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The Symmetries of a Nerve Conduction Equation

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Abstract. The infinitesimal operators of the symmetry group of a special nonlinear nerve conduction equation are determined.

1. INTRODUCTION

Impulse propagation in nerve axons is one of the most fully studied wave phenomena in active media. In [1], the case of a nonuniform axon cable was considered. The equation for wave front propagation in such cables is

$$u_{xx} + r(x) \cdot u_x - u_t - p(u) = 0, \tag{1}$$

where $r(x)$ is a real function of cable diameter, and $p(u) = q_3 \cdot u^3 + q_2 \cdot u^2 + q_1 \cdot u + q_0$ ($q_i \in \mathbf{R}$) is a polynomial of third degree which is motivated by biophysical reasoning [1].

The aim of this paper is to determine the infinitesimal operators of the symmetry group of the partial differential equation (**pde**) (1), using Sophus Lie's theory of symmetry groups [2].

The notations for the infinitesimal operators, derivatives, etc. in this paper are the same as in [2]. For space limitation, we refer to [2]. In addition to these notations, we write

$$\xi_1 = \xi^t, \quad \xi_2 = \xi^x, \quad \eta^1 = \eta, \quad x_1 = t, \quad x_2 = x.$$

2. THE INFINITESIMAL OPERATORS OF THE SYMMETRY GROUP

The way in which the infinitesimal operators U_i of the symmetry group of (1) depend on the diameter function $r(x)$ and the polynomial $p(u)$ is formulated by the following

THEOREM. *Let $c \in \mathbf{R}$ be an arbitrary constant. Then five cases can be differentiated :*

a) *if both $r(x) = x \cdot c$ and*

$$q_0 = \frac{q_2 \cdot (q_2{}^2 - 9 \cdot q_3)}{27 \cdot q^2{}_3}, \quad q_1 = \frac{q_2{}^2 - 3 \cdot q_3}{3 \cdot q_3} \tag{2}$$

hold, then

$$U_1 = -\frac{1}{2} e^{-t} \frac{\partial}{\partial x}, \tag{3}$$

$$U_2 = \frac{1}{2} e^{2t} \frac{\partial}{\partial t} + \frac{x}{2} e^{2t} \frac{\partial}{\partial x} - \left(\frac{u}{2} - \frac{q_2}{6q_3}\right) e^{2t} \frac{\partial}{\partial u}, \tag{4}$$

$$U_3 = \frac{\partial}{\partial t}; \tag{5}$$

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b) *if $r(x) = x \cdot c$ while (2) is not necessarily satisfied, then*

$$U_1 = -e^{-t}\frac{\partial}{\partial x}, \tag{6}$$

$$U_2 = \frac{\partial}{\partial t}; \tag{7}$$

c) *if both $r(x) = c$ and*

$$q_0 = \frac{{q_2}^2}{27 \cdot {q_3}^2}, \quad q_1 = \frac{{q_2}^2}{3 \cdot q_3} \tag{8}$$

hold, then

$$U_1 = \frac{\partial}{\partial t}, \tag{9}$$

$$U_2 = -\frac{1}{2}\frac{\partial}{\partial x}, \tag{10}$$

$$U_3 = t\frac{\partial}{\partial t} + \left(\frac{x-t}{2}\right)\frac{\partial}{\partial x} - \frac{\left(\frac{d_0}{d_1} + u\right)}{2}\frac{\partial}{\partial u}; \tag{11}$$

d) *if $r(x) = c$ while (8) is not necessarily satisfied, then*

$$U_1 = \frac{\partial}{\partial t}, \tag{12}$$

$$U_2 = \frac{\partial}{\partial x} \tag{13}$$

e) *in all other cases only the trivial symmetry operator*

$$U = \frac{\partial}{\partial t} \tag{14}$$

yields.

PROOF: Applying the second prolongation of the infintesimal operator U to Equation (1), we obtain the following system of pde's

$$\xi^t_u = 0, \quad \xi^t_x = 0, \quad \xi^x_u = 0, \quad \eta_{uu} = 0, \tag{15}$$

$$2\,\xi^x_x - \xi^t_t = 0, \tag{16}$$

$$2\,\eta_{ux} + r_x\,\xi^x - \xi^x_{xx} + r\,\xi^x_x + \xi^x_t = 0, \tag{17}$$

$$p\,\eta_u + \eta_{xx} + r\,\eta_x - \eta_t - p_u\eta_t - 2f\xi^x_x = 0, \tag{18}$$

with $r_x = \frac{\partial r}{\partial x}$, $p_u = \frac{\partial p}{\partial u}$. The general solution of (15)–(18) determines the symmetry group of (1). According to (15) and (16) we take

$$\eta = c_2\,(t,x) \cdot u + c_1\,(t,x)\,, \quad \xi^x = \frac{-c_3(t) + x \cdot \xi^t_t}{2} \tag{19}$$

with unkown functions $c_1(t,x)$, $c_2(t,x)$, $c_3(t)$. Then the equations (17) and (18) can be rewritten as

$$r_x c_3 + c_{3t} - 4c_{2x} - x r_x\,\xi^t_t - x\,\xi^t_t - r\,\xi^t_t = 0, \tag{20}$$

$$u c_2\,p_u - c_2\,p + p_u c_1 - u c_{2xx} - u r c_{2x} + u c_{2t} - c_{1xx} - r c_{1x} + c_{1t} + p\,\xi^t_t = 0. \tag{21}$$

Now we split (21) according to powers of u into the equations

$$2c_2 + \xi_t^t = 0, \tag{22}$$

$$q_2 c_2 + 3q_3 c_1 + q_2\,\xi_t^t = 0, \tag{23}$$

$$q_0 c_2 - q_1 c_1 + c_{1xx} + r c_{1x} - c_{1t} - q_0\,\xi_t^t = 0, \tag{24}$$

$$2q_2 - c_{2xx} - r c_{2x} + c_{2t} + q_1\,\xi_t^t = 0, \tag{25}$$

because all variables play the role of independent ones. So it follows that c_1 and c_2 in (22) and (23) are independent of x, and we choose $c_1(t) = -q_2\,\xi_t^t/6q_3$, $c_2(t) = -\xi_t^t/2$. In this way we obtain from (21)

$$\xi_{tt}^t = \frac{(x\,\xi_t^t - c_3)\cdot r_x}{x} + \frac{r\,\xi_t^t - c_{3t}}{x}, \tag{26}$$

$$3q_3\,\xi_{tt}^t - 6q_1 q_3\,\xi_t^t + 2{q_2}^2\,\xi_t^t = 0, \tag{27}$$

$$q_2\,\xi_{tt}^t - 9q_0 q_3\,\xi_t^t + q_1 q_2\,\xi_t^t = 0. \tag{28}$$

Let us consider first the case that both $r(x) \neq c$ and $r(x) \neq c \cdot x$ hold. Comparing the coefficients according to powers of x, it follows from (26) $c_3 \equiv 0$, and so ξ_t^t must also equal zero . The result is the trivial symmetry operator for this case :

$$U = \frac{\partial}{\partial t}. \tag{29}$$

Thus Case e) of the theorem is proved.

Now we take $r(x) = x \cdot c$. Without loss of generality (w.l.o.g.) we assume $c \equiv 1$. Then we derive from (22)

$$c_3 = a_1 e^{-t}, \quad \xi_t^t = a_2 e^{2t}, \qquad (a_i = \text{const.})\,. \tag{30}$$

Now ξ^t has also to satisfy Equations (27) and (28). The conditions for non-vanishing ξ_t^t are

$$q_0 = \frac{q_2\cdot({q_2}^2 - 9\cdot q_3)}{27\cdot q^2{}_3}, \quad q_1 = \frac{{q_2}^2 - 3\cdot q_3}{3\cdot q_3}. \tag{31}$$

We can differentiate two cases:

a) If both $\xi_t^t \neq 0$ and (31) hold, then the infinitesimal operators of the symmetry group are given by (3)–(5), and so Case a) of the theorem is proved.

b) If $\xi_t^t = 0$ while (31) is not necessarily satisfied, then we have as the result for the infinitesimal operators of the symmetry group the assertions (6) and (7), and so Case b) of the theorem is shown.

Last we consider the case $r(x) \equiv c$. We assume w.l.o.g. $c \equiv 1$. Using (26) and (15) it turns out that $\xi_{tt}^t = 0$, and we have

$$\xi_t^t = c_{3t}. \tag{32}$$

ξ_t^t must also satisfy (27) and (28), and we examine in analogy to the above two cases:

c) For $\xi_t^t \neq 0$,

$$q_0 = \frac{{q_2}^2}{27\cdot {q_3}^2}, \quad q_1 = \frac{{q_2}^2}{3\cdot q_3} \tag{33}$$

yields, and this implies that $p(u)$ is of the form

$$p(u) = (d_1\cdot u + d_0)^3, \quad d_1, d_0 \in \mathbb{R}, \quad d_1\cdot d_0 \neq 0. \tag{34}$$

In this case, the infinitesimal operators (9)–(11) of the symmetry group theorem are derived, which proves Assertion c) of the theorem.

d) If $\xi_t^t = 0$ while (33) is not necessarily satisfied, we obtain immediately the infinitesimal operators (12) and (13), i.e., Assertion d) of the theorem.

This completes the proof of the theorem.[1]

[1] A great deal of the calculations was done using the computer algebra system REDUCE with the package SPDE by F. Schwarz [3].

REFERENCES

1. A. Schierwagen, Nerve impulse conduction in non-uniform axons: Analytical treatment, In *Nonlinear Waves in Excitable Media* (Edited by A.V. Holden, M. Markus and H. Othmer), pp. 107–114, Manchester University Press, Manchester, (1991).
2. F. Schwarz, Symmetries of differential equations: From Sophus Lie to computer-algebra, *SIAM Rev.* **30**, 450–481 (1988).
3. F. Schwarz, A REDUCE package for determiming symmetries of ordinary and partial differential equations, *Comput. Phys. Comm.* **27**, 179–186 (1982).

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