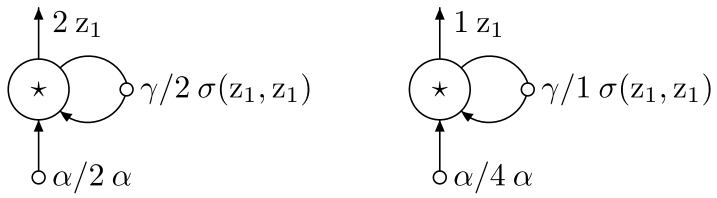

FIGURE 7. Bu-tst $M_{5.8}$ (left) and $M'_{5.8}$ (right) over $\mathbb{R}_+$ (see Example 5.8).

Figure 7(left) illustrates $M_{5.8}$. Note that $M_{5.8} = M^2_{4.24}$ for the semiring $\mathbb{R}_+$ (see Example 4.24). Thus, we know from Lemma 4.25 that for every $t \in T_\Sigma$ we have $\|M_{5.8}\|_{\mathrm{o}}(t) = 2^{2^{n+1}}\, b(n)$ where $n = \mathrm{height}(t)$ and $b(n)$ is the fully balanced tree (over Δ) of height n.

Clearly, 2 is invertible in $\mathbb{R}_+$, so that we can apply Proposition 5.6 to obtain the homomorphism bu-tst $M'_{5.8} = (\{\star\}, \Sigma, \Delta, \mathbb{R}_+, F', \mu')$ with:

- $F'_\star = 1\, \mathrm{z}_1$; *and*
- $\mu'_0(\alpha)_\star = 4\, \alpha$ *and* $\mu'_1(\gamma)_{\star,\star} = 1\, \sigma(\mathrm{z}_1, \mathrm{z}_1)$.

The bu-tst $M'_{5.8}$ is displayed in Figure 7(right). It is easily observed that $\|M'_{5.8}\|_{\mathrm{o}} = \|M_{5.8}\|_{\mathrm{o}}$.

As a second step we show that every η-t-ts transformation computable by a deterministic bu-tst or td-tst can also be computed by some boolean deterministic bu-tst or td-tst provided that the η-t-ts transformation fulfills certain conditions. More precisely, those conditions are that the η-t-ts transformation should be nonzero and boolean everywhere.

The main idea of this construction is a simple one. Let

$$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$$

be a deterministic bu-tst or td-tst. Roughly speaking, we construct a boolean deterministic bu-tst or td-tst by replacing all nonzero coefficients in μ by 1. However, some preparatory steps are required beforehand. Essentially, the construction requires three steps (see Figure 8).

$$\begin{tikzcd}
M \arrow[r, "\text{purge unnecessary states}"] & M_1 \arrow[d, "\text{booleanize top-most outputs}"] \\
M' & M_2 \arrow[l, "\text{booleanize tree representation}"']
\end{tikzcd}$$

FIGURE 8. Illustrating the steps of the construction of Proposition 5.9.

TABLE 2. Preservation of properties for the construction of Proposition 5.9.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Firstly we purge unnecessary states; *i. e.*, if there is a state q such that $F_q \neq \widetilde{0}$ but there is no input tree t such that $F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q) \neq \widetilde{0}$, then clearly this state q need not be final or initial. Secondly we show that the remaining states have invertible nonzero coefficients in F and then apply Proposition 5.6 to obtain a semantically equivalent deterministic bu-tst or td-tst with boolean top-most outputs. Finally in the last step we replace the nonzero coefficients in the obtained tree representation by 1 and show that the such obtained tst computes the same η-t-ts transformation as M.

PROPOSITION 5.9. *Let M be a deterministic bu-tst or td-tst. If $\|M\|_\eta(t)$ is nonzero and boolean for every $t \in T_\Sigma$, then there exists a boolean tst M' such that $\|M'\|_\eta = \|M\|_\eta$.*

PROOF. Let $t \in T_\Sigma$. Since $\|M\|_\eta(t)$ is nonzero and M is deterministic, we have that $F_{q_t} \xleftarrow[\eta]{} (h^\eta_\mu(t)_{q_t})$ is nonzero for some unique $q_t \in Q$ [see Proposition 5.1(3)]. Let $Q' = \{q_t \in Q \mid t \in T_\Sigma\}$; note that Q' is a singleton, if M is top-down. We now prove that $(F_{q'}, u)$ is invertible for every $q' \in Q'$ and $u \in \mathrm{supp}(F_{q'})$.

Let $q' \in Q'$ and $t \in T_\Sigma$ be such that $q_t = q'$. Clearly, such a tree exists because $q' \in Q'$. Moreover, since $\|M\|_\eta(t)$ and $h^\eta_\mu(t)_{q'}$ are monomial by Proposition 5.1(5), let $a', a'' \in A$, $u' \in C_\Delta(\mathrm{Z}_1)$, and $u, u'' \in T_\Delta$ be such that $\|M\|_\eta(t) = 1\, u$ and $F_{q'} = a'\, u'$ and $h^\eta_\mu(t)_{q'} = a''\, u''$.

$$1\, u \;=\; \|M\|_\eta(t) = \sum_{q \in Q} F_q \xleftarrow[\eta]{} \left(h^\eta_\mu(t)_q\right) \;=\; F_{q'} \xleftarrow[\eta]{} \left(h^\eta_\mu(t)_{q'}\right) \;=\; (a' \cdot a'')\, u'[u'']$$

Thus a' is invertible with $(a')^{-1} = a''$.

We obtain the deterministic bu-tst or td-tst $M_1 = (Q, \Sigma, \Delta, \mathcal{A}, F_1, \mu)$ by discarding elements of $Q \setminus Q'$ as final or initial states; *i. e.*, $(F_1)_q = F_q$ for every $q \in Q'$, and $(F_1)_q = \widetilde{0}$ for every $q \in Q \setminus Q'$. We note that $F_1 = F$, if M is top-down. Clearly, $\|M_1\|_\eta = \|M\|_\eta$. Now we can apply Proposition 5.6

to M_1 and obtain the deterministic bu-tst or td-tst $M_2 = (Q, \Sigma, \Delta, \mathcal{A}, F', \mu_2)$, which obeys $\|M_2\|_\eta = \|M_1\|_\eta$, such that F'_q is boolean for every $q \in Q$.

We construct the deterministic bu-tst or td-tst $M' = (Q, \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows. Let $\mu'_k(\sigma)_{q,w} = \chi(\mathrm{supp}((\mu_2)_k(\sigma)_{q,w}))$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$. Clearly, M' is boolean, so it remains to prove that $\|M'\|_\eta = \|M_2\|_\eta$.

(i) We first consider the bottom-up case, so let M be a deterministic bu-tst. It follows that also M_2 and M' are deterministic bu-tst. Since M_2 and M' have the same set of states and the same final outputs, it obviously is sufficient to prove that $h^\eta_{\mu_2}(t)_q = h^\eta_{\mu'}(t)_q$ for every $t \in T_\Sigma$ and $q \in Q$. We prove this by induction, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$. Moreover, for every $q \in Q$ let $a_q \in \{0, 1\}$ and $u_q \in C_\Delta(\mathrm{Z}_1)$ be such that $F'_q = a_q\, u_q$.

$$
\begin{aligned}
& h^\eta_{\mu_2}\big(\sigma(t_1, \ldots, t_k)\big)_q \\
= \quad & \textit{(by Definition 4.7(1) and induction hypothesis)} \\
& \sum_{q_1,\ldots,q_k \in Q} (\mu_2)_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_j)_{q_j}\big)_{j\in[k]} \\
= \quad & \textit{(for some } p_1, \ldots, p_k \in Q \textit{ by Proposition 5.1(2);} \\
& \quad \textit{note that } h^\eta_{\mu'}(t_j)_p = \widetilde{0} \textit{ for all } p \neq p_j \textit{ by Proposition 5.1(1))} \\
& (\mu_2)_k(\sigma)_{q,p_1\cdots p_k} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_j)_{p_j}\big)_{j\in[k]} \\
= \quad & \textit{(} h^\eta_{\mu'}(t_j)_{p_j} \textit{ and } (\mu_2)_k(\sigma)_{q,p_1\cdots p_k} \textit{ are monomial by Proposition 5.1(5),} \\
& \quad \textit{so for every } j \in [k] \textit{ let } a, a_j \in A \textit{ and } u \in T_\Delta(\mathrm{Z}_k) \textit{ and } u_j \in T_\Delta \\
& \quad \textit{be such that } (\mu_2)_k(\sigma)_{q,p_1\cdots p_k} = a\,u \textit{ and } h^\eta_{\mu'}(t_j)_{p_j} = a_j\,u_j\textit{)} \\
& a\,u \xleftarrow[\eta]{} (a_j\,u_j)_{j\in[k]} \\
= \quad & \textit{(let } b = 0 \textit{ if } a = 0 \textit{ and } b = 1 \textit{ otherwise;} \\
& \quad \textit{since } M' \textit{ is boolean, } a_j \in \{0,1\} \textit{ by Observation 5.2, and} \\
& \quad \textit{(1) if } a_j = 0 \textit{ for some } j \in [k]\textit{, then } c\,u \xleftarrow[\eta]{} (a_j\,u_j)_{j\in[k]} = \widetilde{0} \\
& \qquad \textit{for every } c \in A \\
& \quad \textit{(2) if } a = 0\textit{, then } b = 0 \\
& \quad \textit{(3) if } a_j = 1 \textit{ for every } j \in [k] \textit{ and } a \neq 0\textit{, then} \\
& \qquad a\,u \xleftarrow[\eta]{} (1\,u_j)_{j\in[k]} = a\,u[u_j]_{j\in[k]} \textit{ and } h^\eta_{\mu_2}(t)_q \neq \widetilde{0} \textit{ and hence} \\
& \qquad \textit{by Proposition 5.1(1) we have } a_q \neq 0 \textit{ because } \|M_2\|_\eta(t) \neq \widetilde{0}; \\
& \qquad \textit{thus } a_q = 1 \textit{ and } a_q \cdot a = 1 \textit{ by } \|M_2\|_\eta(t) = a_q\,u_q \xleftarrow[\eta]{} (h^\eta_{\mu_2}(t)_q); \\
& \qquad \textit{consequently } a = 1 = b\textit{)} \\
& b\,u \xleftarrow[\eta]{} (a_j\,u_j)_{j\in[k]} \\
= \quad & \textit{(by definition of } \mu'_k(\sigma)_{q,p_1\cdots p_k}\textit{)} \\
& \mu'_k(\sigma)_{q,p_1\cdots p_k} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_j)_{p_j}\big)_{j\in[k]} \\
= \quad & \textit{(because } h^\eta_{\mu'}(t_j)_p = \widetilde{0} \textit{ for every } p \neq p_j\textit{)} \\
& \sum_{q_1,\ldots,q_k \in Q} \mu'_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_j)_{q_j}\big)_{j\in[k]}
\end{aligned}
$$

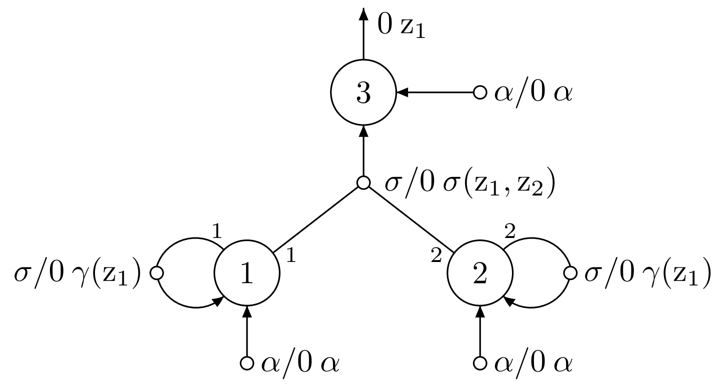

FIGURE 9. Td-tst $M'_{5.10}$ over $\mathbb{T}_{\mathrm{sf}}$ (see Example 5.10).

$=$ *(by Definition 4.7(1))*

$h^\eta_{\mu'}\big(\sigma(t_1, \ldots, t_k)\big)_q$

This proves the statement for deterministic bu-tst M.

(ii) Now let M be top-down. Clearly, $F'_q \neq \widetilde{0}$ for some $q \in Q$, so let $q \in Q$ be the unique state such that $F'_q \neq \widetilde{0}$. Again M_2 and M' have the same set of states and the same initial outputs, so that it is sufficient to prove that $h^\eta_{\mu_2}(t)_q = h^\eta_{\mu'}(t)_q$ for every $t \in T_\Sigma$ (note that q is fixed). However, we first prove that $u \in \mathrm{supp}(h^\eta_{\mu_2}(t)_p)$ implies that $h^\eta_{\mu'}(t)_p = 1\,u$ for every $t \in T_\Sigma$, $p \in Q$, and $u \in T_\Delta$. Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$u \in \mathrm{supp}\big(h^\eta_{\mu_2}(\sigma(t_1, \ldots, t_k))_p\big)$

$\Longrightarrow$ *(for some $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})$ by Proposition 5.1(2))*

$u \in \mathrm{supp}\big((\mu_2)_k(\sigma)_{p,w} \xleftarrow[\eta]{} (h^\eta_{\mu_2}(t_{i_j})_{q_j})_{j\in[n]}\big)$

$\Longrightarrow$ *(there exist $u' \in \mathrm{supp}((\mu_2)_k(\sigma)_{p,w})$ and $u_j \in \mathrm{supp}(h^\eta_{\mu_2}(t_{i_j})_{q_j})$ such that $u = u'[u_j]_{j\in[n]}$; thus $\mu'_k(\sigma)_{p,w} = 1\,u'$ and by induction hypothesis also $h^\eta_{\mu'}(t_{i_j})_{q_j} = 1\,u_j$)*

$\mu'_k(\sigma)_{p,w} \xleftarrow[\eta]{} (h^\eta_{\mu'}(t_{i_j})_{q_j})_{j\in[n]} = 1\,u$

$\Longrightarrow$ *(by Definition 4.7(1) and td-determinism of M')*

$h^\eta_{\mu'}(\sigma(t_1, \ldots, t_k))_p = 1\,u$

Now we turn to the proof of $h^\eta_{\mu_2}(t)_q = h^\eta_{\mu'}(t)_q$ for every $t \in T_\Sigma$. Let $a'' \in A$, $u, u'' \in T_\Delta$, and $u' \in C_\Delta(\mathrm{Z}_1)$ be such that $\|M_2\|_\eta(t) = 1\,u$ and $F'_q = 1\,u'$ and $h^\eta_{\mu_2}(t)_q = a''\,u''$. Since

$$1\,u = \|M_2\|_\eta(t) = F'_q \xleftarrow[\eta]{} \big(h^\eta_{\mu_2}(t)_q\big) = a''\,u'[u''] \ ,$$

we obtain $a'' = 1$. Hence $u'' \in \mathrm{supp}(h^\eta_{\mu_2}(t)_q)$ and also $h^\eta_{\mu'}(t)_q = 1\,u''$ by the previous property, which proves the statement. □

Let us illustrate the construction of Proposition 5.9 by applying it to the deterministic td-tst $M_{5.7}$ of Example 5.7.

$$\begin{array}{ccc}
\tau \in x\text{-BOT}_\eta(\mathcal{A}) & \overset{?}{\wedge} & \tau \notin y\text{-BOT}_\kappa(\mathcal{A}) \\
\Uparrow & & \Updownarrow \\
\tau \in x\text{-BOT}_\eta(\mathbb{B}) & \wedge & \tau \notin y\text{-BOT}_\kappa(\mathbb{B})
\end{array}$$

FIGURE 10. Schematics for the proof of Lemma 5.11.

EXAMPLE 5.10. *Let $M_{5.10} = M_{5.7} = (Q, \Sigma, \Delta, \mathbb{T}_{\mathrm{sf}}, F, \mu)$ be the td-tst of Example 5.7. Applying the construction of Proposition 5.9 stepwise to $M_{5.10}$ we obtain $M_{5.10}$ after purging unnecessary states, $M'_{5.7}$ of Example 5.7 after booleanizing top-most outputs, and finally $M'_{5.10} = (Q, \Sigma, \Delta, \mathbb{T}_{\mathrm{sf}}, F', \mu')$ after booleanizing the tree representation, where:*

- *$F'_1 = F'_2 = \widetilde{\infty}$ and $F'_3 = 0\,\mathrm{z}_1$; and*
- *all entries in μ' are $\widetilde{\infty}$ except*

$$\begin{array}{lll}
\mu'_2(\sigma)_{1,1(\mathrm{x}_1)} = 0\,\gamma(\mathrm{z}_1) & \mu'_2(\sigma)_{3,1(\mathrm{x}_1)2(\mathrm{x}_2)} = 0\,\sigma(\mathrm{z}_1, \mathrm{z}_2) & \mu'_0(\alpha)_2 = 0\,\alpha \\
\mu'_2(\sigma)_{2,2(\mathrm{x}_2)} = 0\,\gamma(\mathrm{z}_1) & \mu'_0(\alpha)_1 = 0\,\alpha & \mu'_0(\alpha)_3 = 0\,\alpha\ .
\end{array}$$

The td-tst $M'_{5.10}$ is illustrated in Figure 9. Again, we easily see that $\|M'_{5.10}\|_\varepsilon = \|M_{5.10}\|_\varepsilon$.

Let us return to the original problem: to lift

$$x\text{–BOT}_\eta(\mathbb{B}) \not\subseteq y\text{–BOT}_\kappa(\mathbb{B})$$
$$x\text{–TOP}_\eta(\mathbb{B}) \not\subseteq y\text{–BOT}_\kappa(\mathbb{B})$$
$$x\text{–BOT}_\eta(\mathbb{B}) \not\subseteq y\text{–TOP}_\kappa(\mathbb{B})$$

from the semiring $\mathbb{B}$ to the semiring $\mathcal{A}$. We explain only the lift of the first inequality. There we take a counterexample for the inclusion in the boolean semiring; *i. e.*, an η-t-ts transformation τ that is in the class x–BOT$_\eta(\mathbb{B})$, but not in the class y–BOT$_\kappa(\mathbb{B})$ for some $x, y \in \Pi_{\mathrm{d}}$. Then we prove that τ is also a counterexample for the inclusion x–BOT$_\eta(\mathcal{A}) \subseteq y$–BOT$_\kappa(\mathcal{A})$; *i. e.*, τ is trivially in x–BOT$_\eta(\mathcal{A})$ because $(\{0,1\}, \cdot)$ is a submonoid, that is isomorphic to $(\{0,1\}, \wedge)$, of $(A, \cdot)$, but still not in y–BOT$_\kappa(\mathcal{A})$.

For the purpose of the next lemma, we restrict the counterexample τ to be computed by a total deterministic bu-tst M. Now assume that $\tau \in y$–BOT$_\kappa(\mathcal{A})$; *i. e.*, there exists a deterministic bu-tst M' such that $\|M'\|_\kappa = \tau$. It follows from the totality of M that for every $t \in T_\Sigma$ there exists a unique $u \in T_\Delta$ such that $\tau(t) = 1\,u$ (see [**58**, Proposition 4.11] and Observation 5.4). Using Proposition 5.9 (applied to M') we obtain a boolean deterministic bu-tst M'' with $\|M''\|_\kappa = \tau$. However, boolean deterministic bu-tst compute solely in $\{0,1\}$ (see Observation 5.2), and therefore, M'' can equivalently be specified as deterministic bu-tst over $\mathbb{B}$, which is a contradiction to the assumption that $\tau \notin y$–BOT$_\kappa(\mathbb{B})$. This approach is illustrated in Figure 10.

LEMMA 5.11. *Let* $x \in \Pi_{\mathrm{dt}}$ *and* $y \in \Pi_{\mathrm{d}}$.

(1) $x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq y\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$, *if* $x\text{–}\mathrm{BOT}_\eta(\mathbb{B}) \not\subseteq y\text{–}\mathrm{BOT}_\kappa(\mathbb{B})$.
(2) $x\text{–}\mathrm{TOP}_\eta(\mathcal{A}) \not\subseteq y\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$, *if* $x\text{–}\mathrm{TOP}_\eta(\mathbb{B}) \not\subseteq y\text{–}\mathrm{BOT}_\kappa(\mathbb{B})$.
(3) $x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq y\text{–}\mathrm{TOP}_\kappa(\mathcal{A})$, *if* $x\text{–}\mathrm{BOT}_\eta(\mathbb{B}) \not\subseteq y\text{–}\mathrm{TOP}_\kappa(\mathbb{B})$.

PROOF. We only prove the first item. The other items are proved analogously by replacing "bottom-up" by "top-down" and "BOT" by "TOP" where appropriate.

Let $\tau \in x\text{–}\mathrm{BOT}_\eta(\mathbb{B}) \setminus y\text{–}\mathrm{BOT}_\kappa(\mathbb{B})$ be an η-t-ts transformation, hence there exists a total deterministic bu-tst $M = (Q, \Sigma, \Delta, \mathbb{B}, F, \mu)$ with the properties x such that $\|M\|_\eta = \tau$. We note that $(\{0,1\}, \wedge)$ is isomorphic to $(\{0,1\}, \cdot)$, which is a submonoid of $(A, \cdot)$. We freely use this isomorphism in the sequel. In this sense, $x\text{–}\mathrm{BOT}_\eta(\mathbb{B}) \subseteq x\text{–}\mathrm{BOT}_\eta(\mathcal{A})$ by Observation 5.3 and hence $\tau \in x\text{–}\mathrm{BOT}_\eta(\mathcal{A})$.

Now we prove by contradiction that $\tau \notin y\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$. Therefore, assume the contrary; *i. e.*, $\tau \in y\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$. Hence there exists a deterministic bu-tst M' with the properties y such that $\|M'\|_\kappa = \tau$. By Observation 5.4 we have that $\|M\|_\eta(t)$ is nonzero and boolean for every $t \in T_\Sigma$. Thus, Proposition 5.9 is applicable to M'. We obtain a boolean deterministic bu-tst M'' with the properties y such that $\|M''\|_\kappa = \tau$. Consequently, $\tau \in y\mathrm{b}\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$ and by Observation 5.3 also $\tau \in y\text{–}\mathrm{BOT}_\kappa(\mathbb{B})$. However, this is a contradiction to the assumption, because τ was chosen such that $\tau \notin y\text{–}\mathrm{BOT}_\kappa(\mathbb{B})$. This proves the lemma. ☐

Thus we can derive inequality for classes of ε-t-ts and o-t-ts transformations over the semiring $\mathcal{A}$ simply by observing inequality for the respective classes of ε-t-ts and o-t-ts transformations over the boolean semiring $\mathbb{B}$. Roughly speaking, these latter inequalities are based solely on a deficiency in the tree output component of one class. For example, the η-t-ts transformation that maps each input tree to a fully balanced binary tree of the same height with whatever nonzero cost cannot be computed by a linear deterministic bu-tst. In order to generate the fully balanced binary trees, one definitely needs the copying of output trees. Another example is the homomorphism property. The η-t-ts transformation that maps every input tree to $\widetilde{0}$ obviously cannot be computed by a homomorphism bu-tst (see [**58**, Proposition 4.11] and Observation 4.23).

The following lemma presents the conclusions drawn from Figure 1 and Lemma 5.11. We also consider the missing case, where the transformation is computed by a deterministic bu-tst that is not necessarily total. Moreover, we are not interested in the boolean property anymore, because boolean deterministic bu-tst and td-tst are essentially deterministic bottom-up and top-down tree transducers (see [**41**, Section 4]). Let

$$\mathrm{P} = \{x \in \Pi \setminus \Pi_{\mathrm{b}} \mid \mathrm{h} \in x \Rightarrow (\mathrm{t} \in x \wedge \mathrm{d} \in x), \mathrm{d} \in x \Rightarrow \mathrm{p} \in x\}$$

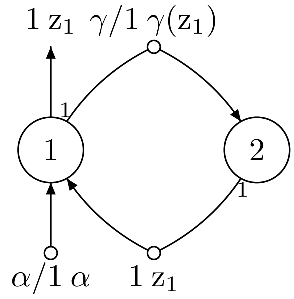

FIGURE 11. Tst M over $\mathcal{A}$ [see Lemma 5.12(ii)].

be the set of reasonable combinations (because a homomorphism bu-tst or td-tst is always deterministic and total). Further, let

$$\mathrm{P}_w = \{x \in \mathrm{P} \mid w \subseteq x\}$$

for every $w \in \mathrm{P}$.

LEMMA 5.12. *For every $x, y \in \mathrm{P_d}$ such that there exists $r \in \mathrm{Pref}$ which occurs in y but not in x (i. e., $r \in y \setminus x$), we have:*

(1) x–$\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq y$–$\mathrm{BOT}_\kappa(\mathcal{A})$;
(2) x–$\mathrm{TOP}_\eta(\mathcal{A}) \not\subseteq y$–$\mathrm{BOT}_\kappa(\mathcal{A})$; *and*
(3) x–$\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq y$–$\mathrm{TOP}_\kappa(\mathcal{A})$.

PROOF. We distinguish two cases.

(i) Let $r \neq \mathrm{t}$. We only prove the first item; the other items can be proved analogously. Obviously, $r \notin x \cup \{\mathrm{t}\}$, so let $x' = x \cup \{\mathrm{t}\}$. From Figure 1, we can check that x'–$\mathrm{BOT}_\eta(\mathbb{B}) \not\subseteq y$–$\mathrm{BOT}_\kappa(\mathbb{B})$ and we conclude x'–$\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq y$–$\mathrm{BOT}_\kappa(\mathcal{A})$ with the help of Lemma 5.11. Trivially,

$$x'\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \subseteq x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \enspace ,$$

which proves the statement.

(ii) Let $r = \mathrm{t}$. Clearly, $\mathrm{h} \notin x$. Further, let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $M = (Q, \Sigma, \Sigma, \mathcal{A}, F, \mu)$ be the boolean, linear, and nondeleting deterministic bu-tst and td-tst with:

- $Q = \{1, 2\}$;
- $F_1 = 1\,\mathrm{z}_1$ and $F_2 = \widetilde{0}$; and
- $\mu_0(\alpha)_1 = 1\,\alpha$ and $\mu_1(\gamma)_{2,1(\mathrm{x}_1)} = 1\,\gamma(\mathrm{z}_1)$ and $\mu_1(\gamma)_{1,2(\mathrm{x}_1)} = 1\,\mathrm{z}_1$.

In Figure 11 we display M. Let $\tau = \|M\|_\eta$. Apparently,

$$\tau \in x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \cap x\text{–}\mathrm{TOP}_\eta(\mathcal{A})$$

and for every $n \in \mathbb{N}$

$$\tau(\gamma^n(\alpha)) = \begin{cases} 1\,\gamma^{(n/2)}(\alpha) & \text{if } n \text{ is even,} \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Now we prove that $\tau \notin y$–$\mathrm{BOT}_\kappa(\mathcal{A})$ and $\tau \notin y$–$\mathrm{TOP}_\kappa(\mathcal{A})$. Assume the contrary; *i. e.*, there is a total deterministic bu-tst or td-tst

$$M' = (Q', \Sigma, \Sigma, \mathcal{A}, F', \mu')$$

such that $\|M'\|_\kappa = \tau$.

(1) First we assume that M' is a total deterministic bu-tst. Since $\tau(t) = \widetilde{0}$ for infinitely many $t \in T_\Sigma$ and also $\tau(t') \neq \widetilde{0}$ for infinitely many $t' \in T_\Sigma$, there exists a state $q \in Q'$ such that $F'_q \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t)_q) = \widetilde{0}$ and $h^\kappa_{\mu'}(t)_q \neq \widetilde{0}$ for infinitely many $t \in T_\Sigma$ because $\|M'\|_\kappa(t) = F'_q \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t)_q)$ by Proposition 5.1(4) [see Observation 4.9]. Let $t_1, t_2 \in T_\Sigma$ with $t_1 \neq t_2$ be two input trees satisfying the previous conditions (note that $\tau(t_1) = \tau(t_2) = \widetilde{0}$). Clearly, there exists a unique $p \in Q'$ such that $\mu'_1(\gamma)_{p,q} \neq \widetilde{0}$. For every $i \in [2]$ let $a, a_p, a_q, a_i \in A_+$, $u_i \in T_\Sigma$, $u \in T_\Sigma(\mathrm{Z}_1)$, and $u_p, u_q \in C_\Sigma(\mathrm{Z}_1)$ be such that

$$F'_p = a_p\, u_p \quad , \quad F'_q = a_q\, u_q \quad , \quad \mu'_1(\gamma)_{p,q} = a\, u \quad , \quad h^\kappa_{\mu'}(t_i)_q = a_i\, u_i \ .$$

This part of M' is displayed in Figure 12(left). Clearly,

$$\|M'\|_\kappa(\gamma(t_i)) = F'_p \xleftarrow[\kappa]{} \big(\mu'_1(\gamma)_{p,q} \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t_i)_q)\big) = (a_p \cdot a \cdot a_i^{\mathrm{sel}(u,1,\kappa)})\, u_p[u[u_i]] \ .$$

If $u \in T_\Sigma$ then $\mathrm{supp}(\|M'\|_\kappa(\gamma(t_i))) \subseteq \{u_p[u]\}$ for every $i \in [2]$. However,

$$\emptyset \neq \mathrm{supp}(\tau(\gamma(t_1))) \neq \mathrm{supp}(\tau(\gamma(t_2))) \neq \emptyset \ ,$$

hence $u \in C_\Sigma(\mathrm{Z}_1)$ because $T_\Sigma(\mathrm{Z}_1) = C_\Sigma(\mathrm{Z}_1) \cup T_\Sigma$. Recall that $\tau(\gamma(t_1)) = 1 u'$ for some $u' \in T_\Sigma$. This allows us to conclude that $1 = a_p \cdot a \cdot a_1$, which shows that a_1 is invertible. Moreover, we also have $F'_q \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t_1)_q) = \widetilde{0}$ hence $a_q \cdot a_1 = 0$, which shows that a_1 is a zero-divisor. This is a contradiction by Observation 2.2.

(2) Now assume that M' is a total deterministic td-tst. Clearly, there exists a unique $p \in Q'$ such that $F'_p \neq \widetilde{0}$. Assume further that $\mu'_1(\gamma)_{p,\varepsilon} \neq \widetilde{0}$. It follows that for every $t \in T_\Sigma$

$$h^\kappa_{\mu'}(\gamma(t))_p = \mu'_1(\gamma)_{p,\varepsilon} \xleftarrow[\kappa]{} () \ .$$

This yields that $\|M'\|_\kappa(\gamma^2(\alpha)) = \|M'\|_\kappa(\gamma(\alpha))$ which is contradictory. Thus, by totality there exists a unique $q \in Q'$ such that $\mu'_1(\gamma)_{p,q(\mathrm{x}_1)} \neq \widetilde{0}$. Moreover, let $a_p, a, a', a'' \in A_+$, and $u', u'' \in T_\Sigma$, $u, u_p \in C_\Sigma(\mathrm{Z}_1)$ be such that

$$F'_p = a_p\, u_p \quad , \quad \mu'_0(\alpha)_p = a'\, u' \quad , \quad \mu'_0(\alpha)_q = a''\, u'' \quad , \quad \mu'_1(\gamma)_{p,q(\mathrm{x}_1)} = a\, u \ .$$

This part of M' is displayed in Figure 12(right).

$$1\, \alpha = \|M'\|_\kappa(\alpha) = F'_p \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(\alpha)_p) = a_p\, u_p \xleftarrow[\kappa]{} (a'\, u') = (a_p \cdot a')\, u_p[u']$$

Hence $a_p \cdot a' = 1$ because $\tau(\alpha) = 1\, \alpha$. It follows that a_p is invertible.

$$\begin{aligned}
\widetilde{0} = \|M'\|_\kappa(\gamma(\alpha)) &= F'_p \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(\gamma(\alpha))_p) \\
&= a_p\, u_p \xleftarrow[\kappa]{} \big(\mu'_1(\gamma)_{p,q(\mathrm{x}_1)} \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(\alpha)_q)\big) \\
&= a_p\, u_p \xleftarrow[\kappa]{} \big(a\, u \xleftarrow[\kappa]{} (a''\, u'')\big) = (a_p \cdot a \cdot a'')\, u_p[u[u'']]
\end{aligned}$$

Thus we showed that $a_p \cdot a \cdot a'' = 0$ because $\tau(\gamma(\alpha)) = \widetilde{0}$. Since a_p, a, and a'' are all nonzero, it follows that a_p or a is a zero-divisor. By Observation 2.2, a_p is no zero-divisor, hence a is a zero-divisor. Finally, since $\tau(\gamma^2(\alpha)) = 1\, \gamma(\alpha)$ and

$$\begin{aligned}
\|M'\|_\kappa(\gamma^2(\alpha)) &= F'_p \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(\gamma^2(\alpha))_p) \\
&= a_p\, u_p \xleftarrow[\kappa]{} \big(\mu'_1(\gamma)_{p,q(\mathrm{x}_1)} \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(\gamma(\alpha))_q)\big) \ ,
\end{aligned}$$

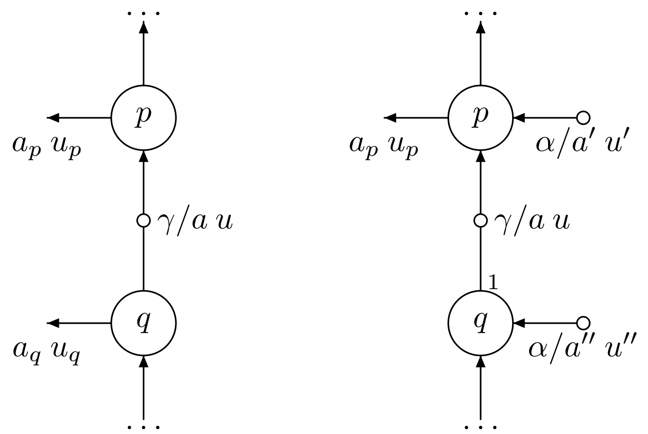

FIGURE 12. Relevant parts of the bu-tst M' (left) and td-tst M' (right) over $\mathcal{A}$ [see Lemma 5.12(ii)].

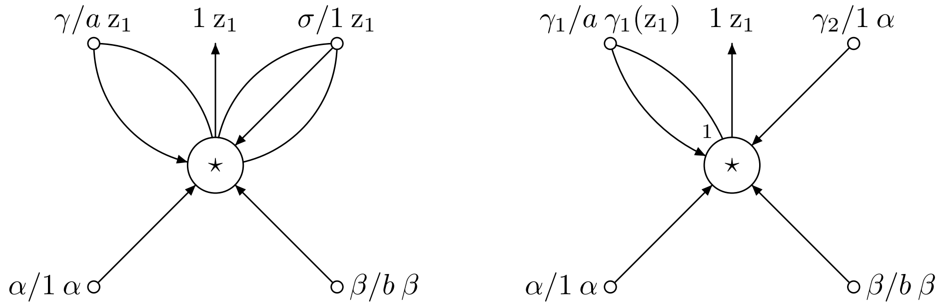

FIGURE 13. Bu-tst M (left) and td-tst $\overline{M}$ (right) over $\mathcal{A}$ used to prove Proposition 5.13.

we have $h^{\kappa}_{\mu'}(\gamma(\alpha))_q \neq \widetilde{0}$. Finally, let $a_q \in A_+$ and $u_q \in T_\Sigma$ be such that $h^{\kappa}_{\mu'}(\gamma(\alpha))_q = a_q\, u_q$.

$$\|M'\|_\kappa(\gamma^2(\alpha)) = a_p\, u_p \xleftarrow[\kappa]{} \big(a\, u \xleftarrow[\kappa]{} (a_q\, u_q)\big) = (a_p \cdot a \cdot a_q)\, u_p[u[u_q]] = 1\, \gamma(\alpha)$$

Clearly, $a_p \cdot a \cdot a_q = 1$. This shows that a is invertible, which is a contradiction (see Observation 2.2). □

Due to the previous lemma, we can restrict our attention to the comparison of classes of transformations, which are related by inclusion in Figure 1. As a first comparison recall the equality of the classes of ε-t-ts and o-t-ts transformations computed by bu-tst for all properties that contain both the nondeletion as well as the linearity property (see Proposition 4.21). Now let us turn to the comparison of classes computed by deterministic bu-tst and td-tst. Provided that $\mathcal{A}$ is zero-divisor free, it is shown in [**58**, Theorem 5.12] that x–$\mathrm{BOT_o}(\mathcal{A}) = x$–$\mathrm{TOP}_\varepsilon(\mathcal{A})$ for every $x \in \mathrm{P_h}$. Next we show that zero-divisor freeness is necessary and sufficient for the statement h–$\mathrm{BOT_o}(\mathcal{A}) =$ h–$\mathrm{TOP}_\varepsilon(\mathcal{A})$.

PROPOSITION 5.13 (*cf.* [**58**, Theorem 5.12]). *Let* $x \in \mathrm{P_h} \setminus \mathrm{P_n}$. *We have* $x\text{–BOT}_\mathrm{o}(\mathcal{A}) = x\text{–TOP}_\varepsilon(\mathcal{A})$ *if and only if* $\mathcal{A}$ *is zero-divisor free. In fact,* $x\text{–BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq \text{d–TOP}_\varepsilon(\mathcal{A})$ *and* $x\text{–TOP}_\varepsilon(\mathcal{A}) \nsubseteq \text{h–BOT}_\mathrm{o}(\mathcal{A})$, *if* $\mathcal{A}$ *is not zero-divisor free.*

PROOF. Sufficiency is shown in [**58**, Theorem 5.12], so it remains to show necessity. We show the contraposition; *i. e.*, the existence of a zero-divisor implies that

$$x\text{–BOT}_\mathrm{o}(\mathcal{A}) \neq x\text{–TOP}_\varepsilon(\mathcal{A}) \enspace .$$

In fact, we even show that

$$x\text{–BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \quad \text{and} \quad x\text{–TOP}_\varepsilon(\mathcal{A}) \nsubseteq \text{h–BOT}_\mathrm{o}(\mathcal{A}) \enspace .$$

Since $\mathcal{A}$ is not zero-divisor free, there exist $a, b \in A_+$ such that $a \cdot b = 0$.

(i) We first show that $\text{hl–BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq \text{d–TOP}_\varepsilon(\mathcal{A})$. To this end, let $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be the linear homomorphism bu-tst with

- $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$;
- $\Delta = \{\alpha^{(0)}, \beta^{(0)}\}$;
- $F_\star = 1\,\mathrm{z}_1$; and

$$\mu_0(\alpha)_\star = 1\,\alpha \quad , \quad \mu_0(\beta)_\star = b\,\beta \quad , \quad \mu_1(\gamma)_{\star,\star} = a\,\mathrm{z}_1 \quad , \quad \mu_2(\sigma)_{\star,\star\star} = 1\,\mathrm{z}_1 \enspace .$$

The bu-tst M is illustrated in Figure 13(left). Let $\tau = \|M\|_\mathrm{o}$. It is easily observed that $\tau(\sigma(\alpha, \beta)) = 1\,\alpha$ and $\tau(\sigma(\beta, \beta)) = b\,\beta$ and $\tau(\sigma(\alpha, \gamma(\beta))) = \widetilde{0}$. Suppose that there exists a deterministic td-tst $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ such that $\|M'\|_\varepsilon = \tau$. Clearly, there exists a unique $p \in Q'$ such that $F'_p \neq \widetilde{0}$. Let $c \in A_+$ be such that $F'_p = c\,\mathrm{z}_1$. With the help of Proposition 5.1(4) we obtain $\|M'\|_\varepsilon(t) = c \cdot h^\varepsilon_{\mu'}(t)_p$ for every $t \in T_\Sigma$, because $F'_p = c\,\mathrm{z}_1$. For every $t_1, t_2 \in T_\Sigma$ we have

$$\begin{aligned} \|M'\|_\varepsilon(\sigma(t_1, t_2)) &= c \cdot h^\varepsilon_{\mu'}(\sigma(t_1, t_2))_p \\ &= c \cdot \sum_{\substack{w \in Q'(\mathrm{X}_2)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu'_2(\sigma)_{p,w} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_{i_j})_{q_j})_{j \in [n]} \enspace . \end{aligned}$$

Let $w \in Q'(\mathrm{X}_2)^*$ be such that $\mu'_2(\sigma)_{p,w} \neq \widetilde{0}$. Such a w exists because $\tau(\sigma(\alpha, \beta)) \neq \widetilde{0}$, and it is unique by determinism. Obviously the choice of the output ranked alphabet Δ limits the number of possibilities for w. In fact, either $w = \varepsilon$ or $w = q(\mathrm{x}_1)$ or $w = q(\mathrm{x}_2)$ for some $q \in Q'$. Now consider $w = \varepsilon$. Then

$$\|M'\|_\varepsilon(\sigma(t_1, t_2)) = c \cdot \bigl(\mu'_2(\sigma)_{p,\varepsilon} \xleftarrow[\varepsilon]{} ()\bigr) = c \cdot \mu'_2(\sigma)_{p,\varepsilon} \enspace ,$$

which is contradictory, because $\tau(\sigma(\alpha, \beta)) \neq \tau(\sigma(\beta, \beta))$. Similarly, assume that $w = q(\mathrm{x}_2)$ for some $q \in Q'$. Then

$$\|M'\|_\varepsilon(\sigma(t_1, t_2)) = c \cdot \bigl(\mu'_2(\sigma)_{p,q(\mathrm{x}_2)} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_2)_q)\bigr) \enspace ,$$

which is also contradictory because $\tau(\sigma(\alpha, \beta)) \neq \tau(\sigma(\beta, \beta))$. Thus, finally let $w = q(\mathrm{x}_1)$ for some $q \in Q'$. In this case

$$\|M'\|_\varepsilon(\sigma(t_1, t_2)) = c \cdot \bigl(\mu'_2(\sigma)_{p,q(\mathrm{x}_1)} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_1)_q)\bigr) \enspace .$$

However, even this is contradictory because $\tau(\sigma(\alpha,\beta)) \neq \tau(\sigma(\alpha,\gamma(\beta)))$. Thus there does not exist a deterministic td-tst M' such that $\|M'\|_\varepsilon = \tau$ and hence $\tau \notin \text{d–TOP}_\varepsilon(\mathcal{A})$.

(ii) Now we show that $\text{hl–TOP}_\varepsilon(\mathcal{A}) \nsubseteq \text{h–BOT}_\text{o}(\mathcal{A})$. To this end, let $\overline{M} = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, F, \mu)$ be the linear homomorphism td-tst with

- $\Sigma = \{\gamma_1^{(1)}, \gamma_2^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$;
- $F_\star = 1\,\mathrm{z}_1$; and

$$\mu_0(\alpha)_\star = 1\,\alpha \qquad\qquad \mu_0(\beta)_\star = b\,\beta$$
$$\mu_1(\gamma_1)_{\star,\star(\mathrm{x}_1)} = a\,\gamma_1(\mathrm{z}_1) \qquad\qquad \mu_1(\gamma_2)_{\star,\varepsilon} = 1\,\alpha \ .$$

The td-tst $\overline{M}$ is illustrated in Figure 13(right). Let $\tau = \|\overline{M}\|_\varepsilon$. Assume that there exists a homomorphism bu-tst $M' = (\{\star\}, \Sigma, \Sigma, \mathcal{A}, F, \mu')$ such that $\|M'\|_\text{o} = \tau$. We observe that $\|M'\|_\text{o}(t) = h^\text{o}_{\mu'}(t)_\star$ for every $t \in T_\Sigma$ by Observation 4.23. It follows that $h^\text{o}_{\mu'}(\gamma_1(\beta))_\star = \widetilde{0}$ because $\tau(\gamma_1(\beta)) = \widetilde{0}$. Consequently, $\|M'\|_\text{o}(\gamma_2(\gamma_1(\beta))) = \widetilde{0}$ by Observation 4.9. However, a straightforward calculation yields $\tau(\gamma_2(\gamma_1(\beta))) = 1\,\alpha$, which is a contradiction. Thus there does not exist a homomorphism bu-tst M' such that $\|M'\|_\text{o} = \tau$ and hence $\tau \notin \text{h–BOT}_\text{o}(\mathcal{A})$. □

We note the asymmetry

$$x\text{–BOT}_\text{o}(\mathcal{A}) \nsubseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \quad \text{and} \quad x\text{–TOP}_\varepsilon(\mathcal{A}) \nsubseteq \text{h–BOT}_\text{o}(\mathcal{A})$$

of the previous proposition in the case that $\mathcal{A}$ is not zero-divisor free. It is this asymmetry that yields a problem in Section 5. Moreover, the previous proposition does not handle nondeletion. We remedy this fact with the next observation, which is independent of zero-divisor freeness.

OBSERVATION 5.14 (see [**58**, Theorem 5.12]). *Let $\mathcal{A}$ be commutative. For every $x \in \Pi_\text{hn}$:*

$$x\text{–BOT}_\text{o}(\mathcal{A}) = x\text{–TOP}_\varepsilon(\mathcal{A}) \quad \textit{and} \quad x\text{–BOT}_\varepsilon(\mathcal{A}) \subseteq \text{dnt–TOP}_\varepsilon(\mathcal{A}) \ .$$

PROOF. Suppose that $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ is a nondeleting homomorphism bu-tst. For every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$, let $a_\sigma \in A_+$ and $u_\sigma \in T_\Delta(\mathrm{Z}_k)$ be such that $\mu_k(\sigma)_{\star,\star\cdots\star} = a_\sigma\ u_\sigma$. Moreover, let $n_\sigma(i) = |u_\sigma|_{\mathrm{z}_i}$ for every $i \in [k]$. Note that $n_\sigma(i) \geqslant 1$ for every $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, and $i \in [k]$. Finally, let $\mathrm{lin}(u_\sigma)$ denote the tree obtained from u_σ by replacing the different occurrences of z_1 in u_σ by $\mathrm{z}_1, \ldots, \mathrm{z}_{n_\sigma(1)}$; the different occurrences of z_2 by $\mathrm{z}_{n_\sigma(1)+1}, \ldots, \mathrm{z}_{n_\sigma(1)+n_\sigma(2)}$; etc.

(i) We construct a nondeleting homomorphism td-tst

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu') \ .$$

For every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$ let $\mu'_k(\sigma)_{\star,w} = a_\sigma\ \mathrm{lin}(u_\sigma)$ where

$$w = \underbrace{\star(\mathrm{x}_1)\cdots\star(\mathrm{x}_1)}_{n_\sigma(1)\text{ times}}\cdots\underbrace{\star(\mathrm{x}_k)\cdots\star(\mathrm{x}_k)}_{n_\sigma(k)\text{ times}} \ .$$

Recall that we assume that all remaining entries in μ' are $\widetilde{0}$.

We prove $\|M'\|_\varepsilon = \|M\|_{\mathrm{o}}$ by proving inductively that $h^\varepsilon_{\mu'}(t)_\star = h^{\mathrm{o}}_\mu(t)_\star$ for every $t \in T_\Sigma$. Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$. By Proposition 5.1(5), $h^{\mathrm{o}}_\mu(t)_\star$ is monomial for every $t \in T_\Sigma$, so for every $i \in [k]$ let $a_i \in A$ and $u_i \in T_\Delta$ be such that $h^{\mathrm{o}}_\mu(t_i)_\star = a_i\, u_i$. Clearly, by induction hypothesis we also have $h^\varepsilon_{\mu'}(t_i)_\star = a_i\, u_i$.

$$
\begin{aligned}
& h^{\mathrm{o}}_\mu(\sigma(t_1, \ldots, t_k))_\star \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \mu_k(\sigma)_{\star,\star\cdots\star} \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_\mu(t_1)_\star, \ldots, h^{\mathrm{o}}_\mu(t_k)_\star) \\
= \quad & \textit{(by definition of } \xleftarrow[\mathrm{o}]{} \textit{ and } n_\sigma(i) \geqslant 1\textit{)} \\
& \big(a_\sigma \cdot \prod_{i\in[k]} a_i^{n_\sigma(i)}\big)\, u_\sigma[u_1, \ldots, u_k] \\
= \quad & \textit{(by definition of } \mathrm{lin}\textit{)} \\
& \Big(a_\sigma \cdot \big(\prod_{j\in[n_\sigma(1)]} a_1\big) \cdot \ldots \cdot \prod_{j\in[n_\sigma(k)]} a_k\Big)\ \mathrm{lin}(u_\sigma)[\underbrace{u_1, \ldots, u_1}_{n_\sigma(1)}, \ldots, \underbrace{u_k, \ldots, u_k}_{n_\sigma(k)}] \\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& \mu'_k(\sigma)_{\star,\star(\mathrm{x}_1)\cdots\star(\mathrm{x}_1)\cdots\star(\mathrm{x}_k)\cdots\star(\mathrm{x}_k)} \xleftarrow[\varepsilon]{} \\
& \quad \xleftarrow[\varepsilon]{} (\underbrace{h^\varepsilon_{\mu'}(t_1)_\star, \ldots, h^\varepsilon_{\mu'}(t_1)_\star}_{n_\sigma(1)\text{ times}}, \ldots, \underbrace{h^\varepsilon_{\mu'}(t_k)_\star, \ldots, h^\varepsilon_{\mu'}(t_k)_\star}_{n_\sigma(k)\text{ times}}) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& h^\varepsilon_{\mu'}(\sigma(t_1, \ldots, t_k))_\star
\end{aligned}
$$

The proof of the converse inclusion is similar and omitted. We note, however, that only for the converse inclusion we need commutativity.

(ii) We construct a nondeleting, total, and deterministic td-tst

$$M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$$

as follows. Let $Q' = \{\star, \dagger\}$ and $F'_\star = 1\, \mathrm{z}_1$ and $F'_\dagger = \widetilde{0}$. Moreover, for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$ let

$$\mu'_k(\sigma)_{\star,w} = a_\sigma\ \mathrm{lin}(u_\sigma) \quad \text{and} \quad \mu'_k(\sigma)_{\dagger,w'} = 1\ \mathrm{lin}(u_\sigma)$$

where

$$
\begin{aligned}
w &= \star(\mathrm{x}_1)\, \underbrace{\dagger(\mathrm{x}_1) \cdots \dagger(\mathrm{x}_1)}_{n_\sigma(1)-1\text{ times}} \cdots \star(\mathrm{x}_k)\, \underbrace{\dagger(\mathrm{x}_k) \cdots \dagger(\mathrm{x}_k)}_{n_\sigma(k)-1\text{ times}} \\
w' &= \underbrace{\dagger(\mathrm{x}_1) \cdots \dagger(\mathrm{x}_1)}_{n_\sigma(1)\text{ times}} \cdots \underbrace{\dagger(\mathrm{x}_k) \cdots \dagger(\mathrm{x}_k)}_{n_\sigma(k)\text{ times}} \ .
\end{aligned}
$$

For every $t \in T_\Sigma$ and $u \in T_\Delta$, it is easily proved that $u \in \mathrm{supp}(h^\varepsilon_\mu(t)_\star)$ implies that $(h^\varepsilon_{\mu'}(t)_\dagger, u) = 1$ [*cf.* the proof of the corresponding statement in (i) with all weights replaced by 1]. Using this fact the proof of $h^\varepsilon_\mu(t)_\star = h^\varepsilon_{\mu'}(t)_\star$ is easily achieved by induction on t. To this end, let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$. By Proposition 5.1(5) we have that for every $i \in [k]$ there exist $a_i \in A$ and $u_i \in T_\Delta$ such that $h^\varepsilon_\mu(t_i)_\star = a_i\, u_i$.

TABLE 3. Preservation of properties for the construction of Observation 5.14(i).

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✗	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓	✓	✓	✗	✓

TABLE 4. Preservation of properties for the construction of Observation 5.14(ii).

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✗	✓	✓	✓	✗	✗	✓	✓	✗	✓	✓	✓	✓	✗	✗

Moreover, by induction hypothesis $h^{\varepsilon}_{\mu'}(t)_\star = a_i\, u_i$.

$$
\begin{aligned}
& h^{\varepsilon}_{\mu}(\sigma(t_1,\dots,t_k))_\star \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \mu_k(\sigma)_{\star,\star\cdots\star} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(t_1)_\star,\dots,h^{\varepsilon}_{\mu}(t_k)_\star) \\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& (a_\sigma \cdot \prod_{i\in[k]} a_i)\; u_\sigma[u_1,\dots,u_k] \\
= \quad & \textit{(by definition of } \mathrm{lin}\textit{)} \\
& \Big(a_\sigma \cdot (\prod_{i\in[k]} a_i) \cdot (\prod_{j\in[n_\sigma(1)-1]} 1) \cdot \ldots \cdot \prod_{j\in[n_\sigma(k)-1]} 1\Big) \\
& \quad \mathrm{lin}(u_\sigma)[\underbrace{u_1,\dots,u_1}_{n_\sigma(1)},\dots,\underbrace{u_k,\dots,u_k}_{n_\sigma(k)}] \\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& \mu'_k(\sigma)_{\star,\star(\mathrm{x}_1)\dagger(\mathrm{x}_1)\cdots\dagger(\mathrm{x}_1)\cdots\star(\mathrm{x}_k)\dagger(\mathrm{x}_k)\cdots\dagger(\mathrm{x}_k)} \xleftarrow[\varepsilon]{} \\
& \quad \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu'}(t_1)_\star, \underbrace{h^{\varepsilon}_{\mu'}(t_1)_\dagger,\dots,h^{\varepsilon}_{\mu'}(t_1)_\dagger}_{n_\sigma(1)-1 \text{ times}},\dots,h^{\varepsilon}_{\mu'}(t_k)_\star, \underbrace{h^{\varepsilon}_{\mu'}(t_k)_\dagger,\dots,h^{\varepsilon}_{\mu'}(t_k)_\dagger}_{n_\sigma(k)-1 \text{ times}}) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& h^{\varepsilon}_{\mu'}(\sigma(t_1,\dots,t_k))_\star
\end{aligned}
$$

□

Preservation of properties for the above constructions is displayed in Tables 3 and 4. Since there is no example for [**58**, Theorem 5.12], we present a small one illustrating the above constructions.

EXAMPLE 5.15. *Let* $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ *and* $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$. *We define the bu-tst* $M_{5.15} = (\{\star\}, \Sigma, \Delta, \mathbb{N}, F, \mu)$ *with* $F_\star = 1\,\mathrm{z}_1$ *and*

$$\mu_1(\gamma)_{\star,\star} = 3\,\sigma(\mathrm{z}_1,\mathrm{z}_1) \quad \textit{and} \quad \mu_0(\alpha)_\star = 3\,\alpha\ .$$

The bu-tst $M_{5.15}$*, which is displayed in Figure 14(left), is a nondeleting homomorphism bu-tst. Thus we can apply the construction found in the proof of Observation 5.14(i) to* $M_{5.15}$ *and obtain the td-tst*

$$M'_{5.15} = (\{\star\}, \Sigma, \Delta, \mathbb{N}, F, \mu')$$

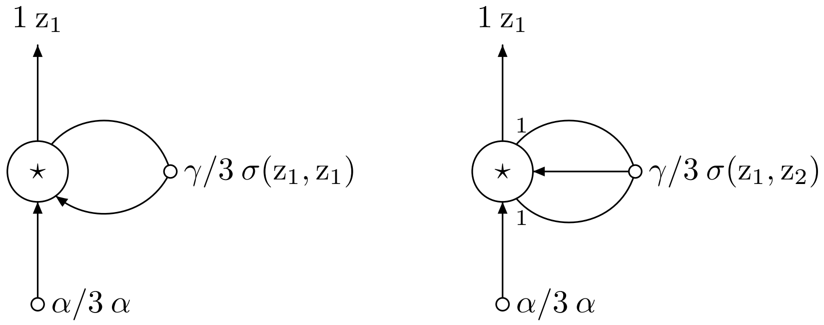

FIGURE 14. Bu-tst $M_{5.15}$ (left) and td-tst $M'_{5.15}$ (right) over $\mathbb{N}$ of Example 5.15.

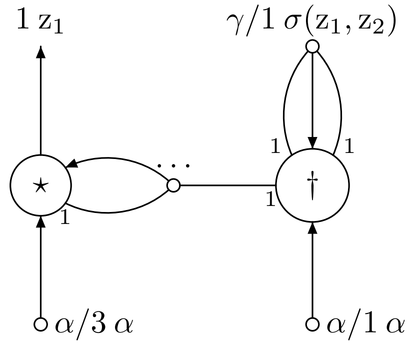

FIGURE 15. Bu-tst $M''_{5.15}$ over $\mathbb{N}$ of Example 5.15 where $\dots$ stands for $\gamma/3\;\sigma(z_1, z_2)$.

with

$$\mu'_1(\gamma)_{\star,\star(x_1)\star(x_1)} = 3\;\sigma(z_1, z_2) \quad \textit{and} \quad \mu'_0(\alpha)_\star = 3\;\alpha \;\; .$$

Clearly, $M'_{5.15}$, illustrated in Figure 14(right), is a nondeleting homomorphism td-tst and $\|M'_{5.15}\|_\varepsilon = \|M_{5.15}\|_o$. Finally, we can also apply the construction of Observation 5.14(ii) to $M_{5.15}$ and obtain the td-tst $M''_{5.15} = (\{\star, \dagger\}, \Sigma, \Delta, \mathbb{N}, F'', \mu'')$ with $F''_\star = 1\,z_1$ and $F''_\dagger = \widetilde{0}$ and

$$\begin{aligned} \mu''_1(\gamma)_{\star,\star(x_1)\dagger(x_1)} &= 3\;\sigma(z_1, z_2) & \qquad \mu''_0(\alpha)_\star &= 3\;\alpha \\ \mu''_1(\gamma)_{\dagger,\dagger(x_1)\dagger(x_1)} &= 1\;\sigma(z_1, z_2) & \qquad \mu''_0(\alpha)_\dagger &= 1\;\alpha \;\; . \end{aligned}$$

We depict $M''_{5.15}$, which is a nondeleting, total, and deterministic td-tst with the property that $\|M''_{5.15}\|_\varepsilon = \|M_{5.15}\|_\varepsilon$, in Figure 15.

The final result of this section shows two inequality results. Essentially, we prove that the classes of ε-t-ts and o-t-ts transformations computed by linear homomorphism bu-tst are incomparable. Due to the HASSE diagram presented in Figure 1, we cannot prove this result for every semiring, but rather we require that the semiring has at least three elements (*i. e.*, $0 \neq 1$ and it is not isomorphic to $\mathbb{B}$ or $\mathbb{Z}_2$).

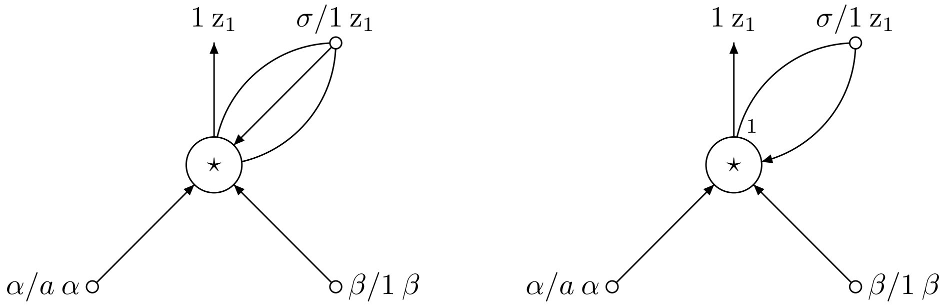

FIGURE 16. Bu-tst M (left) and td-tst M'' (right) over $\mathcal{A}$ used to show Lemma 5.16(1) and (2), respectively.

The result $\mathrm{hl{-}BOT}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{h{-}BOT}_\mathrm{o}(\mathcal{A})$ is proved essentially by an exploit of the property that pure substitution can distinguish two output trees with different weights, although it deletes them. On the other hand, this distinction vanishes in o-substitution, and we cannot use the state to signal the difference, because we consider homomorphism bu-tst. The same properties are used to prove the statement $\mathrm{hl{-}BOT}_\mathrm{o}(\mathcal{A}) \not\subseteq \mathrm{h{-}BOT}_\varepsilon(\mathcal{A})$.

Using the same idea we also prove that the class of ε-t-ts transformations computed by linear homomorphism bu-tst is incomparable with the class of ε-t-ts transformations computed by linear homomorphism td-tst. In fact, part of this is proved by showing that the class of ε-t-ts transformations computed by linear homomorphism bu-tst is not contained in the class of ε-t-ts transformations computed by deterministic td-tst.

LEMMA 5.16. *Let* $A \neq \{0,1\}$.

(1) $\mathrm{hl{-}BOT}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{h{-}BOT}_\mathrm{o}(\mathcal{A})$ *and* $\mathrm{hl{-}BOT}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{d{-}TOP}_\varepsilon(\mathcal{A})$.
(2) $\mathrm{hl{-}BOT}_\mathrm{o}(\mathcal{A}) \not\subseteq \mathrm{h{-}BOT}_\varepsilon(\mathcal{A})$ *and* $\mathrm{hl{-}TOP}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{h{-}BOT}_\varepsilon(\mathcal{A})$.

PROOF. Let $a \in A \setminus \{0,1\}$ be arbitrary. Let $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be the linear homomorphism bu-tst with ranked alphabets $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Delta = \{\alpha^{(0)}, \beta^{(0)}\}$ and $F_\star = 1\,\mathrm{z}_1$ and

$$\mu_2(\sigma)_{\star,\star\star} = 1\,\mathrm{z}_1 \quad , \quad \mu_0(\alpha)_\star = a\,\alpha \quad , \quad \mu_0(\beta)_\star = 1\,\beta \ .$$

The bu-tst M is illustrated in Figure 16(left).

(1) Let $\tau = \|M\|_\varepsilon$. Clearly, $\tau \in \mathrm{hl{-}BOT}_\varepsilon(\mathcal{A})$ and $\tau(\sigma(\alpha,\beta)) = a\,\alpha$ and $\tau(\sigma(\beta,\alpha)) = a\,\beta$ and $\tau(\sigma(\beta,\beta)) = 1\,\beta$. Let us prove that $\tau \notin \mathrm{h{-}BOT}_\mathrm{o}(\mathcal{A})$. We prove this statement by contradiction, so assume that there exists a homomorphism bu-tst

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu')$$

such that $\|M'\|_\mathrm{o} = \tau$. By definition we have $F_\star = 1\,\mathrm{z}_1$ and thus $\mu'_0(\alpha)_\star = a\,\alpha$ and $\mu'_0(\beta)_\star = 1\,\beta$ by Observation 4.23. Let $c \in A_+$ and $u \in T_\Delta(\mathrm{Z}_2)$ be such that $\mu'_2(\sigma)_{\star,\star\star} = c\,u$. We can readily exclude $u = \alpha$ and $u = \mathrm{z}_2$ (respectively, $u = \beta$) because otherwise $\beta \notin \mathrm{supp}(\|M'\|_{\mathrm{o}}(\sigma(\beta,\alpha)))$ (respectively, $\alpha \notin \mathrm{supp}(\|M'\|_{\mathrm{o}}(\sigma(\alpha,\beta))))$. Hence, $u = \mathrm{z}_1$ and for every $t \in T_\Sigma$ such that $h^{\mathrm{o}}_{\mu'}(t)_\star \neq \widetilde{0}$ we have

$$\begin{aligned}\|M'\|_{\mathrm{o}}(\sigma(\beta,t)) = h^{\mathrm{o}}_{\mu'}(\sigma(\beta,t))_\star &= \mu'_2(\sigma)_{\star,\star\star} \xleftarrow[\mathrm{o}]{} (h^{\mathrm{o}}_{\mu'}(\beta)_\star, h^{\mathrm{o}}_{\mu'}(t)_\star) \\ &= (c\cdot 1)\,\mathrm{z}_1[\beta] = c\,\beta \ .\end{aligned}$$

Clearly, $h^{\mathrm{o}}_{\mu'}(\alpha)_\star \neq \widetilde{0} \neq h^{\mathrm{o}}_{\mu'}(\beta)_\star$ and thus $\|M'\|_{\mathrm{o}}(\sigma(\beta,\alpha)) = \|M'\|_{\mathrm{o}}(\sigma(\beta,\beta))$, which is a contradiction. It follows that there does not exist a homomorphism bu-tst M' such that $\|M'\|_{\mathrm{o}} = \tau$ and hence $\tau \notin \mathrm{h\text{–}BOT_o}(\mathcal{A})$.

Let us now show that $\tau \notin \mathrm{d\text{–}TOP}_\varepsilon(\mathcal{A})$. Suppose the contrary; *i. e.*, $\tau \in \mathrm{d\text{–}TOP}_\varepsilon(\mathcal{A})$. Consequently, there exists a deterministic td-tst

$$M'' = (Q'', \Sigma, \Delta, \mathcal{A}, F'', \mu'')$$

such that $\|M''\|_\varepsilon = \tau$. Obviously, there exists a unique state $p \in Q''$ such that $F''_p \neq \widetilde{0}$ (such a state exists because $\|M''\|_\varepsilon(\alpha) \neq \widetilde{0}$). Moreover, let $c \in A_+$ be such that $F''_p = c\,\mathrm{z}_1$, and let $w \in Q''(\mathrm{X}_2)^*$ be the unique word such that $\mu''_k(\sigma)_{p,w} \neq \widetilde{0}$ (such a word exists because $\|M''\|_\varepsilon(\sigma(\beta,\beta)) \neq \widetilde{0}$). Now assume that $w = \varepsilon$. For every $t_1, t_2 \in T_\Sigma$ we then have

$$\begin{aligned}\|M''\|_\varepsilon(\sigma(t_1,t_2)) &= F''_p \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu''}(\sigma(t_1,t_2))_p) \\ &= c \cdot \big(\mu''_2(\sigma)_{p,\varepsilon} \xleftarrow[\varepsilon]{} ()\big) = c \cdot \mu''_2(\sigma)_{p,\varepsilon} \ ,\end{aligned}$$

which is contradictory because $\tau(\sigma(\alpha,\beta)) \neq \tau(\sigma(\beta,\beta))$. Similarly, we obtain a contradiction, if $w = q(\mathrm{x}_2)$ for some $q \in Q''$. Thus, $w = q(\mathrm{x}_1)$ for some $q \in Q''$. Let $c' \in A_+$ and $u' \in C_\Delta(\mathrm{Z}_1)$ be such that $\mu''_2(\sigma)_{p,q(\mathrm{x}_1)} = c'\,u'$. It is immediate that $u' = \mathrm{z}_1$ and hence

$$\|M''\|_\varepsilon(\sigma(\beta,t_2)) = c \cdot \big(c'\,\mathrm{z}_1 \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu''}(\beta)_q)\big) = c \cdot c' \cdot \mu''_0(\beta)_q \ .$$

This yields that $\|M''\|_\varepsilon(\sigma(\beta,\alpha)) = \|M''\|_\varepsilon(\sigma(\beta,\beta))$, which is again a contradiction. We conclude that $\tau \notin \mathrm{d\text{–}TOP}_\varepsilon(\mathcal{A})$.

(2) Let $\tau' = \|M\|_{\mathrm{o}}$. Obviously, $\tau' \in \mathrm{hl\text{–}BOT_o}(\mathcal{A})$ and

$$\tau'(\sigma(\beta,\beta)) = \tau'(\sigma(\beta,\alpha)) = 1\,\beta \quad\text{and}\quad \tau'(\sigma(\alpha,\beta)) = a\,\alpha \ .$$

Let us prove that $\tau' \notin \mathrm{h\text{–}BOT}_\varepsilon(\mathcal{A})$. We prove this statement by contradiction, so suppose that there exists a homomorphism bu-tst

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu')$$

such that $\|M'\|_\varepsilon = \tau'$. Trivially, we see that $\mu'_0(\alpha)_\star = a\alpha$ and $\mu'_0(\beta)_\star = 1\beta$ by Observation 4.23. Let $c \in A_+$ and $u \in T_\Delta(\mathrm{Z}_2)$ be such that $\mu'_2(\sigma)_{\star,\star\star} = c\,u$. Moreover, we again readily observe $u = \mathrm{z}_1$, else $\beta \notin \mathrm{supp}(\|M'\|_\varepsilon(\sigma(\beta,\alpha)))$ or $\alpha \notin \mathrm{supp}(\|M'\|_\varepsilon(\sigma(\alpha,\beta)))$. Due to $\tau'(\sigma(\beta,\beta)) = \tau'(\sigma(\beta,\alpha)) = 1\,\beta$ we obtain

$$\begin{aligned}1\,\beta &= \|M'\|_\varepsilon(\sigma(\beta,\alpha)) = \mu'_2(\sigma)_{\star,\star\star} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(\beta)_\star, h^\varepsilon_{\mu'}(\alpha)_\star) = (c\cdot a)\,\beta \\ 1\,\beta &= \|M'\|_\varepsilon(\sigma(\beta,\beta)) = \mu'_2(\sigma)_{\star,\star\star} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(\beta)_\star, h^\varepsilon_{\mu'}(\beta)_\star) = c\,\beta \ .\end{aligned}$$

This yields that $c = 1$ and hence $a = 1$. This is contrary to the assumption that $a \in A \setminus \{0,1\}$. Thus we conclude that $\tau' \notin \mathrm{h\text{–}BOT}_\varepsilon(\mathcal{A})$.

Finally it remains to prove that $\tau' \in \text{hl–TOP}_\varepsilon(\mathcal{A})$. In fact, the linear homomorphism td-tst $M'' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu'')$ with

$$\mu''_2(\sigma)_{\star,\star(\mathrm{x}_1)} = 1\,\mathrm{z}_1 \quad , \quad \mu''_0(\alpha)_\star = a\,\alpha \quad , \quad \mu''_0(\beta)_\star = 1\,\beta$$

(illustrated in Figure 16[right]) computes τ' as ε-t-ts transformation (*i. e.*, $\|M''\|_\varepsilon = \tau'$). The proof of this fact is standard and omitted. □

Note the asymmetry of the statements

$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \nsubseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \quad \text{and} \quad \text{hl–TOP}_\varepsilon(\mathcal{A}) \nsubseteq \text{h–BOT}_\varepsilon(\mathcal{A})$$

of the previous lemma. In particular, the lemma also proves that the classes of ε-t-ts and o-t-ts transformations computed by homomorphism bu-tst are incomparable for all (nontrivial) semirings different from $\mathbb{B}$ and $\mathbb{Z}_2$. In fact, it can be seen from the proof of the previous lemma that there exists a single homomorphism bu-tst M such that $\|M\|_\varepsilon \notin \text{h–BOT}_\mathrm{o}(\mathcal{A})$ and $\|M\|_\mathrm{o} \notin \text{h–BOT}_\varepsilon(\mathcal{A})$. For the next corollary, recall that $\mathcal{A}$ is supposed to be nontrivial and commutative.

COROLLARY 5.17. *We have $\mathcal{A} = \mathbb{Z}_2$ or $\mathcal{A} = \mathbb{B}$, if and only if for every $x \in \Pi$ the equality $x\text{–BOT}_\varepsilon(\mathcal{A}) = x\text{–BOT}_\mathrm{o}(\mathcal{A})$ holds.*

PROOF. The equality in $\mathbb{B}$ and $\mathbb{Z}_2$ is shown in Observation 4.18 (because the multiplicative monoids of $\mathbb{B}$ and $\mathbb{Z}_2$ are isomorphic and deterministic devices compute solely in the multiplicative monoid; see Proposition 5.1), and Lemma 5.16 proves the incomparability of $\text{hl–BOT}_\varepsilon(\mathcal{A})$ and $\text{hl–BOT}_\mathrm{o}(\mathcal{A})$ in all other (nontrivial) semirings. □

Note that at this point, we can already completely classify the relation between all classes of ε-t-ts transformations computed by deterministic td-tst and all classes of o-t-ts transformations computed by deterministic bu-tst provided that the semiring is zero-divisor free and commutative. However, without additional information about the semiring (more precisely, its multiplicative monoid) we are unable to prove further comparability or incomparability results. Hence we consider semirings with certain properties in subsequent sections. The properties are chosen such that we obtain a HASSE diagram for every commutative and zero-divisor free semiring.

5. Multiplicatively nonperiodic semirings

In this section, we show that for *multiplicatively nonperiodic* (*i. e.*, not multiplicatively periodic) semirings almost all classes of ε-t-ts and o-t-ts transformations (except the ones computed by nondeleting and linear bu-tst) computed by restricted deterministic bu-tst are incomparable with respect to inclusion. The semiring of the natural numbers $\mathbb{N}$ is an example of a multiplicatively nonperiodic semiring. To be precise, we even show that

$$x\text{–BOT}_\varepsilon(\mathcal{A}) \nsubseteq \text{d–BOT}_\mathrm{o}(\mathcal{A}) \quad \text{and} \quad x\text{–BOT}_\mathrm{o}(\mathcal{A}) \nsubseteq \text{d–BOT}_\varepsilon(\mathcal{A})$$

for every $x \in \{\mathrm{hn}, \mathrm{hl}\}$ and multiplicatively nonperiodic and commutative semiring $\mathcal{A}$.

The general idea of the proof is the following. Let $a \in A$ be such that $a^i \neq a^j$, whenever $i \neq j$ where $i, j \in \mathbb{N}$. Such an a exists because $\mathcal{A}$ is multiplicatively nonperiodic. We construct a homomorphism bu-tst M, which computes an ε-t-ts transformation τ in which arbitrarily large powers of a occur as weights in the range. Let us first consider the result hl–$\mathrm{BOT}_\eta(\mathcal{A}) \not\subseteq$ d–$\mathrm{BOT}_\kappa(\mathcal{A})$ where η and κ are different. Our input ranked alphabet will have two unary symbols; encountering γ_1 in the input we stack another a to the weight computed so far and output a prolonged output tree, and encountering γ_2 we delete the computed output tree at no cost [see Figure 17(left)]. Since every deterministic bu-tst $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$, which also computes τ but as a κ-t-ts transformation, has only finitely many states, it must permit at least one final state q, which accepts infinitely many input trees. In particular, the transition from q to some state p reading γ_2 is interesting. In the case of $\kappa = \mathrm{o}$, the weight of the outputted tree is reset to the weight present in the monomial $\mu'_1(\gamma_2)_{p,q}$, which is to be defined. On the other hand, pure substitution stacks another a to the weight of the output tree computed. It can be shown that among those infinitely many input trees which q accepts, there are two for which the weights a^{n_1} and a^{n_2} of their corresponding output trees are different (this is mainly due to the fact that arbitrarily large powers of a can occur). Since all the powers of a are different, there is no consistent way to define $\mu'_1(\gamma_2)_{p,q}$. Similarly, when $\kappa = \varepsilon$ one encounters the problem that o-substitution resets the weight to 1, whenever a γ_2 is read in the input. The above remarks about the weights a^{n_1} and a^{n_2} apply as well, and in order to define $\mu'_1(\gamma_2)_{p,q}$ in this case, there should be an element $b \in A$ such that $a^{n_1} \cdot b = 1 = a^{n_2} \cdot b$ which is contradictory by Observation 2.1.

Summing up, with pure substitution one can remember the number of occurrences of γ_2 encountered in the whole input tree even if a part of the transformation of the input tree is deleted. On the other hand, using o-substitution when deleting a computed output tree, we can easily reset the weight to a determined value irrespective of the weight of the output tree computed so far.

The arguments required for the result on nondeleting homomorphism bu-tst are similar, but use copying instead of deletion. In principle, pure substitution has the problem that it is supposed to square the weight of the computed output tree. However, those output trees may have infinitely many different weights, so that this information cannot be stored in the states, and there is no element $b \in A$ which squares a^{n_1} and a^{n_2} (*i. e.*, $a^{2n_1} = a^{n_1} \cdot b$ and $a^{2n_2} = a^{n_2} \cdot b$) for suitable $n_1, n_2 \in \mathbb{N}$. Conversely, o-substitution squares the weight of the

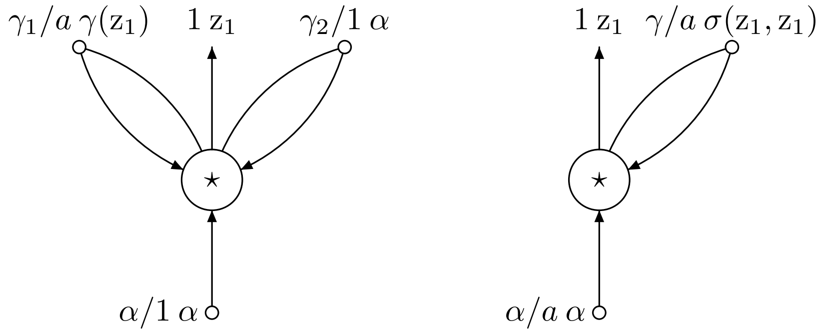

FIGURE 17. Bu-tst M (left) and $\overline{M}$ (right) over $\mathcal{A}$ used in Lemma 5.18(i) and Lemma 5.18(ii), respectively.

computed output tree and therefore needs an element which when multiplied to a^{2n_1} and a^{2n_2} computes their square roots. It is shown that for selected $n_1, n_2 \in \mathbb{N}$ such an element cannot exist.

LEMMA 5.18. *Let $\mathcal{A}$ be multiplicatively nonperiodic and commutative. For every $x \in \{\mathrm{hn}, \mathrm{hl}\}$ and $\{\eta, \kappa\} = \{\varepsilon, \mathrm{o}\}$ we have*

$$x\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \nsubseteq \mathrm{d}\text{–}\mathrm{BOT}_\kappa(\mathcal{A}) \ . \tag{35}$$

PROOF. Since $\mathcal{A}$ is multiplicatively nonperiodic, there exists an $a \in A$ such that for every $i, j \in \mathbb{N}$ we have $a^i = a^j$, if and only if $i = j$. Let us prove the statement by case analysis on x. Case (i) considers the case where $x = \mathrm{hl}$ and Case (ii) supposes that $x = \mathrm{hn}$.

(i) Let $\Sigma = \{\gamma_1^{(1)}, \gamma_2^{(1)}, \alpha^{(0)}\}$ and $\Delta = \{\gamma^{(1)}, \alpha^{(0)}\}$. We construct the linear homomorphism bu-tst $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ with $F_\star = 1\,\mathrm{z}_1$ and

$$\mu_1(\gamma_1)_{\star,\star} = a\,\gamma(\mathrm{z}_1) \quad , \quad \mu_1(\gamma_2)_{\star,\star} = \mu_0(\alpha)_\star = 1\,\alpha \ .$$

We display M in Figure 17(left). Moreover, we define $l \colon T_\Sigma \longrightarrow \mathbb{N}$ recursively for every $t \in T_\Sigma$ as follows.

$$l\big(\gamma_1(t)\big) = l(t) + 1 \quad \text{and} \quad l\big(\gamma_2(t)\big) = l(\alpha) = 0 \ .$$

Roughly speaking, l counts the number of consecutive γ_1-symbols in its argument starting at the root. If the root of t is not labeled γ_1, then $l(t) = 0$. Note that M computes the ε-t-ts transformation $\|M\|_\varepsilon$ mapping every $t \in T_\Sigma$ to the monomial tree series $a^{|t|_{\gamma_1}}\gamma^{l(t)}(\alpha)$ where $|t|_{\gamma_1}$ denotes the number of γ_1-symbols in t, and the o-t-ts transformation $\|M\|_\mathrm{o}$ mapping t to the monomial tree series $a^{l(t)}\,\gamma^{l(t)}(\alpha)$. Note that $\|M\|_\eta$ is nonzero everywhere, because if $a^n = 0$ for some $n \in \mathbb{N}$, then $a^n = a^{n+1}$ which contradicts to our assumption.

We prove $\|M\|_\eta \notin \mathrm{d}\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$ and thus $\mathrm{hl}\text{–}\mathrm{BOT}_\eta(\mathcal{A}) \nsubseteq \mathrm{d}\text{–}\mathrm{BOT}_\kappa(\mathcal{A})$. Suppose that there exists a deterministic bu-tst M' with $\|M'\|_\kappa = \|M\|_\eta$. Let $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$. We observe that $\|M'\|_\kappa(t) \neq \widetilde{0}$ for every $t \in T_\Sigma$. First, we prove that there are $q \in Q'$ and $t_1, t_2 \in T_\Sigma$ such that $h^\kappa_{\mu'}(t_1)_q \neq \widetilde{0} \neq h^\kappa_{\mu'}(t_2)_q$ and $|t_1|_{\gamma_1} \neq |t_2|_{\gamma_1}$ and $l(t_1) \neq l(t_2)$. For this, let $\Gamma = \{\gamma_1^{(1)}, \alpha^{(0)}\} \subset \Sigma$, hence $T_\Gamma \subseteq T_\Sigma$. We show that t_1 and t_2 can actually be chosen from T_Γ. Since $\|M'\|_\kappa(t) = F'_q \underset{\kappa}{\longleftarrow} (h^\kappa_{\mu'}(t)_q)$ for some $q \in Q'$ [see

Proposition 5.1(4)], we conclude that there exist $q \in Q'$ and an infinite set $T \subseteq T_\Gamma$ such that $h^\kappa_{\mu'}(t)_q \neq \widetilde{0}$ for every $t \in T$ (because Q' is finite whereas T_Γ is infinite). For every $t \in T$ we have $\mathrm{size}(t) = |t|_{\gamma_1} + 1 = l(t) + 1$ because $T \subseteq T_\Gamma$. Moreover, $\mathrm{size}(t_1) \neq \mathrm{size}(t_2)$ for every $t_1, t_2 \in T$ with $t_1 \neq t_2$. Thereby also $|t_1|_{\gamma_1} \neq |t_2|_{\gamma_1}$ and $l(t_1) \neq l(t_2)$.

We can safely assume that there exist $q \in Q'$ and $t_1, t_2 \in T_\Sigma$ such that $h^\kappa_{\mu'}(t_1)_q \neq \widetilde{0}$ and $h^\kappa_{\mu'}(t_2)_q \neq \widetilde{0}$ and $|t_1|_{\gamma_1} \neq |t_2|_{\gamma_1}$ and $l(t_1) \neq l(t_2)$. For every $i \in [2]$ we have that $h^\kappa_{\mu'}(t_i)_q$ is monomial by Proposition 5.1(5), so let $a_i \in A_+$ and $u_i \in T_\Delta$ be such that $h^\kappa_{\mu'}(t_i)_q = a_i\, u_i$. Further let $p \in Q'$ be the unique state such that $\mu'_1(\gamma_2)_{p,q} \neq \widetilde{0}$ (such a state exists because $\|M'\|_\kappa(\gamma_2(t_1)) \neq \widetilde{0}$), and let $a_p, a_q, a' \in A$ and $u_p, u_q \in C_\Delta(\mathrm{Z}_1)$ and $u' \in T_\Delta(\mathrm{Z}_1)$ such that $F'_p = a_p\, u_p$, $F'_q = a_q\, u_q$, and $\mu'_1(\gamma_2)_{p,q} = a'\, u'$. Part of the bu-tst M' is displayed in Figure 18(left). Since $\alpha \in \mathrm{supp}(\|M'\|_\kappa(\gamma_2(t_i)))$ and

$$\|M'\|_\kappa\big(\gamma_2(t_i)\big) = F'_p \xleftarrow[\kappa]{} \big(h^\kappa_{\mu'}(\gamma_2(t_i))_p\big) = \sum_{u \in T_\Delta} \Big(a_p \cdot \big(h^\kappa_{\mu'}(\gamma_2(t_i))_p, u\big)\Big)\, u_p[u]$$

for every $i \in [2]$, we have $a_p \neq 0$ and $u_p = \mathrm{z}_1$. Now we prove that $\mathrm{z}_1 \notin \mathrm{var}(u')$. Therefore, we first observe that

$$h^\kappa_{\mu'}(\gamma_2(t_i))_p = \mu'_1(\gamma_2)_{p,q} \xleftarrow[\kappa]{} \big(h^\kappa_{\mu'}(t_i)_q\big) = \big(a' \cdot a_i^{\mathrm{sel}(u',1,\kappa)}\big)\, u'[u_i]$$

and that $t_1 \neq t_2$ implies that there exists a $j \in [2]$ such that $t_j \neq \alpha$. Now assume that $\mathrm{z}_1 \in \mathrm{var}(u')$. Then $u'[u_j] \neq \alpha$, which is a contradiction. Hence $\mathrm{z}_1 \notin \mathrm{var}(u')$, and moreover, $u' = \alpha$. We obtain for every $i \in [2]$

$$\|M'\|_\kappa(\gamma_2(t_i)) = (a_p \cdot a' \cdot a_i^{\mathrm{sel}(u',1,\kappa)})\, \alpha = \begin{cases} (a_p \cdot a' \cdot a_i)\, \alpha & \text{if } \kappa = \varepsilon, \\ (a_p \cdot a')\, \alpha & \text{if } \kappa = \mathrm{o}. \end{cases}$$

Recall that $\eta \neq \kappa$ and

$$\|M\|_\varepsilon(\gamma_2(t_i)) = a^{|t_i|_{\gamma_1}}\, \alpha \quad \text{and} \quad \|M\|_\mathrm{o}(\gamma_2(t_i)) = a^{l(\gamma_2(t_i))}\, \alpha = 1\, \alpha \ .$$

Hence for every $i \in [2]$ we derive the equation

$$\begin{aligned} a_p \cdot a' \cdot a_i = 1 \qquad &= \big(\|M\|_\mathrm{o}(\gamma_2(t_i)), \alpha\big) \quad \text{if } \kappa = \varepsilon, \\ a_p \cdot a' = a^{|t_i|_{\gamma_1}} \quad &= \big(\|M\|_\varepsilon(\gamma_2(t_i)), \alpha\big) \quad \text{if } \kappa = \mathrm{o}. \end{aligned}$$

In case $\kappa = \mathrm{o}$ we have $a_p \cdot a' = a^{|t_1|_{\gamma_1}} = a^{|t_2|_{\gamma_1}}$, which is contradictory due to $a^{|t_1|_{\gamma_1}} \neq a^{|t_2|_{\gamma_1}}$ by $|t_1|_{\gamma_1} \neq |t_2|_{\gamma_1}$. Finally, in the other case (*i. e.*, $\kappa = \varepsilon$) we have that $a_p \cdot a'$ acts as the inverse of a_1 and a_2. So it remains to prove that $a_1 \neq a_2$ in order to derive a contradiction (see Observation 2.1). To prove $a_1 \neq a_2$, we consider

$$\|M'\|_\varepsilon(t_i) = F'_q \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_i)_q) = F'_q \xleftarrow[\varepsilon]{} (a_i\, u_i) = (a_q \cdot a_i)\, u_q[u_i] \ .$$

By $\|M\|_\mathrm{o}(t_i) = a^{l(t_i)}\, \gamma^{l(t_i)}(\alpha)$ and $\|M'\|_\varepsilon = \|M\|_\mathrm{o}$, we obtain $a_q \cdot a_i = a^{l(t_i)}$. Moreover, since $l(t_1) \neq l(t_2)$ we have $a^{l(t_1)} \neq a^{l(t_2)}$. Consequently, $a_1 = a_2$ would be contradictory. Irrespective of κ, we have thus proved that there is no deterministic bu-tst M' such that $\|M'\|_\kappa = \|M\|_\eta$. This proves that $\|M\|_\eta \notin \mathrm{d\text{–}BOT}_\kappa(\mathcal{A})$.

(ii) Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$. We define the nondeleting homomorphism bu-tst $\overline{M} = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ with $F_\star = 1\,\mathrm{z}_1$ and

$$\mu_1(\gamma)_{\star,\star} = a\,\sigma(\mathrm{z}_1, \mathrm{z}_1) \quad \text{and} \quad \mu_0(\alpha)_\star = a\,\alpha \ .$$

Note that $\overline{M}$ is displayed in Figure 17(right). For every $t \in T_\Sigma$ let $u_t \in T_\Delta$ be the fully balanced tree such that $\mathrm{height}(u_t) = \mathrm{height}(t)$. The ε-t-ts transformation $\|\overline{M}\|_\varepsilon$ maps t to $a^{\mathrm{size}(t)}\,u_t$, whereas the o-t-ts transformation $\|\overline{M}\|_\mathrm{o}$ maps t to $a^{\mathrm{size}(u_t)}\,u_t$ (*cf.* Lemma 4.25). Note that $\mathrm{size}(u_t) = 2^{\mathrm{size}(t)} - 1$.

Let us prove $\|\overline{M}\|_\eta \notin \text{d–BOT}_\kappa(\mathcal{A})$ and so $\text{hn–BOT}_\eta(\mathcal{A}) \nsubseteq \text{d–BOT}_\kappa(\mathcal{A})$. Suppose that there exists a deterministic bu-tst $M'' = (Q'', \Sigma, \Delta, \mathcal{A}, F'', \mu'')$ such that $\|M''\|_\kappa = \|\overline{M}\|_\eta$ and show that this is contradictory.

By Proposition 5.1(4) we have that for every $t \in T_\Sigma$ there exists $q \in Q''$ such that $\|M''\|_\kappa(t) = F''_q \xleftarrow[\kappa]{} (h^\kappa_{\mu''}(t)_q)$, and moreover, we observe that $\|M''\|_\kappa(t) \neq \widetilde{0}$ for every $t \in T_\Sigma$. Clearly, T_Σ is infinite, whereas M'' has only a finite set Q'' of states. Hence there must exist $q \in Q''$ and $t_1, t_2 \in T_\Sigma$ such that $h^\kappa_{\mu''}(t_1)_q \neq \widetilde{0}$ and $h^\kappa_{\mu''}(t_2)_q \neq \widetilde{0}$ and $t_1 \neq t_2$. Note that $t_1 \neq t_2$ implies that $\mathrm{size}(t_1) \neq \mathrm{size}(t_2)$ and $u_{t_1} \neq u_{t_2}$. By Proposition 5.1(5) we have that $h^\kappa_{\mu''}(t_i)_q$ is monomial for every $i \in [2]$, so let $a_i \in A$ and $u_i \in T_\Delta$ be such that $h^\kappa_{\mu''}(t_i)_q = a_i\,u_i$. Moreover, let $p \in Q''$ be the unique state such that $\mu''_1(\gamma)_{p,q} \neq \widetilde{0}$ (such a state exists because $\|M''\|_\kappa(\gamma(t_1)) \neq \widetilde{0}$), and let $a_p, a_q, a' \in A$ and $u_p, u_q \in C_\Delta(\mathrm{Z}_1)$ and $u' \in T_\Delta(\mathrm{Z}_1)$ be such that $F''_p = a_p\,u_p$, $F''_q = a_q\,u_q$, and $\mu''_1(\gamma)_{p,q} = a'\,u'$. Part of the bu-tst M'' is displayed in Figure 18(right). Since

$$\|M''\|_\kappa\big(\gamma(t_i)\big) = F''_p \xleftarrow[\kappa]{} \big(h^\kappa_{\mu''}(\gamma(t_i))_p\big) = \sum_{u \in T_\Delta} \Big(a_p \cdot \big(h^\kappa_{\mu''}(\gamma(t_i))_p, u\big)\Big)\,u_p[u] \ ,$$

we obtain $u_{\gamma(t_i)} = u_p[u'_i]$ for some $u'_i \in T_\Delta$. From $t_1 \neq t_2$ follows that $u_{\gamma(t_1)} \neq u_{\gamma(t_2)}$ and hence $u_p = \mathrm{z}_1$ because $u_p \in C_\Delta(\mathrm{Z}_1)$. Moreover, $a_p \neq 0$. In a similar manner, we can show that $u_q = \mathrm{z}_1$. For this we consider

$$\|M''\|_\kappa(t_i) = F''_q \xleftarrow[\kappa]{} (h^\kappa_{\mu''}(t_i)_q) = (a_q \cdot a_i)\,u_q[u_i] \ .$$

Thus we obtain $u_{t_i} = u_q[u_i]$. From $t_1 \neq t_2$ follows $u_{t_1} \neq u_{t_2}$ and hence $u_q = \mathrm{z}_1$ because $u_q \in C_\Delta(\mathrm{Z}_1)$. Further, it is evident that $a_q \neq 0$. Since $u_q = \mathrm{z}_1$ we obtain $u_i = u_{t_i}$. Now we prove that $u' = \sigma(\mathrm{z}_1, \mathrm{z}_1)$. First we observe that

$$h^\kappa_{\mu''}(\gamma(t_i))_p = \mu''_1(\gamma)_{p,q} \xleftarrow[\kappa]{} \big(h^\kappa_{\mu''}(t_i)_q\big) = \big(a' \cdot a_i^{\mathrm{sel}(u',1,\kappa)}\big)\,u'[u_{t_i}] \ ,$$

thus $u_{\gamma(t_i)} = u'[u_{t_i}]$ because $u_p = \mathrm{z}_1$. Since $u_{t_1} \neq u_{t_2}$ this directly yields $u' = \sigma(\mathrm{z}_1, \mathrm{z}_1)$. We obtain for every $i \in [2]$

$$\|M''\|_\kappa(\gamma(t_i)) = \big(a_p \cdot a' \cdot a_i^{\mathrm{sel}(u',1,\kappa)}\big)\,u'[u_{t_i}] = \begin{cases} (a_p \cdot a' \cdot a_i)\,u_{\gamma(t_i)} & \text{if } \kappa = \varepsilon, \\ (a_p \cdot a' \cdot a_i^2)\,u_{\gamma(t_i)} & \text{if } \kappa = \mathrm{o}. \end{cases}$$

Recall that

$$\|\overline{M}\|_\varepsilon(\gamma(t_i)) = a^{\mathrm{size}(t_i)+1}\,u_{\gamma(t_i)} \quad \text{and} \quad \|\overline{M}\|_\mathrm{o}(\gamma(t_i)) = a^{2\cdot\mathrm{size}(u_{t_i})+1}\,u_{\gamma(t_i)} \ .$$

Hence for every $i \in [2]$ we derive the equation

$$a_p \cdot a' \cdot a_i = a^{2\cdot\mathrm{size}(u_{t_i})+1} = (\|\overline{M}\|_\mathrm{o}(\gamma(t_i)), u_{\gamma(t_i)}) \quad \text{if } \kappa = \varepsilon,$$

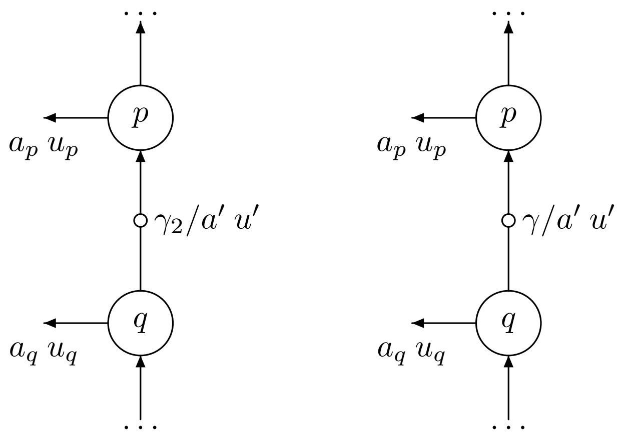

FIGURE 18. Relevant parts of bu-tst M' (left) and M'' (right) over $\mathcal{A}$ of Lemma 5.18.

$$a_p \cdot a' \cdot a_i^2 = a^{\mathrm{size}(t_i)+1} \qquad = (\|\overline{M}\|_\varepsilon(\gamma(t_i)), u_{\gamma(t_i)}) \quad \text{if } \kappa = \mathrm{o}.$$

For every $i \in [2]$ we let $y_i = \mathrm{size}(u_{t_i})$ if $\kappa = \varepsilon$, whereas we let $y_i = \mathrm{size}(t_i)$ in case $\kappa = \mathrm{o}$. Note that in both cases $y_1 \neq y_2$. Also in both cases we have $a_q \cdot a_i = a^{y_i} = (\|\overline{M}\|_\eta(t_i), u_{t_i})$. We continue with the equations

$$\begin{aligned}
\text{if } \kappa = \varepsilon: \quad a^{2y_1+y_2+1} &= (a_p \cdot a' \cdot a_1) \cdot (a_q \cdot a_2) \\
&= (a_p \cdot a' \cdot a_2) \cdot (a_q \cdot a_1) = a^{y_1+2y_2+1} \\
\text{if } \kappa = \mathrm{o}: \quad a^{y_1+2y_2+1} &= (a_p \cdot a' \cdot a_1^2) \cdot (a_q^2 \cdot a_2^2) \\
&= (a_p \cdot a' \cdot a_2^2) \cdot (a_q^2 \cdot a_1^2) = a^{2y_1+y_2+1} \quad .
\end{aligned}$$

Thus in any case $a^{y_1+2y_2+1} = a^{2y_1+y_2+1}$. Since $a^i \neq a^j$ whenever $i \neq j$ for all $i, j \in \mathbb{N}$, we conclude $y_1 + 2y_2 + 1 = 2y_1 + y_2 + 1$ and thereby $y_1 = y_2$ which contradicts to $y_1 \neq y_2$. Consequently, irrespective of κ, we have proved that there is no deterministic bu-tst M'' such that $\|M''\|_\kappa = \|\overline{M}\|_\eta$. Thus $\|\overline{M}\|_\eta \notin \mathrm{d\text{–}BOT}_\kappa(\mathcal{A})$. □

Together with the results of Section 4, we can already derive the HASSE diagram (see Figure 19) for multiplicatively nonperiodic, zero-divisor free, and commutative semirings. We observe that the classes of ε-t-ts and o-t-ts transformations computed by bu-tst are incomparable, whenever inclusion is not trivial by definition or given as a result of Proposition 4.21. However, we note that we cannot present a diagram for multiplicatively nonperiodic and commutative semirings that possess zero-divisors. This is due to the fact that we, *e. g.*, cannot relate the classes $\mathrm{hl\text{–}TOP}_\varepsilon(\mathcal{A})$ and $\mathrm{d\text{–}BOT}_\mathrm{o}(\mathcal{A})$.

THEOREM 5.19. *Figure 19 is the* HASSE *diagram for multiplicatively nonperiodic, zero-divisor free, and commutative semirings $\mathcal{A}$.*

PROOF. All the inclusions are trivial or hold by virtue of Observation 5.14. The equalities are due to Propositions 4.21 and 5.13 and Theorem 4.17 and Observation 5.14. Then for every $\{\eta, \kappa\} = \{\varepsilon, \mathrm{o}\}$ the following

13 statements are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{36}$$
$$\text{dnlt–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{d–BOT}_\eta(\mathcal{A}) \tag{37}$$
$$\text{dnlt–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \tag{38}$$
$$\text{dnl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{dt–BOT}_\eta(\mathcal{A}) \tag{39}$$
$$\text{dnl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dt–TOP}_\varepsilon(\mathcal{A}) \tag{40}$$
$$\text{hn–BOT}_\eta(\mathcal{A}) \not\subseteq \text{dl–BOT}_\eta(\mathcal{A}) \tag{41}$$
$$\text{hn–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{dl–TOP}_\varepsilon(\mathcal{A}) \tag{42}$$
$$\text{hn–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dl–TOP}_\varepsilon(\mathcal{A}) \tag{43}$$
$$\text{hl–BOT}_\eta(\mathcal{A}) \not\subseteq \text{dn–BOT}_\eta(\mathcal{A}) \tag{44}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dn–TOP}_\varepsilon(\mathcal{A}) \tag{45}$$
$$\text{hl–BOT}_\eta(\mathcal{A}) \not\subseteq \text{d–BOT}_\kappa(\mathcal{A}) \tag{46}$$
$$\text{hn–BOT}_\eta(\mathcal{A}) \not\subseteq \text{d–BOT}_\kappa(\mathcal{A}) \tag{47}$$
$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{48}$$

The Inequalities (36) and (37) are shown in Lemma 5.11. The Inequalities (38)–(45) are proved in Lemma 5.12, whereas Inequalities (46) and (47) follow from Lemma 5.18. Finally, Inequality (48) is due to Lemma 5.16. □

6. Multiplicatively periodic semirings

In this section, we consider semirings that are multiplicatively periodic and commutative. For example, the semiring $\mathbb{Z}_4$ is multiplicatively periodic and commutative (without being multiplicatively idempotent). It is easily seen that in multiplicatively periodic, commutative semirings $\mathcal{A} = (A, +, \cdot)$ the carrier set $\langle A' \rangle$ of the least multiplicative submonoid generated from a finite set $A' \subseteq A$ is again finite. This property is essential in the core construction of this section, because it allows us to keep track of the current weight in the states.

PROPOSITION 5.20. *Let $\mathcal{A} = (A, +, \cdot)$ be a multiplicatively periodic and commutative semiring. For every finite $A' \subseteq A$ we have that $\langle A' \rangle$ is finite.*

PROOF. We first observe that $\langle \emptyset \rangle = \{1\}$. Let $A' = \{a_1, \dots, a_k\} \subseteq A$ for some $k \in \mathbb{N}_+$. Then

$$\begin{aligned}\langle A' \rangle &= \{a_1^{i_1} \cdot \ldots \cdot a_k^{i_k} \mid i_1, \dots, i_k \in \mathbb{N}\} \\ &= \{a_1^{i_1} \cdot \ldots \cdot a_k^{i_k} \mid i_1 \in [0, n_1], \dots, i_k \in [0, n_k]\} \ ,\end{aligned}$$

where for every $j \in [k]$ the integer $n_j \in \mathbb{N}$ is the smallest nonnegative integer such that there exists $m_j \in \mathbb{N}$ with $m_j < n_j$ and $a_j^{n_j} = a_j^{m_j}$. Hence $\langle A' \rangle$ is finite. □

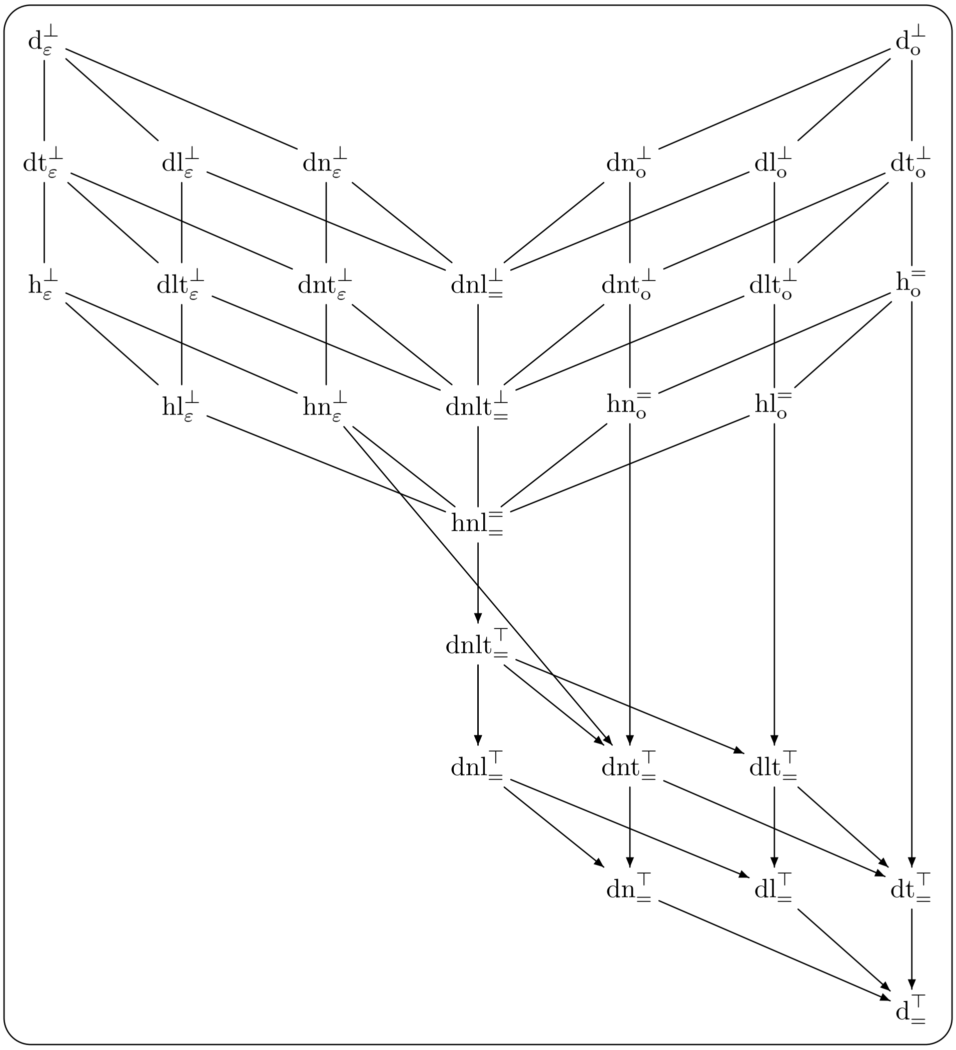

FIGURE 19. HASSE diagram for multiplicatively nonperiodic and commutative semirings $\mathcal{A}$ that are zero-divisor free.

Given a deterministic bu-tst computing an ε-t-ts transformation τ, we construct another deterministic bu-tst computing τ as o-t-ts transformation. Moreover, most of the properties defined for deterministic bu-tst (namely nondeletion, linearity, and totality) are preserved by this construction. However, a homomorphism bu-tst might yield a non-homomorphism bu-tst, because the construction increases the state-space compared to the given bu-tst.

The central idea is the opposite of the one of Proposition 5.6. There we moved the final weight to the transitions, whereas here we move the weight of the transitions to the final weight. This is possible because the semiring is multiplicatively periodic and commutative. Essentially, this means that there is a finite set of possible weights by Proposition 5.20, and we can use the finite set of states to keep track of the current weight. Thus it suffices to use boolean transition weights and apply the weight, which we remember in the states, at the final output.

LEMMA 5.21. *Let $\mathcal{A}$ be multiplicatively periodic and commutative, and let M be a deterministic bu-tst. There exists a deterministic bu-tst M' with a boolean tree representation such that $\|M'\|_\eta = \|M\|_\eta$.*

PROOF. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ and let C be

$$\{(\mu_k(\sigma)_{q,q_1\cdots q_k}, u) \mid k \in \mathbb{N}, \sigma \in \Sigma_k, q, q_1, \ldots, q_k \in Q, u \in \operatorname{supp}(\mu_k(\sigma)_{q,q_1\cdots q_k})\}$$

be the finite set of semiring elements that occur in the monomial tree series in the range of μ. Since $\mathcal{A}$ is multiplicatively periodic and commutative, we conclude that $\langle C\rangle$ is finite by Proposition 5.20. We construct $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows:

- $Q' = Q \times \langle C\rangle$;
- $F'_{(q,c)} = c \cdot F_q$ for every $q \in Q$ and $c \in \langle C\rangle$; and
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q, q_1, \ldots, q_k \in Q$, and $c_1, \ldots, c_k \in \langle C\rangle$ let $c' \in C$ and $u' \in T_\Delta(\mathrm{Z}_k)$ be such that $\mu_k(\sigma)_{q,q_1\cdots q_k} = c'\, u'$, and let
$$c = \begin{cases} 0 & \text{if } (\exists i \in [k])\colon c_i = 0, \\ c' \cdot \prod_{i\in[k]} c_i^{\operatorname{sel}(u',i,\eta)} & \text{otherwise.} \end{cases}$$
Finally, we set $\mu'_k(\sigma)_{(q,c),(q_1,c_1)\cdots(q_k,c_k)} = \chi(\operatorname{supp}(\mu_k(\sigma)_{q,q_1\cdots q_k}))$. Recall that we suppose that all remaining entries in μ' are $\widetilde{0}$.

Obviously, μ' is boolean. It remains to prove that $\|M'\|_\eta = \|M\|_\eta$. For this we first prove that for every $t \in T_\Sigma$, $u \in T_\Delta$, $q \in Q$, and $c \in \langle C\rangle \setminus \{0\}$ we have

$$h^\eta_{\mu'}(t)_{(q,c)} = 1\, u \quad\iff\quad h^\eta_\mu(t)_q = c\, u\ . \tag{49}$$

We prove the statement by induction on t, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$\begin{aligned}
& h^\eta_{\mu'}(\sigma(t_1, \ldots, t_k))_{(q,c)} = 1\, u \\
\iff & \quad \textit{(by Definition 4.7(1) and Proposition 5.1(2))} \\
& (\forall i \in [k])\, (\exists q_i \in Q)\, (\exists c_i \in \langle C\rangle)\colon \\
& \mu'_k(\sigma)_{(q,c),(q_1,c_1)\cdots(q_k,c_k)} \xleftarrow[\eta]{} \big(h^\eta_{\mu'}(t_i)_{(q_i,c_i)}\big)_{i\in[k]} = 1\, u \\
\iff & \quad \textit{(by definition of } \xleftarrow[\eta]{} \textit{ and Proposition 5.1(5) and} \\
& \quad\ \textit{Observation 5.2; note that } c \neq 0 \textit{ implies that } c_i \neq 0\textit{)} \\
& (\forall i \in [k])\, (\exists q_i \in Q)\, (\exists c_i \in \langle C\rangle \setminus \{0\})\, (\exists u' \in T_\Delta(\mathrm{Z}_k))\, (\exists u_i \in T_\Delta)\colon \\
& \mu'_k(\sigma)_{(q,c),(q_1,c_1)\cdots(q_k,c_k)} = 1\, u' \;\wedge\; h^\eta_{\mu'}(t_i)_{(q_i,c_i)} = 1\, u_i \;\wedge\; u = u'[u_i]_{i\in[k]} \\
\iff & \quad \textit{(by definition of } \mu' \textit{ and induction hypothesis)} \\
& (\forall i \in [k])\, (\exists q_i \in Q)\, (\exists c', c_i \in \langle C\rangle \setminus \{0\})\, (\exists u' \in T_\Delta(\mathrm{Z}_k))\, (\exists u_i \in T_\Delta)\colon \\
& \mu_k(\sigma)_{q,q_1\cdots q_k} = c'\, u' \;\wedge\; h^\eta_\mu(t_i)_{q_i} = c_i\, u_i \;\wedge\; u = u'[u_i]_{i\in[k]} \;\wedge \\
& \wedge\; c = c' \cdot \prod_{i\in[k]} c_i^{\operatorname{sel}(u',i,\eta)}
\end{aligned}$$

TABLE 5. Preservation of properties for the construction of Lemma 5.21

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✗	✗

$$\begin{aligned}
\iff\quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& (\forall i \in [k])\,(\exists q_i \in Q)\colon \mu_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\eta]{} \big(h^\eta_\mu(t_i)_{q_i}\big)_{i\in[k]} = c\,u \\
\iff\quad & \textit{(by Definition 4.7(1) and Proposition 5.1(2)} \\
& \quad \textit{because } h^\eta_\mu(t_i)_{q_i} \neq \widetilde{0}\textit{)} \\
& h^\eta_\mu(\sigma(t_1,\dots,t_k))_q = c\,u
\end{aligned}$$

Now we also prove for every $t \in T_\Sigma$, $u \in T_\Delta$, and $q \in Q$ that

$$h^\eta_{\mu'}(t)_{(q,0)} = 1\,u \quad\Longrightarrow\quad h^\eta_\mu(t)_q = \widetilde{0}\ . \tag{50}$$

Suppose that $h^\eta_{\mu'}(t)_{(q,0)} = 1\,u$ and $h^\eta_\mu(t)_q = c\,u'$ for some $c \in \langle C\rangle \setminus \{0\}$ and $u' \in T_\Delta$ (by Proposition 5.1(5), $h^\eta_\mu(t)_q$ is monomial). By (49) we then have $h^\eta_{\mu'}(t)_{(q,c)} = 1\,u'$, which is a contradiction due to Proposition 5.1(1). Now we prove $\|M'\|_\eta = \|M\|_\eta$ as follows. Let $t \in T_\Sigma$. Suppose that there exists exactly one $q \in Q$ and $c \in \langle C\rangle$ such that $h^\eta_{\mu'}(t)_{(q,c)} \neq \widetilde{0}$ [there is at most one such pair (q,c) by Proposition 5.1(1)].

$$\begin{aligned}
\|M'\|_\eta(t) &= F'_{(q,c)} \xleftarrow[\eta]{} (h^\eta_{\mu'}(t)_{(q,c)}) = (c\cdot F_q) \xleftarrow[\eta]{} (h^\eta_{\mu'}(t)_{(q,c)}) \\
&= F_q \xleftarrow[\eta]{} (c\cdot h^\eta_{\mu'}(t)_{(q,c)}) = F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q) = \|M\|_\eta(t)
\end{aligned}$$

Finally, let $h^\eta_{\mu'}(t)_{(q,c)} = \widetilde{0}$ for every $q \in Q$ and $c \in \langle C\rangle$. Thus $\|M'\|_\eta(t) = \widetilde{0}$. Clearly, also $h^\eta_\mu(t)_q = \widetilde{0}$ for every $q \in Q$ by (49). Hence $\|M\|_\eta(t) = \widetilde{0}$. □

Obviously, M' is nondeleting (respectively, linear), if M is nondeleting (respectively, linear). For zero-divisor free semirings also preservation of totality is obvious; an overview of the properties preserved by the construction can be found in Table 5. Let us illustrate the lemma by an example.

EXAMPLE 5.22. *Let* $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$, $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$, *and*

$$\mu_0(\alpha)_\star = 1\,\alpha \quad\textit{and}\quad \mu_1(\gamma)_{\star,\star} = 2\,\sigma(\mathrm{z}_1,\mathrm{z}_1)\ .$$

The bu-tst $M_{5.22} = (\{\star\}, \Sigma, \Delta, \mathbb{Z}_5, F, \mu)$ *with* $F_\star = 1\,\mathrm{z}_1$ *is a nondeleting homomorphism, and we display* $M_{5.22}$ *in Figure 20(left). If we apply the construction of Lemma 5.21, we obtain the bu-tst*

$$M'_{5.22} = (Q', \Sigma, \Delta, \mathbb{Z}_5, F', \mu')$$

with

- $Q' = [4]$ *(note that we renamed the states from* $(\star, i)$ *to just* i *for convenience);*
- $F'_1 = 1\,\mathrm{z}_1$, $F'_2 = 2\,\mathrm{z}_1$, $F'_3 = 3\,\mathrm{z}_1$, *and* $F'_4 = 4\,\mathrm{z}_1$; *and*

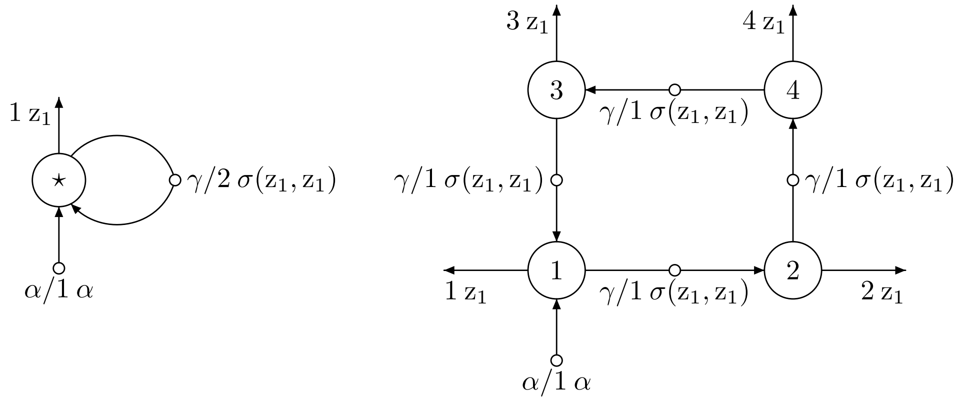

FIGURE 20. Bu-tst $M_{5.22}$ (left) and $M'_{5.22}$ (right) over $\mathbb{Z}_5$ (see Example 5.22).

- $\mu'_0(\alpha)_1 = 1\,\alpha$ *and*

$$\mu'_1(\gamma)_{2,1} = \mu'_1(\gamma)_{4,2} = \mu'_1(\gamma)_{3,4} = \mu'_1(\gamma)_{1,3} = 1\,\sigma(\mathrm{z}_1, \mathrm{z}_1) \enspace .$$

We also display $M'_{5.22}$ in Figure 20(right).

With the help of Observation 4.19 we immediately obtain the following corollary of Lemma 5.21.

COROLLARY 5.23. *For every multiplicatively periodic and commutative semiring $\mathcal{A}$ and $x \in \mathrm{P_d} \setminus \mathrm{P_t}$:*

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{BOT_o}(\mathcal{A}) \enspace .$$

If $\mathcal{A}$ is additionally zero-divisor free, then for every $x \in \Pi_\mathrm{d} \setminus \Pi_\mathrm{h}$:

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{BOT_o}(\mathcal{A}) \enspace .$$

PROOF. Let $\tau \in x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A})$. It follows that there exists a deterministic bu-tst M with properties x such that $\|M\|_\varepsilon = \tau$. By Lemma 5.21 there also exists a deterministic bu-tst M' with properties x and boolean tree representation such that $\|M'\|_\varepsilon = \tau$. Finally, $\|M'\|_\mathrm{o} = \|M'\|_\varepsilon$ due to Observation 4.19; hence $\tau \in x\text{–}\mathrm{BOT_o}(\mathcal{A})$. The converse can be proved in exactly the same manner. □

In order to make Lemma 5.21 also preserve totality, we can exploit the presence of zero-divisors $a, b \in A_+$ with $a \cdot b = 0$ as follows. All transitions $\mu'_k(\sigma)_{(q,0),(q_1,c_1)\cdots(q_k,c_k)}$ with $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q, q_1, \ldots, q_k \in Q$, and $c_1, \ldots, c_k \in \langle C \rangle$ are multiplied with the semiring element a. All other transitions are treated as in the construction of Lemma 5.21. Finally, the final output of state $(q, 0)$ is set to $b \cdot F_q$. Thus all states are final, which remedies the problem with totality. The construction works because we can only enter state $(q, 0)$ with weight a. Thus the weight in state $(q, 0)$ is always a multiple of a. By commutativity the multiplication with b yields 0.

TABLE 6. Preservation of properties for the construction of Lemma 5.24

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✓	✓	✓	✗	✗	✓	✓	✓	✓	✓	✗	✗

LEMMA 5.24. *Let $\mathcal{A}$ be a multiplicatively periodic and commutative semiring, which is not zero-divisor free, and let M be a deterministic bu-tst. There exists a deterministic bu-tst M' such that $\|M'\|_\kappa = \|M\|_\eta$.*

PROOF. Let $a, b \in A_+$ be such that $a \cdot b = 0$. The existence of a and b is given by the fact that $\mathcal{A}$ is not zero-divisor free. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ and

$$C = \{a, b\} \cup \{(\mu_k(\sigma)_{q,q_1\cdots q_k}, u) \mid k \in \mathbb{N}, \sigma \in \Sigma_k, q, q_1, \ldots, q_k \in Q, u \in T_\Delta(\mathrm{Z}_k)\}$$

be the finite set of semiring elements that occur in the monomial tree series in the range of μ. Since $\mathcal{A}$ is multiplicatively periodic and commutative, we conclude that $\langle C \rangle$ is finite by Proposition 5.20. Note that $0 \in \langle C \rangle$. We construct $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows:

- $Q' = Q \times \langle C \rangle$;
- $F'_{(q,0)} = b \cdot F_q$ and $F'_{(q,c)} = c \cdot F_q$ for every $q \in Q$ and $c \in \langle C \rangle \setminus \{0\}$; and
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q, q_1, \ldots, q_k \in Q$, and $c_1, \ldots, c_k \in \langle C \rangle$ let $c' \in C$ and $u' \in T_\Delta(\mathrm{Z}_k)$ be such that $\mu_k(\sigma)_{q,q_1\cdots q_k} = c'\, u'$, and let

$$c = \begin{cases} 0 & \text{if } (\exists i \in [k])\colon c_i = 0, \\ c' \cdot \prod_{i \in [k]} c_i^{\mathrm{sel}(u', i, \eta)} & \text{otherwise.} \end{cases}$$

  Finally, we set

$$\mu'_k(\sigma)_{(q,c),(q_1,c_1)\cdots(q_k,c_k)} = \begin{cases} a \cdot \chi(\mathrm{supp}(\mu_k(\sigma)_{q,q_1\cdots q_k})) & \text{if } c = 0, \\ \chi(\mathrm{supp}(\mu_k(\sigma)_{q,q_1\cdots q_k})) & \text{otherwise.} \end{cases}$$

The proof of the correctness of the construction is similar to the proof of Lemma 5.21. The only major difference is that (50) becomes

$$u \in \mathrm{supp}(h^\eta_{\mu'}(t)_{(q,0)}) \quad \Longrightarrow \quad h^\eta_\mu(t)_q = \widetilde{0} \ . \tag{51}$$

We additionally need to show that $(h^\eta_{\mu'}(t)_{(q,0)}, u) \cdot b = 0$ for every tree $u \in \mathrm{supp}(h^\eta_{\mu'}(t)_{(q,0)})$. This property, however, is clear because a will be a factor in $(h^\eta_{\mu'}(t)_{(q,0)}, u)$. With this knowledge the remaining proof is straightforward. □

This construction preserves all properties except for the homomorphism, td-determinism, and td-totality properties (see Table 6). Let us also illustrate this construction on an example.

EXAMPLE 5.25. *Clearly, 2 is a zero-divisor in $\mathbb{Z}_4$. Suppose that $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$, $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$, and*

$$\mu_0(\alpha)_\star = 1\,\alpha \quad \textit{and} \quad \mu_1(\gamma)_{\star,\star} = 2\,\sigma(\mathrm{z}_1, \mathrm{z}_1) \ .$$

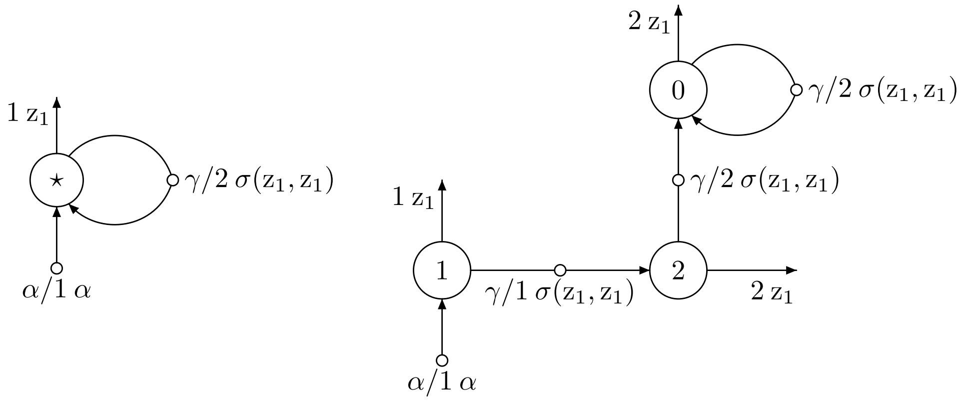

FIGURE 21. Bu-tst $M_{5.25}$ (left) and $M'_{5.25}$ (right) over $\mathbb{Z}_4$ (see Example 5.25).

The bu-tst $M_{5.25} = (\{\star\}, \Sigma, \Delta, \mathbb{Z}_4, F, \mu)$ *with* $F_\star = 1\,\mathrm{z}_1$*, illustrated in Figure 21(left), is a nondeleting homomorphism. If we apply the construction of Lemma 5.24 to* $M_{5.25}$*, we obtain the bu-tst*

$$M'_{5.25} = (Q', \Sigma, \Delta, \mathbb{Z}_4, F', \mu')$$

with

- $Q' = \{0, 1, 2\}$ *(note that we renamed the states from* $(\star, i)$ *to just* i *for convenience);*
- $F'_1 = 1\,\mathrm{z}_1$*,* $F'_2 = 2\,\mathrm{z}_1$*, and* $F'_0 = 2\,\mathrm{z}_1$; *and*
- $\mu'_0(\alpha)_1 = 1\,\alpha$ *and* $\mu'_1(\gamma)_{2,1} = 1\,\sigma(\mathrm{z}_1, \mathrm{z}_1)$ *and*

$$\mu'_1(\gamma)_{0,2} = \mu'_1(\gamma)_{0,0} = 2\,\sigma(\mathrm{z}_1, \mathrm{z}_1)\ .$$

The bu-tst $M'_{5.25}$ *is illustrated in Figure 21(right).*

Next we present a construction that, given a homomorphism td-tst, constructs a semantically equivalent total deterministic bu-tst that uses pure substitution. We thus prove $\text{h–TOP}_\varepsilon(\mathcal{A}) \subseteq \text{dt–BOT}_\varepsilon(\mathcal{A})$. We have already seen in Proposition 5.13 that zero-divisors are problematic. In zero-divisor free semirings the statement is immediate by Proposition 5.13. However, if the semiring is multiplicatively periodic and commutative, then we can keep track of the current weight and avoid that the current weight becomes 0 due to a zero-divisor. The approach closely resembles the one of the construction in Lemma 5.24.

LEMMA 5.26. *Let* $\mathcal{A}$ *be a multiplicatively periodic and commutative semiring, which is not zero-divisor free.*

$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \subseteq \text{dlt–BOT}_\varepsilon(\mathcal{A}) \quad \textit{and} \quad \text{h–TOP}_\varepsilon(\mathcal{A}) \subseteq \text{dt–BOT}_\varepsilon(\mathcal{A})$$

PROOF. Since $\mathcal{A}$ is not zero-divisor free, there exist $a, b \in A_+$ such that $a \cdot b = 0$. Let $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a homomorphism td-tst. Moreover, let

$$C = \{0\} \cup \{(\mu_k(\sigma)_{\star,w}, u) \mid k \in \mathbb{N}, \sigma \in \Sigma_k, w \in \{\star\}(\mathrm{X}_k)^*, u \in T_\Delta(\mathrm{Z}_{|w|})\}$$

be the finite set of semiring elements that occur in the monomial tree series in the range of μ. Since $\mathcal{A}$ is multiplicatively periodic and commutative, we conclude that $\langle C \rangle$ is finite due to Proposition 5.20. We construct $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ as follows:

- $Q' = \langle C \rangle$;
- $F'_0 = b\, \mathrm{z}_1$ and $F'_c = c\, \mathrm{z}_1$ for every $c \in \langle C \rangle \setminus \{0\}$; and
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $w = \star(\mathrm{x}_{i_1}) \cdots \star(\mathrm{x}_{i_n}) \in \{\star\}(\mathrm{X}_k)^*$ and $c_1, \dots, c_k \in \langle C \rangle$ let $c' \in C$ and $u' \in C_\Delta(\mathrm{Z}_n)$ be such that $\mu_k(\sigma)_{\star,w} = c'\, u'$, and let
$$c = \begin{cases} 0 & \text{if } (\exists i \in [k])\colon c_i = 0, \\ c' \cdot \prod_{i \in [k]} c_i^{|w|_{\star(\mathrm{x}_i)}} & \text{otherwise.} \end{cases}$$
Let $c'' = 1$ if $c' \neq 0$ and $c'' = 0$ otherwise. Finally, we set
$$\mu'_k(\sigma)_{c,c_1\cdots c_k} = \begin{cases} (a \cdot c'')\, u'[\mathrm{z}_{i_j}]_{j \in [n]} & \text{if } c = 0, \\ c''\, u'[\mathrm{z}_{i_j}]_{j \in [n]} & \text{otherwise.} \end{cases}$$

Note that $h^\varepsilon_{\mu'}(t)_c$ is boolean for every $t \in T_\Sigma$ and $c \in \langle C \rangle \setminus \{0\}$. It remains to prove that $\|M'\|_\varepsilon = \|M\|_\varepsilon$. For this we first prove that for every $t \in T_\Sigma$, $u \in T_\Delta$, and $c \in \langle C \rangle \setminus \{0\}$ we have
$$h^\varepsilon_{\mu'}(t)_c = 1\, u \quad \Longleftrightarrow \quad h^\varepsilon_\mu(t)_\star = c\, u \ . \tag{52}$$

We prove the statement by induction on t, so let $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$. Moreover, let $w \in \{\star\}(\mathrm{X}_k)^*$ be such that $\mu_k(\sigma)_{\star,w'} = \widetilde{0}$ for every $w' \in \{\star\}(\mathrm{X}_k)^*$ with $w' \neq w$ (such a w exists by determinism). Finally, let $c' \in \langle C \rangle$ and $u' \in C_\Delta(\mathrm{Z}_{|w|})$ be such that $\mu_k(\sigma)_{\star,w} = c'\, u'$.

$$\begin{aligned}
& h^\varepsilon_{\mu'}(\sigma(t_1, \dots, t_k))_c = 1\, u \\
\Longleftrightarrow \quad & \qquad \textit{(by Definition 4.7(1) and Proposition 5.1(2))} \\
& (\forall i \in [k])\, (\exists c_i \in \langle C \rangle)\colon \mu'_k(\sigma)_{c,c_1\cdots c_k} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu'}(t_i)_{c_i}\big)_{i \in [k]} = 1\, u \\
\Longleftrightarrow \quad & \qquad \textit{(by definition of } \xleftarrow[\varepsilon]{}\textit{; note that } c \neq 0 \textit{ implies that } c_i \neq 0\textit{)} \\
& (\forall i \in [k])\, (\exists c_i \in \langle C \rangle \setminus \{0\})\, (\exists u'' \in T_\Delta(\mathrm{Z}_k))\, (\exists u_i \in T_\Delta)\colon \\
& \mu'_k(\sigma)_{c,c_1\cdots c_k} = 1\, u'' \ \wedge\ h^\varepsilon_{\mu'}(t_i)_{c_i} = 1\, u_i \ \wedge\ u = u''[u_i]_{i \in [k]} \\
\Longleftrightarrow \quad & \qquad \textit{(by definition of } \mu' \textit{ and induction hypothesis)} \\
& (\forall i \in [k])\, (\exists c', c_i \in \langle C \rangle \setminus \{0\})\, (\exists u' \in C_\Delta(\mathrm{Z}_n))\, (\exists u_i \in T_\Delta) \\
& (\exists w = \star(\mathrm{x}_{i_1}) \cdots \star(\mathrm{x}_{i_n}) \in \{\star\}(\mathrm{X}_k)^*)\colon \\
& \mu_k(\sigma)_{\star,w} = c'\, u' \ \wedge\ h^\varepsilon_\mu(t_i)_\star = c_i\, u_i \ \wedge\ u = (u'[\mathrm{z}_{i_j}]_{j \in [n]})[u_i]_{i \in [k]} \ \wedge \\
& \ \wedge\ c = c' \cdot \prod_{i \in [k]} c_i^{|w|_{\star(\mathrm{x}_i)}} \\
\Longleftrightarrow \quad & \qquad \textit{(by definition of } \xleftarrow[\varepsilon]{} \textit{ and } (u'[\mathrm{z}_{i_j}]_{j \in [n]})[u_i]_{i \in [k]} = u'[u_{i_j}]_{i \in [n]}\textit{)} \\
& (\forall i \in [k])\, (\exists w = \star(\mathrm{x}_{i_1}) \cdots \star(\mathrm{x}_{i_n}) \in \{\star\}(\mathrm{X}_k)^*)\colon \\
& \mu_k(\sigma)_{\star,w} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_\mu(t_{i_j})_\star\big)_{j \in [n]} = c\, u
\end{aligned}$$

TABLE 7. Preservation of properties for the construction of Lemma 5.26

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✗	✓	✓	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✗

$\iff$ *(by Definition 4.7(1) and Proposition 5.1(2) since $h^\varepsilon_\mu(t_{i_j})_\star \neq \widetilde{0}$)*
$h^\varepsilon_\mu(\sigma(t_1, \ldots, t_k))_\star = c\, u$

Now we also prove for every $t \in T_\Sigma$ and $u \in T_\Delta$ that

$$u \in \operatorname{supp}(h^\varepsilon_{\mu'}(t)_0) \quad \Longrightarrow \quad h^\varepsilon_\mu(t)_\star = \widetilde{0} \ . \tag{53}$$

Suppose that $u \in \operatorname{supp}(h^\varepsilon_{\mu'}(t)_0)$ and $h^\varepsilon_\mu(t)_\star = c\, u'$ for some $c \in \langle C\rangle \setminus \{0\}$ and $u' \in T_\Delta$ (by Proposition 5.1(5), $h^\varepsilon_\mu(t)_\star$ is monomial). By (52) we then have $h^\varepsilon_{\mu'}(t)_c = 1\, u'$, which is a contradiction due to of Proposition 5.1(1). It is easily seen that $b \cdot (h^\varepsilon_{\mu'}(t)_0, u) = 0$ because $a \cdot b = 0$.

Now we prove $\|M'\|_\varepsilon = \|M\|_\varepsilon$ as follows. Let $t \in T_\Sigma$. Suppose that there exists exactly one $c \in \langle C\rangle \setminus \{0\}$ such that $h^\varepsilon_{\mu'}(t)_c \neq \widetilde{0}$ [there is at most one such c by Proposition 5.1(1)].

$$\begin{aligned} &\|M'\|_\varepsilon(t) && = F'_c \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t)_c) && = c \cdot h^\varepsilon_{\mu'}(t)_c \\ = {}& F_\star \xleftarrow[\varepsilon]{} (c \cdot h^\varepsilon_{\mu'}(t)_c) && = F_\star \xleftarrow[\varepsilon]{} (h^\varepsilon_\mu(t)_\star) && = \|M\|_\varepsilon(t) \end{aligned}$$

Now, let $h^\varepsilon_{\mu'}(t)_0 \neq \widetilde{0}$. Then by (53) we have $h^\varepsilon_\mu(t)_\star = \widetilde{0}$ and thereby $\|M\|_\varepsilon(t) = \widetilde{0}$.

$$\|M'\|_\varepsilon(t) = F'_0 \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t)_0) = b \cdot h^\varepsilon_{\mu'}(t)_0 = \widetilde{0} = \|M\|_\varepsilon(t)$$

Finally, let $h^\varepsilon_{\mu'}(t)_c = \widetilde{0}$ for every $c \in \langle C\rangle$. Thus $\|M'\|_\varepsilon(t) = \widetilde{0}$. Clearly, also $h^\varepsilon_\mu(t)_\star = \widetilde{0}$ by (52). Hence $\|M\|_\varepsilon(t) = \widetilde{0}$. □

We show that the class of all ε-t-ts transformations computed by nondeleting homomorphism bu-tst is not contained in the class of all o-t-ts transformations computed by homomorphism bu-tst, *i. e.*,

$$\mathrm{hn\text{–}BOT}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{h\text{–}BOT}_\mathrm{o}(\mathcal{A}) \ ,$$

as long as the semiring $\mathcal{A}$ is not multiplicatively idempotent. Moreover, we show that the class of o-t-ts transformations computed by nondeleting homomorphism bu-tst is not contained in the class of ε-t-ts transformations computed by homomorphism bu-tst. It is clear that both classes of transformations computed by nondeleting homomorphism bu-tst are properly contained in the class of all ε-t-ts transformations computed by deterministic bu-tst due to Lemma 5.21 and Observation 4.19 (on multiplicatively periodic and commutative semirings); *i. e.*, $\mathrm{hn\text{–}BOT}_\eta(\mathcal{A}) \subseteq \mathrm{d\text{–}BOT}_\varepsilon(\mathcal{A})$.

LEMMA 5.27. *Let $\mathcal{A}$ be multiplicatively non-idempotent, and let $\{\eta, \kappa\} = \{\varepsilon, \mathrm{o}\}$.*

$$\mathrm{hn\text{–}BOT}_\eta(\mathcal{A}) \not\subseteq \mathrm{h\text{–}BOT}_\kappa(\mathcal{A}) \quad \textit{and} \quad \mathrm{hn\text{–}BOT}_\varepsilon(\mathcal{A}) \not\subseteq \mathrm{h\text{–}TOP}_\varepsilon(\mathcal{A})$$

PROOF. Since $\mathcal{A}$ is not multiplicatively idempotent, there exists $a \in A$ such that $a \neq a^2$. We construct the bu-tst $M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$ with

- $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Delta = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$;
- $F_\star = 1\, \mathrm{z}_1$; and
- $\mu_0(\alpha)_\star = a\, \alpha$ and $\mu_0(\beta)_\star = 1\, \beta$ and $\mu_1(\gamma)_{\star,\star} = 1\, \sigma(\mathrm{z}_1, \mathrm{z}_1)$.

The bu-tst is depicted in Figure 22(left). Clearly, M is a nondeleting homomorphism. Let $\tau = \|M\|_\eta$. We observe that

$$\tau(\alpha) = a\, \alpha \quad \tau(\beta) = 1\, \beta \quad \tau(\gamma(\alpha)) = a^n\, \sigma(\alpha, \alpha) \quad \tau(\gamma(\beta)) = 1\, \sigma(\beta, \beta) \ ,$$

where $n = 1$ if $\eta = \varepsilon$ and $n = 2$ otherwise.

(i) Suppose that there exists a homomorphism bu-tst

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu')$$

such that $\|M'\|_\kappa = \tau$. Since M' is a homomorphism, we have $\tau(t) = h^\kappa_{\mu'}(t)_\star$ for every $t \in T_\Sigma$ by Observation 4.23. It follows that $\mu'_0(\alpha)_\star = a\, \alpha$ and $\mu'_0(\beta)_\star = 1\, \beta$. Let $c \in A$ and $u \in T_\Delta(\mathrm{Z}_1)$ be such that $\mu'_1(\gamma)_{\star,\star} = c\, u$ [see Figure 22(right) for a graphical representation of M'].

$$\tau(\gamma(t)) = h^\kappa_{\mu'}(\gamma(t))_\star = \mu'_1(\gamma)_{\star,\star} \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t)_\star) = (c\, u) \xleftarrow[\kappa]{} (h^\kappa_{\mu'}(t)_\star)$$

Hence $u = \sigma(\mathrm{z}_1, \mathrm{z}_1)$ because otherwise either $\sigma(\alpha, \alpha) \notin \mathrm{supp}(\|M'\|_\kappa(\gamma(\alpha)))$ or $\sigma(\beta, \beta) \notin \mathrm{supp}(\|M'\|_\kappa(\gamma(\beta)))$. With this knowledge we obtain

$$h^\kappa_{\mu'}(\gamma(\alpha))_\star = (c \cdot a^{3-n})\, \sigma(\alpha, \alpha) \quad \text{and} \quad h^\kappa_{\mu'}(\gamma(\beta))_\star = c\, \sigma(\beta, \beta) \ .$$

The latter equality allows us to conclude that $c = 1$. However, this yields a contradiction because $\tau(\gamma(\alpha)) = a^n\, \sigma(\alpha, \alpha)$ and $\|M'\|_\kappa(\gamma(\alpha)) = a^{3-n}\, \sigma(\alpha, \alpha)$ with $a^n \neq a^{3-n}$. Thus there exists no homomorphism bu-tst M' such that $\|M'\|_\kappa = \tau$, which proves that $\tau \notin \text{h–BOT}_\kappa(\mathcal{A})$.

(ii) Suppose that there exists a homomorphism td-tst

$$M'' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu'')$$

such that $\|M''\|_\varepsilon = \|M\|_\varepsilon$. Let $\tau = \|M\|_\varepsilon$. Because M'' is a homomorphism td-tst, we have $\tau(t) = h^\varepsilon_{\mu''}(t)_\star$ for every $t \in T_\Sigma$ by Observation 4.23. It follows that $\mu''_0(\alpha)_\star = a\alpha$ and $\mu''_0(\beta)_\star = 1\beta$. It is easily seen that $\mu''_1(\gamma)_{\star,\star(\mathrm{x}_1)\star(\mathrm{x}_1)} \neq \widetilde{0}$. Let $c \in A_+$ and $u \in C_\Delta(\mathrm{Z}_2)$ be such that $\mu''_1(\gamma)_{\star,\star(\mathrm{x}_1)\star(\mathrm{x}_1)} = c\, u$.

$$\begin{aligned}\tau(\gamma(t)) = h^\varepsilon_{\mu''}(\gamma(t))_\star &= \mu''_1(\gamma)_{\star,\star(\mathrm{x}_1)\star(\mathrm{x}_1)} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu''}(t)_\star, h^\varepsilon_{\mu''}(t)_\star) \\ &= (c\, u) \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu''}(t)_\star, h^\varepsilon_{\mu''}(t)_\star)\end{aligned}$$

Hence $u = \sigma(\mathrm{z}_1, \mathrm{z}_2)$ because otherwise either $\sigma(\alpha, \alpha) \notin \mathrm{supp}(\|M''\|_\varepsilon(\gamma(\alpha)))$ or $\sigma(\beta, \beta) \notin \mathrm{supp}(\|M''\|_\varepsilon(\gamma(\beta)))$. With this knowledge we obtain

$$h^\varepsilon_{\mu''}(\gamma(\alpha))_\star = (c \cdot a^2)\, \sigma(\alpha, \alpha) \quad \text{and} \quad h^\varepsilon_{\mu''}(\gamma(\beta))_\star = c\, \sigma(\beta, \beta) \ .$$

The latter equality allows us to conclude that $c = 1$. However, this yields a contradiction because $\tau(\gamma(\alpha)) = a\, \sigma(\alpha, \alpha)$ and $\|M''\|_\varepsilon(\gamma(\alpha)) = a^2\, \sigma(\alpha, \alpha)$ with $a \neq a^2$. Thus there exists no homomorphism td-tst M'' such that $\|M''\|_\varepsilon = \tau$, which proves that $\tau \notin \text{h–TOP}_\varepsilon(\mathcal{A})$. $\square$

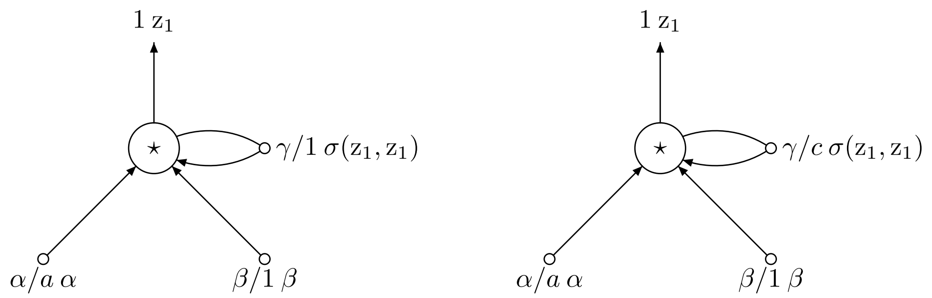

FIGURE 22. Bu-tst M (left) and M' (right) over $\mathcal{A}$ used in the proof of Lemma 5.27.

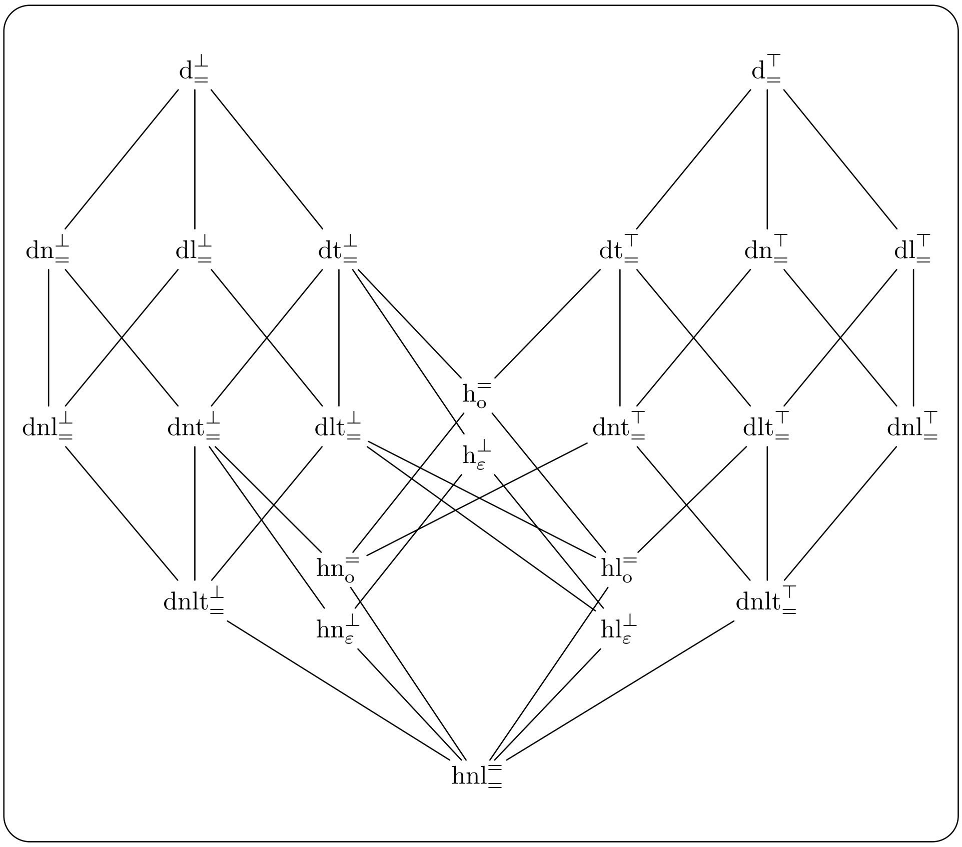

FIGURE 23. HASSE diagram for multiplicatively periodic and commutative semirings $\mathcal{A}$ that are zero-divisor free but not multiplicatively idempotent.

Finally, we are able to present the HASSE diagram for multiplicatively periodic and commutative semirings $\mathcal{A}$ that are not multiplicatively idempotent.

THEOREM 5.28. *Let $\mathcal{A}$ be a multiplicatively periodic and commutative semiring that is not multiplicatively idempotent. Figure 23 is the*

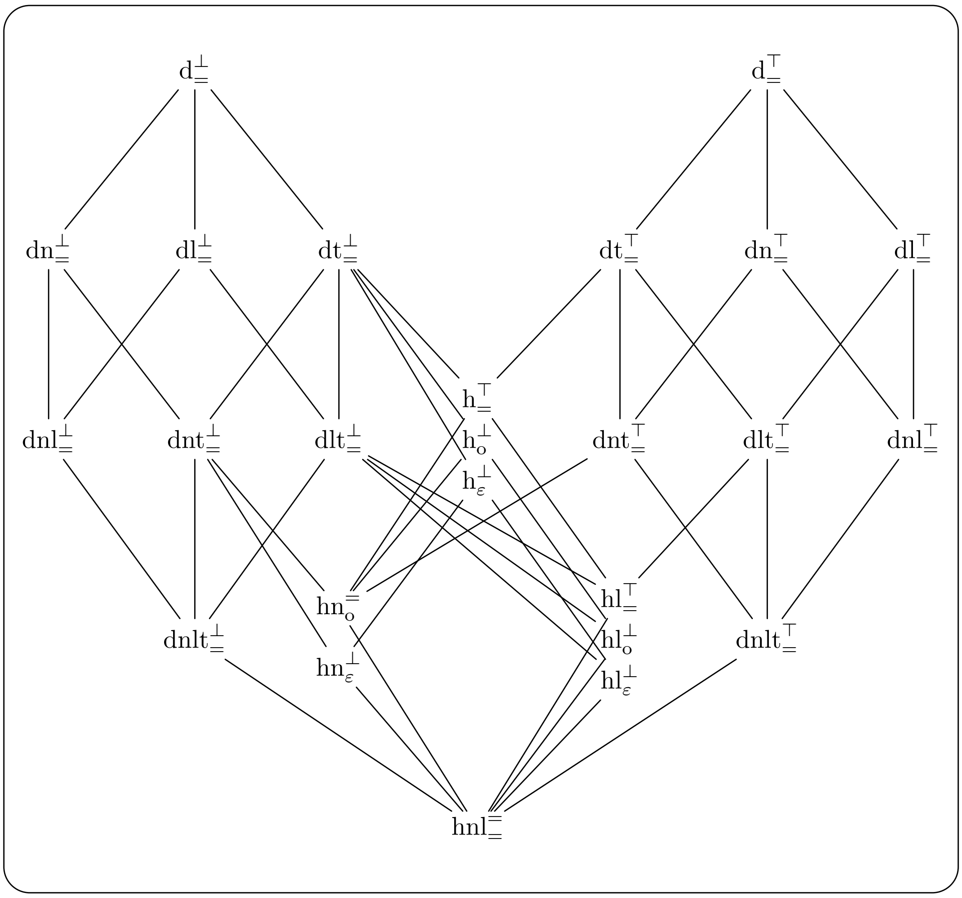

FIGURE 24. HASSE diagram for multiplicatively periodic and commutative semirings $\mathcal{A}$ that are neither zero-divisor free nor multiplicatively idempotent.

HASSE *diagram in case $\mathcal{A}$ is zero-divisor free, and Figure 24 is the* HASSE *diagram in case $\mathcal{A}$ is not zero-divisor free.*

PROOF. (i) Let us first prove that Figure 23 is a HASSE diagram. All the inclusions are trivial or hold by virtue of Observation 5.14. The equalities are due to Propositions 4.21 and 5.13 and Theorem 4.17 and Observation 5.14 and Corollary 5.23. Then for every $\{\eta, \kappa\} = \{\varepsilon, \mathrm{o}\}$ the following 13 statements are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{d–TOP}_{\varepsilon}(\mathcal{A}) \tag{54}$$

$$\text{dnlt–TOP}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{d–BOT}_{\varepsilon}(\mathcal{A}) \tag{55}$$

$$\text{dnlt–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{h–BOT}_{\varepsilon}(\mathcal{A}) \tag{56}$$

$$\text{dnl–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{dt–BOT}_{\varepsilon}(\mathcal{A}) \tag{57}$$

$$\text{dnl–TOP}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{dt–TOP}_{\varepsilon}(\mathcal{A}) \tag{58}$$

$$\text{hn–BOT}_{\eta}(\mathcal{A}) \not\subseteq \text{dl–BOT}_{\varepsilon}(\mathcal{A}) \tag{59}$$

$$\text{hn–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{dl–TOP}_{\varepsilon}(\mathcal{A}) \tag{60}$$

$$\text{hn–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dl–TOP}_\varepsilon(\mathcal{A}) \tag{61}$$
$$\text{hl–BOT}_\eta(\mathcal{A}) \not\subseteq \text{dn–BOT}_\varepsilon(\mathcal{A}) \tag{62}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dn–TOP}_\varepsilon(\mathcal{A}) \tag{63}$$
$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{64}$$
$$\text{hl–BOT}_\text{o}(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \tag{65}$$
$$\text{hn–BOT}_\eta(\mathcal{A}) \not\subseteq \text{h–BOT}_\kappa(\mathcal{A}) \tag{66}$$

The Inequalities (54) and (55) are proved in Lemma 5.11. The Inequalities (56)–(63) are proved in Lemma 5.12, whereas Inequalities (64) and (65) are due to Lemma 5.16. Finally, Inequality (66) holds because of Lemma 5.27.

(ii) Let us now prove that Figure 24 is a HASSE diagram. All the inclusions are trivial or hold by virtue of Observation 5.14 or Lemma 5.26. The equalities are due to Proposition 4.21 and Theorem 4.17 and Observation 5.14 and Lemma 5.24. The above Inequalities (54)–(66) and the following five inequalities are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{67}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dn–BOT}_\varepsilon(\mathcal{A}) \tag{68}$$
$$\text{hl–BOT}_\text{o}(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{69}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{70}$$
$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{71}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \tag{72}$$
$$\text{hn–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–TOP}_\varepsilon(\mathcal{A}) \tag{73}$$

The Inequalities (67) and (68) are due to Lemma 5.12 and Inequalities (69) and (70) are proved in Proposition 5.13. Finally, we have proved Inequalities (71) and (72) in Lemma 5.16 and Inequality (73) in Lemma 5.27. □

7. Multiplicatively idempotent semirings

This section is devoted to the study of multiplicatively idempotent and commutative semirings. The semiring

$$\mathbb{R}_{\min,\max} = (\mathbb{R} \cup \{\infty, -\infty\}, \min, \max)$$

is an example of such a semiring. Clearly, $a^n = a$ for every $n \in \mathbb{N}_+$ and $a \in A$ of a multiplicatively idempotent semiring $(A, +, \cdot)$. Hence we easily derive the following proposition.

PROPOSITION 5.29. *Let $\mathcal{A}$ be multiplicatively idempotent, $k \in \mathbb{N}$, and Δ be a ranked alphabet. For every $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{Z}_k)\rangle\rangle$, which is nondeleting in Z_k and $\psi_1, \dots, \psi_k \in \mathcal{A}\langle\langle T_\Delta\rangle\rangle$ we have*

$$\psi \xleftarrow[\varepsilon]{} (\psi_1, \dots, \psi_k) = \psi \xleftarrow[\text{o}]{} (\psi_1, \dots, \psi_k) \ .$$

This proposition immediately yields the corollary that the class of ε-t-ts transformations computed by nondeleting bu-tst coincides with the class of o-t-ts transformations computed by nondeleting bu-tst.

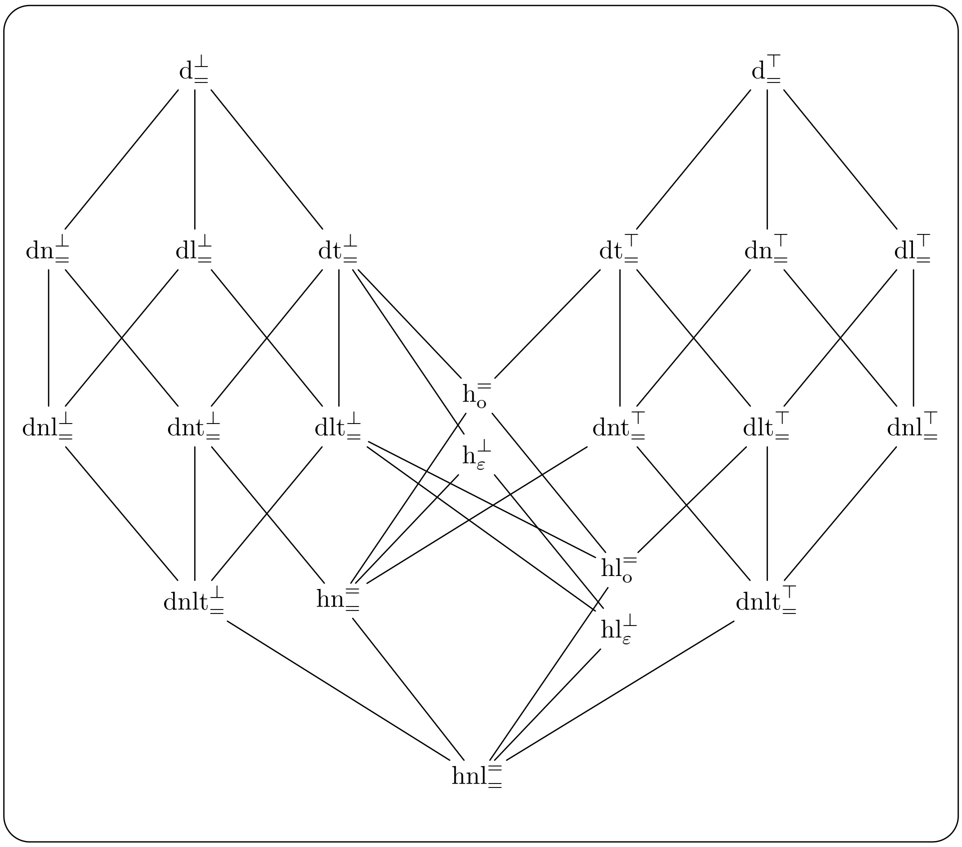

FIGURE 25. HASSE diagram for multiplicatively idempotent and commutative semirings, which are zero-divisor free and have at least three elements.

Moreover this equality is in fact characteristic for multiplicatively idempotent semirings. These two statements are formalized in the next two corollaries.

COROLLARY 5.30. *Let $\mathcal{A}$ be multiplicatively idempotent. For every $x \in \Pi_{\mathrm{n}}$ we have* $x\text{–BOT}_\varepsilon(\mathcal{A}) = x\text{–BOT}_{\mathrm{o}}(\mathcal{A})$.

COROLLARY 5.31. *For every semiring $\mathcal{A}$, we have that $\mathcal{A}$ is multiplicatively idempotent, if and only if* $\mathrm{hn}\text{–BOT}_\varepsilon(\mathcal{A}) = \mathrm{hn}\text{–BOT}_{\mathrm{o}}(\mathcal{A})$.

PROOF. The equality in multiplicatively idempotent semirings is proved in Corollary 5.30 and Lemma 5.27 proves the inequality in all multiplicatively non-idempotent semirings. □

These are indeed all the new results necessary to prove the HASSE diagram for multiplicatively idempotent and commutative semirings. Note that multiplicatively idempotent semirings are trivially multiplicatively periodic, so we apply some of the results derived in Section 6.

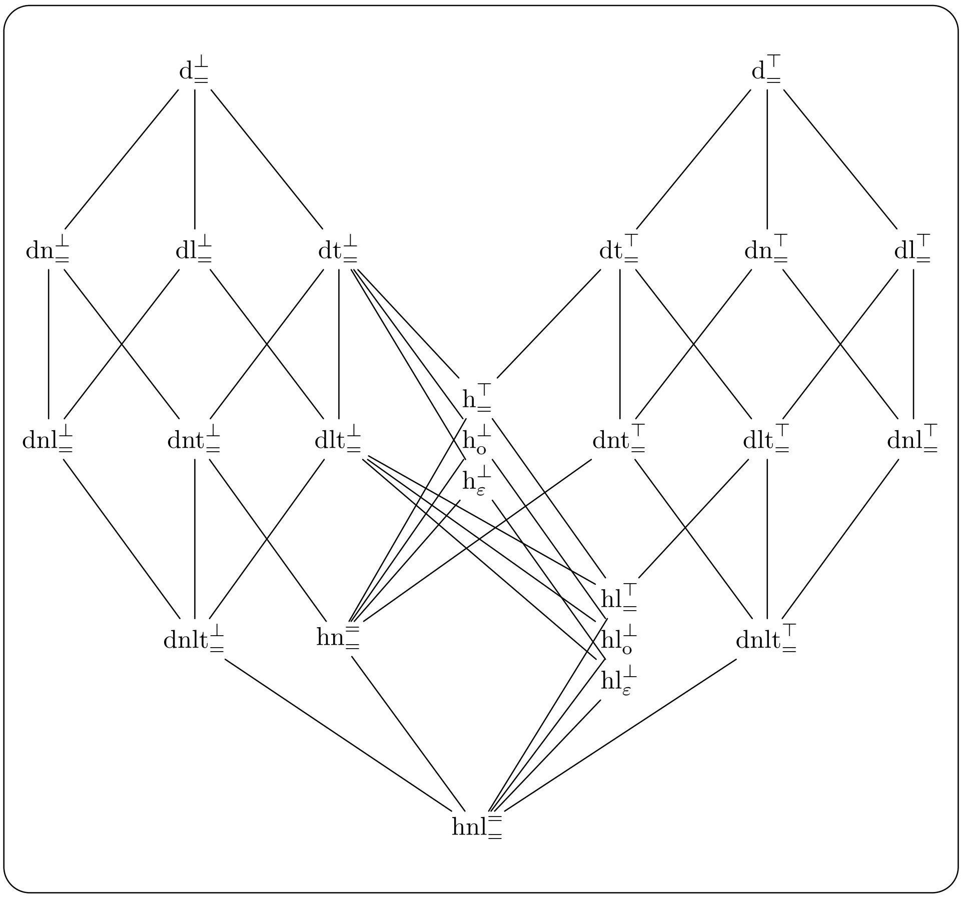

FIGURE 26. HASSE diagram for multiplicatively idempotent and commutative semirings, which are not zero-divisor free and have at least three elements.

THEOREM 5.32. *Let $\mathcal{A}$ be multiplicatively idempotent and commutative with $A \neq \{0, 1\}$. Figure 25 is the* HASSE *diagram in case $\mathcal{A}$ is zero-divisor free, and Figure 26 is the* HASSE *diagram in case $\mathcal{A}$ is not zero-divisor free.*

PROOF. (i) Let us first prove that Figure 25 is a HASSE diagram. All the inclusions are trivial or hold by virtue of Observation 5.14. The equalities are due to Propositions 4.21 and 5.13 and Theorem 4.17 and Observation 5.14 and Corollaries 5.23 and 5.30. Then for every $\{\eta, \kappa\} = \{\varepsilon, \mathrm{o}\}$ the following eleven statements are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{d–TOP}_{\varepsilon}(\mathcal{A}) \tag{74}$$

$$\text{dnlt–TOP}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{d–BOT}_{\varepsilon}(\mathcal{A}) \tag{75}$$

$$\text{dnlt–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{h–BOT}_{\varepsilon}(\mathcal{A}) \tag{76}$$

$$\text{dnl–BOT}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{dt–BOT}_{\varepsilon}(\mathcal{A}) \tag{77}$$

$$\text{dnl–TOP}_{\varepsilon}(\mathcal{A}) \not\subseteq \text{dt–TOP}_{\varepsilon}(\mathcal{A}) \tag{78}$$

$$\text{hn–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{dl–BOT}_\varepsilon(\mathcal{A}) \tag{79}$$
$$\text{hn–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dl–TOP}_\varepsilon(\mathcal{A}) \tag{80}$$
$$\text{hl–BOT}_\eta(\mathcal{A}) \not\subseteq \text{dn–BOT}_\varepsilon(\mathcal{A}) \tag{81}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dn–TOP}_\varepsilon(\mathcal{A}) \tag{82}$$
$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{83}$$
$$\text{hl–BOT}_\text{o}(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \tag{84}$$

The Inequalities (74) and (75) are proved in Lemma 5.11. The Inequalities (76)–(82) are proved in Lemma 5.12, whereas Inequalities (83) and (84) are due to Lemma 5.16.

(ii) Let us now prove that Figure 26 is a HASSE diagram. All the inclusions are trivial or hold by virtue of Observation 5.14 or Lemma 5.26. The equalities are due to Proposition 4.21 and Theorem 4.17 and Observation 5.14 and Lemma 5.24 and Corollary 5.30. The above Inequalities (74)–(84) and the following five inequalities are sufficient to prove strictness and incomparability.

$$\text{dnlt–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{85}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{dn–BOT}_\varepsilon(\mathcal{A}) \tag{86}$$
$$\text{hl–BOT}_\text{o}(\mathcal{A}) \not\subseteq \text{d–TOP}_\varepsilon(\mathcal{A}) \tag{87}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{88}$$
$$\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\text{o}(\mathcal{A}) \tag{89}$$
$$\text{hl–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \tag{90}$$

The Inequalities (85) and (86) are due to Lemma 5.12 and Inequalities (87) and (88) are proved in Proposition 5.13. Finally, we have proved Inequalities (89) and (90) in Lemma 5.16. □

8. Open problems and future work

The most prominent open problem is the missing diagram for commutative and multiplicatively nonperiodic semirings that have zero-divisors. Most of the results needed to prove a HASSE diagram are already available in this thesis. Moreover, the transformational power of deterministic bu-tst and td-tst can also be studied in non-commutative semirings, however most of the semirings that are relevant in applications are commutative.

In addition, more refined restrictions for deterministic tree transducers (like superlinearity [**57, 54**] or relabeling) were studied. Those conditions could be suitably generalized to bu-tst and td-tst and studied on these grounds. Finally, the transformational power of non-deterministic bu-tst and td-tst should be studied. We present some results that contribute to this line of research in Chapter 6.

CHAPTER 6

Polynomial Tree Series Transducers

Man is a masterpiece of creation
if for no other reason than that,
all the weight of evidence for determinism notwithstanding,
he believes he has free will.

Georg Christoph Lichtenberg (1742–1799): Aphorism 249 of "Notebook J"
Aphorisms, translated by R.J. Hollingdale, 1765–1799

1. Bibliographic information

In this chapter we continue the investigations of the previous chapter by examining polynomial tst. In particular, we show that for important semirings like $\mathbb{N}$, $\mathbb{A}$, and $\mathbb{T}$ the choice of pure or o-substitution is relevant for bu-tst. To this end, we prove that there exist ε-t-ts transformations computable by deterministic bu-tst, which cannot be computed as the o-t-ts transformation of any polynomial bu-tst. This line of investigation was started in **[58]**, where the mentioned incomparability was shown for the specific semirings $\mathbb{N}_\infty$ and $\mathbb{T}$ and bu-tst with designated states.

This chapter is a revised and extended version of **[89]**, where the results for bu-tst and td-tst with designated states appeared. We extend these results to polynomial bu-tst and tst with top-most outputs.

2. Coefficient majorization

Throughout this chapter we only consider polynomial tst and their computed η-t-ts transformations, which are well-defined by Observation 4.14(3). In this section we derive mappings, called coefficient majorizations, which allow us to give an upper bound for all nonzero coefficients present in a particular tree series in the range of the η-t-ts transformation computed by a tst. Later we use this bound to show that certain η-t-ts transformations may not belong to a given class of transformations.

At this point we explicitly exclude certain non-interesting tst. We call a tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ *nontrivial*, if $\Sigma_0 \neq \emptyset$ and there exist $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q \in Q$ such that $\mu_k(\sigma)_{q,\varepsilon} \neq \widetilde{0}$. Hence, in particular, for bu-tst non-triviality implies that there exists a $\sigma \in \Sigma_0$ satisfying the condition above. Moreover, non-triviality implies that $\Delta_0 \neq \emptyset$. Let M be trivial (*i. e.*, not nontrivial). Then $\|M\|_\eta(t) = \widetilde{0}$ for every $t \in T_\Sigma$

and $\eta \in \{\varepsilon, \mathrm{o}\}$. Since this particular case is not interesting, we assume that all tst that are considered in the rest of this chapter are nontrivial.

Moreover, we henceforth assume that $\mathrm{mx}_\Sigma \geqslant 1$ for all considered ranked alphabets Σ of input symbols. This is justified because if we restrict ourselves to input alphabets with only nullary symbols then

$$x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{BOT}_\mathrm{o}(\mathcal{A}) = x\text{–}\mathrm{TOP}_\varepsilon(\mathcal{A}) = x\text{–}\mathrm{TOP}_\mathrm{o}(\mathcal{A})$$

for every $x \in \Pi$.

Our coefficient majorization approach uses a partial order on the semiring carrier and derives an upper bound for the nonzero coefficients present in any tree series in the range of the η-t-ts transformation computed by a polynomial bu-tst or td-tst. Next we introduce the notion of a partially ordered semiring. Let $\mathcal{A} = (A, +, \cdot)$ be a semiring and $\leqslant$ be a partial order on A. We say that $\leqslant$ *partially orders* $\mathcal{A}$ (or alternatively: $\mathcal{A}$ *is partially ordered by* $\leqslant$), if for every $a, a', b, b' \in A$ such that $a \leqslant a'$ and $b \leqslant b'$:

(PO+) $a + b \leqslant a' + b'$; and
(PO $\cdot$) $a \cdot b \leqslant a' \cdot b'$.

Note that every naturally ordered semiring is partially ordered by the natural order $\sqsubseteq$. Thus, *e. g.*, the semirings $\mathbb{N}$, $\mathbb{A}$, $\mathbb{T}$, and $\mathbb{B}$ are partially ordered by their respective natural order. Moreover, the following three semirings are also partially ordered by their respective natural order:

- the min-max-*semiring of the reals*
  $$\mathbb{R}_{\min,\max} = (\mathbb{R} \cup \{\infty, -\infty\}, \min, \max)$$
  with the usual operations of minimum and maximum extended to $-\infty$ and ∞ such that $-\infty$ is the smallest element and ∞ is the greatest element;
- for every alphabet S the *formal language semiring*
  $$\mathbb{L}_S = (\mathfrak{P}(S^*), \cup, \circ)$$
  where $V \circ W = \{vw \mid v \in V, w \in W\}$ for every $V, W \subseteq S^*$; and
- the *least common multiple semiring* $\mathrm{Lcm} = (\mathbb{N}, \mathrm{lcm}, \cdot)$ with the usual operations of least common multiple and multiplication.

The next observation is central in this chapter. It lifts the conditions (PO+) and (PO $\cdot$) from sums and products of just two elements to arbitrary (finite) sums and products.

OBSERVATION 6.1. *Let $\mathcal{A}$ be a semiring that is partially ordered by $\leqslant$.*

(1) *Let $n \in \mathbb{N}$, and let $a_i, b_i \in A$ for every $i \in [n]$. If $a_i \leqslant b_i$ for every $i \in [n]$, then*
$$\sum_{i \in [n]} a_i \leqslant \sum_{i \in [n]} b_i \quad \textit{and} \quad \prod_{i \in [n]} a_i \leqslant \prod_{i \in [n]} b_i \enspace .$$

(2) *Let* $a \in A$ *and* $m, n \in \mathbb{N}_+$ *such that* $m \leqslant n$. *If* $1 \leqslant 1+1$, *then* $\sum_{i\in[m]} a \leqslant \sum_{i\in[n]} a$. *Note that* $m = 0$ *is excluded because there may exist an* $a \in A$ *with* $0 \not\leqslant a$.

(3) *Let* $b \in A$ *with* $b \geqslant 1$, *and let* $m, n \in \mathbb{N}$ *such that* $m \leqslant n$. *We then have* $b^m \leqslant b^n$.

PROOF. All three statements are easy consequences of the fact that $\leqslant$ partially orders $\mathcal{A}$. □

For the rest of this chapter, we let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a non-trivial polynomial tst with $\mathrm{mx}_\Sigma \geqslant 1$ and $\eta \in \{\varepsilon, \mathrm{o}\}$. More specifically, M is bottom-up in Section 2.2. The general case including td-tst is considered in Section 2.3.

2.1. The general approach. Our goal is to approximate the coefficient of an output tree that is in the support of a tree series in the range of $\|M\|_\eta$. More precisely, we define coefficient majorizations $f\colon \mathbb{N} \longrightarrow A$, which fulfill $f(n) \in \uparrow\mathcal{C}^\eta_M(n)$ for every $n \in \mathbb{N}$, where $\mathcal{C}^\eta_M(n) \subseteq A$, the set of *coefficients generated by* M *on input trees of height at most* n, is

$$\mathcal{C}^\eta_M(n) = \{(h^\eta_\mu(t)_q, u) \mid q \in Q, t \in T_\Sigma, \mathrm{height}(t) \leqslant n, u \in \mathrm{supp}(h^\eta_\mu(t)_q)\} \ .$$

The existence of such mappings gives rise to a property of polynomial tst. We exploit this property in Section 3 to reprove some recent results and to provide some insight into the relation between the two modes of traversing the input tree (*i. e.*, bottom-up and top-down) and the two types of substitution (*i. e.*, pure and o-substitution).

We start with the definition of some constants that are associated with the polynomial tst M. They provide the abstraction from the concrete tst used in our majorizations.

DEFINITION 6.2. *The following constants represent facts of* M*:*

- *the number* $d_M \in \mathbb{N}_+$ *of* follow-up states *(or* successor states):

$$d_M = \begin{cases} \mathrm{card}(Q) & \text{if } M \text{ is bottom-up,} \\ \max(2, \mathrm{card}(Q)\cdot \mathrm{mx}_\Sigma) & \text{otherwise;} \end{cases}$$

- *the maximal support cardinality* $e_M \in \mathbb{N}_+$ *of the tree representation* μ:

$$e_M = \max\{\mathrm{card}\big(\mathrm{supp}(\mu_k(\sigma)_{q,w})\big) \mid k \in \mathbb{N}, \sigma \in \Sigma_k, q \in Q, w \in Q(\mathrm{X}_k)^*\} \ ;$$

- *the maximal support cardinality* $e'_M \in \mathbb{N}$ *of the top-most output* F*:*

$$e'_M = \max\{\mathrm{card}(\mathrm{supp}(F_q)) \mid q \in Q\} \ ;$$

- *and the maximal variable degree* $u_{M,\eta} \in \mathbb{N}$:

$$u_{M,\eta} = \begin{cases} \max\limits_{\substack{k\in\mathbb{N},\sigma\in\Sigma_k,q\in Q,\\ w\in Q(\mathrm{X}_k)^*,\mu_k(\sigma)_{q,w}\neq\widetilde{0}}} |w| & \text{if } \eta=\varepsilon, \\ \max\limits_{\substack{k\in\mathbb{N},\sigma\in\Sigma_k,q\in Q,\\ w\in Q(\mathrm{X}_k)^*,u\in\mathrm{supp}(\mu_k(\sigma)_{q,w})}} \Big(\sum\limits_{z\in \mathrm{Z}_{|w|}} |u|_z\Big) & \text{if } \eta=\mathrm{o}. \end{cases}$$

Let us discuss those constants in more detail. The constant mx_Σ represents the maximal number of direct subtrees of any node in any given input tree. Clearly, this number coincides with the maximal rank of the input ranked alphabet. Next we consider a state $q \in Q$ and a word $w \in Q(\mathrm{X}_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$. The constant d_M represents the number of possible combinations for a single symbol of the word w. Given that M is bottom-up, we have only $\mathrm{card}(Q)$ choices for the symbol because the variable of X_k is uniquely determined by the position in the word w. Finally, for polynomial tst we have $\mathrm{card}(Q)\cdot k$ choices and thus at most $\mathrm{card}(Q)\cdot \mathrm{mx}_\Sigma$ choices independently of the input symbol, but for technical reasons we take $\max(2, \mathrm{card}(Q)\cdot \mathrm{mx}_\Sigma)$.

The intention of the constants e_M and e'_M, which are well-defined because M is polynomial, is obvious. Lastly, the constant $u_{M,\eta}$ limits the number of factors representing subtree coefficients in any multiplication (see the definition of pure and o-substitution in Definition 3.1). If M is bottom-up, then there are at most mx_Σ factors (if $\eta = \varepsilon$) or at most as many factors as there are variables in the tree selected from the tree representation (if $\eta = \mathrm{o}$). In the general case, there are at most as many factors as the length of the longest word $w \in Q(\mathrm{X}_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ (if $\eta = \varepsilon$) or at most as many factors as there are variables in the tree selected from the tree representation (if $\eta = \mathrm{o}$). Note that if M is top-down, then $u_{M,\varepsilon} = u_{M,\mathrm{o}}$, so there is no difference between pure and o-substitution (see Theorem 4.17). Further note that $u_{M,\eta}$ is well-defined because M is polynomial.

For our coefficient majorizations we need a semiring element which is larger (with respect to some order on A) than any coefficient in the tree representation μ and any coefficient in the top-most outputs F. The next definition introduces the required notions.

DEFINITION 6.3. *Let $\mathcal{A} = (A, +, \cdot)$ be a semiring and $\leqslant$ a partial order on A.*

(1) *An element $c \in A$ is called an* upper bound of the coefficients of μ (with respect to $\leqslant$), *if c is an upper bound of*

$$\{(\mu_k(\sigma)_{q,w}, u) \mid k \in \mathbb{N}, \sigma \in \Sigma_k, q \in Q, w \in Q(\mathrm{X}_k)^*, u \in \mathrm{supp}(\mu_k(\sigma)_{q,w})\}.$$

(2) *An element $c \in A$ is called an* upper bound of the coefficients of F (with respect to $\leqslant$), *if*

$$c \in {\uparrow}\{(F_q, u) \mid q \in Q, u \in \mathrm{supp}(F_q)\} .$$

(3) *Finally, an element* $c \in A$ *is called an* upper bound of the coefficients of M (with respect to $\leqslant$), *if* c *is both an upper bound of the coefficients of* μ *and an upper bound of the coefficients of* F.

Note that such elements need not exist in general. However, the existence can be guaranteed, *e. g.*, by demanding that A is *directed* (with respect to $\leqslant$); *i. e.*, $\uparrow\{a, b\} \neq \emptyset$ for every $a, b \in A$. In the following, we often assume an upper bound c of the coefficients of M with $c \geqslant 1$, and apparently, to obtain the best results, it should be chosen as small as possible; hence it should be the supremum of the nonzero coefficients occurring in M and 1, if this supremum exists.

Next we introduce particular mappings, namely cardinality and coefficient majorizations. Given $n \in \mathbb{N}$, a cardinality majorization is supposed to limit the cardinality of the support of $h^\eta_\mu(t)_q$ for every $q \in Q$ and $t \in T_\Sigma$ of height at most n. Finally, given $n \in \mathbb{N}$, a coefficient majorization f is supposed to limit all nonzero coefficients generated by M on input trees of height at most n; *i. e.*, it must fulfill $f(n) \in \uparrow \mathcal{C}^\eta_M(n)$.

Definition 6.4. *Let* $\mathcal{A} = (A, +, \cdot)$ *be a semiring and* $\leqslant$ *a partial order on* A.

- *A mapping* $l\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ *such that* $\mathrm{card}(\mathrm{supp}(h^\eta_\mu(t)_q)) \leqslant l(n)$ *for every* $n \in \mathbb{N}$, $t \in T_\Sigma$ *of height at most* n, *and* $q \in Q$ *is called a* cardinality majorization (with respect to M and η).
- *Moreover, a mapping* $f\colon \mathbb{N} \longrightarrow A$ *such that* $f(n) \in \uparrow \mathcal{C}^\eta_M(n)$ *for every* $n \in \mathbb{N}$ *is called a* coefficient majorization (with respect to M, $\leqslant$, and η).

Next we provide an example for each of the above defined majorizations. We use the bu-tst $M_{4.6}$ of Example 4.6 (see Chapter 4).

Example 6.5. *Let* $M = M_{4.6} = (Q, \Sigma, \Delta, \mathbb{A}_\infty, F, \mu)$ *be the bu-tst of Example 4.6. Recall that* $\mathbb{A}_\infty$ *is naturally ordered.*

- *The mapping* $l\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ *defined by* $l(n) = 1$ *for every* $n \in \mathbb{N}$ *is a cardinality majorization (with respect to* M *and* o*) because* $\mathrm{card}(T_\Delta) = 1$.
- *The mapping* $f\colon \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty, -\infty\}$ *defined by* $f(n) = n$ *for every* $n \in \mathbb{N}$ *is a coefficient majorization (with respect to* M, $\sqsubseteq$, *and* o*), which can be seen in Example 4.8.*

Next we discuss the general approach used to derive a coefficient majorization. Let $\mathcal{A}$ be partially ordered by $\leqslant$. Moreover, let $t \in T_\Sigma$ and let c be an upper bound (with respect to $\leqslant$) of the coefficients of M with $c \geqslant 1$ (see Definition 6.3). Using a cardinality majorization l, we can introduce a so-called ample coefficient majorization associated with l and c (see Definitions 6.6 and 6.13). The different modifiers (*i. e.*, $\eta = \varepsilon$ or $\eta = \mathrm{o}$) are taken care of by the maximal variable degree $u_{M,\eta}$ (see Definition 6.2). This difference vanishes for polynomial td-tst, because the classes of η-t-ts transformations computed by td-tst using on the one hand $\eta = \varepsilon$ and on the other hand $\eta = \mathrm{o}$ coincide (see Theorem 4.17).

Roughly speaking, if t has height 0, then every support element of $h_\mu^\eta(t)_q$ has a coefficient that is at most c. Given t of height $n+1$, we first compute an upper bound of the coefficients of all subtrees of height at most n. Since those weights are multiplied in the definition of substitution, we take the result of the recursive call to the $u_{M,\eta}$-th power. Recall that $u_{M,\eta}$ is defined such that it holds the maximal number of multiplications in any product generated by one η-substitution. The other factor is provided by the tree representation, and thus c provides a suitable upper bound of this factor.

Finally, by substitution, equal trees might arise such that the coefficients of those are going to be summed up. The cardinality majorization l is going to provide an upper bound of this sum as we will see in Theorems 6.8 and 6.15.

In the sections to follow we distinguish the two modes of traversing the input tree, namely bottom-up and top-down. In particular, in the section on polynomial tst, which also handles td-tst, we casually refer to the bottom-up section because the derived majorizations generally have the same structure and so properties only depending on the structure carry over to the general case.

2.2. The bottom-up case. Recall that in this section M is always a (nontrivial) polynomial bu-tst with $\mathrm{mx}_\Sigma \geqslant 1$. Moreover, let $\eta \in \{\varepsilon, \mathrm{o}\}$. According to the outline just presented, we define the following coefficient majorization. Recall the constants d_M, e_M, and $u_{M,\eta}$ from Definition 6.2.

DEFINITION 6.6. *Let* $l\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ *and* $c \in A$. *The* ample coefficient majorization $f^{\perp}_{M,\eta,l,c}\colon \mathbb{N} \longrightarrow A$ (associated with l and c) *is defined recursively by*

$$f^{\perp}_{M,\eta,l,c}(0) = c$$

$$f^{\perp}_{M,\eta,l,c}(n) = \sum_{i \in [(d_M)^{\mathrm{mx}_\Sigma} \cdot e_M \cdot l(n-1)^{\mathrm{mx}_\Sigma}]} c \cdot f^{\perp}_{M,\eta,l,c}(n-1)^{u_{M,\eta}}$$

for every $n \geqslant 1$.

Thus the ample coefficient majorization depends on the polynomial bu-tst M (or more specifically: a few characteristics of M), the type η of substitution employed, the cardinality majorization l, and the upper bound c. Next we prove that the ample coefficient majorization is indeed a coefficient majorization whenever $\mathcal{A}$ is partially ordered by $\leqslant$, l is a cardinality majorization, and c is an upper bound (with respect to $\leqslant$) of the coefficients of M with $c \geqslant 1$.

For this result we need one additional assumption. Specifically, we assume that $\mathcal{A}$ satisfies $1 \leqslant 1+1$. This condition is trivially satisfied in any additively idempotent semiring. Moreover, also naturally ordered semirings fulfill this restriction with respect to their natural order. First we present an intermediate result that is required in the main theorem.

OBSERVATION 6.7. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ with $1 \leqslant 1+1$. Moreover, let $l\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ and $c \in A$ be such that $c \geqslant 1$. We have $f^{\perp}_{M,\eta,l,c}(n) \geqslant 1$ for every $n \in \mathbb{N}$.*

PROOF. The proof is by induction on n. The induction base is immediate, so we proceed with the induction step. Clearly, $a = c \cdot f^{\perp}_{M,\eta,l,c}(n)^{u_{M,\eta}}$ obeys $a \geqslant 1$ by induction hypothesis, Observation 6.1(3), and (PO $\cdot$). We have $a \leqslant f^{\perp}_{M,\eta,l,c}(n+1)$ by Observation 6.1(2). Thus $1 \leqslant a \leqslant f^{\perp}_{M,\eta,l,c}(n+1)$. □

Now we are ready to state the first theorem. It shows that given a cardinality majorization l and an upper bound of the coefficients of M, the ample coefficient majorization $f^{\perp}_{M,\eta,l,c}$ is indeed a coefficient majorization.

THEOREM 6.8. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ such that $1 \leqslant 1+1$. Moreover, let l be a cardinality majorization, and let c be an upper bound (with respect to $\leqslant$) of the coefficients of M such that $c \geqslant 1$. The ample coefficient majorization $f^{\perp}_{M,\eta,l,c}$ is a coefficient majorization; i. e., $f^{\perp}_{M,\eta,l,c}(n) \in {\uparrow}\, \mathcal{C}^{\eta}_{M}(n)$ for every $n \in \mathbb{N}$. Moreover,*

$$(\|M\|_{\eta}(t), u) \leqslant \sum_{i \in [d_M \cdot e'_M \cdot l(n)]} c \cdot f^{\perp}_{M,\eta,l,c}(n) \tag{91}$$

for every $n \in \mathbb{N}$, $t \in T_{\Sigma}$, and $u \in \operatorname{supp}(\|M\|_{\eta}(t))$ with $\operatorname{height}(t) \leqslant n$.

PROOF. Obviously we have to prove $(h^{\eta}_{\mu}(t)_q, u) \leqslant f^{\perp}_{M,\eta,l,c}(n)$ for every $n \in \mathbb{N}$, $q \in Q$, $t \in T_{\Sigma}$, and $u \in \operatorname{supp}(h^{\eta}_{\mu}(t)_q)$ such that $\operatorname{height}(t) \leqslant n$. We proceed by structural induction on t.

Induction base: Suppose that $t = \alpha$ for some $\alpha \in \Sigma^{(0)}$. Recall that we have $u \in \operatorname{supp}(h^{\eta}_{\mu}(\alpha)_q)$.

$$\begin{aligned}
& (h^{\eta}_{\mu}(\alpha)_q, u) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& (\mu_0(\alpha)_q, u) \\
\leqslant \quad & \textit{(since c is an upper bound of the coefficients of μ)} \\
& c \\
= \quad & \textit{(by Definition 6.6)} \\
& f^{\perp}_{M,\eta,l,c}(0)
\end{aligned}$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_{\Sigma}$ be such that $t = \sigma(t_1, \ldots, t_k)$. Recall that $\operatorname{height}(t) \leqslant n$.

$$(h^{\eta}_{\mu}(\sigma(t_1, \ldots, t_k))_q, u)$$

$$
\begin{aligned}
= \quad & \textit{(by Definition 4.7(1) and the fact that } M \textit{ is bottom-up)} \\
& \Big(\sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\eta]{} (h_\mu^\eta(t_i)_{q_i})_{i\in[k]}, u \Big) \\
= \quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& \sum_{\substack{w=q_1\cdots q_k \in Q^k, \\ u=u'[u_1,\ldots,u_k], u'\in \mathrm{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall i\in[k])\colon u_i \in \mathrm{supp}(h_\mu^\eta(t_i)_{q_i})}} (\mu_k(\sigma)_{q,w}, u') \cdot \prod_{i\in[k]} (h_\mu^\eta(t_i)_{q_i}, u_i)^{\mathrm{sel}(u',i,\eta)} \\
\leqslant \quad & \textit{(by induction hypothesis because } \mathrm{height}(t_i) \leqslant n-1, \\
& (\mu_k(\sigma)_{q,w}, u') \leqslant c, \textit{ and } (PO\cdot); \\
& \textit{in the sequel we use } \ldots \textit{ to abbreviate the index of sum)} \\
& \sum_{\cdots} c \cdot \prod_{i\in[k]} f^{\perp}_{M,\eta,l,c}(n-1)^{\mathrm{sel}(u',i,\eta)} \\
= & \sum_{\cdots} c \cdot f^{\perp}_{M,\eta,l,c}(n-1)^{\mathrm{sel}(u',1,\eta)+\cdots+\mathrm{sel}(u',k,\eta)} \\
\leqslant \quad & \textit{(by Observations 6.1(3) and 6.7)} \\
& \sum_{\cdots} c \cdot f^{\perp}_{M,\eta,l,c}(n-1)^{u_{M,\eta}} \\
\leqslant \quad & \textit{(by Observation 6.1(2) because there exists at least one nonzero} \\
& \textit{summand of the sum by } u \in \mathrm{supp}(h_\mu^\eta(t)_q); \\
& \textit{let } a = c \cdot f^{\perp}_{M,\eta,l,c}(n-1)^{u_{M,\eta}} \textit{)} \\
& \sum_{\substack{w=q_1\cdots q_k \in Q^k, \\ u'\in \mathrm{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall i\in[k])\colon u_i \in \mathrm{supp}(h_\mu^\eta(t_i)_{q_i})}} a \\
\leqslant \quad & \textit{(by Observation 6.1(2) because } d_M = \mathrm{card}(Q), \\
& e_M \geqslant \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w})), \textit{ and } l(n-1) \geqslant \mathrm{card}(\mathrm{supp}(h_\mu^\eta(t_i)_{q_i})) \\
& \textit{because } \mathrm{height}(t_i) \leqslant n-1 \textit{ for every } i \in [k]\textit{)} \\
& \sum_{j\in[(d_M)^k \cdot e_M \cdot l(n-1)^k]} a \\
\leqslant \quad & \textit{(by Observation 6.1(2))} \\
& \sum_{j\in[(d_M)^{\mathrm{mx}_\Sigma} \cdot e_M \cdot l(n-1)^{\mathrm{mx}_\Sigma}]} a \\
= \quad & \textit{(by Definition 6.6)} \\
& f^{\perp}_{M,\eta,l,c}(n)
\end{aligned}
$$

Thus we have proved the first statement of the theorem. This statement allows us to derive the latter statement of the theorem as follows. Note that $e'_M \neq 0$ because $u \in \mathrm{supp}(\|M\|_\eta(t))$.

$$
\begin{aligned}
& (\|M\|_\eta(t), u) \\
= \quad & \textit{(by Definition 4.7(2))} \\
& \sum_{q \in Q} \big(F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q), u\big) \\
= \quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& \sum_{q \in Q} \Big(\sum_{\substack{u = u'[u''],\, u' \in \operatorname{supp}(F_q), \\ u'' \in \operatorname{supp}(h^\eta_\mu(t)_q)}} (F_q, u') \cdot (h^\eta_\mu(t)_q, u'') \Big) \\
\leqslant \quad & \textit{(by (PO}\cdot\textit{), } c \geqslant (F_q, u')\textit{, and the first statement)} \\
& \sum_{q \in Q} \Big(\sum_{\substack{u = u'[u''],\, u' \in \operatorname{supp}(F_q), \\ u'' \in \operatorname{supp}(h^\eta_\mu(t)_q)}} c \cdot f^\perp_{M,\eta,l,c}(n) \Big) \\
\leqslant \quad & \textit{(by Observation 6.1(2) because there exists at least one nonzero} \\
& \quad \textit{summand of the second sum by } u \in \operatorname{supp}(\|M\|_\eta(t))\textit{)} \\
& \sum_{\substack{q \in Q,\, u' \in \operatorname{supp}(F_q), \\ u'' \in \operatorname{supp}(h^\eta_\mu(t)_q)}} c \cdot f^\perp_{M,\eta,l,c}(n) \\
\leqslant \quad & \textit{(by Observation 6.1(2) because } d_M = \operatorname{card}(Q), \\
& \quad e'_M \geqslant \operatorname{card}(\operatorname{supp}(F_q))\textit{, and } l(n) \geqslant \operatorname{card}(\operatorname{supp}(h^\eta_\mu(t)_q)) \\
& \quad \textit{due to } \operatorname{height}(t) \leqslant n\textit{)} \\
& \sum_{i \in [d_M \cdot e'_M \cdot l(n)]} c \cdot f^\perp_{M,\eta,l,c}(n) \qquad \square
\end{aligned}
$$

This theorem admits a trivial corollary for polynomial bu-tst with designated states.

Corollary 6.9. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ with $1 \leqslant 1 + 1$, and let M be a nontrivial polynomial bu-tst with designated states. Moreover, let l be a cardinality majorization, and let c be an upper bound (with respect to $\leqslant$) of the coefficients of μ such that $c \geqslant 1$. Finally, let $Q_{\mathrm{d}} = \{q \in Q \mid F_q \neq \widetilde{0}\}$. For every $n \in \mathbb{N}$, $t \in T_\Sigma$, and $u \in \operatorname{supp}(\|M\|_\eta(t))$ such that* $\operatorname{height}(t) \leqslant n$

$$(\|M\|_\eta(t), u) \leqslant \sum_{i \in [\operatorname{card}(Q_{\mathrm{d}})]} f^\perp_{M,\eta,l,c}(n) \ . \tag{92}$$

Proof.

$$
\begin{aligned}
& (\|M\|_\eta(t), u) \\
= \quad & \textit{(by Definition 4.7(2))} \\
& \sum_{q \in Q} F_q \xleftarrow[\eta]{} (h^\eta_\mu(t)_q)
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \textit{(by Observation 3.4(2))} \\
&\quad \sum_{q \in Q, F_q \neq \widetilde{0}} F_q \xleftarrow[\eta]{} (h_\mu^\eta(t)_q) \\
&= \quad \textit{(because } M \textit{ has designated states)} \\
&\quad \sum_{q \in Q, F_q \neq \widetilde{0}} h_\mu^\eta(t)_q \\
&\leqslant \quad \textit{(by Theorem 6.8 and Observation 6.1(1))} \\
&\quad \sum_{q \in Q, F_q \neq \widetilde{0}} f^\perp_{M,\eta,l,c}(n) \\
&= \quad \textit{(by definition of } Q_\mathrm{d}\textit{)} \\
&\quad \sum_{i \in [\mathrm{card}(Q_\mathrm{d})]} f^\perp_{M,\eta,l,c}(n) \qquad \square
\end{aligned}
$$

Continuing with the running example, we present the ample coefficient majorization for the bu-tst $M_{4.6}$ of Example 4.6.

EXAMPLE 6.10. *Let* $M = M_{4.6} = (Q, \Sigma, \Delta, \mathbb{A}_\infty, F, \mu)$ *be the bu-tst of Example 4.6. The constants of Definition 6.2 are* $\mathrm{mx}_\Sigma = 2$, $d_M = 1$, $e_M = 2$, $e'_M = 1$ *and* $u_{M,\mathrm{o}} = 1$. *We let* l *be the cardinality majorization presented in Example 6.5 (i. e.,* $l(n) = 1$ *for every* $n \in \mathbb{N}$*).*

Finally, 1 *is an upper bound of the coefficients of* M *(see Definition 6.3). We obtain the ample coefficient majorization* $f^\perp_{M,\mathrm{o},l,1}$ *with* $f^\perp_{M,\mathrm{o},l,1}(0) = 1$ *and for every* $n \geqslant 1$

$$f^\perp_{M,\mathrm{o},l,1}(n) = 1 + f^\perp_{M,\mathrm{o},l,1}(n-1) = n + 1 \ .$$

Theorem 6.8 applied to this example yields that $(\|M\|_\mathrm{o}(t), u) \leqslant n + 1$ *for every* $n \in \mathbb{N}$, $t \in T_\Sigma$ *of height at most* n, *and* $u \in \mathrm{supp}(\|M\|_\mathrm{o}(t))$. *Note that* $f^\perp_{M,\mathrm{o},l,1}$ *does not coincide with the coefficient majorization presented in Example 6.5.*

We have derived a mapping $f^\perp_{M,\eta,l,c}$ that limits the coefficients of output subtrees generated by M. By definition, $f^\perp_{M,\eta,l,c}$ depends on a cardinality majorization $l \colon \mathbb{N} \longrightarrow \mathbb{N}_+$. The cardinality majorization l limits the support cardinality of the computed tree series. This mapping was supplied from the outside, but now we derive an easy cardinality majorization $l^\perp_M$.

Given $n \in \mathbb{N}$, we have to limit the cardinality of the support of $h_\mu^\eta(t)_q$ for every $t \in T_\Sigma$ of height at most n and $q \in Q$. The idea is to pessimistically assume that given $k \in \mathbb{N}$, pairs of different trees $(u, u') \in T_\Delta(\mathrm{Z}_k)^2$, and $(u_i, u'_i) \in (T_\Delta)^2$ for every $i \in [k]$, the trees $u[u_1, \dots, u_k]$ and $u'[u'_1, \dots, u'_k]$ are different. This is—of course—not true in general, but it is appropriate for our cardinality majorization because the number of different trees in the support might only be overestimated.

DEFINITION 6.11. *The* ample cardinality majorization associated with M *is the mapping* $l^{\perp}_M \colon \mathbb{N} \longrightarrow \mathbb{N}_+$ *defined for every* $n \in \mathbb{N}$ *by*

$$l^{\perp}_M(n) = (d_M)^{\sum_{i\in[1,n]} \mathrm{mx}^i_\Sigma} \cdot (e_M)^{\sum_{i\in[0,n]} \mathrm{mx}^i_\Sigma}$$
$$= \begin{cases} e_M & \textit{if } n = 0, \\ (d_M)^{\mathrm{mx}_\Sigma} \cdot e_M \cdot l^{\perp}_M(n-1)^{\mathrm{mx}_\Sigma} & \textit{if } n > 0. \end{cases}$$

Now let us show that the ample cardinality majorization associated with M is a cardinality majorization. Once this is achieved, we obtain an ample coefficient majorization $f^{\perp}_{M,\eta,l^{\perp}_M,c}$ that only depends on the constants and c.

LEMMA 6.12. *The ample cardinality majorization associated with the bu-tst* M *is a cardinality majorization; i. e.,*

$$\mathrm{card}(\mathrm{supp}(h^\eta_\mu(t)_q)) \leqslant l^{\perp}_M(n)$$

for every $n \in \mathbb{N}$*,* $q \in Q$*, and* $t \in T_\Sigma$ *of height at most* n*.*

PROOF. We prove the statement by structural induction on t.

Induction base: Suppose that $t = \alpha$ for some $\alpha \in \Sigma_0$.

$$\begin{aligned}
&\mathrm{card}\big(\mathrm{supp}(h^\eta_\mu(\alpha)_q)\big)\\
= \quad & \textit{(by Definition 4.7(1))}\\
&\mathrm{card}\big(\mathrm{supp}(\mu_0(\alpha)_q)\big)\\
\leqslant \quad & \textit{(by definition of } e_M \textit{ in Definition 6.2)}\\
&e_M\\
= \quad & \textit{(by Definition 6.11)}\\
&l^{\perp}_M(0)
\end{aligned}$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ be such that $t = \sigma(t_1, \ldots, t_k)$. Recall that height$(t) \leqslant n$.

$$\begin{aligned}
&\mathrm{card}\big(\mathrm{supp}(h^\eta_\mu(\sigma(t_1,\ldots,t_k))_q)\big)\\
= \quad & \textit{(by Definition 4.7(1) and the fact that } M \textit{ is bottom-up)}\\
&\mathrm{card}\Big(\mathrm{supp}\Big(\sum_{w=q_1\cdots q_k \in Q^k} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h^\eta_\mu(t_i)_{q_i})_{i\in[k]}\Big)\Big)\\
= \quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{)}\\
&\mathrm{card}\Big(\mathrm{supp}\Big(\sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ u'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k])\colon u_i\in\mathrm{supp}(h^\eta_\mu(t_i)_{q_i})}} \Big((\mu_k(\sigma)_{q,w}, u') \cdot \prod_{i\in[k]} (h^\eta_\mu(t_i)_{q_i}, u_i)^{\mathrm{sel}(u',i,\eta)}\Big)\\
&\qquad u'[u_1,\ldots,u_k]\Big)\Big)\\
\leqslant \quad & \textit{(by } e_M \geqslant \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w})) \textit{ and induction hypothesis)}\\
&(d_M)^k \cdot e_M \cdot l^{\perp}_M(n-1)^k\\
\leqslant \quad & \textit{(because } \mathrm{mx}_\Sigma \geqslant k\textit{)}\\
&(d_M)^{\mathrm{mx}_\Sigma} \cdot e_M \cdot l^{\perp}_M(n-1)^{\mathrm{mx}_\Sigma}
\end{aligned}$$

$= \quad$ *(by Definition 6.11)*

$l^{\perp}_M(n)$ □

Now we can combine the ample coefficient majorization and the ample cardinality majorization. If we instantiate $f^{\perp}_{M,\eta,l,c}$ with $l = l^{\perp}_M$, then we obtain $f^{\perp}_{M,\eta,l^{\perp}_M,c}$, which is defined by

$$f^{\perp}_{M,\eta,l^{\perp}_M,c}(0) = c$$

$$f^{\perp}_{M,\eta,l^{\perp}_M,c}(n) = \sum_{i \in [l^{\perp}_M(n)]} c \cdot f^{\perp}_{M,\eta,l^{\perp}_M,c}(n-1)^{u_{M,\eta}}$$

for every $n \geqslant 1$.

2.3. The general case. The main result of this section is also proved under the assumption that $1 \leqslant 1 + 1$. However, in this section we consider polynomial tst and derive similar majorizations for them. Thus, M always denotes a (nontrivial) polynomial tst with $\mathrm{mx}_\Sigma \geqslant 1$ in this section. Moreover, let $\eta \in \{\varepsilon, \mathrm{o}\}$. Recall the constants d_M, e_M, and $u_{M,\eta}$ of Definition 6.2.

DEFINITION 6.13. *Let $l \colon \mathbb{N} \longrightarrow \mathbb{N}_+$ and $c \in A$. The* ample coefficient majorization $f^{\top}_{M,\eta,l,c} \colon \mathbb{N} \longrightarrow A$ (associated with l and c) *is defined recursively by*

$$f^{\top}_{M,\eta,l,c}(0) = c$$

$$f^{\top}_{M,\eta,l,c}(n) = \sum_{i \in [(d_M)^{1+u_{M,\varepsilon}} \cdot e_M \cdot l(n-1)^{u_{M,\varepsilon}}]} c \cdot f^{\top}_{M,\eta,l,c}(n-1)^{u_{M,\eta}}$$

for every $n \in \mathbb{N}_+$.

Note the structural similarity of $f^{\top}_{M,\eta,l,c}$ and the ample coefficient majorization of a polynomial bu-tst. Also note that $f^{\top}_{M,\eta,l,c}$ does not depend on η, if M is top-down. Theorem 6.8, which states that the ample coefficient majorization of a polynomial bu-tst is indeed a coefficient majorization, and its proof can be translated in a straightforward manner to the general (and thus also the top-down) case. The general approach remains the same, though there are some notational changes, so we resupply the proof.

OBSERVATION 6.14. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ such that $1 \leqslant 1 + 1$. Moreover, let $l \colon \mathbb{N} \longrightarrow \mathbb{N}_+$ and $c \in A$ be such that $c \geqslant 1$. We have $f^{\top}_{M,\eta,l,c}(n) \geqslant 1$ for every $n \in \mathbb{N}$.*

PROOF. The proof is literally the same as the proof of Observation 6.7 except that $f^{\perp}_{M,\eta,l,c}$ has to be replaced by $f^{\top}_{M,\eta,l,c}$. □

Now we are ready to state the theorem for polynomial tst. It shows that given a cardinality majorization l and an upper bound of the coefficients of M, the ample coefficient majorization $f^{\top}_{M,\eta,l,c}$ is indeed a coefficient majorization.

THEOREM 6.15. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ such that $1 \leqslant 1+1$. Moreover, let l be a cardinality majorization, and let c be an upper bound (with respect to $\leqslant$) of the coefficients of M such that $c \geqslant 1$. The ample coefficient majorization $f^{\top}_{M,\eta,l,c}$ is a coefficient majorization; i. e., $f^{\top}_{M,\eta,l,c}(n) \in \uparrow \mathcal{C}^{\eta}_{M}(n)$ for every $n \in \mathbb{N}$. Moreover,*

$$(\|M\|_{\eta}(t), u) \leqslant \sum_{i \in [\mathrm{card}(Q) \cdot e'_M \cdot l(n)]} c \cdot f^{\top}_{M,\eta,l,c}(n)$$

for every $n \in \mathbb{N}$, $t \in T_\Sigma$, and $u \in \mathrm{supp}(\|M\|_{\eta}(t))$ with $\mathrm{height}(t) \leqslant n$.

PROOF. The proof of the latter statement is identical to the proof of the corresponding statement of Theorem 6.8. So it remains to prove $(h^{\eta}_{\mu}(t)_q, u) \leqslant f^{\top}_{M,\eta,l,c}(n)$ for every $n \in \mathbb{N}$, $q \in Q$, $t \in T_\Sigma$, and $u \in \mathrm{supp}(h^{\eta}_{\mu}(t)_q)$ such that $\mathrm{height}(t) \leqslant n$. We prove this statement by structural induction on t.

Induction base: Suppose that $t = \alpha$ with $\alpha \in \Sigma_0$. Recall that we have $u \in \mathrm{supp}(h^{\eta}_{\mu}(\alpha)_q)$.

$$\begin{aligned}
& (h^{\eta}_{\mu}(\alpha)_q, u) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& (\mu_0(\alpha)_q, u) \\
\leqslant \quad & \textit{(since c is an upper bound of the coefficients of μ)} \\
& c \\
= \quad & \textit{(by Definition 6.13)} \\
& f^{\top}_{M,\eta,l,c}(0)
\end{aligned}$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ be such that $t = \sigma(t_1, \ldots, t_k)$. Recall that $\mathrm{height}(t) \leqslant n$.

$$\begin{aligned}
& \big(h^{\eta}_{\mu}(\sigma(t_1, \ldots, t_k))_q, u\big) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_l(\mathrm{x}_{i_l})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h^{\eta}_{\mu}(t_{i_j})_{q_j})_{j \in [l]}, u \Big) \\
= \quad & \textit{(by definition of $\xleftarrow[\eta]{}$)} \\
& \sum_{\substack{w = q_1(\mathrm{x}_{i_1}) \cdots q_l(\mathrm{x}_{i_l}) \in Q(\mathrm{X}_k)^*, \\ u = u'[u_1, \ldots, u_l], u' \in \mathrm{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall j \in [l]) \colon u_j \in \mathrm{supp}(h^{\eta}_{\mu}(t_{i_j})_{q_j})}} (\mu_k(\sigma)_{q,w}, u') \cdot \prod_{j \in [l]} (h^{\eta}_{\mu}(t_{i_j})_{q_j}, u_j)^{\mathrm{sel}(u', j, \eta)} \\
\leqslant \quad & \textit{(by $c \geqslant (\mu_k(\sigma)_{q,w}, u')$, induction hypothesis} \\
& \quad \textit{because } \mathrm{height}(t_{i_j}) \leqslant n-1, \textit{ and (PO$\cdot$);} \\
& \quad \textit{we use $\ldots$ to abbreviate the index set of the previous sum)} \\
& \sum_{\cdots} c \cdot f^{\top}_{M,\eta,l,c}(n-1)^{\mathrm{sel}(u',1,\eta) + \cdots + \mathrm{sel}(u',l,\eta)}
\end{aligned}$$

$$
\begin{aligned}
\leqslant \quad & \textit{(by Observations 6.1(3) and 6.14 because} \\
& \operatorname{sel}(u', 1, \eta) + \cdots + \operatorname{sel}(u', l, \eta) \leqslant u_{M,\eta}\textit{)} \\
& \sum_{\cdots} c \cdot f^{\top}_{M,\eta,l,c}(n-1)^{u_{M,\eta}} \\
\leqslant \quad & \textit{(by Observation 6.1(2) because there exists at least one nonzero} \\
& \textit{summand of the sum by } u \in \operatorname{supp}(h^{\eta}_{\mu}(t)_q)\textit{;} \\
& \textit{let } a = c \cdot f^{\top}_{M,\eta,l,c}(n-1)^{u_{M,\eta}}\textit{)} \\
& \sum_{\substack{w=(q_1(\mathrm{x}_{i_1}),\ldots,q_l(\mathrm{x}_{i_l}))\in Q(\mathrm{X}_k)^*, \\ u'\in\operatorname{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall j\in[l])\colon u_j\in\operatorname{supp}(h^{\eta}_{\mu}(t_{i_j})_{q_j})}} a \\
\leqslant \quad & \textit{(by Observation 6.1(2) by } d^l_M \geqslant \operatorname{card}(Q(\mathrm{X}_k)^l)\textit{,} \\
& e_M \geqslant \operatorname{card}(\operatorname{supp}(\mu_k(\sigma)_{q,w}))\textit{, and } l(n-1) \geqslant \operatorname{card}(\operatorname{supp}(h^{\eta}_{\mu}(t_{i_j})_{q_j})) \\
& \textit{for every } j \in [n]\textit{; rationale see below)} \\
& \sum_{j'\in[(d_M)^{1+u_{M,\varepsilon}}\cdot e_M\cdot l(n-1)^{u_{M,\varepsilon}}]} a \\
= \quad & \textit{(by Definition 6.13)} \\
& f^{\top}_{M,\eta,l,c}(n)
\end{aligned}
$$

Let us consider how many $w \in Q(\mathrm{X}_k)^*$ there are such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$. Clearly, there are at most $\sum_{j\in[0,u_{M,\varepsilon}]} d^j_M$ such w, but if M is bottom-up then there are at most d^k_M because $w = q_1(\mathrm{x}_1)\cdots q_k(\mathrm{x}_k)$ for some $q_1, \ldots, q_k \in Q$. Note that $d_M > 1$ by Definition 6.2 except when M is bottom-up. If $d_M > 1$, then $\sum_{j\in[0,u_{M,\varepsilon}]} d^j_M \leqslant (d_M)^{1+u_{M,\varepsilon}}$. However, if M is bottom-up then $d^k_M \leqslant (d_M)^{1+u_{M,\varepsilon}}$ because $k \leqslant u_{M,\varepsilon}$. This concludes the induction step. □

Also this theorem admits a trivial corollary for polynomial tst with designated states.

COROLLARY 6.16. *Let $\mathcal{A}$ be partially ordered by $\leqslant$ with $1 \leqslant 1+1$, and let M be a nontrivial polynomial tst with designated states. Moreover, let l be a cardinality majorization, and let c be an upper bound (with respect to $\leqslant$) of the coefficients of μ such that $c \geqslant 1$. Moreover, let $Q_\mathrm{d} = \{q \in Q \mid F_q \neq \widetilde{0}\}$. For every $n \in \mathbb{N}$, $t \in T_\Sigma$, and $u \in \operatorname{supp}(\|M\|_\eta(t))$ such that $\operatorname{height}(t) \leqslant n$ we have*

$$
(\|M\|_\eta(t), u) \leqslant \sum_{i\in[\operatorname{card}(Q_\mathrm{d})]} f^{\top}_{M,\eta,l,c}(n) \enspace .
$$

PROOF. The proof is analogous to the proof of Corollary 6.9. □

Let us present an example that illustrates the ample coefficient majorization. We demonstrate it on the td-tst $M^2_{4.26}$ on $\mathbb{N}$ of Example 4.26.

EXAMPLE 6.17. *Let $M = M^2_{4.26} = (Q, \Sigma, \Delta, \mathbb{N}, F, \mu)$ be the td-tst of Example 4.26. The constants of Definition 6.2 are $\mathrm{mx}_\Sigma = 1$, $d_M = 2$,*

$e_M = 1$, $e'_M = 1$ and $u_{M,\varepsilon} = 2$. We let $l\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ be the cardinality majorization with $l(n) = 1$ for every $n \in \mathbb{N}$. This is a cardinality majorization because M is td-deterministic [see Proposition 5.1(5)].

Finally, 2 is an upper bound of the coefficients of M (see Definition 6.3). We obtain the ample coefficient majorization $f^{\top}_{M,\varepsilon,l,2}$ with $f^{\top}_{M,\varepsilon,l,2}(0) = 2$ and for every $n \in \mathbb{N}_+$

$$f^{\top}_{M,\varepsilon,l,2}(n) = \sum_{i \in [8]} 2 \cdot f^{\top}_{M,\varepsilon,l,2}(n-1)^2 = 16 \cdot f^{\top}_{M,\varepsilon,l,2}(n-1)^2 \ .$$

Theorem 6.15 applied to this example yields that

$$(\|M\|_{\varepsilon}(t), u) \leqslant 2 \cdot f^{\top}_{M,\varepsilon,l,2}(n)$$

for every $n \in \mathbb{N}$, $t \in T_\Sigma$ of height at most n, and $u \in \operatorname{supp}(\|M\|_{\varepsilon}(t))$. If we compare this with Lemma 4.27, then we see that the statement is true.

Finally, we also derive an ample cardinality majorization for polynomial tst. With the help of this majorization we can then present a coefficient majorization that relies only on the constants associated with M, the modifier η, and the upper bound c.

DEFINITION 6.18. *The* ample cardinality majorization associated with *M is the mapping $l^{\top}_M\colon \mathbb{N} \longrightarrow \mathbb{N}_+$ recursively defined by*

$$\begin{aligned} l^{\top}_M(0) &= e_M \\ l^{\top}_M(n) &= (d_M)^{1+u_{M,\varepsilon}} \cdot e_M \cdot l^{\top}_M(n-1)^{u_{M,\varepsilon}} \end{aligned}$$

for every $n \in \mathbb{N}_+$.

LEMMA 6.19. *The ample cardinality majorization associated with the tst M is a cardinality majorization; i. e.,*

$$\operatorname{card}(\operatorname{supp}(h^{\eta}_{\mu}(t)_q)) \leqslant l^{\top}_M(n)$$

for every $n \in \mathbb{N}$, $q \in Q$, and $t \in T_\Sigma$ of height at most n.

PROOF. The proof proceeds along the lines of the proof of Lemma 6.12 with just minor changes, most of which were already outlined in the proof of Theorem 6.15. We prove the statement by structural induction on t.

Induction base: Suppose that $t = \alpha$ for some $\alpha \in \Sigma_0$.

$$\begin{aligned} & \operatorname{card}\big(\operatorname{supp}(h^{\eta}_{\mu}(\alpha)_q)\big) \\ = & \qquad \textit{(by Definition 4.7(1))} \\ & \operatorname{card}\big(\operatorname{supp}(\mu_0(\alpha)_q)\big) \\ \leqslant & \qquad \textit{(by definition of } e_M \textit{ in Definition 6.2)} \\ & e_M \\ = & \qquad \textit{(by Definition 6.11)} \\ & l^{\top}_M(0) \end{aligned}$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ be such that $t = \sigma(t_1, \ldots, t_k)$. Recall that $\mathrm{height}(t) \leqslant n$.

$$
\begin{aligned}
& \mathrm{card}\big(\mathrm{supp}(h_\mu^\eta(\sigma(t_1, \ldots, t_k))_q)\big) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_l(\mathrm{x}_{i_l})}} \mu_k(\sigma)_{q,w} \xleftarrow[\eta]{} (h_\mu^\eta(t_{i_j})_{q_j})_{j \in [l]} \Big)\Big) \\
= \quad & \textit{(by definition of } \xleftarrow[\eta]{} \textit{)} \\
& \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_l(\mathrm{x}_{i_l}), \\ u' \in \mathrm{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall j \in [l]) \colon u_j \in \mathrm{supp}(h_\mu^\eta(t_{i_j})_{q_j})}} \Big((\mu_k(\sigma)_{q,w}, u') \cdot \prod_{j \in [l]} (h_\mu^\eta(t_{i_j})_{q_j}, u_j)^{\mathrm{sel}(u', j, \eta)} \Big) \; u'[u_1, \ldots, u_l] \Big)\Big) \\
\leqslant \quad & \textit{(by } e_M \geqslant \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w})) \textit{ and induction hypothesis; rationale below)} \\
& (d_M)^{1 + u_{M,\varepsilon}} \cdot e_M \cdot l_M^\top(n-1)^{u_{M,\varepsilon}} \\
= \quad & \textit{(by Definition 6.11)} \\
& l_M^\top(n)
\end{aligned}
$$

Let us consider how many $w \in Q(\mathrm{X}_k)^*$ there are such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$. Clearly, there are at most $\sum_{j \in [0, u_{M,\varepsilon}]} d_M^j$ such w, but if M is bottom-up then there are at most d_M^k because $w = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$ for some $q_1, \ldots, q_k \in Q$. Note that $d_M > 1$ by Definition 6.2 except when M is bottom-up. If $d_M > 1$, then $\sum_{j \in [0, u_{M,\varepsilon}]} d_M^j \leqslant (d_M)^{1 + u_{M,\varepsilon}}$. However, if M is bottom-up then $d_M^k \leqslant (d_M)^{1 + u_{M,\varepsilon}}$ because $k \leqslant u_{M,\varepsilon}$. This concludes the induction step. $\square$

Let us also present an example for the ample cardinality majorization. We again use the deterministic td-tst $M_{4.26}^2$ on $\mathbb{N}$ of Example 4.26.

EXAMPLE 6.20. *Let $M = M_{4.26}^2 = (Q, \Sigma, \Delta, \mathbb{N}, F, \mu)$ be the deterministic td-tst of Example 4.26. We obtain the ample cardinality majorization $l_M^\top \colon \mathbb{N} \longrightarrow \mathbb{N}_+$, which is recursively defined for every $n \in \mathbb{N}_+$ by*

$$
\begin{aligned}
l_M^\top(0) &= 1 \\
l_M^\top(n) &= 8 \cdot l_M^\top(n-1)^2 \enspace .
\end{aligned}
$$

Clearly, this is a cardinality majorization because M is td-deterministic [see Proposition 5.1(5)].

Now we can combine the ample coefficient majorization and the ample cardinality majorization. If we instantiate $f_{M,\eta,l,c}^\top$ with $l = l_M^\top$, then we obtain $f_{M,\eta,l_M^\top,c}^\top$, which is defined by

$$
f_{M,\eta,l_M^\top,c}^\top(0) = c
$$

$$f^{\top}_{M,\eta,l^{\top}_M,c}(n) = \sum_{i \in [l^{\top}_M(n)]} c \cdot f^{\top}_{M,\eta,l^{\top}_M,c}(n-1)^{u_{M,\eta}}$$

for every $n \geqslant 1$. Note the structural similarity of $f^{\perp}_{M,\eta,l^{\perp}_M,c}$ and $f^{\top}_{M,\eta,l^{\top}_M,c}$.

3. Incomparability results

In the first part of this section we reprove two recent results of [**58**] concerning growth properties of polynomial bu-tst with designated states using our coefficient majorization approach (*i. e.*, using Corollary 6.9). The second part then focuses on some simplified coefficient majorization that allows us to derive incomparability results for classes of η-t-ts transformations computed by polynomial bu-tst as well as td-tst.

3.1. Polynomial tst with designated states. In this section let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, F, \mu)$ be a polynomial bu-tst with designated states. First we reprove a slightly less general version of [**58**, Lemma 5.14]. In [**58**], ∞ may not occur as coefficient in $\|M\|_\eta(t)$ for every $t \in T_\Sigma$. However, if ∞ occurs in μ but not in $\|M\|_\eta(t)$, then ∞ can be eliminated from μ. Furthermore, note that height is defined differently in [**58**].

LEMMA 6.21. *Let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, F, \mu)$ be a polynomial bu-tst with designated states and $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta = \{\alpha^{(0)}\}$ such that ∞ does not occur as coefficient in any tree series of μ. There exists a $b \in \mathbb{N}$ such that $(\|M\|_{\mathrm{o}}(t), \alpha) \leqslant b^{\mathrm{height}(t)+1}$ for every $t \in T_\Sigma$.*

PROOF. We can instantiate Corollary 6.9, because $\mathbb{N}_\infty$ is partially ordered by the total order $\leqslant$. Moreover, we trivially have $1 \leqslant 1 + 1$. The constants of Definition 6.2 are: $\mathrm{mx}_\Sigma = 2$, $e_M \leqslant \mathrm{card}(T_\Delta(\mathrm{Z}_2)) = 3$, and $u_{M,\mathrm{o}} \leqslant 1$. Finally, an upper bound $c \in \mathbb{N}$ of the coefficients of μ with $c \geqslant 1$ clearly exists.

We choose the cardinality majorization $l \colon \mathbb{N} \longrightarrow \mathbb{N}_+$ with $l(n) = 1$ for every $n \in \mathbb{N}$, which is a cardinality majorization due to $\mathrm{card}(T_\Delta) = 1$. Hence

$$\begin{aligned} f^{\perp}_{M,\mathrm{o},l,c}(0) &= c \\ f^{\perp}_{M,\mathrm{o},l,c}(n) &= d_M^2 \cdot e_M \cdot c \cdot f^{\perp}_{M,\mathrm{o},l,c}(n-1)^{u_{M,\mathrm{o}}} \end{aligned}$$

for every $n \in \mathbb{N}_+$ and thus $f^{\perp}_{M,\mathrm{o},l,c}(n) \leqslant (3 \cdot d_M^2 \cdot c)^n \cdot c$. Because $c \neq \infty$, we have $b = 3 \cdot d_M^3 \cdot c \neq \infty$, and we obtain $(\|M\|_{\mathrm{o}}(t), \alpha) \leqslant b^{\mathrm{height}(t)+1}$ by Corollary 6.9. □

Similarly we can prove a variant of [**58**, Lemma 5.16].

LEMMA 6.22. *Let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, F, \mu)$ be a polynomial bu-tst with designated states such that $\mathrm{mx}_\Sigma = 1$ and ∞ does not occur as coefficient in any tree series of μ. Then there exists a $b \in \mathbb{N}$ such that $(\|M\|_\varepsilon(t), u) \leqslant b^{(\mathrm{height}(t)+1)^2}$ for every $t \in T_\Sigma$ and $u \in T_\Delta$.*

PROOF. Corollary 6.9 is applicable, because $\mathbb{N}_\infty$ fulfills the general restrictions imposed on the semiring. Obviously, the constants of Definition 6.2 are: $\mathrm{mx}_\Sigma = 1$ and $u_{M,\varepsilon} = 1$. Clearly, there exists an upper bound $c \in \mathbb{N}$ of the coefficients of μ with the property that $c \geqslant 1$. The ample cardinality majorization $l = l_M^\perp$ associated with M of Definition 6.11 is a cardinality majorization due to Lemma 6.12. We have

$$l(n) = d_M^n \cdot e_M^{n+1}$$

for every $n \in \mathbb{N}$. We obtain

$$\begin{aligned} f^\perp_{M,\varepsilon,l,c}(0) &= c \\ f^\perp_{M,\varepsilon,l,c}(n) &= d_M^n \cdot e_M^{n+1} \cdot c \cdot f^\perp_{M,\varepsilon,l,c}(n-1) \end{aligned}$$

for every $n \in \mathbb{N}_+$ and thus

$$f^\perp_{M,\varepsilon,l,c}(n) = (d_M)^{\sum_{i\in[1,n]} i} \cdot (e_M)^{\sum_{i\in[2,n+1]} i} \cdot c^{n+1} \leqslant (d_M \cdot e_M \cdot c)^{\frac{(n+1)\cdot(n+2)}{2}},$$

which implies the required bound by setting $b = d_M^2 \cdot e_M \cdot c_M$ as follows. Since

$$d_M \cdot (d_M \cdot e_M \cdot c)^{\frac{(n+1)\cdot(n+2)}{2}} \leq b^{(n+1)^2},$$

we obtain $(\|M\|_\varepsilon(t), u) \leqslant b^{(\mathrm{height}(t)+1)^2}$ by Corollary 6.9. □

The result in [**58**, Corollary 5.18] is proved using essentially Lemmata 6.21 and 6.22 together with some examples required to show incomparability.

COROLLARY 6.23 (see [**58**, Corollary 5.18]).

$$\text{p–BOT}_\varepsilon(\mathbb{N}_\infty) \bowtie \text{p–BOT}_\mathrm{o}(\mathbb{N}_\infty)$$

Using the same approach we can also reprove [**58**, Lemmata 5.19 and 5.21]. They are used to prove [**58**, Corollary 5.23], which essentially states the above for the tropical semiring $\mathbb{T}$.

3.2. Additively idempotent semirings. Next let us consider additively idempotent semirings. Certainly, such semirings fulfill the inequality $1 \leqslant 1 + 1$ irrespective of the partial order $\leqslant$. Moreover, additively idempotent semirings are partially ordered by their natural order $\sqsubseteq$.

Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a (nontrivial) polynomial tst. The following theorem shows that, provided that $\mathcal{A}$ is additively idempotent, a very simple mapping, called coefficient approximation, is a coefficient majorization. Let us first define the coefficient approximation.

DEFINITION 6.24. *For every $a \in A$ and $y \in \mathbb{N}$ we define the* coefficient approximation *$f_{a,y} \colon \mathbb{N} \longrightarrow A$ by $f_{a,y}(n) = a^{\sum_{i\in[0,n]} y^i}$ for every $n \in \mathbb{N}$.*

THEOREM 6.25. *Let $\eta \in \{\varepsilon, \mathrm{o}\}$, and let $\mathcal{A}$ be an additively idempotent semiring partially ordered by $\leqslant$. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial tst, and let c be an upper bound of the coefficients of M such that $c \geqslant 1$.*

(1) *The coefficient approximation* $f_{c,y} \colon \mathbb{N} \longrightarrow A$ *with* $y = u_{M,\eta}$ *is a coefficient majorization.*
(2) *Moreover* $(\|M\|_\eta(t), u) \leqslant c \cdot f_{c,y}(n)$ *for every* $n \in \mathbb{N}$, $t \in T_\Sigma$ *of height at most* n, *and* $u \in \operatorname{supp}(\|M\|_\eta(t))$.
(3) *If* M *has designated states, then* $(\|M\|_\eta(t), u) \leqslant f_{c,y}(n)$ *for every* $n \in \mathbb{N}$, $t \in T_\Sigma$ *of height at most* n, *and* $u \in \operatorname{supp}(\|M\|_\eta(t))$.

PROOF. To show that $f_{c,y}$ is a coefficient majorization, we show that $f_{c,y}$ is equal to the ample coefficient majorization $f^{\perp}_{M,\eta,l,c}$ or $f^{\top}_{M,\eta,l,c}$ (depending on whether M is bottom-up or not) for any cardinality majorization l; *e. g.*, we could set $l = l^{\perp}_M$ if M is bottom-up, and $l = l^{\top}_M$ otherwise.

We continue to show that $f_{c,y} = h$, where $h = f^{\perp}_{M,\eta,l,c}$ if M is bottom-up, and $h = f^{\top}_{M,\eta,l,c}$ otherwise. Obviously, $h(0) = c = f_{c,y}(0)$ and otherwise

$$
\begin{aligned}
& h(n) \\
= \quad & \textit{(by definition of } h\textit{; let } i = j = \mathrm{mx}_\Sigma \textit{ if } M \textit{ is bottom-up} \\
& \quad \textit{otherwise } i = 1 + u_{M,\varepsilon} \textit{ and } j = u_{M,\varepsilon}\textit{)} \\
& \sum_{m \in [d^i_M \cdot e_M \cdot l(n-1)^j]} c \cdot h(n-1)^y \\
= \quad & \textit{(because } \mathcal{A} \textit{ is additively idempotent)} \\
& c \cdot h(n-1)^y \\
= \; & c^{\sum_{i \in [0,n]} y^i} \\
= \; & f_{c,y}(n)
\end{aligned}
$$

for every $n \in \mathbb{N}_+$. Thus $h = f_{c,y}$ and by Theorems 6.8 and 6.15 it follows that $f_{c,y}$ is a coefficient majorization. Next we show the second statement of the theorem.

$$
\begin{aligned}
& (\|M\|_\eta(t), u) \\
\leqslant \quad & \textit{(by Theorems 6.8 and 6.15)} \\
& \sum_{i \in [\mathrm{card}(Q) \cdot e'_M \cdot l(n)]} c \cdot f_{c,y}(n) \\
= \quad & \textit{(because } \mathcal{A} \textit{ is additively idempotent and} \\
& \quad e'_M \neq 0 \textit{ by } u \in \operatorname{supp}(\|M\|_\eta(t))\textit{)} \\
& c \cdot f_{c,y}(n)
\end{aligned}
$$

It remains to show the third statement of the theorem.

$$
\begin{aligned}
& (\|M\|_\eta(t), u) \\
\leqslant \quad & \textit{(by Corollaries 6.9 and 6.16)} \\
& \sum_{i \in [\mathrm{card}(Q)]} f_{c,y}(n) \\
= \quad & \textit{(because } \mathcal{A} \textit{ is additively idempotent)} \\
& f_{c,y}(n)
\end{aligned}
$$

□

The following observation shows that $f_{a,y}(n) \leqslant f_{a,y'}(n)$ whenever $y \leqslant y'$. This allows us to use an upper bound of the parameter y in order to obtain an upper bound of the coefficient of an output tree.

OBSERVATION 6.26. *Let $a \in A$ with $a \geqslant 1$ and $y, y' \in \mathbb{N}_+$ and $n \in \mathbb{N}$ with $y \leqslant y'$. Then $f_{a,y}(n) \leqslant f_{a,y'}(n)$.*

PROOF. This statement is immediate from Observation 6.1(3). □

Next we establish that the coefficient approximation for polynomial tst over additively idempotent semirings (*i. e.*, in the case when it is a coefficient majorization according to Theorem 6.25) gives an upper bound that can be reached by a homomorphism bu-tst or td-tst. We use this result in our main incomparability result (see Lemma 6.29).

LEMMA 6.27. *Let $\eta \in \{\varepsilon, \mathrm{o}\}$ and $\Sigma' = \{\gamma^{(1)}, \alpha^{(0)}\}$, $\Delta' = \{\delta^{(2)}, \alpha^{(0)}\}$, and $\Delta'' = \{\alpha^{(0)}\}$. Moreover, let $y \in \mathbb{N}_+$, $c \in A$ with $c \geqslant 1$, and $\Sigma'' = \{\sigma^{(y)}, \alpha^{(0)}\}$. There exists a homomorphism bu-tst or td-tst*

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F', \mu')$$

with properties x such that c is an upper bound of the coefficients of M', $u_{M',\eta} = y$, and for every $n \in \mathbb{N}$ there exist $t \in T_\Sigma$ of height n and $u \in \operatorname{supp}(\|M'\|_\eta(t))$ such that $(\|M'\|_\eta(t), u) = f_{c,y}(n)$, where:

(1) *$\eta = \varepsilon$, $\Sigma = \Sigma''$, $\Delta = \Delta''$, and $x =$ bottom-up;*
(2) *$\eta = \mathrm{o}$, $\Sigma = \Sigma'$, $\Delta = \Delta'$, and $x =$ bottom-up; or*
(3) *$\eta = \varepsilon$, $\Sigma = \Sigma'$, $\Delta = \Delta'$, and $x =$ top-down.*

PROOF. We prove the statements individually. We depict the tst in Figure 1.

(1) Let $\mu'_0(\alpha)_\star = c\ \alpha$ and $\mu'_y(\sigma)_{\star,\star\cdots\star} = c\ \alpha$ and $F'_\star = 1\ \mathrm{z}_1$. Note that $u_{M',\varepsilon} = y$. Moreover, let $t \in T_{\Sigma''}$ be the fully balanced tree of height $n \in \mathbb{N}$. A straightforward structural induction shows that $(\|M'\|_\varepsilon(t), \alpha) = f_{c,y}(n)$ as follows. The induction base is $(\|M'\|_\varepsilon(\alpha), \alpha) = c = f_{c,y}(0)$. In the induction step we have, for every $n \in \mathbb{N}_+$ and $t = \sigma(t', \dots, t')$ with $t' \in T_{\Sigma''}$ being a fully balanced tree of height $n-1$,

$$\begin{aligned}
& (\|M'\|_\varepsilon\big(\sigma(t', \dots, t')\big), \alpha) \\
= \quad & \textit{(by Observation 4.23)} \\
& (h^\varepsilon_{\mu'}(\sigma(t', \dots, t'))_\star, \alpha) \\
= \quad & \textit{(by Definition 4.7(1))} \\
& (\mu'_y(\sigma)_{\star,\star\cdots\star}, \alpha) \cdot \prod_{i \in [y]} (h_{\mu'}(t')_\star, \alpha) \\
= \quad & \textit{(by Observation 4.23)} \\
& c \cdot (\|M'\|_\varepsilon(t'), \alpha)^y \\
= \quad & \textit{(by induction hypothesis)} \\
& c \cdot f_{c,y}(n-1)^y
\end{aligned}$$

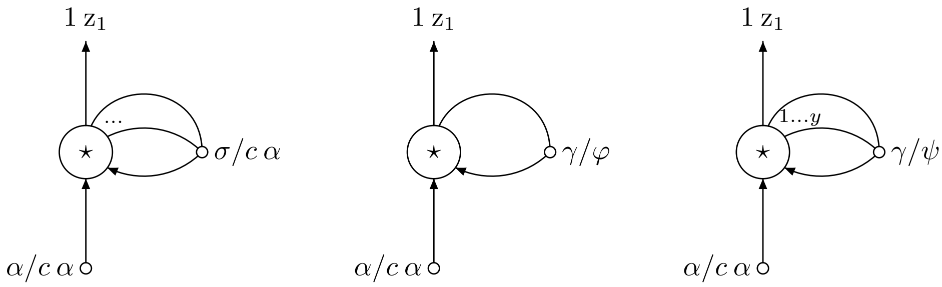

FIGURE 1. Tst over $\mathcal{A}$ of Lemma 6.27 [left: Item (1), middle: Item (2), right: Item(3)], where $\varphi = c\ \delta(\mathrm{z}_1, \delta(\dots, \delta(\mathrm{z}_1, \alpha)\dots))$ and $\psi = c\,\delta(\mathrm{z}_1, \delta(\dots, \delta(\mathrm{z}_y, \alpha)\dots))$.

$$= \quad \text{(by Definition 6.24)}$$
$$f_{c,y}(n) \ .$$

(2) Let $\mu'_0(\alpha)_\star = c\,\alpha$ and $\mu'_1(\gamma)_{\star,\star} = c\,\delta(\mathrm{z}_1, \delta(\dots, \delta(\mathrm{z}_1, \alpha)\dots))$ such that z_1 occurs y times in the latter tree. Moreover, let $F'_\star = 1\,\mathrm{z}_1$. Clearly, $u_{M',\mathrm{o}} = y$. Moreover, one can easily show by a similar induction as in Item (1) that for every $t \in T_{\Sigma'}$ of height $n \in \mathbb{N}$ there exists $u \in T_{\Delta'}$ such that $(\|M'\|_{\mathrm{o}}(t), u) = f_{c,y}(n)$.

(3) Let $\mu'_0(\alpha)_\star = c\alpha$ and $\mu'_1(\gamma)_{\star,\star(\mathrm{x}_1)\cdots\star(\mathrm{x}_1)} = c\delta(\mathrm{z}_1, \delta(\dots, \delta(\mathrm{z}_y, \alpha)\dots))$. Moreover, let $F'_\star = 1\,\mathrm{z}_1$. Clearly $u_{M',\varepsilon} = y$ and the proof of (3) is analogous to the previous ones and omitted. □

The main theorem states the incomparability of the classes of η-t-ts transformations computed by polynomial bu-tst for $\eta = \varepsilon$ and $\eta = \mathrm{o}$ over an additively idempotent semiring $\mathcal{A}$ with an additional property, which we introduce next. Roughly speaking, we require that $\mathcal{A}$ is partially ordered by a partial order $\leqslant$ such that for some $a \in A$ we have $a^i < a^j$ whenever $i < j$. Moreover, we require that every element that occurs in a decomposition of a^n can be bounded from above by a power of a.

DEFINITION 6.28. *A semiring $\mathcal{A} = (A, +, \cdot)$ that is partially ordered by the partial order $\leqslant$ is called* weakly growing, *if:*

(1) *there exists an $a \in A$ such that $a^i < a^j$ for all $i, j \in \mathbb{N}$ with $i < j$; and*
(2) *for every $a_1, a_2, b \in A_+$, $d \in A$, and $n \in \mathbb{N}$, if $a^n = a_1 \cdot b \cdot a_2 + d$, then there exists an $m \in \mathbb{N}$ such that $b \leqslant a^m$.*

The first condition ensures that $a^0 < a^1 < a^2 < \cdots$. The second condition intuitively requires that the growth is not too slow; *i. e.*, we should at least be able to bound (from above) elements that occur in decompositions. Stronger conditions than (2) can be obtained, for example, by requiring that whenever $b \neq 0$, then $b \leqslant a^m$ for some $m \in \mathbb{N}$, which is an Archimedian type property for the element a.

This would essentially state that the growth of a is unbounded; *i. e.*, $\uparrow\{a^n \mid n \in \mathbb{N}\} = \emptyset$. Certainly, the natural numbers (without infinity) $\mathbb{N} = (\mathbb{N}, +, \cdot)$ fulfill this property for $a = 2$ as well as $\mathbb{A}$ does for $a = 1$. However, already $\mathbb{N}_\infty$ does not satisfy it.

Another strong notion of growth can be obtained by requiring that (i) $\mathcal{A}$ is naturally ordered, (ii) $a^i \sqsubset a^j$ whenever $i < j$, and (iii) $a \sqsubseteq a \cdot b$ and $a \sqsubseteq b \cdot a$ for every $a, b \in A$ with $b \neq 0$. The semirings $\mathbb{N}$, $\mathbb{N}_\infty$, and $\mathbb{A}$ fulfill this property, but $\mathbb{T}$ does not. For our incomparability results we only need the weakly growing property.

The following semirings are weakly growing:

- $\mathbb{N}_\infty$ with the partial order $\leqslant$, $a = 2$, and $m = n$;
- $\mathbb{T}$ with the partial order $\leqslant$, $a = 1$, and $m = \max(n, d)$;
- $\mathbb{A}$ with the partial order $\leqslant$, $a = 1$, and $m = n$;
- $\mathbb{L}_S$ (S an alphabet) with the partial order $\subseteq$, $a = \{\varepsilon, s\}$ for some $s \in S$, and $m = n$.

The above statements are easily checked. On the other hand, the semirings $\mathbb{B}$ and $\mathbb{R}_{\min,\max}$ are not weakly growing because they are multiplicatively periodic.

Next we show that given an additively idempotent and weakly growing semiring, the classes of ε-t-ts and o-t-ts transformations computed by polynomial bu-tst are incomparable. Moreover, we also obtain the incomparability of the class of ε-t-ts transformations computed by polynomial bu-tst and the class of ε-t-ts transformations computed by polynomial td-tst.

Before stating the incomparability theorem, we provide a sketch of the proof. Informally speaking, we show both directions by constructing a specific homomorphism tst M' using the particular coefficient $a \in A$ that fulfills the conditions of Definition 6.28. The approximation mapping can be applied to every polynomial tst M that is supposed to compute the same η-t-ts transformation. By a careful choice of the input and output ranked alphabets we limit the constant $u_{M,\eta}$. We then proceed to show that M' has a higher growth rate than M. This growth argument yields the desired contradiction.

LEMMA 6.29. *Let $\mathcal{A}$ be a weakly growing and additively idempotent semiring.*

(1) $\text{h–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–BOT}_\text{o}(\mathcal{A})$ *and* $\text{h–BOT}_\text{o}(\mathcal{A}) \not\subseteq \text{p–BOT}_\varepsilon(\mathcal{A})$.
(2) $\text{h–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–TOP}_\varepsilon(\mathcal{A})$ *and* $\text{h–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–BOT}_\varepsilon(\mathcal{A})$.

PROOF. Let $\mathcal{A}$ be weakly growing with respect to the partial order $\leqslant$ and the element $a \in A$ (see Definition 6.28).

(i) First we prove

$$\text{h–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–BOT}_\text{o}(\mathcal{A}) \quad \text{and} \quad \text{h–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–TOP}_\varepsilon(\mathcal{A}) \ .$$

We consider the ranked alphabets $\Sigma'' = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta'' = \{\alpha^{(0)}\}$ as input and output ranked alphabet, respectively. Then by Lemma 6.27(1) [with $y = 2$] there is a homomorphism bu-tst $M' = (\{\star\}, \Sigma'', \Delta'', \mathcal{A}, F', \mu')$ such that a is an upper bound of the coefficients of M', $u_{M',\varepsilon} = \mathrm{mx}_{\Sigma''} = 2$, and for every $n \in \mathbb{N}$ there exist $t \in T_{\Sigma''}$ of height n and $u \in \mathrm{supp}(\|M'\|_\varepsilon(t))$ such that

$$(\|M'\|_\varepsilon(t), u) = f_{a,2}(n) = a^{2^{n+1}-1} \quad .$$

Assume that there exists a polynomial bu-tst or td-tst

$$M = (Q, \Sigma'', \Delta'', \mathcal{A}, F, \mu)$$

with $\|M\|_\mathrm{o} = \|M'\|_\varepsilon$. Since M is polynomial, there are only finitely many nonzero coefficients $c_1, \dots, c_k \in A$ for some $k \in \mathbb{N}$ that occur in the tree series in the range of μ. Obviously, we can assume that for every c_j with $j \in [k]$ there exist $a_j, \bar{a}_j \in A_+$, $b_j \in A$, and $m_j \in \mathbb{N}$ such that $a^{m_j} = a_j \cdot c_j \cdot \bar{a}_j + b_j$. If there is a c_j that does not obey this property, then it cannot influence $\|M\|_\mathrm{o}$ (see Definition 4.7), because $\|M\|_\mathrm{o} = \|M'\|_\varepsilon$ and every coefficient that appears in a tree series in the range of $\|M'\|_\varepsilon$ is a power of a. Thus, such coefficients c_j can be changed in μ to 1 without effect on $\|M\|_\mathrm{o}$.

Since $\mathcal{A}$ is weakly growing with respect to a, there is an $e_j \in \mathbb{N}$ such that $c_j \leqslant a^{e_j}$. Consequently, $\max_{i \in [k]} a^{e_i} = a^{\max_{i \in [k]} e_i}$ is an upper bound of the coefficients of μ. Let $e = \max_{i \in [k]} e_i$ and $c' = a^e$. By Theorem 6.25 and Observation 6.26, for every $t \in T_{\Sigma''}$ and every $u \in \mathrm{supp}(\|M\|_\mathrm{o}(t))$,

$$(\|M\|_\mathrm{o}(t), u) \leqslant c' \cdot f_{c',1}(\mathrm{height}(t)) = (c')^{\mathrm{height}(t)+2} = (a^e)^{\mathrm{height}(t)+2} \quad ,$$

because $u_{M,\mathrm{o}} \leqslant 1$ due to the specific form of Δ''. However, there exists an $n' \in \mathbb{N}$ such that $e \cdot (n' + 2) < 2^{n'+1} - 1$. With this height n' there also exist $t' \in T_{\Sigma''}$ and $u' \in \mathrm{supp}(\|M'\|_\varepsilon(t'))$ such that

$$(\|M'\|_\varepsilon(t'), u') = f_{a,2}(n') = a^{2^{n'+1}-1} \quad ,$$

whereas $(\|M\|_\mathrm{o}(t'), u') \leqslant a^{e \cdot (n'+2)}$ and $a^{e \cdot (n'+2)} < a^{2^{n'+1}-1}$ (because $\mathcal{A}$ is weakly growing with respect to a), which yields a contradiction to the assumption that $\|M\|_\mathrm{o} = \|M'\|_\varepsilon$. Hence, $\|M'\|_\varepsilon$ is neither in p–$\mathrm{BOT}_\mathrm{o}(\mathcal{A})$ nor in p–$\mathrm{TOP}_\varepsilon(\mathcal{A})$.

(ii) The statements

$$\text{h–BOT}_\mathrm{o}(\mathcal{A}) \not\subseteq \text{p–BOT}_\varepsilon(\mathcal{A}) \quad \text{and} \quad \text{h–TOP}_\varepsilon(\mathcal{A}) \not\subseteq \text{p–BOT}_\varepsilon(\mathcal{A})$$

are established using the input ranked alphabet $\Sigma' = \{\gamma^{(1)}, \alpha^{(0)}\}$ and output ranked alphabet $\Delta' = \{\delta^{(2)}, \alpha^{(0)}\}$. By Lemma 6.27(2) there is a homomorphism bu-tst M' such that a is an upper bound of the coefficients of M' and $u_{M',\mathrm{o}} = 2$ and by Lemma 6.27(3) there is a homomorphism td-tst M'' such that a is an upper bound of the coefficients of M'' and $u_{M'',\varepsilon} = 2$. Moreover, for every $n \in \mathbb{N}$ there exist $t, t' \in T_{\Sigma'}$ of height n and $u \in \mathrm{supp}(\|M'\|_\mathrm{o}(t))$ and $u' \in \mathrm{supp}(\|M''\|_\varepsilon(t'))$ such that

$$(\|M'\|_\mathrm{o}(t), u) = f_{a,2}(n) = a^{2^{n+1}-1}$$

$$(\|M''\|_\varepsilon(t'), u') = f_{a,2}(n) = a^{2^{n+1}-1} \quad .$$

Let $M = (Q, \Sigma', \Delta', \mathcal{A}, F, \mu)$ be a polynomial bu-tst with $\|M\|_\varepsilon = \|M'\|_\mathrm{o}$. An argumentation analogous to the one in the first part of the proof (using $u_{M,\varepsilon} = \mathrm{mx}_{\Sigma'} = 1$) shows that $(\|M\|_\varepsilon(t), u) \leqslant (c')^{\mathrm{height}(t)+2}$ for every $t \in T_{\Sigma'}$

and $u \in \mathrm{supp}(\|M\|_\varepsilon(t))$, where $c' = a^e$ for some $e \in \mathbb{N}$. This again yields the desired contradiction. $\square$

THEOREM 6.30. *Let $\mathcal{A}$ be a weakly growing and additively idempotent semiring.*

$$\text{p–BOT}_\varepsilon(\mathcal{A}) \bowtie \text{p–BOT}_\mathrm{o}(\mathcal{A}) \quad \textit{and} \quad \text{p–BOT}_\varepsilon(\mathcal{A}) \bowtie \text{p–TOP}_\varepsilon(\mathcal{A}) \ .$$

PROOF. The theorem is an immediate consequence of Lemma 6.29. $\square$

Consider the additively idempotent semirings $\mathbb{T}$, $\mathbb{A}$, and $\mathbb{L}_S$. For those we derive the following statements.

COROLLARY 6.31. *Let S be an alphabet. For every $\mathcal{A} \in \{\mathbb{T}, \mathbb{A}, \mathbb{L}_S\}$*

$$\text{p–BOT}_\varepsilon(\mathcal{A}) \bowtie \text{p–BOT}_\mathrm{o}(\mathcal{A}) \quad \textit{and} \quad \text{p–BOT}_\varepsilon(\mathcal{A}) \bowtie \text{p–TOP}_\varepsilon(\mathcal{A}) \ .$$

PROOF. Both results are immediate consequences of Theorem 6.30. $\square$

Note that for $\mathcal{A} = \mathbb{T}$ the first part of Corollary 6.31 restates [**58**, Corollary 5.23].

4. Open problems and future work

We derived our incomparability results for additively idempotent semirings. Such semirings are naturally ordered and in fact we use the order to prove incomparability. It remains an open problem to prove a general incomparability result for semirings that are not additively idempotent (and not even partially ordered). For example, all rings are not naturally ordered. A potential start could be the use of finiteness and closure instead of the use of a partial order. If we consider the specific tst, which are used in Lemma 6.29, then the following property could be a suitable starting point.

DEFINITION 6.32. *A semiring $\mathcal{A} = (A, +, \cdot)$ has Property (P), if there exists an $a \in A$ such that for every $m \in \mathbb{N}_+$ and finite $B \subseteq A$ there exists an $n \in \mathbb{N}_+$ such that $a^{2^n - 1} \notin \langle B^{(n)} \rangle_{m^n}$ where for every $C \subseteq A$ and $k \in \mathbb{N}$*

$$C^{(k)} = \{c_1 \cdot \ldots \cdot c_k \mid c_1, \ldots, c_k \in C\}$$
$$\langle C \rangle_k = \{c_1 + \cdots + c_r \mid r \in [0, k], c_1, \ldots, c_r \in C\} \ .$$

For example, the natural number semiring $\mathbb{N}$ has Property (P). To show this let $a = 2$ and $m \in \mathbb{N}_+$ be arbitrary. Moreover, let $B \subseteq \mathbb{N}$ be finite with greatest element $b \in B$. Clearly, the greatest element in $\langle B^{(n)} \rangle_{m^n}$ is $(mb)^n$. It is well-known that there exists an $n \in \mathbb{N}_+$ such that $a^{2^n - 1} > (mb)^n$.

Finally, we should also consider equality or inclusion results, especially in important classes of semirings like additively idempotent semirings. A general study was already started in [**58**], but for restricted classes of semirings results are still missing.

CHAPTER 7

Composition of Tree Series Transducers

It's not what you look at that matters,
it's what you see.
Henry David Thoreau (1817-1862)

1. Bibliographic information

We consider compositions of ε-ts-ts transformations. Given two tst M' and M'', we ask whether there exists a tst M such that

$$\|M\|_{\varepsilon}^{\mathrm{ts}} = \|M'\|_{\varepsilon}^{\mathrm{ts}} \,; \|M''\|_{\varepsilon}^{\mathrm{ts}}$$

where ; denotes functional composition. In particular, we would expect M to have properties similar to those shared by M' and M''. The first section introduces some common terminology and notations. In Section 3 we consider compositions where M, M', and M'' are bottom-up, whereas Section 4 is devoted to top-down devices.

Section 3 is a revised and extended version of [**84**]. In Section 4 we present the results of [**84**] concerning td-tst.

2. General definitions and remarks

In this chapter we study compositions of ε-ts-ts transformations computed by tst. Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ for some sets A, B, and C. The *composition of f with g*, denoted $f\,; g$, is the mapping $(f\,; g)\colon A \longrightarrow C$ such that $(f\,; g)(a) = g(f(a))$ for every $a \in A$. Note that ; is associative.

For the rest of the chapter, let $\mathcal{A} = (A, +, \cdot)$ be a semiring. Provided with tst

$$M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu') \quad \text{and} \quad M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'') \ ,$$

we investigate whether there exists a tst $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ such that $\|M\|_{\varepsilon}^{\mathrm{ts}} = \|M'\|_{\varepsilon}^{\mathrm{ts}} \,; \|M''\|_{\varepsilon}^{\mathrm{ts}}$. Note that in the last equation we refer to the η-ts-ts transformation computed by the tst [see Definition 4.7(3)] and ; denotes functional composition; *i. e.*, rephrased the equation is $\|M\|_{\varepsilon}^{\mathrm{ts}}(\psi) = \|M''\|_{\varepsilon}^{\mathrm{ts}}(\|M'\|_{\varepsilon}^{\mathrm{ts}}(\psi))$ for every $\psi \in \mathcal{A}\langle\!\langle T_{\Sigma} \rangle\!\rangle$.

Since our introduced notation focusses on the ε-t-ts transformation computed by tst, we define the following composition of ε-t-ts transformations. Let Σ, Γ, and Δ be ranked alphabets and $\mathcal{A}$ be a semiring.

Moreover, let $\tau_1 \colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Gamma \rangle\rangle$ and $\tau_2 \colon T_\Gamma \longrightarrow \mathcal{A}\langle\langle T_\Delta \rangle\rangle$. We define the *composition of* τ_1 *and* τ_2, denoted by $\tau_1 \,;\, \tau_2$, for every $t \in T_\Sigma$ by

$$(\tau_1 \,;\, \tau_2)(t) = \sum_{u \in T_\Gamma} (\tau_1(t), u) \cdot \tau_2(u) \enspace .$$

Essentially, this means that $\|M'\|_\varepsilon \,;\, \|M''\|_\varepsilon = \|M'\|_\varepsilon \,;\, \|M''\|_\varepsilon^{\mathrm{ts}}$ where in the left hand side ; is the newly introduced composition and in the right hand side ; is ordinary function composition. This newly defined composition is extended to classes of ε-t-ts transformations in the obvious manner. Thus we may henceforth write $\mathrm{nl–BOT}_\varepsilon(\mathcal{A}) \,;\, \mathrm{nl–BOT}_\varepsilon(\mathcal{A})$ and thereby avoid the introduction of a special denotation for classes of ε-ts-ts transformations. By this we also avoid a well-definedness problem. Let M' be polynomial. In fact, $\|M'\|_\varepsilon^{\mathrm{ts}}(\psi)$ need not be well-defined for every $\psi \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$, but $\|M'\|_\varepsilon^{\mathrm{ts}}(\psi)$ is well-defined if ψ is polynomial. From [**55**, Proposition 3.4] (in conjunction with [**55**, Proposition 3.7]) we can easily conclude that $\|M'\|_\varepsilon(t)$ is polynomial for every $t \in T_\Sigma$ (the proof essentially uses the first statement of Observation 3.6). Thus the composition $\|M'\|_\varepsilon \,;\, \|M''\|_\varepsilon$ is well-defined, whenever M' and M'' are polynomial.

Actually, there are ε-t-ts transformations that behave neutral with respect to our composition of ε-t-ts transformations. Suppose that $\tau \colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta \rangle\rangle$. For every ranked alphabet Γ let $\tau_\Gamma \colon T_\Gamma \longrightarrow \mathcal{A}\langle\langle T_\Gamma \rangle\rangle$ be such that $\tau_\Gamma(t) = 1\; t$ for every $t \in T_\Gamma$. We easily check that $\tau_\Sigma \,;\, \tau = \tau = \tau \,;\, \tau_\Delta$. The following observation shows that every introduced class of η-t-ts transformations contains at least τ_Γ for every ranked alphabet Γ. This fact deserves mention, but in the sequel we apply the observation freely without explicit mention.

Observation 7.1. *Let* Γ *be a ranked alphabet and* $x \in \Pi$:

$$\tau_\Gamma \in x\text{–}\mathrm{BOT}_\varepsilon(\mathcal{A}) \cap x\text{–}\mathrm{TOP}_\varepsilon(\mathcal{A}) \enspace . \tag{93}$$

Proof. The construction of a tst with those properties is straightforward and hence omitted. □

Finally, $\|M\|_\varepsilon = \|M'\|_\varepsilon \,;\, \|M''\|_\varepsilon$ trivially implies that

$$\|M\|_\varepsilon^{\mathrm{ts}} = \|M'\|_\varepsilon^{\mathrm{ts}} \,;\, \|M''\|_\varepsilon^{\mathrm{ts}}$$

(see [**41**, Lemma 2.14]). Thus all the results of this chapter (which are for classes of ε-t-ts transformations) automatically yield the corresponding results for ε-ts-ts transformations.

Moreover, we mostly prove our results for classes of transformations computed by polynomial tst. In the theorems we also state the corresponding results for non-polynomial tst with the additional condition that the underlying semiring is $\aleph_0$-complete with respect to $\sum$. This condition implies that the semantics of bu-tst is well-defined in general (see Observation 4.14). However, the proofs for those statements are absolutely analogous to the polynomial case and hence we generally omit them. Note that some proofs are no longer constructive once we consider non-polynomial tst.

3. Bottom-up tree series transducers

First let us review what is known about compositions of classes of ε-t-ts transformations computed by bu-tst. The class of ε-t-ts transformations computed by polynomial bu-tst over the boolean semiring (*i. e.*, bottom-up tree transducers [**41**, Section 4]) is closed under left-composition with the class of ε-t-ts transformations computed by linear polynomial bu-tst over $\mathbb{B}$ (see [**5**, Theorem 6] and [**35**, Theorem 4.5]); *i. e.*

$$\text{lp–BOT}_\varepsilon(\mathbb{B})\,;\text{p–BOT}_\varepsilon(\mathbb{B}) = \text{p–BOT}_\varepsilon(\mathbb{B}) \enspace .$$

This composition result was generalized to bu-tst over commutative and $\aleph_0$-complete semirings in [**78, 41**]. More precisely, [**78**, Theorem 2.4] yields that

$$\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{nlp–BOT}_\varepsilon(\mathcal{A}) = \text{nlp–BOT}_\varepsilon(\mathcal{A}) \enspace .$$

In fact it is shown for nondeleting, linear td-tst [**41**], but nondeleting, linear td-tst and nondeleting, linear bu-tst are equally powerful (see [**41**, Theorem 5.24] and Proposition 4.21). Moreover, the statement is shown for continuous semirings in [**78**], but can easily be shown for $\aleph_0$-complete semirings. Finally, the construction of [**78**] preserves the polynomial property, and when only polynomial bu-tst are considered, then the semiring need not be $\aleph_0$-complete.

In [**41**, Corollary 5.5] it is shown that

$$\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{h–BOT}_\varepsilon(\mathcal{A}) \subseteq \text{p–BOT}_\varepsilon(\mathcal{A}) \enspace .$$

So taking those results together and the decomposition [**41**, Lemma 5.6]

$$\text{p–BOT}_\varepsilon(\mathcal{A}) \subseteq \text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{h–BOT}_\varepsilon(\mathcal{A}) \enspace ,$$

we obtain the following result.

THEOREM 7.2. *For every commutative semiring $\mathcal{A}$*

$$\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{p–BOT}_\varepsilon(\mathcal{A}) = \text{p–BOT}_\varepsilon(\mathcal{A}) \enspace . \tag{94}$$

PROOF. The part $\text{p–BOT}_\varepsilon(\mathcal{A}) \subseteq \text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{p–BOT}_\varepsilon(\mathcal{A})$ is trivial (by Observation 7.1), so it remains to prove

$$\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{p–BOT}_\varepsilon(\mathcal{A}) \subseteq \text{p–BOT}_\varepsilon(\mathcal{A}) \enspace .$$

$$\begin{aligned}
&\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{p–BOT}_\varepsilon(\mathcal{A}) && \\
&\subseteq \text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{h–BOT}_\varepsilon(\mathcal{A}) && [\mathbf{41}, \text{Lemma 5.6}] \\
&\subseteq \text{nlp–BOT}_\varepsilon(\mathcal{A})\,;\text{h–BOT}_\varepsilon(\mathcal{A}) && [\mathbf{78}, \text{Theorem 2.4}] \\
&\subseteq \text{p–BOT}_\varepsilon(\mathcal{A}) && [\mathbf{41}, \text{Corollary 5.5}] \quad \square
\end{aligned}$$

Our aim is a result like $\text{lp–BOT}_\varepsilon(\mathcal{A})\,;\text{p–BOT}_\varepsilon(\mathcal{A}) = \text{p–BOT}_\varepsilon(\mathcal{A})$ for all commutative semirings $\mathcal{A}$. We try to follow the classical (unweighted) construction, so we first extend h_μ^ε such that it can treat variables (of Z). We extend h_μ^ε to $T_\Sigma(\mathrm{Z})$ by supplying, for some $J \subseteq \mathbb{N}_+$, a mapping $\overline{q} \in Q^J$, which associates a state $\overline{q}(j)$, usually written as $\overline{q}_j$, to the variable z_j for $j \in J$. Intuitively speaking, the state $\overline{q}_j$ represents the initial state, with which the computation should be started at the leaves labeled z_j in the input tree. For all states $q \in Q$ different from $\overline{q}_j$ it should not be possible to start a (meaningful) computation at z_j (*i. e.*, $h_{\mu,\overline{q}}^\varepsilon(\mathrm{z}_j)_q = \widetilde{0}$). This mapping is then extended to $T_\Sigma(\mathrm{Z})$ in a manner analogous to h_μ^ε.

Definition 7.3. *Let $(Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial bu-tst. For every finite $J \subseteq \mathbb{N}_+$ and $\overline{q} \in Q^J$ we define the mapping*

$$h_{\mu,\overline{q}}^\varepsilon \colon T_\Sigma(\mathrm{Z}) \longrightarrow \mathcal{A}\langle\langle T_\Delta(\mathrm{Z})\rangle\rangle^Q$$

inductively for every $q \in Q$ as follows.

- *For every $n \in \mathbb{N}_+$*

$$h_{\mu,\overline{q}}^\varepsilon(\mathrm{z}_n)_q = \begin{cases} 1\,\mathrm{z}_n & \text{if } n \notin J \text{ or } (n \in J \text{ and } q = \overline{q}_n), \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

- *For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma(\mathrm{Z})$*

$$h_{\mu,\overline{q}}^\varepsilon(\sigma(t_1, \dots, t_k))_q = \sum_{q_1,\dots,q_k \in Q} \mu_k(\sigma)_{q,q_1\cdots q_k} \xleftarrow[\varepsilon]{} (h_{\mu,\overline{q}}^\varepsilon(t_i)_{q_i})_{i \in [k]} \ .$$

The mapping $h_{\mu,\overline{q}}^\varepsilon \colon \mathcal{A}\langle T_\Sigma(\mathrm{Z})\rangle \longrightarrow \mathcal{A}\langle T_\Delta(\mathrm{Z})\rangle^Q$ is given for every tree series $\psi \in \mathcal{A}\langle T_\Sigma(\mathrm{Z})\rangle$ by

$$h_{\mu,\overline{q}}^\varepsilon(\psi)_q = \sum_{t \in T_\Sigma(\mathrm{Z})} (\psi, t) \cdot h_{\mu,\overline{q}}^\varepsilon(t)_q \ .$$

Let us illustrate the previous definition on an example.

Example 7.4. *Assume the ranked alphabets $\Sigma = \{\sigma^{(3)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Gamma = \Sigma \cup \{\mathrm{l}^{(1)}, \mathrm{m}^{(1)}, \mathrm{r}^{(1)}\}$. Moreover, let $M_{7.4} = (Q, \Gamma, \Sigma, \mathbb{R}_+, F, \mu)$ with:*

- $Q = \{\top, \mathrm{l}, \mathrm{m}, \mathrm{r}\}$;
- $F_\top = 1\,\mathrm{z}_1$ *and* $F_\gamma = \widetilde{0}$ *for every* $\gamma \in \Gamma_1$; *and*
- *tree representation μ given by:*

$$\mu_0(\alpha)_\gamma = \mu_0(\alpha)_\top = 1\,\alpha \qquad \mu_3(\sigma)_{\mathrm{l},\top\top\top} = 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3$$

$$\mu_0(\beta)_\gamma = \mu_0(\beta)_\top = 1\,\beta \qquad \mu_3(\sigma)_{\mathrm{m},\top\top\top} = 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3$$

$$\mu_1(\gamma)_{\top,\gamma} = 1\,\mathrm{z}_1 \qquad \mu_3(\sigma)_{\mathrm{r},\top\top\top} = 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$$

for every $\gamma \in \Gamma_1$.

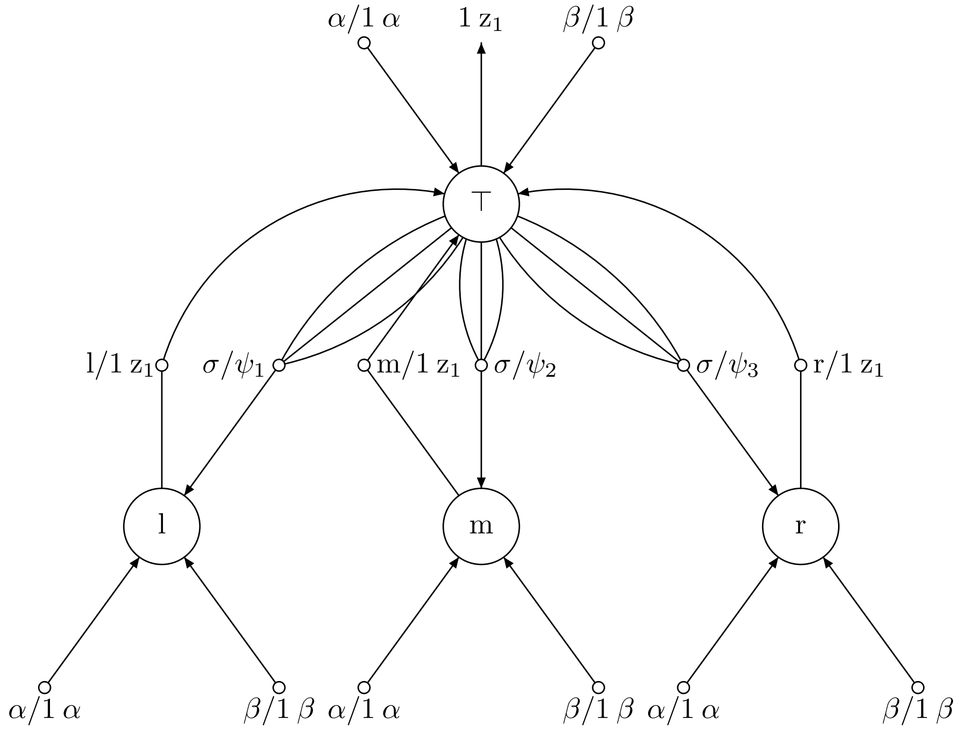

FIGURE 1. Bu-tst $M_{7.4}$ over the semiring $\mathbb{R}_+$ of Example 7.4 where $\psi_1 = 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3$, $\psi_2 = 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3$, and $\psi_3 = 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$.

The bu-tst $M_{7.4}$ is illustrated in Figure 1. Let

$$\psi = 0.1\,\mathrm{l}(\mathrm{z}_1) + 0.3\,\mathrm{m}(\mathrm{z}_2) + 0.6\,\mathrm{r}(\mathrm{z}_3) \ ,$$

and let us compute $h^{\varepsilon}_{\mu,\mathrm{lml}}(\psi)_\top$.

$$h^{\varepsilon}_{\mu,\mathrm{lml}}(\psi)_\top = 0.1 \cdot h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{l}(\mathrm{z}_1))_\top + 0.3 \cdot h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{m}(\mathrm{z}_2))_\top + 0.6 \cdot h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{r}(\mathrm{z}_3))_\top$$

We present only the calculation of the first summand. The other two summands can be calculated similarly.

$$0.1 \cdot h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{l}(\mathrm{z}_1))_\top \ = \ 0.1 \cdot \sum_{p \in Q} \mu_1(\mathrm{l})_{\top,p} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{z}_1)_p)$$
$$= 0.1 \cdot h^{\varepsilon}_{\mu,\mathrm{lml}}(\mathrm{z}_1)_{\mathrm{l}} \ = \ 0.1\,\mathrm{z}_1$$

Altogether we obtain that $h^{\varepsilon}_{\mu,\mathrm{lml}}(\psi)_\top = 0.1\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$.

Let $M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu')$ and $M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$ be polynomial bu-tst. Then, similar to the (unweighted) product construction of bottom-up tree transducers [**5**, p. 199], we translate the entries of μ' with the help of μ''. Let $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $p, p_1, \ldots, p_k \in Q'$, and $q, q_1, \ldots, q_k \in Q''$. Roughly speaking, we obtain the entry

$$\mu_k(\sigma)_{(p,q),(p_1,q_1)\cdots(p_k,q_k)}$$

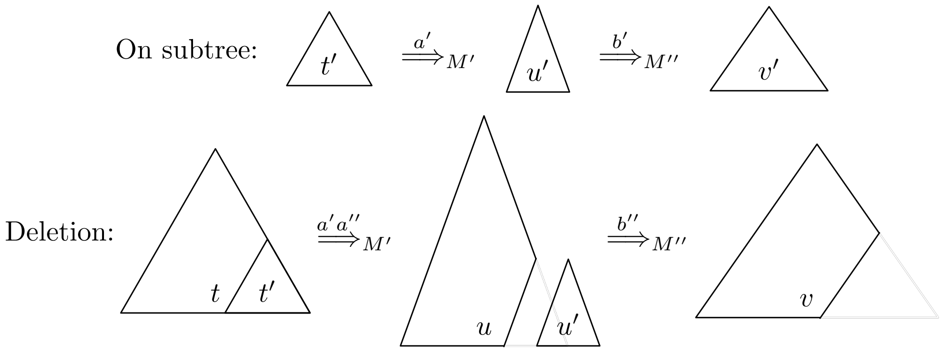

FIGURE 2. Computation of M' followed by M'' (for pure substitution).

in the tree representation μ of the composition of M' and M'' by taking the q-component of the result of the application of the extended mapping $h^{\eta}_{\mu'',q_1\cdots q_k}$ to the entry $\mu'_k(\sigma)_{p,p_1\cdots p_k}$. Thereby, we process the output trees of $\mathrm{supp}(\mu'_k(\sigma)_{p,p_1\cdots p_k})$ with the help of μ'' starting the computation at the variables $\mathrm{z}_1, \ldots, \mathrm{z}_k$ in states $q_1, \ldots, q_k$, respectively. The transition $\mu'_k(\sigma)_{p,p_1\cdots p_k}$ changes into the state p of M' and we consider the q-component of the translation. Consequently, the new transition should change into the state (p, q).

This approach reveals a small problem which does not arise in the unweighted case. We depict the problem in Figures 2 and 3. Let us suppose that M' translates an input tree $t \in T_\Sigma$ into an output tree $u \in T_\Gamma$ with weight $a \in A$. During the translation, M' decides to delete the translation $u' \in T_\Gamma$ with weight $a' \in A$ of an input subtree $t' \in T_\Sigma$. Then due to the definition of pure substitution the weight a' of u' contributes to the weight a of u, whereas u' does not contribute to u. Furthermore, let us suppose that M'' would transform u into $v \in T_\Delta$ at weight $b \in A$ and u' into $v' \in T_\Delta$ at weight $b' \in A$. Since M'' does not process u', the weight b' does not contribute to b. However, the composition of M' and M'', when processing the input subtree t', transforms t' into u' at weight a' using the rules of M' and immediately also transforms u' into v' at weight b' using the rules of M''. If the composition tst now deletes the translation v' of t', then a' and b' still contribute to the weight of the overall transformation. This contrasts the situation encountered when M' and M'' run separately, because there only a' contributed to the weight of the overall transformation.

In the classical case of tree transducers, b' could only be 0 or 1, so that we just have to avoid that $b' = 0$. In principle, this is achieved by requiring M'' to be total (however, by adjoining a dummy state, each bottom-up tree transducer can be turned into a total one computing the same tree transformation). The construction we propose here is similar, but has the major disadvantage that, for example, determinism is not preserved.

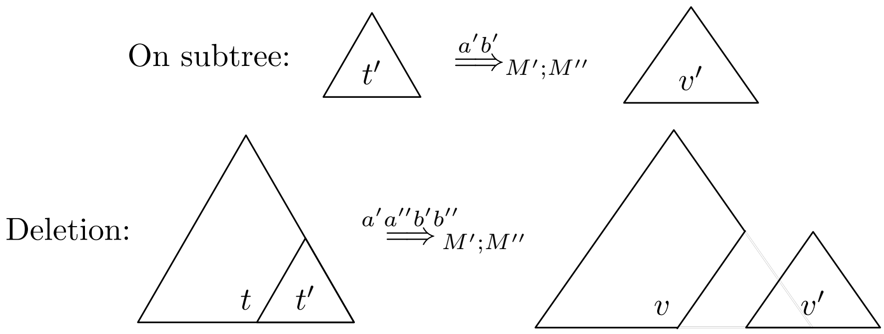

FIGURE 3. Computation of $M'\,;M''$.

Specifically, we address the aforementioned problem by manipulating the second transducer M'' such that it has a state $\perp$ which transforms each input tree into some output tree $\alpha \in \Delta_0$ at weight 1. Note that $\perp$ is no final state; *i. e.*, its top-most output is $\widetilde{0}$. Then in the composition of M' and M'' we process those subtrees, which M' decided to delete, in the state $\perp$. Let us first define a type of state that is suited for this purpose. These states are called blind, because they do not distinguish between different input trees.

DEFINITION 7.5. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial bu-tst. A state $\perp \in Q$ is called* blind, *if there exists an $\alpha \in \Delta_0$ such that:*

- $F_\perp = \widetilde{0}$;
- *for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$ we have $\mu_k(\sigma)_{\perp,\perp\cdots\perp} = 1\,\alpha$; and*
- *for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q_1, \ldots, q_k \in Q$ whenever $\mu_k(\sigma)_{\perp,q_1\cdots q_k} \neq \widetilde{0}$ then $q_i = \perp$ for every $i \in [k]$.*

We already noted that a blind state should transform every input tree into $1\,\alpha$ for some $\alpha \in \Delta_0$. Let us formally prove this property.

OBSERVATION 7.6. *Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial bu-tst with the blind state $\perp$. There exists an $\alpha \in \Delta_0$ such that for every $t \in T_\Sigma$ we have $h^\varepsilon_\mu(t)_\perp = 1\,\alpha$.*

PROOF. Let $\alpha \in \Delta_0$ be such that $\mu_0(\beta)_\perp = 1\,\alpha$ for every $\beta \in \Sigma_0$. We prove the statement inductively, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$
\begin{aligned}
& h^\varepsilon_\mu(\sigma(t_1, \ldots, t_k))_\perp \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{\perp,q_1,\ldots,q_k} \xleftarrow[\varepsilon]{} (h^\varepsilon_\mu(t_i)_{q_i})_{i\in[k]} \\
= \quad & \textit{(by definition of } \mu \textit{ because } \perp \textit{ is blind)} \\
& \mu_k(\sigma)_{\perp,\perp\cdots\perp} \xleftarrow[\varepsilon]{} (h^\varepsilon_\mu(t_i)_\perp)_{i\in[k]} \\
= \quad & \textit{(by induction hypothesis and definition of } \mu\textit{)} \\
& 1\,\alpha \xleftarrow[\varepsilon]{} (1\,\alpha)_{i\in[k]}
\end{aligned}
$$

TABLE 1. Preservation of properties for the construction of Observation 7.7.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✗	✓	✓	✗	✗	✓	✓	✓	✓	✓	✗	✓	✗	✗

$$= \quad \text{(by definition of } \underset{\varepsilon}{\longleftarrow}\text{)}$$
$$1\,\alpha$$
□

Note that in $M_{7.4}$ of Example 7.4 no state is blind. To every polynomial bu-tst M we can adjoin a blind state $\perp$ and thereby obtain a polynomial bu-tst M' such that $\|M'\|_\eta = \|M\|_\eta$.

OBSERVATION 7.7. *For every polynomial bu-tst M, there exists a polynomial bu-tst M' that possesses a blind state with $\|M'\|_\varepsilon = \|M\|_\varepsilon$.*

PROOF. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, $\perp \notin Q$ be a fresh state, and $\alpha \in \Delta_0$. We construct $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ with $Q' = Q \cup \{\perp\}$, $F'_q = F_q$ for every $q \in Q$ and $F'_\perp = \widetilde{0}$. The tree representation μ' is defined for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $q, q_1, \ldots, q_k \in Q$ by

$$\mu'_k(\sigma)_{q,q_1\cdots q_k} = \mu_k(\sigma)_{q,q_1\cdots q_k}$$
$$\mu'_k(\sigma)_{\perp,\perp\ldots\perp} = 1\,\alpha\ .$$

Clearly, $\perp$ is a blind state of M' and also $\|M'\|_\varepsilon = \|M\|_\varepsilon$. □

Note that the construction does not preserve determinism. In summary the preservation of properties is displayed in Table 1. The following example adjoins a blind state to the bu-tst $M_{7.4}$ of Example 7.4.

EXAMPLE 7.8. *Let $M_{7.4} = (Q, \Gamma, \Sigma, \mathbb{R}_+, F, \mu)$ be the polynomial bu-tst of Example 7.4. Adjoining a blind state $\perp$ to $M_{7.4}$ yields the polynomial bu-tst*

$$M_{7.8} = (Q', \Gamma, \Sigma, \mathbb{R}_+, F', \mu')$$

with

- $Q' = \{\perp, \top, \mathrm{l}, \mathrm{m}, \mathrm{r}\}$;
- $F'_\top = 1\,\mathrm{z}_1$ *and* $F'_\perp = F'_\gamma = \widetilde{0}$ *for every* $\gamma \in \Gamma_1$; *and*
- *tree representation* μ' *given by:*

$$\mu'_0(\alpha)_\gamma = \mu'_0(\alpha)_\top = 1\,\alpha \qquad \mu'_3(\sigma)_{\mathrm{l},\top\top\top} = 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3$$
$$\mu'_0(\beta)_\gamma = \mu'_0(\beta)_\top = 1\,\beta \qquad \mu'_3(\sigma)_{\mathrm{m},\top\top\top} = 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3$$
$$\mu'_0(\alpha)_\perp = \mu'_0(\beta)_\perp = 1\,\alpha \qquad \mu'_3(\sigma)_{\mathrm{r},\top\top\top} = 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$$
$$\mu'_1(\gamma)_{\top,\gamma} = 1\,\mathrm{z}_1 \qquad \mu'_3(\sigma)_{\perp,\perp\perp\perp} = 1\,\alpha$$
$$\mu'_1(\gamma)_{\perp,\perp} = 1\,\alpha$$

for every $\gamma \in \Gamma_1$.

We display $M_{7.8}$ in Figure 4.

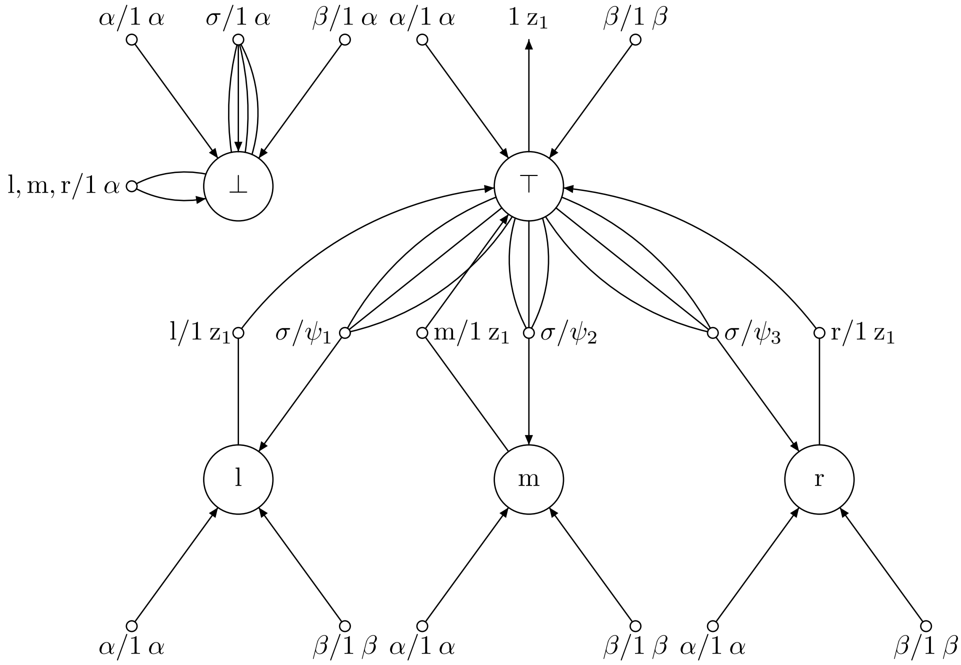

FIGURE 4. Bu-tst $M_{7.8}$ over the semiring $\mathbb{R}_+$ of Example 7.8 where $\psi_1 = 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3$, $\psi_2 = 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3$, and $\psi_3 = 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$.

With the help of a blind state we can now present our first composition result. Instead of composing two arbitrary polynomial bu-tst, we compose a polynomial bu-tst with designated states with a polynomial bu-tst that possesses a blind state. Lemma 4.16 tell us that for every polynomial bu-tst there exists an equivalent one with designated states and the previous observation essentially states that for any polynomial bu-tst an equivalent polynomial bu-tst with a blind state can be constructed. Altogether these restrictions do not limit the generality of the composition construction.

DEFINITION 7.9. *Let*

$$M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu') \quad \textit{and} \quad M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$$

be two polynomial bu-tst such that M' has designated states and $\perp$ is a blind state of M''. The composition of M' and M'', *denoted by $M'\,;M''$, is defined to be the bu-tst*

$$M'\,;M'' = (Q' \times Q'', \Sigma, \Delta, \mathcal{A}, F, \mu)$$

with

- $F_{(p,q)} = \sum_{q' \in Q''} F''_{q'} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu'',q}(F'_p)_{q'})$ *for every $(p,q) \in Q' \times Q''$; and*

TABLE 2. Preservation of properties for the construction of Definition 7.9.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✗	✗	✗

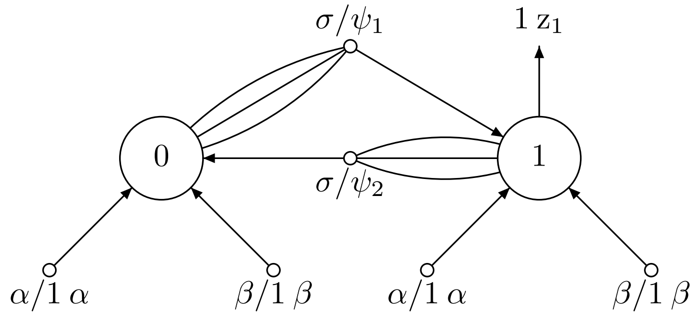

FIGURE 5. Bu-tst $M'_{7.10}$ over the semiring $\mathbb{R}_+$ of Example 7.10 where $\psi_1 = 0.1\,\mathrm{l}(\mathrm{z}_1) + 0.3\,\mathrm{m}(\mathrm{z}_2) + 0.6\,\mathrm{r}(\mathrm{z}_3)$ and $\psi_2 = 1\,\sigma(\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3)$.

- *tree representation μ given for every $k \in \mathbb{N}$, symbol $\sigma \in \Sigma_k$, $p, p_1, \ldots, p_k \in Q'$, $q \in Q'' \setminus \{\bot\}$, and $q_1, \ldots, q_k \in Q''$ by:*

$$\mu_k(\sigma)_{(p,q),(p_1,q_1)\cdots(p_k,q_k)} = h^{\varepsilon}_{\mu'',q_1\cdots q_k}\Big(\sum_{\substack{u \in T_\Gamma(\mathrm{Z}_k),\\ (\forall i \in [k]):\, i \notin \mathrm{var}(u) \iff q_i = \bot}} \big(\mu'_k(\sigma)_{p,p_1\cdots p_k}, u\big)\, u \Big)_q$$

$$\mu_k(\sigma)_{(p,\bot),(p_1,\bot)\cdots(p_k,\bot)} = h^{\varepsilon}_{\mu'',\bot\cdots\bot}(\mu'_k(\sigma)_{p,p_1\cdots p_k})_\bot \ .$$

All the remaining entries in μ are $\widetilde{0}$.

Note that $F_{(p,\bot)} = \widetilde{0}$ for every $p \in Q'$ because $\bot$ is blind. Preservation of properties is displayed in Table 2; note that here we say that a property is preserved, if it holds that the composition bu-tst has the property whenever both input bu-tst have the property. Let us now show the composition on our running example bu-tst $M_{7.8}$ of Example 7.8. For this, we compose another polynomial bu-tst $M'_{7.10}$ of Example 7.10 (see Figure 5) with $M_{7.8}$ (see Figure 4).

EXAMPLE 7.10. *Let $M_{7.8} = (Q'', \Gamma, \Sigma, \mathbb{R}_+, F'', \mu'')$ be the polynomial bu-tst of Example 7.8. Moreover, let $M'_{7.10} = (Q', \Sigma, \Gamma, \mathbb{R}_+, F', \mu')$ be the polynomial bu-tst with:*

- $Q' = \{0, 1\}$;
- $F'_1 = 1\,\mathrm{z}_1$ *and* $F'_0 = \widetilde{0}$; *and*
- *tree representation μ' given by:*

$$\mu'_0(\alpha)_0 = \mu'_0(\alpha)_1 = 1\,\alpha \qquad \mu'_3(\sigma)_{1,000} = 0.1\,\mathrm{l}(\mathrm{z}_1) + 0.3\,\mathrm{m}(\mathrm{z}_2) + 0.6\,\mathrm{r}(\mathrm{z}_3)$$

$$\mu'_0(\beta)_0 = \mu'_0(\beta)_1 = 1\,\beta \qquad \mu'_3(\sigma)_{0,111} = 1\,\sigma(\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3) \ .$$

The bu-tst $M'_{7.10}$ is illustrated in Figure 5. Now let us compose $M'_{7.10}$ and $M_{7.8}$. The composition

$$M_{7.10} = M'_{7.10} \, ; M_{7.8} = (Q' \times Q'', \Sigma, \Sigma, \mathbb{R}_+, F, \mu)$$

is given by:

- $F_{(1,\top)} = 1 \, \mathrm{z}_1$ *and* $F_q = \widetilde{0}$ *for all* $q \in (Q' \times Q'') \setminus \{(1,\top)\}$; *and*
- *for every* $p \in Q'$, $q \in Q'' \setminus \{\bot\}$, *and* $\gamma \in \Gamma_1$

$$\begin{aligned}
\mu_0(\alpha)_{(p,q)} = \mu_0(\alpha)_{(p,\bot)} = \mu_0(\beta)_{(p,\bot)} &= 1 \, \alpha \\
\mu_0(\beta)_{(p,q)} &= 1 \, \beta \\
\mu_3(\sigma)_{(1,\top),(0,\mathrm{l})(0,\bot)(0,\bot)} &= 0.1 \, \mathrm{z}_1 \\
\mu_3(\sigma)_{(1,\top),(0,\bot)(0,\mathrm{m})(0,\bot)} &= 0.3 \, \mathrm{z}_2 \\
\mu_3(\sigma)_{(1,\top),(0,\bot)(0,\bot)(0,\mathrm{r})} &= 0.6 \, \mathrm{z}_3 \\
\mu_3(\sigma)_{(0,\gamma),(1,\top)(1,\top)(1,\top)} &= \begin{cases} 0.4 \, \mathrm{z}_2 + 0.6 \, \mathrm{z}_3 & \textit{if } \gamma = \mathrm{l}, \\ 0.5 \, \mathrm{z}_1 + 0.5 \, \mathrm{z}_3 & \textit{if } \gamma = \mathrm{m}, \\ 0.7 \, \mathrm{z}_1 + 0.3 \, \mathrm{z}_2 & \textit{if } \gamma = \mathrm{r}; \end{cases} \\
\mu_3(\sigma)_{(0,\bot),(1,\bot)(1,\bot)(1,\bot)} &= 1 \, \alpha \ .
\end{aligned}$$

It is quite clear that in general the composition $M = M' \, ; M''$ might be such that $\|M\|_\varepsilon \neq \|M'\|_\varepsilon \, ; \|M''\|_\varepsilon$. This is true because already for bottom-up tree transducers (*i. e.*, polynomial bu-tst over $\mathbb{B}$) it can be shown that the computed transformations are not closed under composition [**35**, Theorem 2.5]. However, we have already mentioned that p–BOT$_\varepsilon(\mathbb{B})$ is closed under left-composition with lp–BOT$_\varepsilon(\mathbb{B})$ and under right-composition with d–BOT$_\varepsilon(\mathbb{B})$. The next proposition shows the central property needed for the correctness of the composition construction in Definition 7.9 and a forthcoming composition construction in Definition 7.15. Roughly speaking, it presents sufficient conditions that imply that h^ε_μ distributes over substitutions $t[u_1, \dots, u_k]$ for $t \in T_\Sigma(\mathrm{Z}_k)$ and $u_1, \dots, u_k \in T_\Sigma$.

PROPOSITION 7.11. *Let $V \subseteq \mathrm{Z}$, and let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial bu-tst, $q \in Q$, $t \in T_\Sigma(V)$, and $u_i \in T_\Sigma$ for every $i \in \mathrm{var}(t)$.*

$$h^\varepsilon_\mu(t[u_i]_{i \in \mathrm{var}(t)})_q = \sum_{\overline{q} \in Q^{\mathrm{var}(t)}} h^\varepsilon_{\mu,\overline{q}}(t)_q \xleftarrow[\varepsilon]{} (h^\varepsilon_\mu(u_i)_{\overline{q}_i})_{i \in \mathrm{var}(t)} \ ,$$

provided that:

(a) *M is boolean and deterministic; or*
(b) *t is linear.*

PROOF. We prove the statement by induction on t.

(i) First, let $t = \mathrm{z}_j$ for some $j \in \mathbb{N}_+$. Clearly, $\mathrm{var}(t) = \{j\}$.

$$h^\varepsilon_\mu(\mathrm{z}_j[u_i]_{i \in \mathrm{var}(t)})_q$$

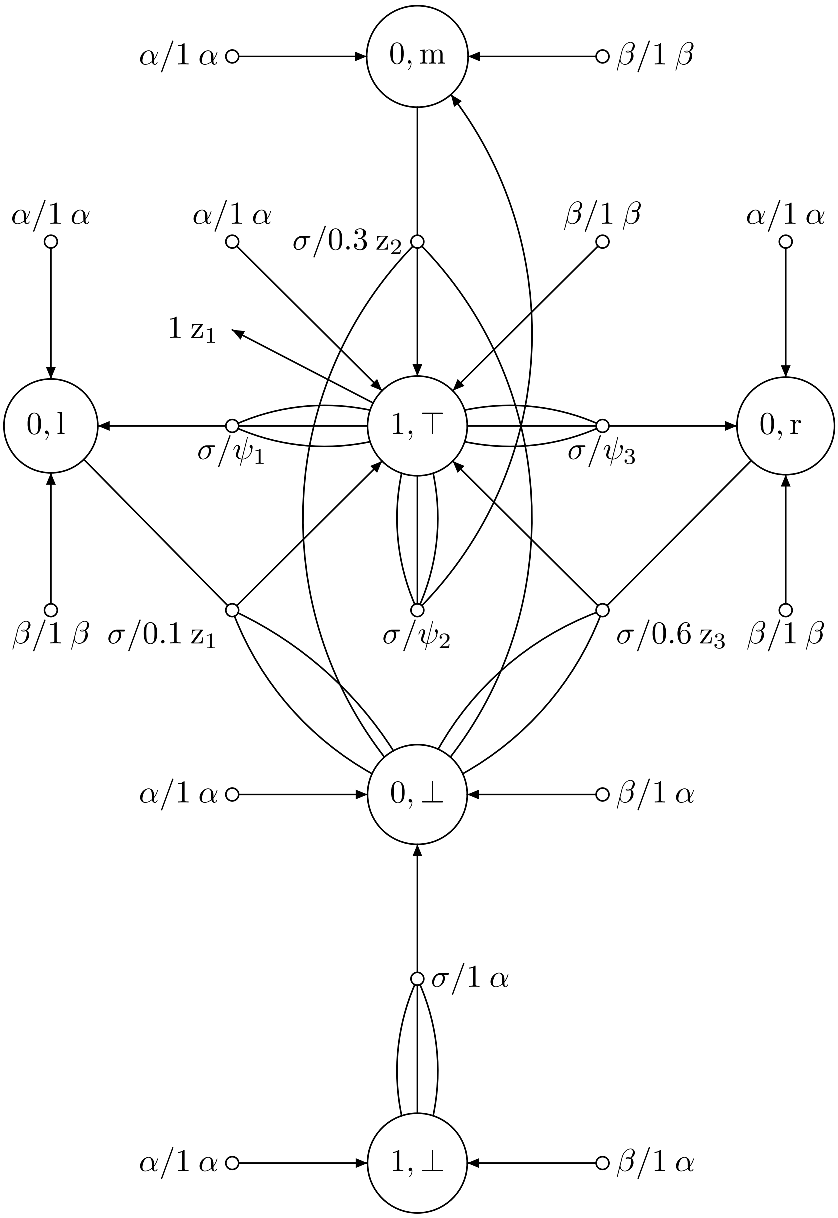

FIGURE 6. Relevant part of bu-tst $M_{7.10}$ over $\mathbb{R}_+$ of Example 7.10 where $\psi_1 = 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3$, $\psi_2 = 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3$, and $\psi_3 = 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2$ (note that we omitted the parentheses).

$$
\begin{aligned}
= &\qquad \textit{(by tree substitution)} \\
& h^{\varepsilon}_{\mu}(u_j)_q \\
= &\qquad \textit{(by definition of } \xleftarrow[\varepsilon]{} \textit{)} \\
& 1\,\mathrm{z}_j \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_q)_{i \in \mathrm{var}(t)}
\end{aligned}
$$

$$= \quad \textit{(because } h^{\varepsilon}_{\mu,\overline{q}}(\mathrm{z}_j)_q = \widetilde{0} \textit{ for every } \overline{q} \textit{ such that } \overline{q}_j \neq q\textit{)}$$

$$\sum_{\overline{q} \in Q^{\mathrm{var}(t)}} h^{\varepsilon}_{\mu,\overline{q}}(\mathrm{z}_j)_q \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_{\overline{q}_i})_{i \in \mathrm{var}(t)}$$

(ii) Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma(V)$.

$$h^{\varepsilon}_{\mu}(\sigma(t_1, \ldots, t_k)[u_i]_{i \in \mathrm{var}(t)})_q$$

$$= \quad \textit{(by tree substitution)}$$

$$h^{\varepsilon}_{\mu}(\sigma(t_1[u_i]_{i \in \mathrm{var}(t_1)}, \ldots, t_k[u_i]_{i \in \mathrm{var}(t_k)}))_q$$

$$= \quad \textit{(by Definition 4.7(1))}$$

$$\sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\varepsilon]{} \big(h^{\varepsilon}_{\mu}(t_j[u_i]_{i \in \mathrm{var}(t_j)})_{q_j}\big)_{j \in [k]}$$

$$= \quad \textit{(by induction hypothesis)}$$

$$\sum_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\varepsilon]{} \Big(\sum_{\overline{q_j} \in Q^{\mathrm{var}(t_j)}} h^{\varepsilon}_{\mu,\overline{q_j}}(t_j)_{q_j} \xleftarrow[\varepsilon]{} \big(h^{\varepsilon}_{\mu}(u_i)_{\overline{q_j}(i)}\big)_{i \in \mathrm{var}(t_j)}\Big)_{j \in [k]}$$

$$= \quad \textit{(by Proposition 3.8)}$$

$$\sum_{\substack{q_1, \ldots, q_k \in Q, \\ (\forall j \in [k]) \colon \overline{q_j} \in Q^{\mathrm{var}(t_j)}}} \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\varepsilon]{} \big(h^{\varepsilon}_{\mu,\overline{q_j}}(t_j)_{q_j} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_{\overline{q_j}(i)})_{i \in \mathrm{var}(t_j)}\big)_{j \in [k]}$$

$= \quad$ *(because* $\bigcup_{j \in [k]} \mathrm{var}(t_j) = \mathrm{var}(t)$ *and by:*
(a) determinism because $h^{\varepsilon}_{\mu}(u_i)_p \neq \widetilde{0}$ *for at most one* $p \in Q$ *by Proposition 5.1; or*
(b) linearity of t *because* $\mathrm{var}(t_{j_1}) \cap \mathrm{var}(t_{j_2}) = \emptyset$ *for* $j_1 \neq j_2$*)*

$$\sum_{\substack{q_1, \ldots, q_k \in Q, \\ \overline{q} \in Q^{\mathrm{var}(t)}}} \mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\varepsilon]{} \big(h^{\varepsilon}_{\mu,\overline{q}}(t_j)_{q_j} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_{\overline{q}_i})_{i \in \mathrm{var}(t_j)}\big)_{j \in [k]}$$

$= \quad$ *(by*
(a) Lemma 3.21 because $h^{\varepsilon}_{\mu}(u_i)_{\overline{q}_i}$ *is boolean and monomial by Observation 5.2; or*
(b) Proposition 3.19 because $(\mathrm{var}(t_j))_{j \in [k]}$ *is the partition)*

$$\sum_{\overline{q} \in Q^{\mathrm{var}(t)}} \sum_{q_1, \ldots, q_k \in Q} \big(\mu_k(\sigma)_{q, q_1 \cdots q_k} \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu,\overline{q}}(t_j)_{q_j})_{j \in [k]}\big) \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_{\overline{q}_i})_{i \in \mathrm{var}(t)}$$

$$= \quad \textit{(by Definition 7.3 and Proposition 3.8)}$$

$$\sum_{\overline{q} \in Q^{\mathrm{var}(t)}} h^{\varepsilon}_{\mu,\overline{q}}(\sigma(t_1, \ldots, t_k))_q \xleftarrow[\varepsilon]{} (h^{\varepsilon}_{\mu}(u_i)_{\overline{q}_i})_{i \in \mathrm{var}(t)} \qquad \square$$

With the help of this proposition we can show the correctness of the construction in Definition 7.9 for linear M'; *i. e.*, we show that $\|M' \mathbin{;} M''\|_\varepsilon = \|M'\|_\varepsilon \mathbin{;} \|M''\|_\varepsilon$ for linear polynomial bu-tst M' and polynomial bu-tst M''.

Lemma 7.12. *Let* $\mathcal{A}$ *be commutative,*

$$M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu') \quad \textit{and} \quad M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$$

be polynomial bu-tst, of which M' is linear and has designated states and M'' has a blind state $\perp$. Moreover, let $M = M' \,; M''$ (see Definition 7.9). Then for every $t \in T_\Sigma$, $p \in Q'$, and $q \in Q''$

$$h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t)_p)_q = h^\varepsilon_\mu(t)_{(p,q)} \tag{95}$$

and $\|M\|_\varepsilon = \|M'\|_\varepsilon \,; \|M''\|_\varepsilon$.

Proof. We claim that there exists an $\alpha \in \Delta_0$ such that $h^\varepsilon_{\mu''}(u)_\perp = 1\,\alpha$ for every $u \in T_\Gamma$. The proof of this claim is in Observation 7.6. The remaining proof is done by induction on t and case analysis. Let $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$.

(i) Let $q = \perp$.

$$\begin{aligned}
& h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(\sigma(t_1, \dots, t_k))_p)_\perp \\
= \quad & \textit{(by Definition 4.7(1) and definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& \sum_{p_1,\dots,p_k \in Q'} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_k),\\ (\forall i \in [k]):\, u_i \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u) \cdot \prod_{i \in [k]} (h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i) \Big) \cdot h^\varepsilon_{\mu''}(u[u_i]_{i \in [k]})_\perp \\
= \quad & \textit{(by } h^\varepsilon_{\mu''}(u[u_i]_{i\in[k]})_\perp = 1\,\alpha\textit{; see Observation 7.6)} \\
& \sum_{p_1,\dots,p_k \in Q'} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_k),\\ (\forall i \in [k]):\, u_i \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u) \cdot \prod_{i \in [k]} (h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i) \Big)\, \alpha \\
= \quad & \textit{(by Observation 7.6 and definition of } \xleftarrow[\varepsilon]{}\textit{)} \\
& \sum_{p_1,\dots,p_k \in Q'} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_k),\\ (\forall i \in [k]):\, u_i \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u) \cdot \prod_{i \in [k]} (h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i) \Big) \cdot \\
& \qquad \cdot \Big(h^\varepsilon_{\mu'',\perp\cdots\perp}(u)_\perp \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu''}(u_i)_\perp)_{i \in [k]} \Big) \\
= \quad & \textit{(by Definition 7.3 and Propositions 3.8 and 3.9)} \\
& \sum_{p_1,\dots,p_k \in Q'} h^\varepsilon_{\mu'',\perp\cdots\perp}(\mu'_k(\sigma)_{p,p_1\cdots p_k})_\perp \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t_i)_{p_i})_\perp \big)_{i \in [k]} \\
= \quad & \textit{(by definition of } \mu \textit{ and induction hypothesis)} \\
& \sum_{p_1,\dots,p_k \in Q'} \mu_k(\sigma)_{(p,\perp),(p_1,\perp)\cdots(p_k,\perp)} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_\mu(t_i)_{(p_i,\perp)} \big)_{i \in [k]} \\
= \quad & \textit{(since } \mu_k(\sigma)_{(p,\perp),(p_1,q_1)\dots(p_k,q_k)} \neq \widetilde{0}\textit{, only if } q_1 = \cdots = q_k = \perp\textit{)} \\
& \sum_{\substack{p_1,\dots,p_k \in Q',\\ q_1,\dots,q_k \in Q''}} \mu_k(\sigma)_{(p,\perp),(p_1,q_1)\cdots(p_k,q_k)} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_\mu(t_i)_{(p_i,q_i)} \big)_{i \in [k]} \\
= \quad & \textit{(by Definition 4.7(1))} \\
& h^\varepsilon_\mu(\sigma(t_1, \dots, t_k))_{p,\perp}
\end{aligned}$$

(ii) Now let $q \neq \bot$.

$$
\begin{aligned}
& h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(\sigma(t_1,\ldots,t_k))_p)_q \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{p_1,\ldots,p_k\in Q'} h^\varepsilon_{\mu''}\big(\mu'_k(\sigma)_{p,p_1\cdots p_k} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_i)_{p_i})_{i\in[k]}\big)_q \\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{} \textit{ and Definition 4.7(1))} \\
& \sum_{p_1,\ldots,p_k\in Q'} \sum_{\substack{u\in T_\Gamma(\mathrm{Z}_k),\\ (\forall i\in[k])\colon u_i\in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u)\cdot \prod_{i\in[k]}(h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i)\Big)\cdot h^\varepsilon_{\mu''}(u[u_i]_{i\in[k]})_q \\
= \quad & \textit{(by Proposition 7.11)} \\
& \sum_{p_1,\ldots,p_k\in Q'} \sum_{\substack{u\in T_\Gamma(\mathrm{Z}_k),\\ (\forall i\in[k])\colon u_i\in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u)\cdot \prod_{i\in[k]}(h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i)\Big)\cdot \\
& \qquad \cdot\Big(\sum_{\overline{q}\in(Q'')^{\mathrm{var}(u)}} h^\varepsilon_{\mu'',\overline{q}}(u)_q \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(u_i)_{\overline{q}_i}\big)_{i\in\mathrm{var}(u)}\Big) \\
= \quad & \textit{(by Proposition 3.18 because } h^\varepsilon_{\mu''}(u_i)_\bot = 1\,\alpha \textit{ by Observation 7.6)} \\
& \sum_{\substack{p_1,\ldots,p_k\in Q',\\ q_1,\ldots,q_k\in Q''}} \sum_{\substack{u\in T_\Gamma(\mathrm{Z}_k),\\ (\forall i\in[k])\colon u_i\in T_\Gamma,\\ i\notin\mathrm{var}(u) \iff q_i=\bot}} \Big((\mu'_k(\sigma)_{p,p_1\cdots p_k}, u)\cdot \prod_{i\in[k]}(h^\varepsilon_{\mu'}(t_i)_{p_i}, u_i)\Big)\cdot \\
& \qquad \cdot\Big(h^\varepsilon_{\mu'',q_1\cdots q_k}(u)_q \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(u_i)_{q_i}\big)_{i\in[k]}\Big) \\
= \quad & \textit{(by Propositions 3.8 and 3.9)} \\
& \sum_{\substack{p_1,\ldots,p_k\in Q',\\ q_1,\ldots,q_k\in Q''}} h^\varepsilon_{\mu'',q_1\cdots q_k}\Big(\sum_{\substack{u\in T_\Gamma(\mathrm{Z}_k),\\ i\notin\mathrm{var}(u)\iff q_i=\bot}} (\mu'_k(\sigma)_{p,p_1\cdots p_k}, u)\, u\Big)_q \\
& \qquad \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t_i)_{p_i})_{q_i}\big)_{i\in[k]} \\
= \quad & \textit{(by definition of } \mu\textit{)} \\
& \sum_{\substack{p_1,\ldots,p_k\in Q',\\ q_1,\ldots,q_k\in Q''}} \mu_k(\sigma)_{(p,q),(p_1,q_1)\ldots(p_k,q_k)} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t_i)_{p_i})_{q_i}\big)_{i\in[k]} \\
= \quad & \textit{(by induction hypothesis)} \\
& \sum_{\substack{p_1,\ldots,p_k\in Q',\\ q_1,\ldots,q_k\in Q''}} \mu_k(\sigma)_{(p,q),(p_1,q_1)\ldots(p_k,q_k)} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu}(t_i)_{(p_i,q_i)}\big)_{i\in[k]} \\
= \quad & \textit{(by Definition 4.7(1))} \\
& h^\varepsilon_\mu(\sigma(t_1,\ldots,t_k))_{(p,q)}
\end{aligned}
$$

Now we can prove the main statement. Let $t \in T_\Sigma$ be arbitrary.

$$(\|M'\|_\varepsilon\,;\|M''\|_\varepsilon)(t)$$

$$
\begin{aligned}
= \quad & \textit{(by Definition 4.7(2) and Proposition 3.8)} \\
& \sum_{p\in Q', q'\in Q''} F''_{q'} \underset{\varepsilon}{\longleftarrow} \big(h^{\varepsilon}_{\mu''}\big(F'_p \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu'}(t)_p)\big)_{q'} \big) \\
= \quad & \textit{(by Definition 4.7(1) and definition of } \underset{\varepsilon}{\longleftarrow}\textit{)} \\
& \sum_{p\in Q', q'\in Q''} \sum_{\substack{u\in T_\Gamma(\mathrm{Z}_1),\\ u'\in T_\Gamma}} \big((F'_p, u)\cdot(h^{\varepsilon}_{\mu'}(t)_p, u')\big)\cdot\big(F''_{q'} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu''}(u[u'])_{q'})\big) \\
= \quad & \textit{(by Proposition 7.11)} \\
& \sum_{p\in Q', q'\in Q''} \sum_{\substack{u\in T_\Gamma(\mathrm{Z}_1),\\ u'\in T_\Gamma}} \big((F'_p, u)\cdot(h^{\varepsilon}_{\mu'}(t)_p, u')\big) \\
& \qquad \cdot \Big(F''_{q'} \underset{\varepsilon}{\longleftarrow} \big(\sum_{q\in Q''} h^{\varepsilon}_{\mu'',q}(u)_{q'} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu''}(u')_q)\big)\Big) \\
= \quad & \textit{(by Definition 4.7(1) and Propositions 3.8 and 3.9)} \\
& \sum_{p\in Q', q'\in Q''} F''_{q'} \underset{\varepsilon}{\longleftarrow} \big(\sum_{q\in Q''} h^{\varepsilon}_{\mu'',q}(F'_p)_{q'} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu''}(h^{\varepsilon}_{\mu'}(t)_p)_q)\big) \\
= \quad & \textit{(by } (95)\textit{)} \\
& \sum_{p\in Q', q, q'\in Q''} F''_{q'} \underset{\varepsilon}{\longleftarrow} \big(h^{\varepsilon}_{\mu'',q}(F'_p)_{q'} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu}(t)_{p,q})\big) \\
= \quad & \textit{(by Proposition 3.19)} \\
& \sum_{p\in Q', q, q'\in Q''} \big(F''_{q'} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu'',q}(F'_p)_{q'})\big) \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu}(t)_{p,q}) \\
= \quad & \textit{(by definition of } F_{(p,q)}\textit{; see Definition 7.9)} \\
& \sum_{p\in Q', q\in Q''} F_{(p,q)} \underset{\varepsilon}{\longleftarrow} (h^{\varepsilon}_{\mu}(t)_{p,q}) \\
= \quad & \textit{(by Definition 4.7(2))} \\
& \|M\|_{\varepsilon}(t)
\end{aligned}
$$

□

It is easy to see that whenever M' is nondeleting, then the blind state $\perp$ is not required. If M' and M'' are nondeleting, we can thus drop the states $(p, \perp)$ from $M' \,; M''$ (and their transitions) and obtain a nondeleting polynomial bu-tst M such that $\|M\|_\varepsilon = \|M' \,; M''\|_\varepsilon$ (see Definition 7.15). Moreover, if M' and M'' are linear, then also $M' \,; M''$ is linear. Together with Lemma 7.12 this yields the first main theorem.

THEOREM 7.13. *Let $\mathcal{A}$ be commutative.*

$$\text{lp–BOT}_\varepsilon(\mathcal{A}) \,; \text{p–BOT}_\varepsilon(\mathcal{A}) = \text{p–BOT}_\varepsilon(\mathcal{A}) \tag{96}$$

$$\text{lp–BOT}_\varepsilon(\mathcal{A}) \,; \text{lp–BOT}_\varepsilon(\mathcal{A}) = \text{lp–BOT}_\varepsilon(\mathcal{A}) \tag{97}$$

$$\text{nlp–BOT}_\varepsilon(\mathcal{A}) \,; \text{nlp–BOT}_\varepsilon(\mathcal{A}) = \text{nlp–BOT}_\varepsilon(\mathcal{A}) \tag{98}$$

Moreover, if $\mathcal{A}$ is commutative and $\aleph_0$-complete, then the above equations even hold without the polynomial restriction.

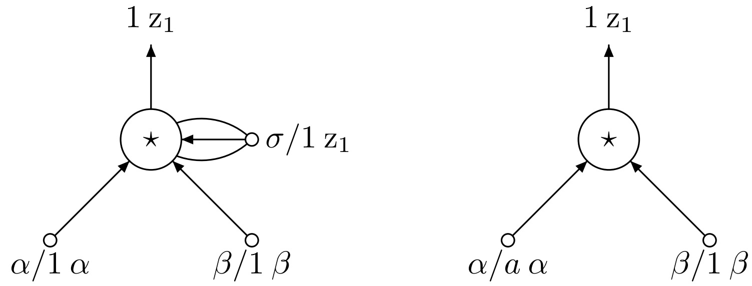

FIGURE 7. Bu-tst M' (left) and M'' (right) over $\mathcal{A}$ used to prove Lemma 7.14.

PROOF. The statements follow directly from Lemma 4.16, Observation 7.7, and Lemma 7.12. $\square$

Note that our construction does not preserve determinism [**41**, Corollary 5.5]. Thus, the statement

$$\text{hl–BOT}_\varepsilon(\mathcal{A})\,;\text{h–BOT}_\varepsilon(\mathcal{A}) = \text{h–BOT}_\varepsilon(\mathcal{A})$$

cannot be shown with the help of our construction due to the introduction of the blind state $\perp$. In the next lemma we prove that for almost all semirings except for the trivial semiring (in which $0 = 1$) and for $\mathbb{B}$ and $\mathbb{Z}_2$ we have that the above equality does not hold.

LEMMA 7.14. *For every semiring $\mathcal{A}$ with at least 3 elements we have*

$$\text{hlb–BOT}_\varepsilon(\mathcal{A})\,;\text{hl–BOT}_\varepsilon(\mathcal{A}) \not\subseteq \text{h–BOT}_\varepsilon(\mathcal{A}) \enspace .$$

PROOF. Since A has at least 3 elements, let $a \in A \setminus \{0, 1\}$. Suppose that $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\Delta = \{\alpha^{(0)}, \beta^{(0)}\}$. Moreover, let

$$M' = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu') \quad \text{and} \quad M'' = (\{\star\}, \Delta, \Delta, \mathcal{A}, F, \mu'')$$

be the homomorphism bu-tst with:

- $F_\star = 1\,\mathrm{z}_1$; and
- the tree representations μ' and μ'' are specified by:

$$\mu_0'(\alpha)_\star = 1\,\alpha \qquad \mu_0''(\alpha)_\star = a\,\alpha$$
$$\mu_0'(\beta)_\star = 1\,\beta \qquad \mu_0''(\beta)_\star = 1\,\beta$$
$$\mu_2'(\sigma)_{\star,\star\star} = 1\,\mathrm{z}_1 \enspace .$$

Clearly, M' and M'' are both linear homomorphism bu-tst, which are illustrated in Figure 7. Let $\tau = \|M'\|_\varepsilon\,; \|M''\|_\varepsilon$. It is easily observed that

$$\tau(\alpha) = a\,\alpha \quad \tau(\beta) = 1\,\beta \quad \tau(\sigma(\beta, \alpha)) = 1\,\beta \quad \tau(\sigma(\beta, \beta)) = 1\,\beta \enspace .$$

Now suppose that there exists a homomorphism bu-tst

$$M = (\{\star\}, \Sigma, \Delta, \mathcal{A}, F, \mu)$$

such that $\|M\|_\varepsilon = \tau$. We can immediately conclude that $\mu_0(\alpha)_\star = a\,\alpha$ and $\mu_0(\beta)_\star = 1\,\beta$ by Observation 4.23. Moreover, let $c \in A_+$ and $u \in T_\Delta(\mathrm{Z}_2)$

be such that $\mu_2(\sigma)_{\star,\star\star} = c\,u$. Now let us calculate $\|M\|_\varepsilon(\sigma(\beta,\alpha))$ and $\|M\|_\varepsilon(\sigma(\beta,\beta))$.

$$\|M\|_\varepsilon(\sigma(\beta,\beta)) = h^\varepsilon_\mu(\sigma(\beta,\beta))_\star = c\,u \underset{\varepsilon}{\longleftarrow} (h^\varepsilon_\mu(\beta)_\star, h^\varepsilon_\mu(\beta)_\star) = c\,u[\beta,\beta]$$
$$\|M\|_\varepsilon(\sigma(\beta,\alpha)) = h^\varepsilon_\mu(\sigma(\beta,\alpha))_\star = c\,u \underset{\varepsilon}{\longleftarrow} (h^\varepsilon_\mu(\beta)_\star, h^\varepsilon_\mu(\alpha)_\star) = (c\cdot a)\,u[\beta,\alpha]$$

Thus it follows that $c = 1$ because $\tau(\sigma(\beta,\beta)) = 1\,\beta$. With this knowledge we obtain that $a\,u[\beta,\alpha] = 1\,\beta$. However, we chose $a \in A \setminus \{0,1\}$, hence there is no homomorphism bu-tst M such that $\|M\|_\varepsilon = \tau$. □

Finally, let us consider the second result, which states that the class of ε-t-ts transformations computed by polynomial bu-tst over $\mathbb{B}$ is closed under composition (from the right) with the class of ε-t-ts transformations computed by deterministic bu-tst over $\mathbb{B}$ (see [**35**, Theorem 4.6] and [**5**, Theorem 6]). This result was also generalized to $\mathrm{BOT}_\varepsilon(\mathcal{A})\,;\mathrm{bh–BOT}_\varepsilon(\mathcal{A}) = \mathrm{BOT}_\varepsilon(\mathcal{A})$ [**41**, Corollary 5.5] for commutative and $\aleph_0$-complete semirings. Since we have already seen that our previous construction destroys determinism due to the introduction of the blind state, we simplify the construction to obtain a construction which is the analogue of the construction for the unweighted case and avoids the blind state. Moreover, this construction will also preserve nondeletion and thereby remedy the second main problem with the construction of Definition 7.9. Note that without loss of generality we may assume a polynomial bu-tst to have a total tree representation, because there is a standard construction which shows that for each polynomial bu-tst M there exists a polynomial bu-tst M' with total tree representation such that $\|M'\|_\eta = \|M\|_\eta$. The construction required to show this is well-known: add a transition into a trap state, if no transition is present.

Definition 7.15. *Let $M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu')$ be a polynomial tst with designated states, and let $M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$ be polynomial bu-tst. The* (simple) composition of M' and M'', *which is denoted by $M'\,;_{\mathrm{S}}\,M''$, is defined to be the polynomial tst*

$$M'\,;_{\mathrm{S}}\,M'' = (Q' \times Q'', \Sigma, \Delta, \mathcal{A}, F, \mu)$$

with

- $F_{(p,q)} = \sum_{q' \in Q''} F''_{q'} \underset{\varepsilon}{\longleftarrow} (h^\varepsilon_{\mu'',q}(F'_p)_{q'})$ *for every* $(p,q) \in Q' \times Q''$; *and*
- *for every* $k, n \in \mathbb{N}$, *symbol* $\sigma \in \Sigma_k$, *states* $p, p_1, \dots, p_n \in Q'$, $q, q_1, \dots, q_n \in Q''$, *and* $i_1, \dots, i_n \in [k]$:

$$\mu_k(\sigma)_{(p,q),(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})} = h^\varepsilon_{\mu'',q_1\cdots q_n}\big(\mu'_k(\sigma)_{p,p_1(\mathrm{x}_{i_1})\cdots p_n(\mathrm{x}_{i_n})}\big)_q\ . \tag{99}$$

Note that M' need not be bottom-up. The freedom of having also polynomial td-tst as M' is used in Theorem 7.19. It is easily seen that $M'\,;_{\mathrm{S}}\,M''$ is a deterministic bu-tst, whenever M' and M'' are deterministic bu-tst. Moreover, $M'\,;_{\mathrm{S}}\,M''$ is a homomorphism bu-tst, if

Table 3. Preservation of properties for the construction of Definition 7.15.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✓	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✗	✗	✗

M' and M'' are homomorphism bu-tst and M'' is boolean. Note that, in general, the restriction that M'' is boolean is necessary in the last statement, because otherwise the composition $M' \mathbin{;_{\mathrm{S}}} M''$ might not be total. Let us illustrate the definition on an example.

Example 7.16. *Using the construction of Definition 7.15 let us compose the polynomial bu-tst $M'_{7.10}$ and $M_{7.4}$ of Examples 7.10 and 7.4, respectively. We obtain*

$$M_{7.16} = M'_{7.10} \mathbin{;_{\mathrm{S}}} M_{7.4} = (Q, \Sigma, \Sigma, \mathbb{R}_+, F, \mu)$$

with:

- $Q = \{(0,\top), (0,\mathrm{l}), (0,\mathrm{m}), (0,\mathrm{r}), (1,\top), (1,\mathrm{l}), (1,\mathrm{m}), (1,\mathrm{r})\}$;
- $\Sigma = \{\sigma^{(3)}, \alpha^{(0)}, \beta^{(0)}\}$;
- $F_{(1,\top)} = 1\,\mathrm{z}_1$ *and* $F_{(p,q)} = \widetilde{0}$ *for every* $(p,q) \in Q \setminus \{(1,\top)\}$; *and*
- *for every* $p \in \{0,1\}$, $q, q_1, q_2, q_3 \in \{\top, \mathrm{l}, \mathrm{m}, \mathrm{r}\}$, *and* $\gamma \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$

$$\mu_0(\alpha)_{(p,q)} = 1\,\alpha$$
$$\mu_0(\beta)_{(p,q)} = 1\,\beta$$

$$\mu_2(\sigma)_{(1,\top),(0,q_1)(0,q_2)(0,q_3)}$$
$$= \begin{cases} 0.1\,\mathrm{z}_1 & \text{if } q_1 = \mathrm{l}, \\ \widetilde{0} & \text{otherwise}; \end{cases} + \begin{cases} 0.3\,\mathrm{z}_2 & \text{if } q_2 = \mathrm{m}, \\ \widetilde{0} & \text{otherwise}; \end{cases} + \begin{cases} 0.6\,\mathrm{z}_3 & \text{if } q_3 = \mathrm{r}, \\ \widetilde{0} & \text{otherwise}; \end{cases}$$

$$\mu_2(\sigma)_{(0,\gamma),(1,\top)(1,\top)(1,\top)}$$
$$= \begin{cases} 0.4\,\mathrm{z}_2 + 0.6\,\mathrm{z}_3 & \text{if } \gamma = \mathrm{l}, \\ 0.5\,\mathrm{z}_1 + 0.5\,\mathrm{z}_3 & \text{if } \gamma = \mathrm{m}, \\ 0.7\,\mathrm{z}_1 + 0.3\,\mathrm{z}_2 & \text{if } \gamma = \mathrm{r}. \end{cases}$$

However, $\|M_{7.16}\|_\varepsilon \neq \|M_{7.10}\|_\varepsilon$ *because*

$$\|M_{7.16}\|_\varepsilon(\sigma(\alpha,\beta,\alpha)) = 0.4\,\alpha + 1.2\,\beta + 2.4\,\alpha = 2.8\,\alpha + 1.2\,\beta$$
$$\|M_{7.10}\|_\varepsilon(\sigma(\alpha,\beta,\alpha)) = 0.1\,\alpha + 0.3\,\beta + 0.6\,\alpha = 0.7\,\alpha + 0.3\,\beta \ .$$

Now we show the correctness of the simple composition $M' \mathbin{;_{\mathrm{S}}} M''$ provided that M' and M'' are bu-tst, of which M'' is boolean, total, and deterministic. Moreover, we prove the correctness also for particular td-tst.

Lemma 7.17. *Let*

$$M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu') \quad \text{and} \quad M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$$

be tst, of which M' has designated states and M'' is bottom-up. Moreover, let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be the simple composition of M' and M''. Then for every $t \in T_\Sigma$, $p \in Q'$, and $q \in Q''$

$$h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t)_p)_q = h^\varepsilon_\mu(t)_{(p,q)}$$

and $\|M'\|_\varepsilon \,;\, \|M''\|_\varepsilon = \|M\|_\varepsilon$ provided that:

(a) *M' is bottom-up and M'' is boolean, total, and deterministic; or*

(b) *M' is top-down.*

PROOF. We prove the statement inductively, so let $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \dots, t_k \in T_\Sigma$.

$$\begin{aligned}
& h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(\sigma(t_1, \dots, t_k))_p)_q \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{\substack{w' \in Q'(\mathrm{X}_k)^*, \\ w' = p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n})}} h^\varepsilon_{\mu''}\big(\mu'_k(\sigma)_{p,w'} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu'}(t_{i_j})_{p_j})_{j \in [n]}\big)_q \\
= \quad & \textit{(by definition of } \xleftarrow[\varepsilon]{} \textit{ and Definition 4.7(1))} \\
& \sum_{\substack{w' \in Q'(\mathrm{X}_k)^*, \\ w' = p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n})}} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_n), \\ (\forall j \in [n]) \colon u_j \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,w'}, u) \cdot \prod_{j \in [n]} (h^\varepsilon_{\mu'}(t_{i_j})_{p_j}, u_j)\Big) \\
& \qquad \cdot h^\varepsilon_{\mu''}(u[u_j]_{j \in [n]})_q \\
= \quad & \textit{(by Proposition 7.11(a) in Case (a) and Proposition 7.11(b))} \\
& \sum_{\substack{w' \in Q'(\mathrm{X}_k)^*, \\ w' = p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n})}} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_n), \\ (\forall j \in [n]) \colon u_j \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,w'}, u) \cdot \prod_{j \in [n]} (h^\varepsilon_{\mu'}(t_{i_j})_{p_j}, u_j)\Big) \cdot \\
& \qquad \cdot \Big(\sum_{\overline{q} \in (Q'')^{\mathrm{var}(u)}} h^\varepsilon_{\mu'', \overline{q}}(u)_q \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(u_j)_{\overline{q}_j}\big)_{j \in \mathrm{var}(u)}\Big) \\
= \quad & \textit{(because} \\
& \quad \textit{(a) Proposition 3.18 is applicable due to Observation 5.4} \\
& \quad \textit{(b) } M' \textit{ is top-down; i. e., } \mathrm{var}(u) = [n] \textit{ for } u \in \mathrm{supp}(\mu'_k(\sigma)_{p,w'})\textit{)} \\
& \sum_{\substack{w' \in Q'(\mathrm{X}_k)^*, \\ w' = p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n})}} \sum_{\substack{u \in T_\Gamma(\mathrm{Z}_n), \\ (\forall j \in [n]) \colon u_j \in T_\Gamma}} \Big((\mu'_k(\sigma)_{p,w'}, u) \cdot \prod_{j \in [n]} (h^\varepsilon_{\mu'}(t_{i_j})_{p_j}, u_j)\Big) \cdot \\
& \qquad \cdot \Big(\sum_{q_1, \dots, q_n \in Q''} h^\varepsilon_{\mu'', q_1 \cdots q_n}(u)_q \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(u_j)_{q_j}\big)_{j \in [n]}\Big) \\
= \quad & \textit{(by Propositions 3.8 and 3.9)} \\
& \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = (p_1, q_1)(\mathrm{x}_{i_1}) \cdots (p_n, q_n)(\mathrm{x}_{i_n})}} h^\varepsilon_{\mu'', q_1 \cdots q_n}(\mu'_k(\sigma)_{p, p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n})})_q \\
& \qquad \xleftarrow[\varepsilon]{} \big(h^\varepsilon_{\mu''}(h^\varepsilon_{\mu'}(t_{i_j})_{p_j})_{q_j}\big)_{j \in [n]}
\end{aligned}$$

$$= \quad \textit{(by definition of } \mu_k(\sigma)_{(p,q),w} \textit{ and induction hypothesis)}$$

$$\sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{(p,q),w} \xleftarrow[\varepsilon]{} \big(h^\varepsilon_\mu(t_{i_j})_{(p_j,q_j)}\big)_{j\in[n]}$$

$$= \quad \textit{(by Definition 4.7(1))}$$

$$h^\varepsilon_\mu(\sigma(t_1,\dots,t_k))_{(p,q)}$$

The proof of the second statement is literally the same as the proof of the second statement of Lemma 7.12. □

Thus we obtain the following theorem for bu-tst [**41**, Corollary 5.5]. It remains open to prove stronger statements for restricted semirings; *e. g.*, for additively idempotent semirings [**66**].

THEOREM 7.18. *Let $\mathcal{A}$ be commutative, and let $x \subseteq \{\mathrm{n},\mathrm{l},\mathrm{h}\}$.*

$$x\text{p–BOT}_\varepsilon(\mathcal{A})\,;x\text{bd–BOT}_\varepsilon(\mathcal{A}) = x\text{p–BOT}_\varepsilon(\mathcal{A}) \tag{100}$$

If $\mathcal{A}$ is also $\aleph_0$-complete, then even

$$x\text{–BOT}_\varepsilon(\mathcal{A})\,;x\text{bd–BOT}_\varepsilon(\mathcal{A}) = x\text{–BOT}_\varepsilon(\mathcal{A}) \ . \tag{101}$$

PROOF. The statement follows from Lemma 7.17. □

4. Top-down tree series transducers

Let us first review the known results about compositions of classes of transformations computed by td-tst. Note that top-down tree transducers are essentially polynomial td-tst over $\mathbb{B}$ (see [**41**, Section 4.3]). In [**5**, Theorem 1] it is shown that

$$\begin{aligned}
\text{p–TOP}_\varepsilon(\mathbb{B})\,;\text{pnl–TOP}_\varepsilon(\mathbb{B}) &\subseteq \text{p–TOP}_\varepsilon(\mathbb{B}) \\
\text{pt–TOP}_\varepsilon(\mathbb{B})\,;\text{pl–TOP}_\varepsilon(\mathbb{B}) &\subseteq \text{p–TOP}_\varepsilon(\mathbb{B}) \\
\text{d–TOP}_\varepsilon(\mathbb{B})\,;\text{pn–TOP}_\varepsilon(\mathbb{B}) &\subseteq \text{p–TOP}_\varepsilon(\mathbb{B}) \\
\text{dt–TOP}_\varepsilon(\mathbb{B})\,;\text{p–TOP}_\varepsilon(\mathbb{B}) &\subseteq \text{p–TOP}_\varepsilon(\mathbb{B}) \ .
\end{aligned}$$

Some results were extended to arbitrary commutative and $\aleph_0$-complete semirings $\mathcal{A}$ in [**78**, Theorem 2.4], which shows that

$$\text{nl–TOP}_\varepsilon(\mathcal{A})\,;\text{nl–TOP}_\varepsilon(\mathcal{A}) = \text{nl–TOP}_\varepsilon(\mathcal{A}) \ ,$$

and in [**41**, Theorem 5.18] that shows for every $x \in \{\mathrm{d},\mathrm{dn},\mathrm{dl},\mathrm{dnl}\}$

$$\begin{aligned}
x\text{–TOP}_\varepsilon(\mathcal{A})\,;\text{dnl–TOP}_\varepsilon(\mathcal{A}) &= x\text{–TOP}_\varepsilon(\mathcal{A}) \\
x\text{bt–TOP}_\varepsilon(\mathcal{A})\,;x\text{–TOP}_\varepsilon(\mathcal{A}) &= x\text{–TOP}_\varepsilon(\mathcal{A}) \ .
\end{aligned}$$

Without any additional construction we can already generalize the first statement of [**41**, Theorem 5.18]. We basically exploit the fact that nondeleting, linear td-tst are as powerful as nondeleting, linear bu-tst (see [**41**, Theorem 5.24] and Proposition 4.21). We note that td-determinism is preserved in the construction of [**41**, Lemma 5.22]. Thus given two td-tst M' and M'', of which M'' is nondeleting and linear, we first construct a nondeleting, linear bu-tst M_2 such that $\|M_2\|_\varepsilon = \|M''\|_\varepsilon$. Note that M_2 is td-deterministic (but not necessarily bu-deterministic) whenever M'' is td-deterministic. Next we apply Lemma 4.16 to M' to obtain an equivalent td-tst M_1 with designated states (note that td-determinism is preserved). Then we can apply the simple composition to M_1 and M_2 (see Definition 7.15) and obtain a tst M. It is easily seen that M is top-down, because M_2 is nondeleting and linear. Moreover, M is td-deterministic if M_1 and M_2 are td-deterministic.

THEOREM 7.19. *Let $\mathcal{A}$ be commutative, and let $x \subseteq \{\mathrm{d}, \mathrm{n}, \mathrm{l}\}$.*

$$x\text{p–TOP}_\varepsilon(\mathcal{A}) \,;\, x\text{nlp–TOP}_\varepsilon(\mathcal{A}) = x\text{p–TOP}_\varepsilon(\mathcal{A}) \tag{102}$$

If $\mathcal{A}$ is $\aleph_0$-complete, then the above equation even holds without the restriction to polynomial td-tst.

PROOF. The decomposition is trivial, so it remains to show the composition. Let M' and M'' be polynomial td-tst such that M'' is nondeleting and linear. By Proposition 4.21 there exists a nondeleting, linear bu-tst M_2 such that $\|M_2\|_\varepsilon = \|M''\|_\varepsilon$. Moreover, M_2 is td-deterministic whenever M'' is td-deterministic. By Lemma 4.16 there exists a td-tst M_1 with designated states such that $\|M_1\|_\varepsilon = \|M'\|_\varepsilon$. Again the td-determinism property is preserved by this construction. Let $M = M_1 \,;_{\mathrm{S}}\, M_2$. By Lemma 7.17 we have $\|M\|_\varepsilon = \|M_1\|_\varepsilon \,;\, \|M_2\|_\varepsilon$. Moreover, it is easily observed that M is in fact top-down, because M_2 is nondeleting and linear. Moreover, M is td-deterministic (respectively, nondeleting, linear), if M_1 and M_2 are td-deterministic (respectively, nondeleting, linear). □

Using the same apparatus, we should also like to generalize the second statement of [**41**, Theorem 5.18]; *i. e.*, for every $x \in \{\mathrm{d}, \mathrm{dn}, \mathrm{dl}, \mathrm{dnl}\}$

$$x\text{bt–TOP}_\varepsilon(\mathcal{A}) \,;\, x\text{–TOP}_\varepsilon(\mathcal{A}) = x\text{–TOP}_\varepsilon(\mathcal{A}) \ .$$

So let M' and M'' be td-tst. The first step is to construct a bu-tst M_2, which is semantically equivalent to M''. However, if M'' is not linear, then, in general, such a tst need not exist [because we also have $\text{p–TOP}_\varepsilon(\mathbb{B}) \not\subseteq \text{p–BOT}_\varepsilon(\mathbb{B})$]. Thus we restrict ourselves to linear M''. Consequently, let M' be boolean, deterministic, and total, and let M'' be linear. We first construct a linear bu-tst M_2 that computes the same ε-t-ts transformation as M'' (we follow the construction found in [**58**, Theorem 5.26]). Secondly, we apply the construction of Lemma 4.16 to M' and obtain the td-tst M_1. The advantage of M_2 is that Proposition 7.11 is applicable to it. We apply the simple composition to M_1 and M_2 and obtain a tst M_3 that computes the ε-t-ts transformation $\|M_3\|_\varepsilon = \|M_1\|_\varepsilon \,;\, \|M_2\|_\varepsilon$. Finally, we observe an important property (namely, that "checking followed by deletion" is not possible)

TABLE 4. Preservation of properties for the construction of Proposition 7.20.

bu	td	p	m	bu-d	bu-t	td-d	td-t	i-l	i-n	o-l	o-n	b	bu-h	td-h
✓	✗	✓	✓	✗	✗	✓	✓	✓	✓	✓	✗	✓	✗	✗

and manipulate M_3 such that we obtain a td-tst M that computes $\|M\|_\varepsilon = \|M_3\|_\varepsilon$.

We restate [**58**, Definition 5.24 and Lemma 5.25], because the construction is essential in the forthcoming theorem.

PROPOSITION 7.20 (see [**58**, Definition 5.24 and Lemma 5.25]). *Let $\mathcal{A}$ be commutative.*

$$\text{lp–TOP}_\varepsilon(\mathcal{A}) \subseteq \text{lp–BOT}_\varepsilon(\mathcal{A})$$

PROOF. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a linear polynomial td-tst and $\perp \notin Q$ be a new state. For every $k \in \mathbb{N}$ and $w = p_1(\mathrm{x}_{i_1}) \cdots p_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*$ such that every $x \in \mathrm{X}_k$ occurs at most once in w, let $\overline{w} = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$ where for every $j \in [k]$

$$q_j = \begin{cases} p_l & \text{if } \mathrm{x}_{i_l} = \mathrm{x}_j, \\ \perp & \text{otherwise.} \end{cases}$$

Note that $\overline{w}$ is well-defined. Let $\alpha \in \Delta_0$. We construct the linear polynomial bu-tst $M' = (Q', \Sigma, \Delta, \mathcal{A}, F', \mu')$ with

- $Q' = Q \cup \{\perp\}$;
- $F'_q = F_q$ for every $q \in Q$ and $F'_\perp = \widetilde{0}$;
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $q_1, \ldots, q_k \in Q'$:

$$\mu'_k(\sigma)_{q,q_1\cdots q_k} = \sum_{\substack{w = p_1(\mathrm{x}_{i_1})\cdots p_n(\mathrm{x}_{i_n}) \in Q(\mathrm{X}_k)^*,\\ \overline{w} = q_1(\mathrm{x}_1)\cdots q_k(\mathrm{x}_k)}} \Big(\sum_{u \in T_\Delta(\mathrm{Z}_n)} (\mu_k(\sigma)_{q,w}, u)\, u[\mathrm{z}_{i_j}]_{j \in [n]} \Big)$$

- $\mu'_k(\sigma)_{\perp,\perp\cdots\perp} = 1\ \alpha$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma_k$.

The proof of $\|M'\|_\varepsilon = \|M\|_\varepsilon$ can be found in [**58**, Lemma 5.25]. □

Note that $\perp$, which is introduced as an additional state in the previous construction, is a blind state. The properties that are preserved by this construction are shown in Table 4. Now we are ready to state the second composition theorem for td-tst.

THEOREM 7.21. *Let $\mathcal{A}$ be commutative.*

$$\text{bdt–TOP}_\varepsilon(\mathcal{A})\,;\text{lp–TOP}_\varepsilon(\mathcal{A}) \subseteq \text{p–TOP}_\varepsilon(\mathcal{A})$$

If $\mathcal{A}$ is $\aleph_0$-complete, then the above result also holds without the polynomial restriction.

PROOF. Let $M' = (Q', \Sigma, \Gamma, \mathcal{A}, F', \mu')$ be a boolean, deterministic, and total td-tst, and let $M'' = (Q'', \Gamma, \Delta, \mathcal{A}, F'', \mu'')$ be a linear polynomial td-tst. First we construct the td-tst $M_1 = (Q_1, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ from M'

using the construction of Lemma 4.16. It is proved in Lemma 4.16 that $\|M_1\|_\varepsilon = \|M'\|_\varepsilon$. Second we construct from M'' the linear polynomial bu-tst $M_2 = (Q_2, \Gamma, \Delta, \mathcal{A}, F_2, \mu_2)$ as presented in Proposition 7.20. Clearly, $\|M_2\|_\varepsilon = \|M''\|_\varepsilon$. Moreover, it is noteworthy that we have the following two properties. There is a blind state $\perp \in Q_2$ and an $\alpha \in \Delta_0$ such that:

(a) $h^\varepsilon_{\mu_2}(t)_\perp = 1\,\alpha$ for every $t \in T_\Gamma$ (see Observation 7.6); and
(b) for every $k \in \mathbb{N}$, symbol $\gamma \in \Gamma_k$, states $q, q_1, \dots, q_k \in Q_2$, output tree $u \in \mathrm{supp}((\mu_2)_k(\gamma)_{q,q_1\cdots q_k})$, and $i \in [k]$

$$i \notin \mathrm{var}(u) \quad \Longleftrightarrow \quad q_i = \perp \ .$$

Now we may compose M_1 with M_2 using the simple composition (see Definition 7.15). We obtain the tst $M_3 = M_1 ;_{\mathrm{S}} M_2$ (actually M_3 is a tst of type II [**86**]). Let

$$M_3 = (Q_3, \Sigma, \Delta, \mathcal{A}, F_3, \mu_3) \ .$$

We show that M_3 has the following properties (*cf.* [**86**, Lemma 2]):

(i) $h^\varepsilon_{\mu_3}(t)_{(p,\perp)} = 1\,\alpha$ for every $t \in T_\Sigma$ and $p \in Q'$;
(ii) $\mathrm{supp}((\mu_3)_k(\sigma)_{q,w})$ is linear for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q_3$, and $w \in Q_3(\mathrm{X}_k)^*$; and
(iii) for every $k \in \mathbb{N}$, $w = (p_1, q_1)(\mathrm{x}_{i_1}) \cdots (p_n, q_n)(\mathrm{x}_{i_n}) \in Q_3(\mathrm{X}_k)^*$, $i \in [n]$, $\sigma \in \Sigma_k$, $(p, q) \in Q_3$, and $u \in \mathrm{supp}((\mu_3)_k(\sigma)_{(p,q),w})$

$$i \notin \mathrm{var}(u) \quad \Longleftrightarrow \quad q_i = \perp \ .$$

(i) By the proof of Lemma 7.17 we know that $h^\varepsilon_{\mu_3}(t)_{(p,\perp)} = h^\varepsilon_{\mu_2}(h^\varepsilon_{\mu_1}(t)_p)_\perp$. By Observation 5.4 we know that $h^\varepsilon_{\mu_1}(t)_p = 1\,u$ for some $u \in T_\Gamma$. Moreover, by Property (a) we have that $h^\varepsilon_{\mu_2}(1\,u)_\perp = 1\,\alpha$; thus $h^\varepsilon_{\mu_3}(t)_{(p,\perp)} = 1\,\alpha$.

(ii–iii) These properties are easily observed because M_1 is output-linear and output-nondeleting and M_2 is linear. For Property (iii) one also needs Statement (b).

Let $n \in \mathbb{N}$. We define $\mathrm{norm}_n \colon T_\Delta(\mathrm{Z}_n) \longrightarrow T_\Delta(\mathrm{Z}_n)$ by

$$\mathrm{norm}_n(u) = \mathrm{norm}_n(u, 1)$$

for every $u \in T_\Delta(\mathrm{Z}_n)$ where

$$\mathrm{norm}_n(u, n) = u$$
$$\mathrm{norm}_n(u, i) = \begin{cases} \mathrm{norm}_n(u, i+1) & \text{if } i \in \mathrm{var}(u), \\ \mathrm{norm}_{n-1}(u[\mathrm{z}_{j-1}]_{j \in [n] \setminus [i]}, i) & \text{otherwise;} \end{cases}$$

for every $i \in [n-1]$. Intuitively speaking, norm_n normalizes a tree u, in which at most the variables $\mathrm{z}_1, \dots, \mathrm{z}_n$ may occur, by renaming the variables such that only the variables $\mathrm{z}_1, \dots, \mathrm{z}_k$ occur, where $k = \mathrm{card}(\mathrm{var}(u))$. Essentially, this normalizes scattered blocks of variables into one block of variables. Thus, *e. g.*, $\mathrm{norm}_3(\mathrm{z}_3) = \mathrm{z}_1$. Further, we define the mapping $\mathrm{del} \colon Q_3(\mathrm{X})^* \longrightarrow Q_3(\mathrm{X})^*$ for every $(p, q) \in Q_3$, $i \in \mathbb{N}_+$, and $w \in Q_3(\mathrm{X})^*$ by

$$\mathrm{del}(\varepsilon) = \varepsilon$$
$$\mathrm{del}((p, q)(\mathrm{x}_i)\, w) = \begin{cases} \mathrm{del}(w) & \text{if } q = \perp, \\ (p, q)(\mathrm{x}_i)\ \mathrm{del}(w) & \text{if } q \neq \perp. \end{cases}$$

Given an input word w, the del-mapping applied to w deletes all those symbols of w whose state has $\perp$ in the second component.

We obtain $M = (Q_3, \Sigma, \Delta, \mathcal{A}, F_3, \mu)$ as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q_3$, and $w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n}) \in Q_3(\mathrm{X}_k)^*$ let

$$\mu_k(\sigma)_{q,w} = \sum_{\substack{w' \in Q_3(\mathrm{X}_k)^*,\\ \mathrm{del}(w')=w}} \Big(\sum_{u' \in T_\Delta(\mathrm{Z})} ((\mu_3)_k(\sigma)_{q,w'}, u')\ \mathrm{norm}_{|w'|}(u') \Big) .$$

Clearly, M is a td-tst. We prove $h^\varepsilon_\mu(t)_{(p,q)} = h^\varepsilon_{\mu_3}(t)_{(p,q)}$ for every $t \in T_\Sigma$ and $(p,q) \in Q_3$ such that $q \neq \perp$. Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$\begin{aligned}
& h^\varepsilon_\mu(\sigma(t_1,\ldots,t_k))_{(p,q)} \\
= \quad & \textit{(by Definition 4.7(1))} \\
& \sum_{\substack{w \in Q_3(\mathrm{X}_k)^*,\\ w=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{(p,q),w} \xleftarrow[\varepsilon]{} (h^\varepsilon_\mu(t_{i_j})_{(p_j,q_j)})_{j\in[n]} \\
= \quad & \textit{(by induction hypothesis because } q_j \neq \perp\textit{)} \\
& \sum_{\substack{w \in Q_3(\mathrm{X}_k)^*,\\ w=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{(p,q),w} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu_3}(t_{i_j})_{(p_j,q_j)})_{j\in[n]} \\
= \quad & \textit{(by definition of } \mu_k(\sigma)_{(p,q),w}\textit{)} \\
& \sum_{\substack{w \in Q_3(\mathrm{X}_k)^*,\\ w=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} \Big(\sum_{\substack{w' \in Q_3(\mathrm{X}_k)^*,\\ \mathrm{del}(w')=w}} \\
& \Big(\sum_{u' \in T_\Delta(\mathrm{Z})} ((\mu_3)_k(\sigma)_{(p,q),w'}, u')\ \mathrm{norm}_{|w'|}(u') \Big) \Big) \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu_3}(t_{i_j})_{(p_j,q_j)})_{j\in[n]} \\
= \quad & \textit{(by Proposition 3.8)} \\
& \sum_{\substack{w' \in Q_3(\mathrm{X}_k)^*,\\ \mathrm{del}(w')=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} \Big(\sum_{u' \in T_\Delta(\mathrm{Z})} ((\mu_3)_k(\sigma)_{(p,q),w'}, u')\ \mathrm{norm}_{|w'|}(u') \Big) \\
& \qquad \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu_3}(t_{i_j})_{(p_j,q_j)})_{j\in[n]} \\
= \quad & \textit{(by Proposition 3.18 because } h^\varepsilon_{\mu_3}(t_{i_j})_{(p_j,\perp)} = 1\ \alpha\textit{)} \\
& \sum_{\substack{w' \in Q_3(\mathrm{X}_k)^*,\\ w'=(p_1,q_1)(\mathrm{x}_{i_1})\cdots(p_n,q_n)(\mathrm{x}_{i_n})}} (\mu_3)_k(\sigma)_{(p,q),w'} \xleftarrow[\varepsilon]{} (h^\varepsilon_{\mu_3}(t_{i_j})_{(p_j,q_j)})_{j\in[n]} \\
= \quad & \textit{(by Definition 4.7(1))} \\
& h^\varepsilon_{\mu_3}(\sigma(t_1,\ldots,t_k))_{(p,q)}
\end{aligned}$$

It follows that $\|M\|_\varepsilon = \|M_3\|_\varepsilon$ and thus the main statement is proved. □

5. Open problems and future work

In this chapter we concentrated on pure substitution, but composition results for devices that use o-substitution are equally interesting. Moreover, we can also study compositions of devices that are not necessarily bottom-up or top-down. A few results on such compositions can be found in [**86**]. In particular the results presented there are again only for devices that use pure substitution.

Another direction of future research are composition results for particular classes of semirings. Here we considered commutative semirings, but another very interesting class of semirings is given by all commutative and additively idempotent semirings. For them less restrictive composition results are highly likely. Finally, we can consider mixed compositions, where one device is top-down and the other is bottom-up. Lemma 7.17 shows that results in this direction are possible and such results could provide valuable insight into the power of tree series transducers.

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