

Cryptography

Lecture 2: Vigenère Cipher & Perfect Secrecy

October 23, 2026

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Monoalphabetic Cipher

Key (bijection) b

σ	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
$b(\sigma)$	k	d	g	f	n	s	l	v	b	w	a	h	e	x	j	m	q	c	p	z	r	t	y	i	u	o

Illustration

- Ciphertext $\text{enc}_b(\text{"hello"}) = \text{"vnhhj"}$
- Message $\text{dec}_b(\text{"vnhhj"}) = \text{"hello"}$

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Monoalphabetic Cipher

§2.1 Example (Monoalphabetic cipher)

- Key space $K = \{b: \Sigma \rightarrow \Sigma \mid b \text{ bijective}\}$ with $\Sigma = \{'a', 'b', \dots, 'z'\}$
- Message space $\mathcal{M} = \Sigma^*$
- Ciphertext space $\mathcal{C} = \Sigma^*$
- Encryption $\text{enc}_b(m_1, \dots, m_n) = (b(m_1), \dots, b(m_n))$
- Decryption $\text{dec}_b(c_1, \dots, c_n) = (b^{-1}(c_1), \dots, b^{-1}(c_n))$

Note

- Caesar & shift cipher are monoalphabetic

Illustration of bijection b

σ	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
$b(\sigma)$	k	d	g	f	n	s	l	v	b	w	a	h	e	x	j	m	q	c	p	z	r	t	y	i	u	o

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Monoalphabetic Cipher

Size of key space

$$26! = 4.03 \cdot 10^{26} \geq 2^{88}$$

Sufficiently large key space

- Suppose decrypt/analysis takes 1 as per key candidate
 - (1,000 AMD Threadripper 7980X [64,000 cores]: $6.8 \cdot 10^9$ MIPS)
 - (Fujitsu Fugaku [7,630,848 cores]: $537 \cdot 10^9$ MIPS)
 - (HPE Frontier [591,872 cores]: $1,686 \cdot 10^9$ MIPS)
 - ($\rightarrow$ AMD: 6.8 instructions per fs Frontier: 1.686 instructions per as)
- Brute-force attack takes $4.03 \cdot 10^8 \text{ s} \approx 12.8 \text{ years}$

So is this scheme secure?

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Monoalphabetic Cipher

All monoalphabetic ciphers are insecure

Attacks

- Shift-cipher attack fails in general
- Frequency analysis still successful (→ exercise)

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Vigenère Cipher

§2.2 Example (Vigenère cipher; actually Giovan Battista Bellaso 1553)

- Key space $K = \Sigma^+$ with $\Sigma = \{'a', 'b', \dots, 'z'\}$
- Message and ciphertext space $M = C = \Sigma^*$
- Encryption $\text{enc}_k(m_1, \dots, m_n) = (m_1 \oplus k_0 \bmod \ell, \dots, m_n \oplus k_{(n-1) \bmod \ell})$ with key $k = (k_0, \dots, k_{\ell-1})$
- Decryption $\text{dec}_k(c_1, \dots, c_n) = (c_1 \ominus k_0 \bmod \ell, \dots, m_n \ominus k_{(n-1) \bmod \ell})$

Blaise de Vigenère (* 1523; † 1596)

- French diplomat, cryptographer & alchemist
- Diplomatic mission to Rome
- Vigenère cipher misattributed in 19th century; he developed stronger autokey cipher



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Vigenère Cipher

Notes

- $\oplus$ and $\ominus$ modular arithmetic on letters according to alphabet position
- Considered unbreakable until 1863 (Kasiski attack)

Illustration

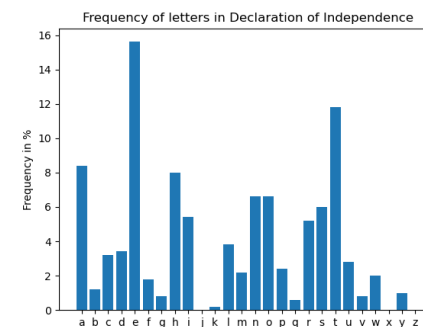
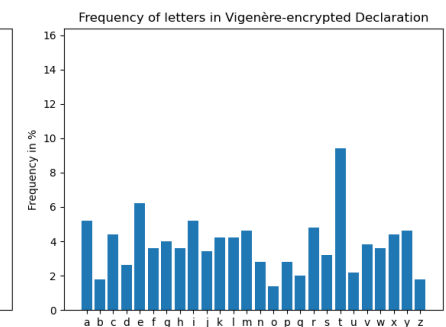
message	t	h	i	s	i	s	t	h	e	s	e	c	r	e	t	m	e	s	s	a	g	e
repeated key	p	a	r	t	y	p	a	r	t	y	p	a	r	t	y	p	a	r	t	y	p	a
ciphertext	i	h	z	l	g	h	t	y	x	q	t	c	i	x	r	b	e	j	l	y	v	e

- Sample computation $'t' \oplus 'p' = 'i'$

$$19 + 15 \bmod 26 = 8$$

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Cryptanalysis of Vigenère Cipher



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Cryptanalysis of Vigenère Cipher

Main insights

- No monoalphabetic cipher (unless key length 1)
- Suppose key length $|k| = \ell$ and cipher text $c = c_1 \cdots c_n$
 - ▶ $w_i^{(\ell)} = c_{i+0\ell} c_{i+1\ell} c_{i+2\ell} \cdots$ encrypted by shift k_{i-1} for all $1 \leq i \leq \ell$
 - ▶ Provided c is long enough $\rightarrow w_i^{(\ell)}$ long enough
 - ▶ Break shift cipher on $w_i^{(\ell)}$ to determine k_{i-1}
 - ▶ Determine all key letters $k = k_0 \cdots k_{\ell-1}$

Illustration

message	t	h	i	s	i	s	t	h	e	s	e	c	r	e	t	m	e	s	s	a	g	e
repeated key	p	a	r	t	y	p	a	r	t	y	p	a	r	t	y	p	a	r	t	y	p	a
ciphertext	i	h	z	l	g	h	t	y	x	q	t	c	i	x	r	b	e	i	l	y	v	e

How to determine key length?

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Cryptanalysis of Vigenère Cipher

Determine key length

- Guess upper bound $\hat{\ell}$ on key length
- Determine $w_i^{(\ell)}$ for all $1 \leq \ell \leq \hat{\ell}$ and $1 \leq i \leq \ell$
- Perform frequency analysis on $w_i^{(\ell)}$; i.e. determine $I(w_i^{(\ell)})$
- Select smallest i' such that $I(w_i^{(i')}) \approx I = 1.706926$ for all $1 \leq i \leq i'$
- Suppose key length is i' and proceed to break i' shift ciphers

Questions

- Why does this work?
- Why smallest i' and not “best” (best-match) i' ?
- What is expected value $I(w_i^{(i')})$ for wrong choice of i' ?

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Cryptanalysis of Vigenère Cipher

Letter counts b_i^m of message m with $n = |m|$

$$I(m) = 26 \sum_{i=0}^{25} \frac{b_i^m(b_i^m-1)}{n(n-1)}$$

$I(c_i^{(\ell)})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$\ell = 1$	1.128					
$\ell = 2$	1.119	1.109				
$\ell = 3$	1.047	1.240	1.105			
$\ell = 4$	1.188	1.047	1.060	1.091		
$\ell = 5$	2.017	2.064	1.917	1.943	1.629	
$\ell = 6$	1.059	1.352	1.276	0.955	1.154	1.054

$\rightarrow$ Key length probably 5

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Classic Ciphers

Outlook

- Multiple additional ciphers & attacks (autokey ciphers, Enigma, etc.)
- Classical cryptography was art
 - ▶ Ciphers developed & deployed (kept secret)
 - ▶ Tested only by initiated few
 - ▶ Eventually reverse-engineered & vulnerabilities discovered (often grave consequences; e.g. Enigma)
 - ▶ Cipher adjusted to prevent existing attacks (rinse & repeat)
- Perfect secrecy & one-time pad

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Perfectly Secret Encryption

§2.3 Definition (Probabilistic algorithm)

Probabilistic algorithm is multi-tape deterministic Turing machine (DTM) that additionally has access to infinite read-only tape of (truly) random bits

Notes

- Compute computable **probabilistic functions** $f: \{0,1\}^\omega \times \Sigma^* \rightarrow \Delta^*$ for alphabets Σ and Δ with access to uniformly random bits $r \in \{0,1\}^\omega$
- We use $f_r: \Sigma^* \rightarrow \Delta^*$ when random bits r are known (rarely so)
- $f_r(x)$ essentially computable sampler from distribution over Δ^* (returns $f_r(x)$ based on random r)
- We use $P[w \leftarrow f(x)]$ for $P(\{r \mid f_r(x) = w\})$ i.e. probability that $f_r(x)$ with uniformly random r returns w
- Sources of true randomness discussed later

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Perfectly Secret Encryption

Additional notes & simplifications

- Modern notions & definitions (slightly overkill at this point)
- Random sequences $R = \{0,1\}^\omega$
- Message space $\emptyset \subsetneq \mathcal{M} \subseteq \{0,1\}^*$
- Key space $\emptyset \subsetneq \mathcal{K} \subseteq \{0,1\}^*$
- Ciphertext space $\emptyset \subsetneq \mathcal{C} \subseteq \{0,1\}^*$
- Messages, keys & ciphertexts coded in binary

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Perfectly Secret Encryption

§2.4 Definition (Private-key encryption scheme)

Private-key encryption scheme is triple $(\text{gen}, \text{enc}, \text{dec})$ of computable probabilistic functions

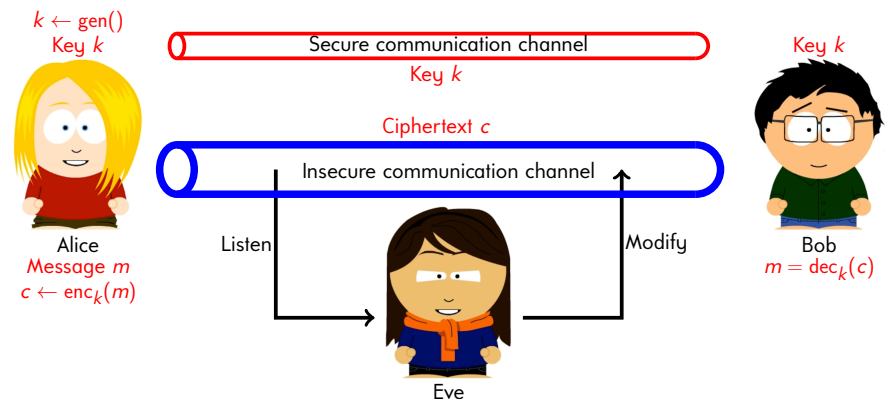
- $\text{gen}: R \rightarrow K$ yielding distribution $P_K: K \rightarrow [0,1]$
- $\text{enc}: R \times K \times \mathcal{M} \rightarrow \mathcal{C}$ and $\text{dec}: R \times K \times \mathcal{C} \rightarrow \mathcal{M}$
- $\text{dec}(r, k, c) = m$ for all $r \in R$, $k \in \text{ran}(\text{gen})$, $m \in \mathcal{M}$, and $c \in \mathcal{C}$ with $P[c \leftarrow \text{enc}_k(m)] \neq 0$ (perfect correctness)

Notes

- Wlog. dec is not probabilistic, so $\text{dec}: K \times \mathcal{C} \rightarrow \mathcal{M}$
- Scheme contains no distribution over messages (scheme is generic, i.e. agnostic to message distribution)

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Perfectly Secret Encryption



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Perfectly Secret Encryption

§2.5 Definition (Perfectly secret)

Encryption scheme $\mathcal{E} = (\text{gen}, \text{enc}, \text{dec})$ is **perfectly secret** if for all messages $m, m' \in \mathcal{M}$ and ciphertexts $c \in \mathcal{C}$

$$\sum_{k \in K} P_K(k) \cdot P[c \leftarrow \text{enc}_k(m)] = \sum_{k \in K} P_K(k) \cdot P[c \leftarrow \text{enc}_k(m')]$$

Notes

- Probability of ciphertext c is invariant under given message m
- We let $O_{\mathcal{E}}(c, m) = \sum_{k \in K} P_K(k) \cdot P[c \leftarrow \text{enc}_k(m)]$
- Simplifies to $\sum_{k \in K} P[c \leftarrow \text{enc}_k(m)] = \sum_{k \in K} P[c \leftarrow \text{enc}_k(m')]$ for uniform key distribution P_K

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Perfectly Secret Encryption

§2.6 Definition (Scenario)

Scenario is pair $(\mathcal{E}, P_M)$ of private-key encryption scheme $\mathcal{E}$ and probability distribution $P_M: \mathcal{M} \rightarrow [0, 1]$. Define $P_C: \mathcal{C} \rightarrow [0, 1]$ for all $c \in \mathcal{C}$ by

$$P_C(c) = \sum_{k \in K, m \in \mathcal{M}} P_K(k) \cdot P_M(m) \cdot P[c \leftarrow \text{enc}_k(m)]$$

Note for perfectly secret encryption scheme and some $m' \in \mathcal{M}$

$$\begin{aligned} P_C(c) &= \sum_{k \in K, m \in \mathcal{M}} P_K(k) \cdot P_M(m) \cdot P[c \leftarrow \text{enc}_k(m)] \\ &= \sum_{m \in \mathcal{M}} P_M(m) \cdot \left(\sum_{k \in K} P_K(k) \cdot P[c \leftarrow \text{enc}_k(m)] \right) \\ &= \sum_{m \in \mathcal{M}} P_M(m) \cdot O_{\mathcal{E}}(c, m) = O_{\mathcal{E}}(c, m') \cdot \sum_{m \in \mathcal{M}} P_M(m) = O_{\mathcal{E}}(c, m') \end{aligned}$$

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Perfectly Secret Encryption

Vigenère cipher with uniformly distributed keys of length 2

- Let $P_K(b_1 b_2) = \frac{1}{4}$ (uniform key distribution)
- Consider ciphertext $c = 11$ and messages $m = 00$ and $m' = 01$

$$\sum_{k \in K} P[c \leftarrow \text{enc}_k(m)] = 1 = \sum_{k \in K} P[c \leftarrow \text{enc}_k(m')]$$

- Consider ciphertext $c = 000$ and messages $m = 000$ and $m' = 001$

$$\sum_{k \in K} P[c \leftarrow \text{enc}_k(m)] = 1 \neq 0 = \sum_{k \in K} P[c \leftarrow \text{enc}_k(m')]$$

- Encryption scheme **not perfectly secret**

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Perfectly Secret Encryption

Vigenère cipher with uniformly distributed keys of length 2

- Let $P_M(m_1 \cdots m_n) = \prod_{i=1}^n p(m_i)$ with $p(0) = \frac{2}{5}$ and $p(1) = \frac{3}{5}$
- Let $P_K(b_1 b_2) = \frac{1}{4}$
- 4 ways to obtain ciphertext $c = 000$

$$\begin{aligned} P_C(c) &= \sum_{k \in K, m \in \mathcal{M}} P_K(k) \cdot P_M(m) \cdot P[c \leftarrow \text{enc}_k(m)] \\ &= P_K(00) \cdot P_M(000) + P_K(01) \cdot P_M(010) + \\ &\quad P_K(10) \cdot P_M(101) + P_K(11) \cdot P_M(111) \\ &= \frac{1}{4} \cdot \left(\frac{2^3 + 2^2 \cdot 3 + 2 \cdot 3^2 + 3^3}{5^3} \right) = \frac{1}{4} \cdot \frac{8 + 12 + 18 + 27}{125} \\ &= \frac{1}{4} \cdot \frac{65}{125} = \frac{13}{100} \end{aligned}$$

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