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Wave Transmission in Cable Structures of Tree Type

J. FRANCŮ[1] AND A.K. SCHIERWAGEN[2]

[1]Chair for Mathematical Analysis, Purkyne University Brno
[2]Carl Ludwig Institute for Physiology, Karl Marx University Leipzig

Abstract. The continuum approach is employed for modeling wave transmission in branching cable systems of tree type

1. INTRODUCTION

In this note a linear theory of wave transmission in cable systems of tree type is described. Due to the familiar analogies between electrical and mechanical quantities, this theory is relevant to transport phenomena in dichotomous, tree-like branching physiological systems.

We have used a combination of linear transmission line theory with the continuum approach. The method consists in replacing the branching cable structure of tree type by an equivalent locally homogeneous electrical medium described by averaged "macroscopic" parameters.

For the system state variables partial differential equations are derived which represent natural generalizations of the case of a single, uniform transmission line. Assertions on solutions depend on the presence of certain branching conditions for the tree structures.

2. PROBLEM FORMULATION

We assume that the root segment of the tree structure emanates from the origin O of a system of polar coordinates r, φ.

Let the tree be contained in a space angle θ. Since the embedding medium surrounding the tree is considered isotropic of zero resistance, the polar coordinate φ becomes a parameter and can be omitted. The tree structure is regarded as composed of cylindrical cable segments joining at branch points. In order to determine characteristic bulk properties for the tree, an averaging procedure must be specified [1] which employs the "microscopic" structural and electrical parameters of the cable structure (mean segment diameter $d(r)$, specific electrical parameters such as resistance ρ, capacitance c inductance l and leakance g). For example, the "macroscopic" value of inductance $L(r)$ is calculated as

$$L(r) = \frac{4l}{n(r) \cdot \pi \cdot d(r)^2},$$

with $n(r)$ being the number of segments at distance r and taking all cable segments at location r as parallel. Analogously, macroscopic values of resistances $R(r)$, capacitance $C(r)$ and conductance (leakage) $G(r)$ are derived.

If we consider a volume element V at time t within the space angle θ bounded by the sphere segments S_1, S_2, the application of Kirchhoff's law yields

$$I_1 + I_2 + I_C + I_G = 0 \tag{1}$$

where I_1, I_2 are the (opposite) currents crossing the surfaces S_1, S_2 at distances r_1, r_2; I_C and I_G denote the summed capacitive and lost currents in V.

The currents in (1) can be expressed as

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$$\int_{\theta}\int_{r_1}^{r_2}\left(\frac{\partial I(r,t)}{\partial r}+C(r)\frac{\partial V(r,t)}{\partial t}+G(r)\cdot V(r,t)\right)drd\theta=0. \tag{2}$$

Dividing (2) by $|\theta|\cdot|r_2-r_1|$ and passing to the limit $|\theta|\to 0,\quad |r_2-r_2|\to 0$, we obtain

$$\frac{\partial I}{\partial r}+C(r)\cdot\frac{\partial V}{\partial t}+G(r)\cdot V=0. \tag{3}$$

The voltage drop in V, on the other hand, consists of an inductive part and ohmic one, i.e.,

$$-\int_{r_1}^{r_2}\frac{\partial V(r.t)}{\partial r}dr=\int_{r_1}^{r_2}L(r)\cdot\frac{\partial I(r,t)}{\partial t}dr+\int_{r_1}^{r_2}R(r)\cdot I(r,t)dr \tag{4}$$

Differentiating (4) with respect to r gives

$$\frac{\partial V}{\partial r}+L(r)\cdot\frac{\partial I}{\partial t}+R(r)\cdot I=0. \tag{5}$$

Equations (3) and (5) are the telegraph equations with space-dependent coefficients.

3. Analysis of the Equations

Because of space limitations only the hyperbolic case ($L>0,\ C>0$) will be discussed here. Let

$$\frac{R(r)}{L(r)}=\alpha,\quad \frac{G(r)}{C(r)}=\beta \tag{6}$$

where α,β are positive constants. Then (3) and (5) can be transformed into two second-order hyperbolic equations

$$\frac{\partial^2 V}{\partial t^2}-\frac{1}{C(r)}\left(\frac{\partial}{\partial r}\frac{1}{L(r)}\frac{\partial V}{\partial r}\right)+(\alpha+\beta)\frac{\partial V}{\partial t}+\alpha\beta\cdot V=0 \tag{7}$$

$$\frac{\partial^2 I}{\partial t^2}-\frac{1}{L(r)}\left(\frac{\partial}{\partial r}\frac{1}{C(r)}\frac{\partial I}{\partial r}\right)+(\alpha+\beta)\frac{\partial I}{\partial t}+\alpha\beta\cdot I=0 \tag{8}$$

which can be solved separately.

Applying the transformations

$$x(r)=\int_0^r(L(s)C(s))^{\frac{1}{2}}\,ds, \tag{9}$$

$$V(x,t)=\exp\left(-(\alpha+\beta)\frac{t}{2}\right)\cdot\exp\left(-\frac{1}{2}\int\gamma(x)dx\right)\cdot v(x,t) \tag{10}$$

$$I(x,t)=\exp\left(-(\alpha+\beta)\frac{t}{2}\right)\cdot\exp\left(\frac{1}{2}\int\gamma(x)dx\right)\cdot i(x,t), \tag{11}$$

we obtain the canonical forms

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-\delta_1\cdot v=0 \tag{12}$$

$$\frac{\partial^2 i}{\partial t^2}-\frac{\partial^2 i}{\partial x^2}-\delta_2\cdot i=0 \tag{13}$$

where $\delta_1=\frac{(\alpha-\beta)^2}{4}-\frac{\gamma}{4}-\frac{\gamma'}{2},\quad \delta_2=\delta_1+\gamma'$ and

$$\gamma=\frac{LC'-L'C}{2LC}. \tag{14}$$

Assertions on solutions of (12), (13) are only possible when $\delta_1 = \delta_2 = \delta =$ const. If so, the original equations (7), (8) admit damped wave solutions travelling with velocity $a = 1/\sqrt{LC}$ for $\delta = 0$, as well as damped "harmonic wave"solutions moving with velocity $a = \sqrt{1 - \delta/\omega}$ depending on frequency $f = \omega/(2\pi)$ in the case of $\delta > 0$.

The condition $\delta_1 = \delta_2$ implies $\gamma' = 0$, i.e., γ =const. imposing some restrictions on the trees. Inserting expressions for L and C into (14) derived from the microscopic parameters, we obtain

$$\gamma = \frac{n'}{n} + \frac{3}{2}\frac{d'}{d} = \text{const.} \tag{15}$$

Hence by integration of (15)

$$n(x) \cdot dx^{\frac{3}{2}} = d(0)^{\frac{3}{2}} \cdot \exp(Ax). \tag{16}$$

Thus, the class of tree structures which can be described by the present approach is characterized through the condition (16).

4. Concluding Remarks

The results derived in Sections 2 and 3 enable us to formulate explicit expressions relating the solutions of the constant parameter case to those in the present case of spatially varying parameters. Using (16), we can rewrite (10), (11) as

$$V(x,t) = d_o^{-\frac{3}{4}} \cdot \exp\left(-(\alpha+\beta)\frac{t}{2}\right) \cdot \exp\left(-\frac{Ax}{2}\right) \cdot v(x,t) \tag{17}$$

$$I(x,t) = d_o^{\frac{3}{4}} \cdot \exp\left(-(\alpha+\beta)\frac{t}{2}\right) \cdot \exp\left(\frac{Ax}{2}\right) \cdot i(x,t). \tag{18}$$

The transformation (9) is equivalent to

$$x(r) = 2\sqrt{lc} \cdot \int_o^r \frac{ds}{\sqrt{d(s)}} \tag{19}$$

which shows that the course of $d(r)$ must be specified if the back-transformation $x \rightarrow r$ is to carry out. If (19) is known, the various specific solutions of the generalized wave equations (12), (13) can be applied to give solutions of the continuum equations (7), (8), provided the boundary conditions are changed accordingly. These results will be presented in a forthcoming, full paper in detail [2].

References

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[1] Chair for Mathematical Analysis, Purkyne University Brno,
Janačkovo nam.2a, 662 95 Brno, Czechoslovakia
[2] Carl Ludwig Institute for Physiology, Karl Marx University Leipzig,
Leibigstr. 27, 7010 Leipzig, German Democratic Republic