The aim of this paper is to prove the exactness of Bresinsky's resolution [1] for monomial curves in $\mathbf{P}^3$ using Gröbner bases. Further we construct a resolution for monomial Gorenstein curves in $\mathbf{A}^4$.

1. MONOMIAL CURVES IN $\mathbf{P}^3$

A monomial curve in $\mathbf{P}^3_k$, k a field, is the projective closure of the affine curve $(t^{n_1}, t^{n_2}, t^{n_3})$, $n_1 < n_2 < n_3$ and gcd $(n_1, n_2, n_3) = 1$. In [3] an algorithm was developed to construct a minimal generating set for the corresponding prime ideal $P(n_1, n_2, n_3) = P \subseteq S = k\,[x_0, x_1, x_2, x_3]$. For this purpose let

$$f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, \quad \alpha_1 \text{ minimal},$$

$$f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, \quad \alpha_2 \text{ minimal}, \ \alpha_{21} < \alpha_1 \text{ if } f_1 \neq f_2,$$

$$f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, \quad \alpha_3 \text{ minimal}, \ \alpha_{31} < \alpha_1 \text{ if } f_1 \neq f_3$$

be the generators of the defining ideal P' of the affine curve. Two of them may coincide up to sign. Following [1] or [3] set $\{i, j\} = \{2,3\}$. Then

$$\beta_j n_j = \beta_{j1} n_1 + \beta_{ji} n_i$$

$$\beta_1 n_i = \beta_{i1} n_1 + \beta_{ij} n_j \ \beta_{i1} \leqslant \beta_{j1}$$

produce a new relation

$$(\beta_j + \beta_{ij})\, n_j = (\beta_{j1} - \beta_{i1})\, n_1 + (\beta_{ji} + \beta_i)\, n_i.$$

If all the generators f_1, f_2, f_3 are distinct f_3 can be derived from f_1 and f_2 by this procedure. Indeed, assume

$$f_3' = x_3^{\alpha_{13}+\alpha_{23}} - x_1^{\alpha_1 - \alpha_{21}} x_2^{\alpha_2 - \alpha_{12}}$$

is not $f_3(\alpha_2 > \alpha_{12}$ otherwise α_1 would not be minimal). Hence $\alpha_3 < \alpha_{13} + \alpha_{23}$. Choose q such that $0 \leqslant \alpha_{13} + \alpha_{23} - q\,\alpha_3 < \alpha_3$. Then $(\alpha_1 - \alpha_{21} - q\,\alpha_{31})\ n_1 + (\alpha_2 - \alpha_{12} - q\,\alpha_{32})\ n_2 + (q\alpha_3 - \alpha_{13} - \alpha_{23})\ n_3 = 0$. The coefficients can't have the same sign. But this contradicts the minimality of either α_1 or α_2 or α_3.

Hence if one starts with two distinct generators, e.g. f_1 and f_2, the above procedure produces new elements of P'. If we proceed as in the Euclidean algorithm for the powers of x_1 successively and homogenize

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e resulting binoms we get a minimal generating set of P if we continue
itil a homogenized binom with pure x_2-power arises. This way f_{i+1} is
oduced using f_i and a certain $f_{a(i)}, a(i) < i$, for $i \geqslant 2$.

2. THE GRÖBNER BASE

Order monomials degreewise and monomials with equal degree
icographically assuming $x_1 < x_2 < x_3$. For a polynomial f denote $M(f)$ the greatest monomial appearing as a summand in f, and $M(I) = \langle M(f) : f \in I \rangle$ the ideal of leading monomials of I. A set of elements $g_2, \ldots, g_s \in I$ is called a *Gröbner base* if $M(g_1), \ldots, M(g_s)$ generate $I)$, see [4] or [6]. In our case the basis $f_1, \ldots, f_n$ constructed by the
orithm above yields a Gröbner base of P' as will be shown later.
have

$$M(f_1) = x_{1_4}^{\alpha_1}$$

$$M(f_2) = x_1^{\alpha_{21}} x_3^{\alpha_{23}} \quad (\text{if } f_1 \neq f_2 \text{ and } n > 2)$$

for $M(f_{a(k)}) = x_1^{\beta_{j_1}} x_i^{\beta_{ji}}$ and $M(f_k) = x_1^{\beta_{i_1}} x_j^{\beta_{ij}}$ $\Big\}$ or viceversa, $\{i, j\} = \{2,3\}$

llows $M(f_{k+1}) = x_1^{\beta_{j_1} - \beta_{i_1}} x_i^{\beta_{ji} + \beta_i}$, $k + 1 < n$,

$$M(f_n) = x_2^e \text{ with } e = \beta_2 + \beta_{32}$$

So $M(f_i)$ is either $x_1^{\gamma_i} x_2^{\delta_i}$ or $x_1^{\gamma_i} x_3^{\varepsilon_i}$ $(i < n)$ and if $M(f_i)$ and $M(f_j)$ of the same kind $(i < j)$ then $\gamma_i > \gamma_j$, $\delta_i < \delta_j$ or $\varepsilon_i < \varepsilon_j$. Note $a(i) = a(k)$ for $a(k) < i < k$.

3. THE RESOLUTION

As was shown in [5] or [6] a minimal resolution of $S/M(I)$ can be
l to a resolution of S/I. Moreover $g_1, \ldots, g_s$ is a Gröbner base if and
if this lifting is possible. So let's construct a minimal resolution
$/M(I)$. For this purpose one should take Taylor's resolution and
mize it as described in [5]. Recall some basic facts and definitions.
Consider an ideal $J = (M_1, \ldots, M_n)$ generated by monomials. Let
be the set of all k-tuples of elements from $\{1, \ldots, n\}$ and Ind $=$
$\bigcup$ Ind_k. Set

$$M(I) := \text{lcm}(M_i : i \in I)$$

$$M\binom{I}{I-k} := (-1)^{a(I,k)} \frac{M(I)}{M(I-k)} \text{ for } k \in I \in \text{Ind}$$

$$\text{and } a(I, k) = \#\{i \in I : i < k\}.$$

$$0 \to S^{\text{Ind}_n} \xrightarrow{d} \ldots\ldots \xrightarrow{d} S^{\text{Ind}_0} = S \to S/J \to 0$$

with

$$d(e_I) = \sum_{k \in I} M\binom{I}{I-k} e_{I-k} \tag{1}$$

is a resolution for S/J, see [7]. This resolution is not minimal in general. To minimize it one has to delete dependent syzygies. They correspond to basic elements with indices I and $I + k$ $(k \notin I)$ with $M(I) = M(I + k)$. If all $I + j$, $j \neq k$ $(j \in I)$ have been deleted earlier this procedure looks very pleasant because no substitution arises, see [5]. This can be attained in our situation as we will show in the sequel. More precisely we show that

$$\text{Ind}_2' = \{(12), I_0\} \cup \{(a(k), k+1), (k, k+1) : k = 2, \ldots, n-2\}$$

form a basis of the second syzygy module (I_0 to be defined later) while

$$\text{Ind}_3' = \{(123)\} \cup \{(a(k), k, k+1) : k = 3, \ldots, n-2\}$$

for the third syzygy module of J. The proof is by induction on k. Assume that all $I \subset \{1, \ldots, k\}$, $|I| \geqslant 2$ not listed above, have been deleted from the base without substitution as described in [5]. Assume w. l. o. g.

$$M(f_{a(k)}) = x_1^{\gamma a(k)} x_3^{\varepsilon a(k)},\ M(f_k) = x_1^{\gamma k} x_2^{\delta k},\ M(f_{k+1}) = x_1^{\gamma_{k+1}} x_2^{\delta_{k+1}}.$$

If $M(f_i) = x_1^{\gamma_i} x_2^{\delta_i} (i < k)$ then $M(i, k, k+1) = M(i, k+1)$. Hence there is a one-to-one correspondence between $I > (i, k+1)$ not containing k and $I + k$. None of them has been deleted earlier. Hence all of them can be deleted as in [5] without substitution. If $M(f_i) = x_1^{\gamma_i} x_3^{\varepsilon_i}$ then by construction $i \leqslant a(k)$. Assume $i < a(k)$. Then $M(i, a(k), k+1) = M(i, k+1)$ and $I \supset (i, k+1)$ can be deleted without substitution as above. The remaining basic elements are $(a(k), k+1)$, $(k, k+1)$ and $(a(k), k, k+1)$. The other possible cases for $M(f_{a(k)})$, $M(f_k)$ and $M(f_{k+1})$ can be treated in the same manner.

If $k + 1 = n$ we get $M(f_n) = x_2^e$. Assume $M(f_{a(n-1)}) = x_1^{\gamma_{a(n-1)}} x_2^{\delta_{a(n-1)}}$, $M(f_{n-1}) = x_1^{\gamma_{n-1}} x_3^{\varepsilon_{n-1}}$. Then $e > \delta_{a(n-1)}$, $\gamma_{a(n-1)} \leqslant \gamma_{n-1}$. If $M(f_i) = x_1^{\gamma_i} x_2^{\delta_i}$ we must have $i \leqslant a(n-1)$ and this way for $i \neq a(n-1)$ $M(i, a(n-1), n) = M(i, n) = x_1^{\gamma_i} x_2^e$.

If $M(f_i) = x_1^{\gamma_i} x_3^{\varepsilon_i}$ then $\gamma_i \geqslant \gamma_n \geqslant \gamma_{a(n-1)}$ hence $M(i, a(n-1), n) = M(i, n) = x_1^{\gamma_i} x_2^e x_3^{\varepsilon_i}$. Thus all $I \supset (n)$ except of $I_0 := (a(n-1), n)$ (and (n) of course) can be deleted without substitution in this case. The other case $(M(f_{n-1}) = x_1^{\gamma_{n-1}} x_2^{\delta_{n-1}}$, $I_0 = (n-1, n))$ can be treated in the same manner. We proved the following

$$0 \to S^{\text{Ind}_3'} \xrightarrow{d} S^{\text{Ind}_2'} \xrightarrow{d} S^{\text{Ind}_1} \xrightarrow{d} S \to S/J \to 0 \tag{2}$$

THEOREM. *with d as in (1) is a minimal resolution of S/J.*

The resolution given in [1] "fulfills" the complex (2) with respect to the basic straightening relations. Hence it is exact and $f_1, \ldots, f_n$ is a Gröbner base for P' as claimed, by [6]. Moreover the homogenized forms $F_1, \ldots, F_n$ are a Gröbner base for P, too. The homogenizing variable x_0 appears as the deformation parameter of [5].

A monomial curve in $\mathbf{A}^4$ is a curve with coordinate ring $R = k[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}] \subset k[t]$, gcd $(n_1, n_2, n_3, n_4) = 1$. The Gorenstein curves (i.e. R – Gorenstein) among them are classified in [2] to be either complete intersections or generated by 5 elements. The former ones are resolved by the Koszul complex. For the latter ones and $S = k[x_1, x_2, x_3, x_4]$ we have $R = S/P$ with P generated by

$$f_1 = x_3^{\alpha_{13}} x_4^{\alpha_{14}} - x_1^{\alpha_1}$$

$$f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}$$

$$f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}$$

$$f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$$

$f_5 = x_2^{\alpha_{32}} x_4^{\alpha_{14}} - x_3^{\alpha_{43}} x_1^{\alpha_{21}}$, ([2], Theorem 3), with

$0 < \alpha_{ji} < \alpha_i \quad 1 \leqslant i, j \leqslant 4, \quad \alpha_1 = \alpha_{31} + \alpha_{21}, \alpha_2 = \alpha_{32} + \alpha_{42},$

$\alpha_3 = \alpha_{13} + \alpha_{43}, \alpha_4 = \alpha_{24} + \alpha_{14}$ (and the n_i specified, too), ([2], Theorem 5).

Take $H = (1 < 2 < 3 < 4)$ and the lexicographic order corresponding to the grading deg $x_i = n_i$. Then for the discrete RSL, i.e. the ring corresponding to the monomial ideal of leading forms of P, we get

$M(24) = M(245), \qquad M(34) = M(134),$

$M(35) = M(135), \qquad M(12) = M(125).$

Delete the corresponding basic elements as described in [5]. Substitution will arise at the last step only. Filling up the resolution obtained this way we get

$$0 \to S^2 \xrightarrow{B} S^6 \xrightarrow{A} S^5 \to S \to R \to 0$$

A	13	14	15	23	25	45
1	$x_3^{\alpha_{42}}$	$x_4^{\alpha_{24}}$	$x_2^{\alpha_{32}}$	0	0	0
2	0	$-\mathbf{x}_1^{\alpha_{31}}$	0	f_3	$x_4^{\alpha_{14}}$	$x_3^{\alpha_{43}}$
3	$-x_4^{\alpha_{14}}$	$-\mathbf{x}_2^{\alpha_{42}}$	$-\mathbf{x}_1^{\alpha_{21}}$	$-\mathbf{f}_2$	0	0
4	0	$-x_3^{\alpha_{13}}$	0	0	$\mathbf{x}_1^{\alpha_{21}}$	$x_2^{\alpha_{32}}$
5	$-\mathbf{x}_1^{\alpha_{31}}$	0	$-x_3^{\alpha_{13}}$	0	$-x_2^{\alpha_{42}}$	$-x_4^{\alpha_{24}}$

B	123	145
13	$-\mathbf{f}_2$	0
14	$-\mathbf{x}_3^{\alpha_{43}} \mathbf{x}_1^{\alpha_{21}}$	$x_2^{\alpha_{32}}$
15	$x_2^{\alpha_{42}} x_3^{\alpha_{43}}$	$-x_4^{\alpha_{24}}$
23	$x_4^{\alpha_{14}}$	$-\mathbf{1}$
25	$-\mathbf{f}_3$	0
45	$-\mathbf{x}_1^{\alpha_1}$	$x_3^{\alpha_{13}}$

The terms filled up are printed in boldface. For f_i the straightening part is filled up.

A further minimization deletes e_{23}

$$0 \to S \xrightarrow{B'} S^5 \xrightarrow{A'} S^5 \to S \to R \to 0.$$

For A' remove the column (23) in A. B' is the transpose of the matrix

	13	14	15	25	45
123	$-f_2$	f_5	$-f_4$	$-f_3$	$-f_1$

The resolution is symmetric as expected for Gorenstein rings.

Referees's remark. If one takes the columns of the matrix A in the order 45, 13, 25, 15, 14 and one changes the sign of 45 and 15, then one obtains a skew symmetric 5×5 matrix Φ, whose skew symmetric 4×4 minors give 5 pfaffians which generate the ideal I. This way, the resolution found in this paper is isomorphic to what we expect in view of D. A. Buchsbaum, D. Eisenbud, *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension* 3, Amer. I. of Math., Vol. 99, No. 3, pp. 447–485.

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