Hierarchies of Tree Series Transformations Revisited

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Abstract. Tree series transformations computed by polynomial top-down and bottom-up tree series transducers are considered. The hierarchy of tree series transformations obtained in [Fülöp, Gazdag, Vogler: *Hierarchies of Tree Series Transformations*. Theoret. Comput. Sci. 314(3), p. 387–429, 2004] for commutative izz-semirings (izz abbreviates idempotent, zero-sum and zero-divisor free) is generalized to arbitrary positive (*i. e.*, zero-sum and zero-divisor free) commutative semirings. The latter class of semirings includes prominent examples such as the natural numbers semiring and the least common multiple semiring, which are not members of the former class.

1 Introduction

Tree series transducers were introduced in [1,2,3] as a generalization of top-down and bottom-up tree transducers. With the advent of tree series [4,5,6,7,8], especially recognizable tree series [9,10], in formal language theory also transducing devices capable of (finitely) representing transformations on tree series became interesting. For example, in [11] the power of (top-down) tree series transducers for natural language processing was recognized.

In the seminal paper [12] the hierarchy of top-down tree transformation classes was proved to be proper. This result lead to the hierarchy of top-down and bottom-up tree transformation classes (as, *e. g.*, displayed in [13]). This hierarchy was generalized to classes of top-down and bottom-up tree series transformations over izz-semirings (izz abbreviates idempotent, zero-divisor and zero-sum free) in [14]. Let us explain this generalization in some more detail.

By $\mathrm{p\text{–}TOP}_\varepsilon(\mathcal{A})$ and $\mathrm{p\text{–}BOT}_\varepsilon(\mathcal{A})$ we denote the classes of tree-to-tree-series transformations computable by polynomial top-down and bottom-up tree series transducers [2] over the semiring $\mathcal{A}$ [15,16], respectively. Such a tree-to-tree-series transformation is a mapping $\tau\colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta\rangle\rangle$ for some ranked alphabets Σ and Δ. Given ranked alphabets Σ, Δ, and Γ and $\tau_1\colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta\rangle\rangle$ and $\tau_2\colon T_\Delta \longrightarrow \mathcal{A}\langle\langle T_\Gamma\rangle\rangle$, the composition of τ_1 with τ_2 is denoted by $\tau_1 \circ \tau_2$ and is a mapping $\tau\colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Gamma\rangle\rangle$ (an output tree u produced by τ_1 is subjected to τ_2, and the result is multiplied by the weight of u in the series produced

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by τ_1). This composition is lifted to classes of transformations, and we write p–TOP$^n_\varepsilon(\mathcal{A})$ and p–BOT$^n_\varepsilon(\mathcal{A})$ for the n-fold composition of p–TOP$_\varepsilon(\mathcal{A})$ and p–BOT$_\varepsilon(\mathcal{A})$, respectively.

In [14] it is first proved that

$$\text{p–TOP}^n_\varepsilon(\mathcal{A}) \subseteq \text{p–BOT}^{n+1}_\varepsilon(\mathcal{A}) \quad \text{and} \quad \text{p–BOT}^n_\varepsilon(\mathcal{A}) \subseteq \text{p–TOP}^{n+1}_\varepsilon(\mathcal{A})$$

for every commutative semiring and $n \geqslant 1$ (see Theorems 5.1 and 5.7 in [14], respectively). Then in [14, Theorem 6.20] it is proved that

$$\text{p–TOP}^n_\varepsilon(\mathcal{A}) \not\subseteq \text{p–BOT}^n_\varepsilon(\mathcal{A}) \quad \text{and} \quad \text{p–BOT}^n_\varepsilon(\mathcal{A}) \not\subseteq \text{p–TOP}^n_\varepsilon(\mathcal{A})$$

for every izz-semiring and $n \geqslant 1$. Thus the hierarchy that is obtained in [14] is proved for commutative izz-semirings. We generalize the incomparability result to positive (*i. e.*, zero-sum and zero-divisor free) semirings and thereby obtain the hierarchy for all positive and commutative semirings (see Figure 1 for the Hasse diagram).

Our approach used to prove the incomparability is (in essence) similar to the one presented in [14]. However, we carefully avoid the introduction of idempotency by a simpler proof method. We furthermore claim that our method of proof is more illustrative than the one of [14].

Apart from this introduction, the paper has 3 sections. Section 2 introduces the essential notation, Section 3 generalizes the mentioned incomparability result, and Section 4 presents the obtained hierarchy (see Figure 1).

2 Preliminaries

We use $\mathbb{N}$ to represent the nonnegative integers and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. In the sequel, let $k, n \in \mathbb{N}$ and $[k]$ be an abbreviation for $\{i \in \mathbb{N} \mid 1 \leqslant i \leqslant k\}$. A set Σ that is nonempty and finite is also called an *alphabet*, and the elements thereof are called *symbols*. As usual, Σ^* denotes the set of all finite sequences of symbols of Σ (also called Σ-words). Given $w \in \Sigma^*$, the *length of* w is denoted by $|w|$.

A *ranked alphabet* is an alphabet Σ with a mapping $\mathrm{rk}_\Sigma \colon \Sigma \longrightarrow \mathbb{N}$, which associates to each symbol a *rank*. We use Σ_k to represent the set of symbols of Σ that have rank k. Moreover, we use the set $\mathrm{X} = \{\mathrm{x}_i \mid i \in \mathbb{N}_+\}$ of *(formal) variables* and $\mathrm{X}_k = \{\mathrm{x}_i \mid i \in [k]\}$. Given a ranked alphabet Σ and $V \subseteq \mathrm{X}$, the set of *Σ-trees indexed by V*, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T$ also $\sigma(t_1, \ldots, t_k) \in T$. Since we generally assume that $\Sigma \cap \mathrm{X} = \emptyset$, we write α instead of $\alpha()$ whenever $\alpha \in \Sigma_0$. Moreover, we also write T_Σ to denote $T_\Sigma(\emptyset)$.

Given $t_1, \ldots, t_n \in T_\Sigma(\mathrm{X})$, the expression $t[t_1, \ldots, t_n]$ denotes the result of substituting in t every x_i by t_i for every $i \in [n]$. Let $V \subseteq \mathrm{X}$. We say that $t \in T_\Sigma(\mathrm{X})$ is *linear* and *nondeleting* in V, if every $x \in V$ occurs at most once and at least once in t, respectively.

A *semiring* is an algebraic structure $\mathcal{A} = (A, +, \cdot, 0, 1)$ consisting of a commutative monoid $(A, +, 0)$ and a monoid $(A, \cdot, 1)$ such that $\cdot$ distributes over $+$

and 0 is absorbing with respect to $\cdot$. The semiring is called commutative, if $\cdot$ is commutative. As usual we use $\sum_{i \in I} a_i$ for sums of families $(a_i)_{i \in I}$ of $a_i \in A$ where for only finitely many $i \in I$ we have $a_i \neq 0$. Let $\mathcal{A} = (A, +, \cdot, 0_{\mathcal{A}}, 1_{\mathcal{A}})$ and $\mathcal{B} = (B, \oplus, \odot, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ be semirings and $h \colon A \longrightarrow B$. The mapping h is called *homomorphism from $\mathcal{A}$ to $\mathcal{B}$*, if

- $h(0_{\mathcal{A}}) = 0_{\mathcal{B}}$ and $h(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, and
- $h(a + b) = h(a) \oplus h(b)$ and $h(a \cdot b) = h(a) \odot h(b)$ for every $a, b \in A$.

A semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$ is called *idempotent*, if $1 + 1 = 1$. Moreover, we say that a semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$ is *zero-sum free*, if $a + b = 0$ implies that $a = 0 = b$ for every $a, b \in A$. Moreover, $\mathcal{A}$ is *zero-divisor free*, if $a \cdot b = 0$ implies that $0 \in \{a, b\}$ for every $a, b \in A$. A zero-sum and zero-divisor free semiring is also called *positive*. The Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ with the usual disjunction $\vee$ and conjunction $\wedge$ is an example of a positive semiring.

Let S be a set and $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring. A *(formal) power series* ψ is a mapping $\psi \colon S \longrightarrow A$. Given $s \in S$, we denote $\psi(s)$ also by (ψ, s) and write the series as $\sum_{s \in S} (\psi, s)\, s$. The *support* of ψ is $\operatorname{supp}(\psi) = \{s \in S \mid (\psi, s) \neq 0\}$. Power series with finite support are called *polynomials*. We denote the set of all power series by $\mathcal{A}\langle\langle S \rangle\rangle$ and the set of polynomials by $\mathcal{A}\langle S \rangle$. The polynomial with empty support is denoted by $\widetilde{0}$. Power series $\psi, \psi' \in \mathcal{A}\langle\langle S \rangle\rangle$ are added componentwise; *i. e.*, $(\psi + \psi', s) = (\psi, s) + (\psi', s)$ for every $s \in S$, and we multiply ψ with a coefficient $a \in A$ componentwise; *i. e.*, $(a \cdot \psi, s) = a \cdot (\psi, s)$ for every $s \in S$.

In this paper, we only consider power series in which the set S is a set of trees. Such power series are also called *tree series*. Let Δ be a ranked alphabet. A tree series $\psi \in \mathcal{A}\langle\langle T_\Delta(\mathrm{X}) \rangle\rangle$ is said to be *linear* and *nondeleting* in $V \subseteq \mathrm{X}$, if every $t \in \operatorname{supp}(\psi)$ is linear and nondeleting in V, respectively. Let $\psi \in \mathcal{A}\langle T_\Delta(\mathrm{X}) \rangle$ and $\psi_1, \ldots, \psi_n \in \mathcal{A}\langle T_\Delta(\mathrm{X}) \rangle$. The *pure IO tree series substitution* (for short: pure substitution) (of $\psi_1, \ldots, \psi_n$ into ψ) [17,2], denoted by $\psi \underset{\varepsilon}{\longleftarrow} (\psi_1, \ldots, \psi_n)$, is defined by

$$\psi \underset{\varepsilon}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in T_\Delta(\mathrm{X}), \\ t_1, \ldots, t_n \in T_\Delta(\mathrm{X})}} (\psi, t) \cdot (\psi_1, t_1) \cdot \ldots \cdot (\psi_n, t_n)\, t[t_1, \ldots, t_n] \enspace .$$

Let Q be an alphabet. We write $Q(V)$ for $\{q(v) \mid q \in Q, v \in V\}$. We use the notions of linearity and nondeletion in V accordingly also for $w \in Q(\mathrm{X})^*$. Let $\mathcal{A} = (A, +, \cdot, 0, 1)$ be a semiring and Σ and Δ be ranked alphabets. A *tree representation* μ (over Q, Σ, Δ, and $\mathcal{A}$) [2] is a family $(\mu(\sigma))_{\sigma \in \Sigma}$ of matrices $\mu(\sigma) \in \mathcal{A}\langle\langle T_\Delta(\mathrm{X}) \rangle\rangle^{Q \times Q(\mathrm{X}_k)^*}$ where $k = \operatorname{rk}_\Sigma(\sigma)$ such that for every $q \in Q$ and $w \in Q(\mathrm{X}_k)^*$ it holds that $\mu(\sigma)_{q,w} \in \mathcal{A}\langle\langle T_\Delta(\mathrm{X}_n) \rangle\rangle$ with $n = |w|$, and $\mu(\sigma)_{q,w} \neq \widetilde{0}$ for only finitely many $(q, w) \in Q \times Q(\mathrm{X}_k)^*$. A tree representation μ is said to be

- *polynomial*, if $\mu(\sigma)_{q,w}$ is polynomial for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$;
- *linear*, if $\mu(\sigma)_{q,w}$ is linear in $\mathrm{X}_{|w|}$ and w is linear in X_k for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ such that $\mu(\sigma)_{q,w} \neq \widetilde{0}$;

- *top-down* (respectively, *top-down with regular look-ahead*), if $\mu(\sigma)_{q,w}$ is linear and nondeleting (respectively, linear) in $\mathrm{X}_{|w|}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$; and
- *bottom-up*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$ such that $\mu(\sigma)_{q,w} \neq \widetilde{0}$ we have $w = q_1(\mathrm{x}_1) \cdots q_k(\mathrm{x}_k)$ for some $q_1, \ldots, q_k \in Q$.

A *tree series transducer* [2,6] (with designated states), in the sequel abbreviated to tst, is a sixtuple $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ consisting of

- an alphabet Q of *states*,
- ranked alphabets Σ and Δ, also called *input* and *output ranked alphabet*, respectively,
- a semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$,
- a subset $F \subseteq Q$ of *designated states*, and
- a tree representation μ over Q, Σ, Δ, and $\mathcal{A}$.

Tst inherit the properties from their tree representation; *e. g.*, a tst with a polynomial bottom-up tree representation is called a polynomial bottom-up tst. Additionally, we abbreviate bottom-up tst to bu-tst and top-down tst to td-tst.

We introduce the semantics only for polynomial tst because we defined pure substitution only for polynomial tree series (in order to avoid a well-definedness issue related to infinite sums). Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a polynomial tst. Then M induces a mapping $\|M\| \colon T_\Sigma \longrightarrow \mathcal{A}\langle T_\Delta \rangle$ as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$ we define the mapping $h_\mu \colon T_\Sigma \longrightarrow \mathcal{A}\langle T_\Delta \rangle^Q$ componentwise for every $q \in Q$ by

$$h_\mu(\sigma(t_1, \ldots, t_k))_q = \sum_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu_k(\sigma)_{q,w} \xleftarrow[\varepsilon]{} (h_\mu(t_{i_1})_{q_1}, \ldots, h_\mu(t_{i_n})_{q_n}) \enspace .$$

For every $t \in T_\Sigma$ the *tree-to-tree-series (for short: ε-t-ts) transformation computed by M* is $\|M\|(t) = \sum_{q \in F} h_\mu(t)_q$.

By $\mathrm{p\text{–}TOP}_\varepsilon(\mathcal{A})$ and $\mathrm{p\text{–}BOT}_\varepsilon(\mathcal{A})$ we denote the class of ε-t-ts transformations computable by polynomial td-tst and bu-tst over the semiring $\mathcal{A}$, respectively. Likewise we use the prefix l for the linearity property and the stems $\mathrm{TOP}^{\mathrm{R}}_\varepsilon$ and GST_ε for td-tst with regular look-ahead and unrestricted tst, respectively.

We compose ε-t-ts transformations as follows. Let $\tau_1 \colon T_\Sigma \longrightarrow \mathcal{A}\langle T_\Delta \rangle$ and $\tau_2 \colon T_\Delta \longrightarrow \mathcal{A}\langle T_\Gamma \rangle$ then $(\tau_1 \circ \tau_2)(t) = \sum_{u \in T_\Delta} (\tau_1(t), u) \cdot \tau_2(u)$ for every $t \in T_\Sigma$. This composition is extended to classes of ε-t-ts transformations in the usual manner. By $\mathrm{p\text{–}TOP}^n_\varepsilon(\mathcal{A})$ and $\mathrm{p\text{–}BOT}^n_\varepsilon(\mathcal{A})$ with $n \in \mathbb{N}_+$ we denote the n-fold composition $\mathrm{p\text{–}TOP}_\varepsilon(\mathcal{A}) \circ \cdots \circ \mathrm{p\text{–}TOP}_\varepsilon(\mathcal{A})$ and $\mathrm{p\text{–}BOT}_\varepsilon(\mathcal{A}) \circ \cdots \circ \mathrm{p\text{–}BOT}_\varepsilon(\mathcal{A})$, respectively.

3 Incomparability Results

We show the incomparability of $\mathrm{p\text{–}TOP}^n_\varepsilon(\mathcal{A})$ and $\mathrm{p\text{–}BOT}^n_\varepsilon(\mathcal{A})$ for every $n \in \mathbb{N}_+$ and positive semiring $\mathcal{A}$. Together with the results of [14] this yields the Hasse diagram (see Figure 1) that displays the top-down, bottom-up, and alternating hierarchy of tree series transformations. We arrive at the same HASSE diagram as [14], but we can prove it for a distinctively larger class of semirings; namely positive commutative semirings instead of positive, idempotent, and commutative semirings as in [14].

First we show the main property that we exploit in the sequel. Roughly speaking, given a positive semiring $\mathcal{A}$ we present a specific homomorphism from $\mathcal{A}$ to the Boolean semiring $\mathbb{B}$. We later use this homomorphism to lift the incomparability of the top-down and bottom-up tree transformation classes to the level of ε-t-ts transformation classes.

Lemma 1. *Let $\mathcal{A} = (A, +, \cdot, 0_\mathcal{A}, 1_\mathcal{A})$ be a positive semiring. Let $\chi \colon A \longrightarrow \{0, 1\}$ be such that $\chi(0_\mathcal{A}) = 0$ and $\chi(a) = 1$ for every $a \in A \setminus \{0_\mathcal{A}\}$. Then χ is a homomorphism from $\mathcal{A}$ to $\mathbb{B}$.*

Let $\mathcal{A} = (A, +, \cdot, 0_\mathcal{A}, 1_\mathcal{A})$ and $\mathcal{B} = (B, \oplus, \odot, 0_\mathcal{B}, 1_\mathcal{B})$ be two semirings and $\tau \colon T_\Sigma \longrightarrow \mathcal{A}\langle\langle T_\Delta\rangle\rangle$, and $h \colon A \longrightarrow B$. The *image of τ under h*, denoted by $h(\tau)$, is defined by $(h(\tau)(t), u) = h((\tau(t), u))$ for every $t \in T_\Sigma$ and $u \in T_\Delta$. Clearly, $h(\tau) \colon T_\Sigma \longrightarrow \mathcal{B}\langle\langle T_\Delta\rangle\rangle$. If h is a homomorphism, then we also call $h(\tau)$ the *homomorphic image* of τ. This notion of (homomorphic) image is lifted to classes of ε-t-ts transformations in the usual manner.

Next we show that, given an ε-t-ts transformation τ computed by a polynomial td-tst or bu-tst M over the semiring $\mathcal{A}$ and a homomorphism h from $\mathcal{A}$ to $\mathcal{B}$, there exists a polynomial td-tst or bu-tst M' over the semiring $\mathcal{B}$ such that M' computes the homomorphic image of τ; *i. e.*, h is applied to all coefficients in the range of the ε-t-ts transformation τ. This is also the main idea of the construction; we simply apply the homomorphism to all coefficients in the tree representation of M to obtain the tree representation of M'.

Moreover, we show that computable ε-t-ts transformations are also closed under inverse homomorphisms. For this we need the following definition. Let $h \colon A \longrightarrow B$ and $\tau' \colon T_\Sigma \longrightarrow \mathcal{B}\langle\langle T_\Delta\rangle\rangle$. By $h^{-1}(\tau')$ we denote the set

$$\{\tau \in \mathcal{A}\langle\langle T_\Delta\rangle\rangle^{T_\Sigma} \mid h(\tau) = \tau'\} \enspace .$$

This is again lifted to classes as usual.

Lemma 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be semirings and h be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

$$h(\text{p–TOP}_\varepsilon(\mathcal{A})) \subseteq \text{p–TOP}_\varepsilon(\mathcal{B}) \quad \textit{and} \quad h(\text{p–BOT}_\varepsilon(\mathcal{A})) \subseteq \text{p–BOT}_\varepsilon(\mathcal{B})$$

If h is surjective, then also

$$h^{-1}(\text{p–TOP}_\varepsilon(\mathcal{B})) \subseteq \text{p–TOP}_\varepsilon(\mathcal{A}) \quad \textit{and} \quad h^{-1}(\text{p–BOT}_\varepsilon(\mathcal{B})) \subseteq \text{p–BOT}_\varepsilon(\mathcal{A})$$

Proof. Let $\mathcal{C} = (C, +, \cdot, 0_\mathcal{C}, 1_\mathcal{C})$ and $\mathcal{D} = (D, \oplus, \odot, 0_\mathcal{D}, 1_\mathcal{D})$. Let $f \colon C \longrightarrow D$ and $M = (Q, \Sigma, \Delta, \mathcal{C}, F, \mu)$ be a tst. We construct the tst $f(M) = (Q, \Sigma, \Delta, \mathcal{D}, F, \mu')$ as follows. For every $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $q \in Q$, and $w \in Q(\mathrm{X}_k)^*$

$$\mu'(\sigma)_{q,w} = \bigoplus_{u \in \mathrm{supp}(\mu(\sigma)_{q,w})} f((\mu(\sigma)_{q,w}, u))\, u \enspace .$$

Clearly, $f(M)$ is top-down and bottom-up whenever M is top-down and bottom-up, respectively.

Let us prove the former statement. Let $\tau \in$ p–TOP$_\varepsilon(\mathcal{A})$ or $\tau \in$ p–BOT$_\varepsilon(\mathcal{A})$. There exists a polynomial td-tst or bu-tst M such that $\|M\| = \tau$. We claim that $\|h(M)\| = h(\|M\|)$. The proof of this statement can be found below.

For the second statement, let $\tau \in$ p–TOP$_\varepsilon(\mathcal{B})$ or $\tau \in$ p–BOT$_\varepsilon(\mathcal{B})$. There exists a polynomial td-tst or bu-tst M such that $\|M\| = \tau$. Moreover, let $f\colon B \longrightarrow A$ be such that $h(f(b)) = b$ for every $b \in B$. Such an f exists, because h is surjective. The claim $\|f(M)\| \in h^{-1}(\|M\|)$ follows from $h(\|f(M)\|) = \|M\|$, whose proof can also be found below.

Now we prove the mentioned result. Let h be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ with $\mathcal{A} = (A, +, \cdot, 0_\mathcal{A}, 1_\mathcal{A})$ and $\mathcal{B} = (B, \oplus, \odot, 0_\mathcal{B}, 1_\mathcal{B})$. Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a tst. Then $\|h(M)\| = h(\|M\|)$. Let $h(M) = (Q, \Sigma, \Delta, \mathcal{B}, F, \mu')$. We first prove the auxiliary statement that $(h_{\mu'}(t)_q, u) = h((h_\mu(t)_q, u))$ for every $q \in Q$, $t \in T_\Sigma$, and $u \in T_\Delta$. This is proved inductively, so let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T_\Sigma$.

$$
\begin{aligned}
& (h_{\mu'}(\sigma(t_1, \ldots, t_k))_q, u) \\
= \quad & \text{(by definition of } h_{\mu'}) \\
& \Big(\bigoplus_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu'(\sigma)_{q,w} \xleftarrow[\varepsilon]{} (h_{\mu'}(t_{i_1})_{q_1}, \ldots, h_{\mu'}(t_{i_n})_{q_n}), u \Big) \\
= \quad & \text{(by definition of } \xleftarrow[\varepsilon]{}) \\
& \Big(\bigoplus_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \bigoplus_{\substack{u' \in T_\Delta(\mathrm{X}_n), \\ u_1, \ldots, u_n \in T_\Delta}} (\mu'(\sigma)_{q,w}, u') \odot \\
& \quad \odot (h_{\mu'}(t_{i_1})_{q_1}, u_1) \odot \cdots \odot (h_{\mu'}(t_{i_n})_{q_n}, u_n)\, u'[u_1, \ldots, u_n], u \Big) \\
= \quad & \text{(by definition of } \mu' \text{ and induction hypothesis)} \\
& \Big(\bigoplus_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \bigoplus_{\substack{u' \in T_\Delta(\mathrm{X}_n), \\ u_1, \ldots, u_n \in T_\Delta}} h((\mu(\sigma)_{q,w}, u')) \odot \\
& \quad \odot h((h_\mu(t_{i_1})_{q_1}, u_1)) \odot \cdots \odot h((h_\mu(t_{i_n})_{q_n}, u_n))\, u'[u_1, \ldots, u_n], u \Big) \\
= \quad & \text{(by homomorphism property)} \\
& \bigoplus_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \Big(\bigoplus_{\substack{u' \in T_\Delta(\mathrm{X}_n), \\ u_1, \ldots, u_n \in T_\Delta}} h\Big((\mu(\sigma)_{q,w}, u') \cdot \\
& \quad \cdot (h_\mu(t_{i_1})_{q_1}, u_1) \cdot \ldots \cdot (h_\mu(t_{i_n})_{q_n}, u_n) \Big)\, u'[u_1, \ldots, u_n], u \Big) \\
= \quad & \text{(by homomorphism property and definition of } \xleftarrow[\varepsilon]{}) \\
& \bigoplus_{\substack{w \in Q(\mathrm{X}_k)^*, \\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} h\big(\mu(\sigma)_{q,w} \xleftarrow[\varepsilon]{} (h_\mu(t_{i_1})_{q_1}, \ldots, h_\mu(t_{i_n})_{q_n}), u \big)
\end{aligned}
$$

$$= \quad \text{(by homomorphism property)}$$
$$h\Big(\sum_{\substack{w \in Q(\mathrm{X}_k)^*,\\ w = q_1(\mathrm{x}_{i_1}) \cdots q_n(\mathrm{x}_{i_n})}} \mu(\sigma)_{q,w} \xleftarrow[\varepsilon]{} (h_\mu(t_{i_1})_{q_1}, \ldots, h_\mu(t_{i_n})_{q_n}), u\Big)$$
$$= \quad \text{(by definition of } h_\mu\text{)}$$
$$h((h_\mu(\sigma(t_1, \ldots, t_k))_q, u))$$

With this statement the proof is easy. We observe that for every $t \in T_\Sigma$ and $u \in T_\Delta$

$$(\|h(M)\|(t), u) \;=\; \Big(\bigoplus_{q \in F} h_{\mu'}(t)_q, u\Big) \;=\; \bigoplus_{q \in F} (h_{\mu'}(t)_q, u)$$
$$= \quad \text{(by the auxiliary statement)}$$
$$\bigoplus_{q \in F} h((h_\mu(t)_q, u)) \;=\; h\Big(\sum_{q \in F} (h_\mu(t)_q, u)\Big) \;=\; h\Big(\Big(\sum_{q \in F} h_\mu(t)_q, u\Big)\Big)$$
$$= h((\|M\|(t), u)) \;\; .$$

This lemma admits an important corollary, which will form the basis of our new lifting result. Roughly, the corollary states that every ε-t-ts transformation computed by a polynomial td-tst or bu-tst over $\mathbb{B}$ can also be computed as the homomorphic image (under χ) of the ε-t-ts transformation computed by a polynomial td-tst or bu-tst over the positive semiring $\mathcal{A}$. The statement also holds vice versa.

Corollary 1. *Let $\mathcal{A}$ be a positive semiring.*

$$\chi(\text{p–TOP}_\varepsilon(\mathcal{A})) = \text{p–TOP}_\varepsilon(\mathbb{B}) \quad \textit{and} \quad \chi(\text{p–BOT}_\varepsilon(\mathcal{A})) = \text{p–BOT}_\varepsilon(\mathbb{B})$$

Proof. We have seen in Lemma 1 that χ is a homomorphism from $\mathcal{A}$ to $\mathbb{B}$. Consequently, the statement holds by Lemma 2 because χ is surjective.

Next we show that homomorphisms are compatible with the composition introduced for ε-t-ts transformations.

Lemma 3. *Let h be a homomorphism from the semiring $\mathcal{A}$ to the semiring $\mathcal{B}$. Moreover, let $\tau_1 \colon T_\Sigma \longrightarrow \mathcal{A}\langle T_\Delta \rangle$ and $\tau_2 \colon T_\Delta \longrightarrow \mathcal{A}\langle T_\Gamma \rangle$.*

$$h(\tau_1 \circ \tau_2) = h(\tau_1) \circ h(\tau_2)$$

Proof. Let $t \in T_\Sigma$ and $u' \in T_\Gamma$ be an input and output tree, respectively. Further, let $\mathcal{A} = (A, +, \cdot, 0_\mathcal{A}, 1_\mathcal{A})$ and $\mathcal{B} = (B, \oplus, \odot, 0_\mathcal{B}, 1_\mathcal{B})$.

$$h\big(((\tau_1 \circ \tau_2)(t), u')\big) \;=\; h\Big(\Big(\sum_{u \in T_\Delta} (\tau_1(t), u) \cdot \tau_2(u), u'\Big)\Big)$$
$$= \bigoplus_{u \in T_\Delta} h\big(((\tau_1(t), u) \cdot \tau_2(u), u')\big) \;=\; \bigoplus_{u \in T_\Delta} h((\tau_1(t), u)) \odot h((\tau_2(u), u'))$$
$$= \bigoplus_{u \in T_\Delta} (h(\tau_1)(t), u) \odot (h(\tau_2)(u), u') \;=\; \big((h(\tau_1) \circ h(\tau_2))(t), u'\big)$$

Now we ready to state our main theorem, which states the incomparability of p–$\mathrm{TOP}^n_\varepsilon(\mathcal{A})$ and p–$\mathrm{BOT}^n_\varepsilon(\mathcal{A})$ in all positive semirings.

Theorem 1. *Let $\mathcal{A}$ be a positive semiring and $n \in \mathbb{N}_+$.*

$$\text{p–TOP}^n_\varepsilon(\mathcal{A}) \nsubseteq \text{p–BOT}^n_\varepsilon(\mathcal{A}) \qquad \text{p–BOT}^n_\varepsilon(\mathcal{A}) \nsubseteq \text{p–TOP}^n_\varepsilon(\mathcal{A})$$

Proof. We prove the statement by contradiction. To this end, suppose that p–$\mathrm{TOP}^n_\varepsilon(\mathcal{A}) \subseteq$ p–$\mathrm{BOT}^n_\varepsilon(\mathcal{A})$. Then

$$\begin{aligned} & \chi(\text{p–TOP}^n_\varepsilon(\mathcal{A})) & \\ &= \chi(\text{p–TOP}_\varepsilon(\mathcal{A})) \circ \cdots \circ \chi(\text{p–TOP}_\varepsilon(\mathcal{A})) && \text{by Lemma 3} \\ &= \text{p–TOP}_\varepsilon(\mathbb{B}) \circ \cdots \circ \text{p–TOP}_\varepsilon(\mathbb{B}) && \text{by Corollary 1} \\ &= \text{p–TOP}^n_\varepsilon(\mathbb{B}) && \text{by definition} \end{aligned}$$

Analogously we obtain $\chi(\text{p–BOT}^n_\varepsilon(\mathcal{A})) = \text{p–BOT}^n_\varepsilon(\mathbb{B})$. It follows that we also have p–$\mathrm{TOP}^n_\varepsilon(\mathbb{B}) \subseteq$ p–$\mathrm{BOT}^n_\varepsilon(\mathbb{B})$. This, however, contradicts the famous tree transducer hierarchy [18] due to [2, Corollaries 4.7 and 4.14]. The second statement is proved analogously.

4 Hierarchy Results

In this section we state the hierarchy result that can be obtained with the new incomparability result. First we recall the inclusion results of [14].

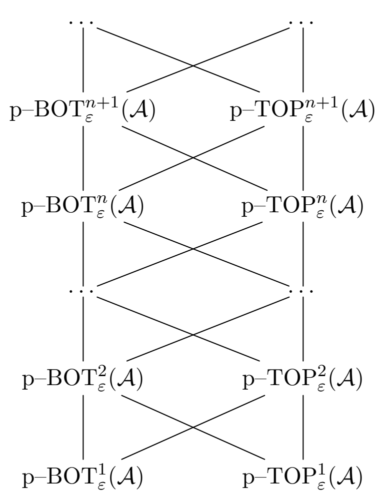

Fig. 1. HASSE diagram of the hierarchies.

Proposition 1 (Theorems 5.1 and 5.7 of [14]). *Let $\mathcal{A}$ be commutative and $n \in \mathbb{N}_+$.*

$$\text{p–BOT}^n_\varepsilon(\mathcal{A}) \subseteq \text{p–TOP}^{n+1}_\varepsilon(\mathcal{A}) \qquad \text{p–TOP}^n_\varepsilon(\mathcal{A}) \subseteq \text{p–BOT}^{n+1}_\varepsilon(\mathcal{A})$$

With these inclusions and the incomparability results of Theorem 1 we obtain the following hierarchy result for positive and commutative semirings. Important semirings like

- the semiring of nonnegative integers $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$,
- the least common multiple semiring $\mathrm{Lcm} = (\mathbb{N}, \mathrm{lcm}, \cdot, 0, 1)$, and
- the matrix semiring $\mathrm{Mat}_n(\mathbb{N}_+) = (\mathbb{N}_+^{n \times n} \cup \{\underline{0}, \underline{1}\}, +, \cdot, \underline{0}, \underline{1})$ over $\mathbb{N}_+$ (where $\underline{0}$ is the $n \times n$ zero matrix and $\underline{1}$ is the $n \times n$ unit matrix)

are all positive, but not idempotent. However, the matrix semiring is not commutative.

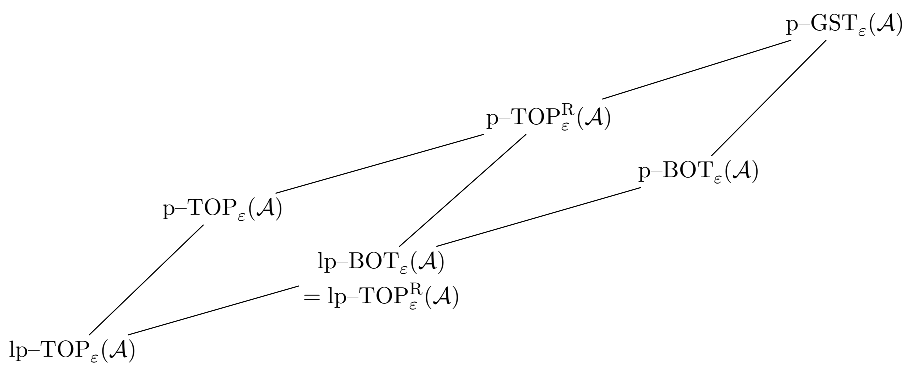

Fig. 2. Hasse diagram of general tst.

Theorem 2. *Let $\mathcal{A}$ be a positive and commutative semiring. Figure 1 is the* Hasse *diagram for the depicted classes of transformations (ordered by inclusion).*

Proof. The inclusions are trivial or follow from Proposition 1. Incomparability is shown in Theorem 1.

Similarly, we can use the approach also for other incomparability results. For example, in [19] a diagram of inclusions is presented (for commutative semirings, *cf.* Section 6 of [20]), however the properness of the inclusions remained open. Using our approach we can now prove this diagram to be a Hasse diagram.

Theorem 3. *Let $\mathcal{A}$ be a positive and commutative semiring. Figure 2 is the* Hasse *diagram for the depicted classes of transformations (ordered by inclusion).*

Proof. Note that the inclusions are proved in [19]. It remains to prove strictness and incomparability.

First we note that the construction of Lemma 2 preserves all introduced properties (thus also linearity and top-down with regular look-ahead). Thus we obtain the following statements.

$$\begin{aligned} \chi(\text{p–TOP}^{\text{R}}_\varepsilon(\mathcal{A})) &= \text{p–TOP}^{\text{R}}_\varepsilon(\mathbb{B}) & \chi(\text{p–GST}_\varepsilon(\mathcal{A})) &= \text{p–GST}_\varepsilon(\mathbb{B}) \\ \chi(\text{lp–TOP}^{\text{R}}_\varepsilon(\mathcal{A})) &= \text{lp–TOP}^{\text{R}}_\varepsilon(\mathbb{B}) & \chi(\text{lp–GST}_\varepsilon(\mathcal{A})) &= \text{lp–GST}_\varepsilon(\mathbb{B}) \\ \chi(\text{lp–TOP}_\varepsilon(\mathcal{A})) &= \text{lp–TOP}_\varepsilon(\mathbb{B}) & \chi(\text{lp–BOT}_\varepsilon(\mathcal{A})) &= \text{lp–BOT}_\varepsilon(\mathbb{B}) \end{aligned}$$

In Section 5 of [20] the diagram is proved to be Hasse diagram for the Boolean semiring and we lift the incomparability results of this diagram using the approach used in the proof of Theorem 2. This proves the correctness of the diagram presented in Figure 2.

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