Foundations of Complexity Theory

Lecture 3: Foundational Relations II

October 28, 2026

Relations between Complexity Classes

§3.1 Theorem

Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq n$ for all $n \in \mathbb{N}$.
Then $\mathrm{DTIME}(f) \subseteq \mathrm{NTIME}(f) \subseteq \mathrm{DSPACE}(f)$

Proof

$\mathrm{DTIME}(f) \subseteq \mathrm{NTIME}(f)$ trivial. Let $M = (Q, \Sigma, \Gamma, \Sigma', \delta, \square, q_0, q_+, q_-)$ be f-time-bounded TM. Input $w \in \Sigma^*$ accepted by M iff exists protocol $(\rho_1, \ldots, \rho_m)$ with $m \leq f(|w| + 2)$ such that computation is accepting. Store those f-length-bounded protocols in space $O(f)$, length-lexicographically enumerate them, and execute them to check validity and acceptance. Using linear tape compression (Theorem §2.5) we achieve $L(M) \in \mathrm{DSPACE}(f)$. □

Space Hierarchy Theorem

§3.2 Theorem (Space hierarchy theorem)

Let $f : \mathbb{N} \to \mathbb{N}$ space-constructible with $f(n) \geq \log n$ for all $n \in \mathbb{N}$.
Then $\mathrm{DSPACE}(o(f)) \neq \mathrm{DSPACE}(O(f))$

Notes

- $g \in o(f)$ if and only if (iff) $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$
- Wlog. assume TMs only utilize tapes right of starting position

Space Hierarchy Theorem

Proof (1/3)

Select suitable coding $\mathrm{bin}(M) \in \{0,1\}^*$ of deterministic 1-TM M.
Use diagonalization with space-bounded TMs.

$$L = \Big\{ \langle \mathrm{bin}(M), w \rangle \mid M \text{ rejects } \langle \mathrm{bin}(M), w \rangle \text{ within space } f(|\langle \mathrm{bin}(M), w \rangle| + 2) \text{ and utilizes only 4 tape symbols} \Big\}$$

We first prove $L \notin \mathrm{DSPACE}(o(f))$. Let M be det. TM with $L(M) = L$ requiring only $o(f)$ space. There also exists equivalent det. 1-TM M' that utilizes only 4 tape symbols and $o(f)$ space such that $L(M') = L(M) = L$. Hence TM M' utilizes strictly fewer space than $f(|\langle \mathrm{bin}(M'), w \rangle| + 2)$ on input $\langle \mathrm{bin}(M'), w \rangle$ for sufficiently long w. However, this yields contradiction

$$\langle \mathrm{bin}(M'), w \rangle \in L \iff \langle \mathrm{bin}(M'), w \rangle \notin L(M')$$

Space Hierarchy Theorem

$$L = \Big\{ \langle \mathrm{bin}(M), w \rangle \mid M \text{ rejects } \langle \mathrm{bin}(M), w \rangle \text{ within space } f(|\langle \mathrm{bin}(M), w \rangle| + 2) \text{ and utilizes only 4 tape symbols} \Big\}$$

Proof (2/3)

It remains to prove $L \in \mathrm{DSPACE}(O(f))$. Use universal TM U that decodes input $\langle \mathrm{bin}(M), w \rangle$. Let $n = |\langle \mathrm{bin}(M), w \rangle|$. Since f is space-constructible we mark $f(n+2)$ cells on tape within space $f(n+2)$. By Lemma §1.7 there exist constants $a, b \in \mathbb{N}$ such that at most $m = a \cdot n \cdot b^{f(n+2)}$ configurations are reachable from initial configuration. Counter ℓ initialized to m stored in binary on another tape requires space $O(f(x) + \log x) = O(f)$.

Start simulation of M inside marked tape cells.
Decrement counter ℓ after each simulated step.

Space Hierarchy Theorem

$$L = \Big\{ \langle \mathrm{bin}(M), w \rangle \mid M \text{ rejects } \langle \mathrm{bin}(M), w \rangle \text{ within space } f(|\langle \mathrm{bin}(M), w \rangle| + 2) \text{ and utilizes only 4 tape symbols} \Big\}$$

Proof (3/3)

Distinguish 3 outcomes

- If simulation leaves marked cells (before ℓ reaches 0), then reject because more than $f(n+2)$ space utilized
- If counter ℓ reaches 0, then accept because configuration was repeated, so M is in infinite loop
- Otherwise invert decision of M

Det. TM U thus accepts language L with space consumption $O(f)$. □

Theorem of Hennie & Stearns

§3.3 Theorem (Theorem of Hennie & Stearns)

Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n) \geq n$ for all $n \in \mathbb{N}$ and $L \in \mathrm{DTIME}(f)$.
There exists $(f(x) \cdot \lceil \log f(x) \rceil)$-time-bounded det. 2-TM M' with $L(M') = L$

Notes

- Know simulation of k-TM by 1-TM (quadratic overhead)
- Collection of tape symbols causes unnecessary runs over tape
- Avoid head divergence (keep all heads reasonably close)
- Proof by Pippenger & Fischer (1979)

Theorem of Hennie & Stearns

Proof (1/3)

Let k-TM M and w input of length n. We construct 2-TM with tape of k-tuples (as in Theorem §2.4) and additionally mark head positions.
2nd tape only needed for fast copying and as storage for call stack.
We call initial position on 1st tape *origin*. We keep heads close to origin.
We use Simulate(m) with m power of 2 that simulates m steps of M.

- If $m = 1$, then simulate step of M and return to origin
- Otherwise execute

$$\mathrm{Center}(m)\ ;\ \mathrm{Simulate}(\tfrac{m}{2})\ ;\ \mathrm{Center}(m)\ ;\ \mathrm{Simulate}(\tfrac{m}{2})\ ;\ \mathrm{Rewind}(m)$$

(second $\frac{m}{2}$ steps continue simulation after first $\frac{m}{2}$ steps)

Theorem of Hennie & Stearns

Proof (2/3)

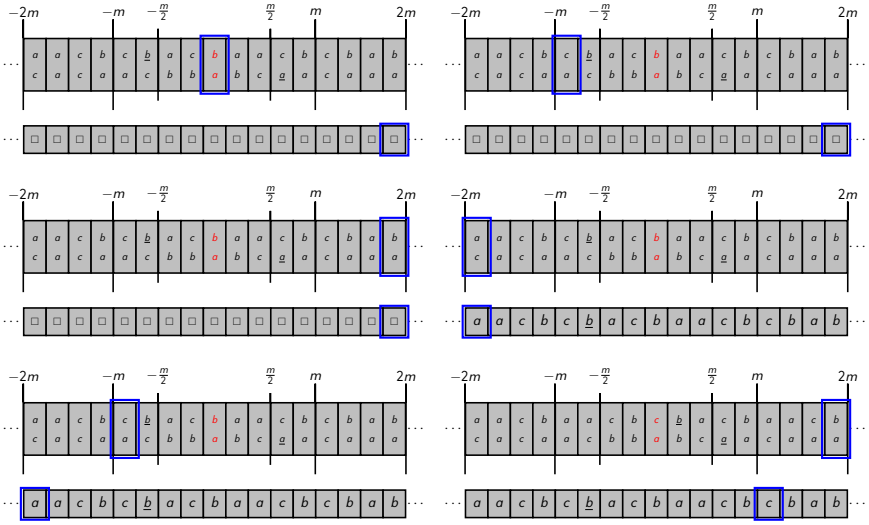
Call Simulate(m) requires heads in interval $[-m, m]$ around origin.
Center(m) moves heads on all simulated tapes to interval $[-\frac{m}{2}, \frac{m}{2}]$.
We achieve this in linear time.

On each simulated tape search head position in interval $[-m, m]$ and decide direction of rotation of interval $[-2m, 2m]$ by m if necessary. With 2nd tape cyclic rotation by m can be implemented in linear time (per tape and overall).

After simulation of $\frac{m}{2}$ steps heads remain in interval $[-m, m]$, but potentially outside $[-\frac{m}{2}, \frac{m}{2}]$ forcing another call to Center(m).
Rewind(m) undoes all rotations (requires remembering which rotations were performed to which tapes).

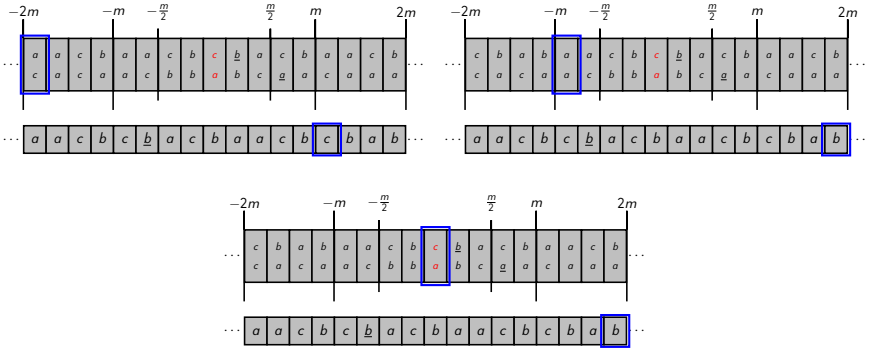
Theorem of Hennie & Stearns

Illustration of centering by rotation of 1st simulated tape to right



Theorem of Hennie & Stearns

Illustration of centering by rotation of 1st simulated tape to right



Theorem of Hennie & Stearns

Proof (3/3)

Correctness of Simulate(m) obvious. We use SimulateFromStart(m) to indicate calls of Simulate(m) on the initial configuration and run

1. Let $m = 1$
2. While M not stopped, execute SimulateFromStart(m) ; $m \leftarrow 2m$

Let $t(m)$ be number of steps for call Simulate(m). For sufficiently large m there is $k \in \mathbb{N}$ such that $t(m) \leq 2t(\frac{m}{2}) + km$ and thus $t \in O(x \log x)$. Hence for large n there exists $k' \in \mathbb{N}$ such that overall time-consumption is

$$\sum_{i=0}^{\lceil \log f(n+2) \rceil} t(2^i) \leq k' \cdot \sum_{i=0}^{\lceil \log f(n+2) \rceil} 2^i \cdot i \leq 2k' \cdot f(n+2) \cdot \lceil \log f(n+2) \rceil$$

and thus in $O(f(x)\lceil \log f(x) \rceil)$. Applying linear speed-up, we obtain $f(x)\lceil \log f(x) \rceil$-time-consumption. □

Foundational Relations

§3.4 Theorem

Let $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq \log n$ for all $n \in \mathbb{N}$.
Then $\mathrm{DSPACE}(f) \subseteq \mathrm{NSPACE}(f) \subseteq \mathrm{DTIME}(2^{O(f)})$

Proof

Only 2nd inclusion needs proof. Let M be f-space-bounded TM and w input of length n. By Lemma §1.7 there exists $g \in O(f)$ such that number of configurations reachable from initial configuration I_w is at most $2^{g(n+2)}$. We close set $\{I_w\}$ under successors and accept if and only if accepting configuration reached. Time-consumption is eventually proportional to $2^{g(n+2)} \cdot 2^{g(n+2)} \cdot g(n+2)$, which proves time-bound $2^{O(f)}$.
(closure after $\leq 2^{g(n+2)}$ steps, search $\leq 2^{g(n+2)}$ elements for configuration with outside successor, configuration length $\leq g(n+2)$) □

Foundational Relations

First conclusions

- $\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathrm{DTIME}(2^{O(\log x)}) = \mathbf{P}$
  (1st inclusion trivial and 2nd inclusion by Theorem §3.4)
- $\mathbf{NL} \subsetneq \mathbf{CSL} = \mathbf{LBA} = \mathrm{NSPACE}(x) \subseteq \mathrm{DTIME}(2^{O(x)})$
  (**CSL** are context-sensitive languages &
  LBA languages of linear-bounded TM
  strictness of 1st inclusion by space hierarchy — Theorem §3.2
  last inclusion by Theorem §3.4)
- $\mathrm{DSPACE}(x^2) \subseteq \mathrm{DTIME}(2^{O(x^2)})$
  (by Theorem §3.4)

Time Hierarchy Theorem

§3.5 Theorem (Time hierarchy theorem)

Let f time-constructible with $f(n) \geq n$ for all $n \in \mathbb{N}$.
Then $\mathrm{DTIME}(o(f)) \subsetneq \mathrm{DTIME}(O(f\,;g))$ with $g(n) = n \log n$ for all $n \in \mathbb{N}$

Proof (1/2)

Select suitable coding $\mathrm{bin}(M) \in \{0,1\}^*$ of det. k-TM M and diagonalize with time-bounded TMs.

$$L = \left\{ \langle \mathrm{bin}(M), w \rangle \mid M \text{ rejects } \langle \mathrm{bin}(M), w \rangle \text{ in time } f(|\langle \mathrm{bin}(M), w \rangle| + 2) \right\}$$

First prove $L \notin \mathrm{DTIME}(o(f))$. Let M det. TM with $L(M) = L$ that runs in time $o(f)$. Hence M runs in time $< f(|\langle \mathrm{bin}(M), w \rangle| + 2)$ on input $\langle \mathrm{bin}(M), w \rangle$ for sufficiently long w. Then we obtain contradiction

$$\langle \mathrm{bin}(M), w \rangle \in L \quad \Longleftrightarrow \quad \langle \mathrm{bin}(M), w \rangle \notin L(M)$$

Time Hierarchy Theorem

$$L = \left\{ \langle \mathrm{bin}(M), w \rangle \mid M \text{ rejects } \langle \mathrm{bin}(M), w \rangle \text{ in time } f(|\langle \mathrm{bin}(M), w \rangle| + 2) \right\}$$

Proof (2/2)

It remains to prove $L \in \mathrm{DTIME}(O(f\,;g))$. Use universal 3-TM U that decodes input $\langle \mathrm{bin}(M), w \rangle$ and then simulates M according to Theorem §3.3 (Hennie & Stearns) on 2 tapes. Let $n = |\langle \mathrm{bin}(M), w \rangle|$. Since f time-constructible, we setup counter with value $f(n+2)$ in time $f(n+2)$ on 3rd tape. We simulate M and decrease counter on 3rd tape after each simulated step.

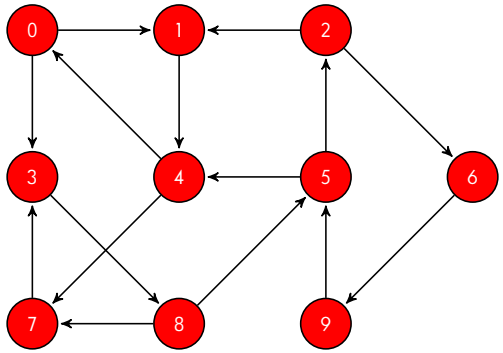
- Reject if counter reaches 0 without halting since time-consumption exceeded
- Otherwise invert decision of M

Det. TM U accepts L in time $O(f\,;g)$ by Theorem §3.3. □

Graph Reachability

Decision variant GAP

- Input: finite directed graph (V, E) and vertices $s, t \in V$
- Output: Is there path from s to t?



Graph Reachability

Algorithm of Savitch

- Every path of length $\leq 2^m$
  splits into 2 paths of length $\leq 2^{m-1}$ and middle vertex
- Space-efficient (but time-inefficient) solution via recursion

Walter Savitch (* 1943)

- US computer scientist
- Complexity theory & programming languages
- Ph.D. student of Stephen Cook at UC Berkeley

Graph Reachability

Finite directed graph (V, E) and vertices $s, t \in V$

We implement predicate $\text{Reach}(v, i, v')$ for $v, v' \in V$ and $i \in \mathbb{N}$, which tells whether v' is reachable from v in 2^i steps

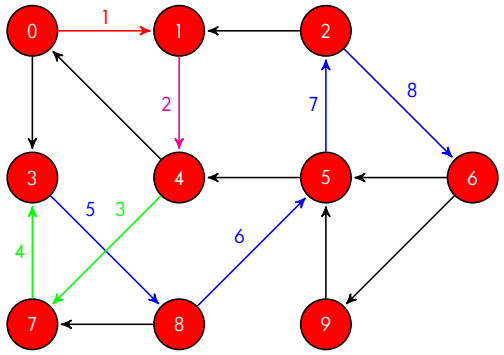
1. If $i = 0$, then return $(v = v') \vee ((v, v') \in E)$
2. Otherwise for all $v'' \in V$ (in arbitrary order)
   - Return true, if $\text{Reach}(v, i-1, v'') \wedge \text{Reach}(v'', i-1, v')$
3. Return false

Initial call $\text{Reach}(s, \lceil \log |V| \rceil, t)$

Input size

- $|V|^2 + 2 \log |V|$ as binary matrix (matrix & 2 vertices)
- $|E| \cdot (2 \log |V|) + 2 \log |V|$ as list of edges (list of edges & 2 vertices)

Graph Reachability



Open calls

Reach(0, 3, 6) Reach(0, 2, 3) ~~Reach(0, 1, 4)~~ ~~Reach(0, 0, 1)~~
~~Reach(1, 0, 4)~~ ~~Reach(4, 1, 3)~~ ~~Reach(4, 0, 7)~~ ~~Reach(7, 0, 3)~~

Graph Reachability

Proof of correctness

Let (V, E) graph and $s, t \in V$. There exists path from s to t iff such path of length $\leq |V|$ exists (visit at most each vertex once). We show invariant of call $\text{Reach}(v, i, v')$ by induction on i.

Induction base at $i = 0$ obvious. Let $i \geq 1$ and $v_0 \to v_1 \to \cdots \to v_m$ path of length $m \leq 2^i$ from $v_0 = v$ to $v_m = v'$. Then

$$v_0 \to v_1 \to \cdots \to v_{\lfloor \frac{m}{2} \rfloor} \quad \text{and} \quad v_{\lfloor \frac{m}{2} \rfloor} \to v_{\lfloor \frac{m}{2} \rfloor + 1} \to \cdots \to v_m$$

path from v to $v'' = v_{\lfloor \frac{m}{2} \rfloor}$ and from v'' to v' of length $\leq 2^{i-1}$ because

$$\lfloor \tfrac{m}{2} \rfloor \leq \tfrac{m}{2} \leq \tfrac{2^i}{2} = 2^{i-1} \quad \text{and} \quad m - \lfloor \tfrac{m}{2} \rfloor = \lceil \tfrac{m}{2} \rceil \leq \lceil \tfrac{2^i}{2} \rceil = 2^{i-1}$$

By IH calls $\text{Reach}(v, i-1, v'')$ and $\text{Reach}(v'', i-1, v')$ return true, which proves invariant. If no path exists, then invariant also true. □

Graph Reachability

Space complexity

- Store inputs of call v, i, v' in space $3 \log |V|$ since $i \leq \lceil \log |V| \rceil$
- When call successful, inspect last 2 calls on stack $\text{Reach}(v, k, v')$, $\text{Reach}(w, k-1, w')$
  - If $v = w$, then delete successful call $\text{Reach}(w, k-1, w')$ and continue with call $\text{Reach}(w', k-1, v')$
  - If $v \neq w$, then delete successful call $\text{Reach}(w, k-1, w')$ and mark call $\text{Reach}(v, k, v')$ as successful and reinspect call stack
- When call not successful, inspect last 2 calls on stack $\text{Reach}(v, k, v')$, $\text{Reach}(w, k-1, w')$
  - If $v = w$ and $w' + 1$ valid, then replace failed call $\text{Reach}(w, k-1, w')$ by $\text{Reach}(w, k-1, w'+1)$
  - If $v = w$ and $w' + 1$ invalid, then delete failed call $\text{Reach}(w, k-1, w')$ and mark call $\text{Reach}(v, k, v')$ as failed and reinspect call stack
  - If $v \neq w$ and $w + 1$ valid, then replace failed call $\text{Reach}(w, k-1, w')$ by $\text{Reach}(v, k-1, w+1)$
  - If $v \neq w$ and $w + 1$ invalid, then delete failed call $\text{Reach}(w, k-1, w')$ and mark call $\text{Reach}(v, k, v')$ as failed and reinspect call stack

Graph Reachability

Space complexity

- Store inputs of call v, i, v' in space $3 \log |V|$ since $i \leq \lceil \log |V| \rceil$
- Neighboring entries differ in depth i
- At most $\log |V| + 1$ open calls
- Space complexity $3 \log |V| \cdot (\log |V| + 1) \in O(\log^2 x)$

§3.6 Theorem

$\text{GAP} \in \text{DSPACE}(\log^2 x)$