Uncountable cofinalities of automorphism groups of linear and partial orders

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Abstract

We demonstrate the uncountable cofinality of the automorphism groups of various linear and partial orders. We also relate this to the 'Bergman' property, and discuss cases where this may fail even though the cofinality is uncountable.

1 Introduction

In [5, 6], three related properties which an infinite group G may have are discussed. These are the so-called *Bergman property* (after [2]), uncountable cofinality and strong uncountable cofinality. These say the following:

Bergman property: for any generating set E for G which contains the identity and is closed under inverses, there is $n \in \mathbb{N}$ such that $G = E^n$.

Uncountable cofinality: G cannot be expressed as $\bigcup_{i\in\omega} G_i$ where $G_0 \subset G_1 \subset G_2 \subset \ldots$ and each G_i is a proper subgroup.

Strong uncountable cofinality: G cannot be expressed as $\bigcup_{i\in\omega} U_i$ where $U_0 \subset U_1 \subset U_2 \subset \ldots$ and each U_i is closed under inverses, and $\forall i \exists j (U_i U_i \subseteq U_j)$.

In fact these are related, and it is shown in [6] that G has strong uncountable cofinality if and only if it has both the Bergman property, and uncountable cofinality.

Special cases of groups which are known to have uncountable cofinality are the symmetric group on an infinite set [16], the automorphism group $A(\mathbb{Q})$ of the rational numbers [12], and an infinite direct power of a finite perfect group [15]. The Bergman property has also been verified in a number of cases, starting with the infinite symmetric group [2] and $A(\mathbb{Q})$ [6], and going on to many more general classes in [5, 9]. The equivalence mentioned above at once assures us that these also all have strong uncountable cofinality.

In this paper we look at three further cases where we can find out when uncountable cofinality and its variants do or do not hold. The first is that

[1]2000 Mathematics subject classification 06A07, 06F15, 20B22, 20E15;
keywords: cofinality, partial order, cycle-free
This work was supported by a grant from the EPSRC.

of the automorphism groups of linear orders (chains). Since we usually make some transitivity assumption on a structure whose automorphism group we are considering (to guarantee that it is sufficiently rich), the minimum here is 1-transitivity. We focus particularly on the linear orders in Morel's classification of the countable 1-transitive linear orders [17], but prove a more general result based on the ideas in [6]. Notice that most of the ones in Morel's list are discrete, but this does not prevent us proving strong uncountable cofinality for those embedding $\mathbb{Q}$. This contrasts with the results of [3], where it is shown that the small index property fails for all of these structures except $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for $\alpha \leq 1$. We also make remarks about those of the form $\mathbb{Z}^\alpha$ which in all cases fail to have the Bergman property, and for limit α, even fail uncountable cofinality (meaning that in this case the automorphism group is the union of a countable chain of proper subgroups).

Next we look at 'trees' (or 'semilinear orders') and their automorphism groups, as studied for instance in [1, 7, 8]. We demonstrate in section 3 that the automorphism group of any weakly 2-transitive tree with countable coinitiality has strong uncountable cofinality. This uses an adaptation of the methods of [6], and some of the technical material from [7], which we recall below.

Finally we consider cycle-free partial orders ($CFPO$s). Viewing these as generalizations of trees, Warren classified a class of sufficiently transitive $CFPO$s in [21], and this was extended to a classification of all the countable k-CS-transitive $CFPO$s for $k \geq 2$ in [4, 19, 13]. One point here is that these structures were shown in [10] to fall into three distinct classes, those whose automorphism groups are not simple, and those whose automorphism groups are simple, with or without a bound on the number of conjugates required (where this means that there is some fixed number m such that for any non-identity elements g and h of G, h may be written as the product of at most m conjugates of g or g^{-1}). This last case at once provides us with natural examples where the cofinality is uncountable, but Bergman's property fails. Here however, even those structures whose automorphism groups are simple, but where a bounded number of conjugates suffices, have uncountable cofinality but not strong uncountable cofinality, and this is for a slightly different (if related) reason, namely that there is a notion of 'distance' in any $CFPO$, and the distance between two vertices can be arbitrarily large. In fact this observation enables us to conclude that a much larger class of $CFPO$s, even including many 1-transitive ones, fail to have the Bergman property.

We conclude this introduction by recalling some of the notation and definitions we shall require.

For any partially ordered set $P = (P, <)$ we write $A(P)$ for the automorphism group of P. This is the notation adopted in [11] for instance, when P is linearly ordered, and also for trees in [7, 8]. The only other types of partially ordered set that we consider here are $CFPO$s, and for consistency we retain the notation for them as well.

A structure is said to be *1-transitive* if its automorphism group acts singly transitively on points. Morel [17] classified all the countable 1-transitive linear orders, and they are $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for countable ordinals α. Here $\mathbb{Z}^\alpha$ is the restricted lexicographic power of α copies of $\mathbb{Z}$ (which may be taken to be the set of functions from α to $\mathbb{Z}$ of finite support ordered lexicographically), and $\mathbb{Q}.\mathbb{Z}^\alpha$ is the lexicographic product of $\mathbb{Q}$ with that, $\mathbb{Q}$ 'copies' of $\mathbb{Z}^\alpha$. A chain is said to be *doubly homogeneous* if its automorphism group acts transitively on its 2-element subsets. There is only one non-trivial countable doubly homogeneous chain, namely $\mathbb{Q}$, but there are many more uncountable ones.

A *tree*, or semilinear order, is a partially ordered set $(T, <)$ in which any two elements have a common lower bound, and for any element x, $\{y \in T : y \leq x\}$ is a chain. A tree is said to be *weakly 2-transitive* if its automorphism group acts transitively on its set of 2-element chains. The countable weakly 2-transitive trees were classified in [7], and properties of their automorphism groups described. In particular, they have $2^{2^{\aleph_0}}$ normal subgroups. In [8] more information was given about these, even not assuming countability.

The precise definition of *cycle-free partial order* ($CFPO$) is given in [21]. The main points are these. Any partially ordered set M has a so-called *Dedekind–MacNeille completion* written M^D, which may be characterized by saying that in M^D, any non-empty bounded above subset has a least upper bound, and it is the minimal such containing M. We then say that M is a cycle-free partial order if between any two of its points there is a unique path in M^D (in a natural analogue of the notion for graphs). This entails that any $CFPO$ is necessarily connected. Note that this gives rise to a notion of 'distance', since we may say that points x and y of M are at *distance* n if there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that for each $i < n$, x_i and x_{i+1} are comparable, and the set I_i of points of M^D lying between x_i and x_{i+1} is a chain for each i, such that if $I_i \cap I_j \neq \emptyset$ and $i < j$, then $j = i + 1$ and the intersection is equal to $\{x_{i+1}\}$. In practice we more often work with M^+ rather than M^D, which is the union of M and all the ramification points, which are the elements of M^D which may be expressed as either the least upper bound or greatest lower bound of two members of M (these are called 'downward' and 'upward ramification points' respectively). This has the advantage that if M is countable, so is M^+.

For trees the appropriate notions of transitivity are k-transitivity, weak k-transitivity, or k-homogeneity for $k \leq 2$. For cycle-free partial orders, the fact that there is a notion of distance as described above means that in non-trivial cases, k-transitivity can never hold for $k \geq 2$ since there will be incomparable points at different distances. So in this context the appropriate notions are k-CS-transitivity (or homogeneity) for $k \leq 3$ where this means that for any two isomorphic *connected* k-element substructures, there is an isomorphism taking the first to the second. A classification of all the countable $2-$ or 3-CS-transitive $CFPO$s is given in the papers [21, 4, 19, 13].

We write $\overline{\Omega}$ for the Dedekind-completion of a chain Ω and $\overline{\overline{\Omega}}$ for its 'total completion', that is, also including endpoints. It is clear that the automorphism group of Ω acts naturally on $\overline{\Omega}$ (and $\overline{\overline{\Omega}}$), and the same applies to the Dedekind–MacNeille completion of a partially ordered set.

2 Uncountable cofinality for some linear orders

The main result in this section arose as an attempt to determine whether or not the automorphism groups of the linear orders in Morel's classification of all the countable 1-transitive linear orders have uncountable cofinality. In fact we found that we were able to prove a considerably stronger result corresponding to the structures in Morel's list of the form $\mathbb{Q}.\mathbb{Z}^\alpha$, by adapting the methods of [6] (which in turn were derived from [12]). The situation for the other structures in her list, $\mathbb{Z}^\alpha$ for $\alpha \geq 2$, is as yet not completely resolved—see below. We say that a chain Ω has *countable coterminality* if there are points $a_n \in \Omega$ for $n \in \mathbb{Z}$ such that $a_n < a_{n+1}$ and $(\forall a \in \overline{\Omega})(\exists m, n(a_m \leq a \leq a_n))$. Note that it makes no difference whether these points are required to lie in Ω or its Dedekind-completion (though the definition as it stands would refer just to Ω). A chain is *scattered* if it does not embed $\mathbb{Q}$.

Before stating and proving the main result of this section, we prove two lemmas which are needed in the proof.

Lemma 2.1. *Let $(Y, <)$ and $(Z, <)$ be linear orders such that Y is doubly homogeneous with countable coterminality and Z is scattered. Then any two positive elements h_1 and h_2 of $A(Y.Z)$ with coterminal orbits are conjugate (where by h_i 'positive' we mean that $h_i(x) > x$ for all x).*

Proof: It is clear that as h_i has a coterminal orbit, it follows that every orbit is coterminal, so we may pick orbits $\{h_1^i(y, z) : i \in \mathbb{Z}\}$ and $\{h_2^i(y, z) : i \in \mathbb{Z}\}$. Now $[(y, z), h_1(y, z)) \cong [(y, z), h_2(y, z))$, since $[(y, z), h_1(y, z)) = \{y\} \times [z, \infty) \cup (y, y_1) \times Z \cup \{y_1\} \times (-\infty, z_1)$ where $h_1(y, z) = (y_1, z_1)$, and similarly for $[(y, z), h_2(y, z))$, and $(y, y_1) \times Z \cong (y, y_2) \times Z$ since $(y, y_1) \cong (y, y_2)$, and $\{y_1\} \times (-\infty, z_1) \cong \{y_2\} \times (-\infty, z_2)$ by the isomorphism $h_2 h_1^{-1}$. (Note that it follows from the facts that Z is scattered and Y is doubly homogeneous that $h_2 h_1^{-1}$ maps $\{y_1\} \times (-\infty, z_1)$ onto $\{y_2\} \times (-\infty, z_2)$.) Let θ be an isomorphism from $[(y, z), h_1(y, z))$ to $[(y, z), h_2(y, z))$, and extend to the whole of $Y.Z$ by letting $\theta(h_1^i(t)) = h_2^i \theta(t)$ for each $t \in [(y, z), h_1(y, z))$ and $i \in \mathbb{Z}$. Then we deduce that $\theta h_1 = h_2 \theta$ and so h_1 and h_2 are conjugate. □

Lemma 2.2. *For Y and Z as in the previous lemma, any element h of $A(Y.Z)$ may be written in the form $h_1 h_2^{-1}$ where h_1 and h_2 are positive with coterminal orbits.*

Proof: Choose $z \in Z$ and $y_i \in Y$ for $i \in \mathbb{Z}$ such that $y_i < y_{i+1}$ and $y_i \to \pm\infty$ as $i \to \pm\infty$. Let $t_i = (y_i, z)$. Then $t_i < t_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. By passing to a coterminal subset, we may assume that for each i, $t_{i-1} < h(t_i) < t_{i+1}$. Let $h_1 \in A(Y.Z)$ satisfy $h_1(t_i) = t_{i+2}$ for each i. Then $h^{-1}h_1(t_i) = h^{-1}(t_{i+2}) > t_{i+1}$, from which it follows that h_1 and $h_2 = h^{-1}h_1$ are both positive with a coterminal orbit. Hence $h = h_1 h_2^{-1}$ is expressed in the desired form. □

Theorem 2.3. *Let Z be a scattered linear order, and $\Omega = X.Z$ be the lexicographic product of X copies of Z for some infinite doubly homogeneous chain*

X. We assume further that X may be expressed as the disjoint union of open intervals having countable coterminality. Then $A(\Omega)$ has strong uncountable cofinality.

Proof: Note that $G = A(\Omega)$ preserves the copies of Z (as Z is scattered), and so is equal to the wreath product of $A(X)$ and $A(Z)$. The same proof would work without the assumption of Z scattered, but then we would have to work with the subgroup of $A(\Omega)$ preserving the copies setwise, which could be a proper subgroup, rather than $A(\Omega)$ itself. We may allow G to act on X if we wish, since any automorphism will carry each copy of Z to a copy of Z (using again the fact that it is scattered).

Suppose for a contradiction that G may be written as $\bigcup_{i\in\omega} U_i$ where $U_0 \subset U_1 \subset U_2 \subset \ldots$, $U_i^{-1} = U_i$, and $\forall i \exists j (U_i U_i \subseteq U_j)$. We shall follow the proof from [6], indicating modifications where necessary.

A key point is to characterize the fixed point sets in the order-completion $\overline{\Omega}$ of Ω of members of G. In [6] these were called *clans*. Since Ω has a more complicated structure than in [6], we just use the word for a special kind of fixed point set. Recall that $\overline{\overline{\Omega}}$ is the total completion of Ω (with endpoints).

Now Ω consists of X copies of Z, and we note that its Dedekind completion is equal to $X.\overline{\overline{Z}} \cup (\overline{X} - X)$, where this is ordered by $(x, z) < x' \Leftrightarrow x < x'$ for $x \in X, x' \in \overline{X} - X$, $z \in \overline{\overline{Z}}$. This is because the Dedekind cuts of $X.Z$ are of two possible kinds, those that 'intersect' or 'abut' some copy of Z, in which case they are determined by a member of $\overline{\overline{Z}}$, and those which do not, in which case they are determined by a member of $\overline{X} - X$. We then define a *clan* to be a closed subset of $\overline{\Omega}$ which is a subset of $\overline{X} - X$ and such that all intervals I making up its complement are unbounded with countable coterminality.

In [6] a subset of $\overline{\Omega}$ was defined to be a clan if and only if it is the fixed point set of some automorphism. In our case, we can see analogously that a subset of $\overline{\Omega}$ is a clan if and only if it is the fixed point set of some automorphism $g \in A(\Omega)$ that moves every point of X under the action of g on X mentioned above. A key point is that if Y is the fixed point set of some automorphism of Ω, then Y contains some clan, and this is a good enough substitute for the property which applied in [6] to make the argument go through. The see the truth of this, let Y be the fixed point set of $g \in G$. By the hypothesis of the theorem, there is a family $\mathcal{I}$ of pairwise disjoint open intervals of countable coterminality in $\overline{\Omega}$ such that $\Omega \subseteq \bigcup \mathcal{I}$. Then $Y \cap Y'$, where Y' is the complement of $\bigcup \mathcal{I}$ in $\overline{\Omega}$, is closed. Now Y' is the fixed point set of some $g' \in G$. We may assume that both g and g' are increasing, meaning that they map each element of Ω to itself or a larger one. Then $gg' \in G$ has precisely $Y \cap Y'$ as its fixed point set, and gg' moves every point of X, since g' does. Hence $Y \cap Y'$ is a clan contained in Y.

We now need to define what is meant by a *moiety* in this context. The word is meant to suggest that it is a subset which is 'half' the set. For the rationals for instance, it is taken to be a subset of the form $\bigcup_{i\in\mathbb{Z}} (x_{2i}, x_{2i+1})$ where x_i are irrationals such that $x_i < x_{i+1}$ and $x_i \to \pm\infty$ as $i \to \pm\infty$, since the complement of such a set is of exactly the same form. More generally, in a chain of countable coterminality it is taken to be a set of the same form where each x_i lies in the Dedekind-completion, the x_{2i} lie in the same orbit, and so do the x_{2i+1}, and (x_{2i}, x_{2i+1}) has countable coterminality. Note that now the complement of such a set no longer need have the same form, since the cofinalities of the intervals may be uncountable; however the complement *contains* such a set of the correct form, so this is good enough. In the case where the coterminality of Ω need not be countable, we have to cut Ω into pieces of countable coterminality, which is what is done in [6], and that is where clans come in. If κ is a clan, then a *κ-moiety* is a subset of Ω which is the union of moieties M_I of I for each open interval in the complement of κ, where for I, 'moiety' has the above meaning. Thus a κ-moiety is a whole collection of moieties. The proof works because we can handle all of these simultaneously. For the time being we work inside $D(\kappa) = \{g \in G : g \text{ fixes all members of } \kappa\}$ where κ is a fixed clan.

We shall say that U_n is *full* on a subset Σ of Ω if each element of $G(\Sigma)$ is equal to the restriction to Σ of some member of U_n where $G(\Sigma)$ is the set of all members of G whose support is contained in Σ. The next stage in the proof is to show that some U_n is full on some κ-moiety. For this, we choose countably many pairwise disjoint κ-moieties M_n, and we show that for some n, U_n is full on M_n. If not, then for each n there is $g_n \in G(M_n)$ which does not agree on M_n with any member of U_n. Let $g \in G$ be obtained by gluing all the g_n together, so that g agrees with g_n on M_n for each n and fixes all other points. Since $G = \bigcup_{n\in\omega} U_n$, $g \in U_n$ for some n. But now g *does* agree with g_n on U_n after all, which is a contradiction. This shows that there is some κ-moiety M_n on which some U_n is full. Let us write M in place of M_n.

Next we show that there is m such that $G(M) \subseteq U_m$. Let $g \in G(M)$. Pick $h \in G(M)$ which on each interval of M acts positively with a coterminal orbit. Applying Lemmas 2.1 and 2.2 to each such interval, and patching the conjugacies together, we see that g may be written in the form $h^{k_1}(h^{-1})^{k_2}$ where $k_i \in G(M)$. Since U_n is full on M, there is $k_i' \in U_n$ agreeing with k_i on M. Then $h^{k_1'}(h^{-1})^{k_2'} = h^{k_1}(h^{-1})^{k_2}$ as one sees by examining separately the action of this map on M and its complement in $\overline{\Omega} - \kappa$. Now h is fixed (though g varies), so there is i such that $h \in U_i$. Find $m \geq n$ such that $(U_n U_i U_n)^2 \subseteq U_m$, and $g \in U_m$. Thus $G(M) \subseteq U_m$.

We now make a fresh choice of y_i in each open interval of $\overline{\Omega} - \kappa$ having the same properties as the x_i did, so that, in addition, y_i is from the same orbit that x_i is, and $x_{2i} < y_{2i-1} < y_{2i} < x_{2i+1}$ for each i. The point of this choice is that the union of the κ-moieties M and M' which are the unions over all the intervals I of $\bigcup_{i\in\mathbb{Z}}(x_{2i}, x_{2i+1})$ and $\bigcup_{i\in\mathbb{Z}}(y_{2i}, y_{2i+1})$ respectively is the whole of $\overline{\Omega} - \kappa$, and their intersection is also a κ-moiety. Let $h \in G(\overline{\Omega} - \kappa)$ map x_i to y_i for each i, and on each interval I. Then $G(M') = G(hM) = hG(M)h^{-1}$, so $U = G(M).G(M') = G(M)hG(M)h^{-1} \subseteq U_m U_j U_m U_j$ where $h \in U_j$, and this is contained in $U_{m'}$ for some $m' \geq m$.

We observe that any member of $D(\kappa)$ is conjugate to some member of U. That is why it was important to ensure that the intervals making up M and M' overlapped. It follows that $D(\kappa) = \bigcup_{i\in\omega} \bigcup_{g\in U_i} gUg^{-1}$.

The remainder of the argument may now be read off from [6]. We suppose for a contradiction that $G = \bigcup_{i\in\omega} U_i$ as above. Then for each n there is $g_n \in A(\Omega) - U_n$. By the proof of [6] Lemma 2.17, and the remarks above, the intersection of the fixed point sets of the g_n contains a clan κ say. By the argument of [6], there is k such that $D(\kappa) \subseteq U_k$, but this contradicts $g_k \in D(\kappa)$. $\Box$

We remark that the result we would really like would be as for Theorem 2.3 but without the extra hypothesis that X is expressible as the disjoint union of open intervals of countable coterminality. The difficulty which arises in doing this seems similar to the situation in investigating the uncountable cofinality of $A(\mathbb{Z}^\alpha)$ for $\alpha \geq 2$. One appears to be forced to study the subgroup comprising those elements which fix each block of imprimitivity setwise. Thus for $\alpha = 2$, the automorphism group is again a wreath product, this time just of $\mathbb{Z}$ with itself, so is generated by the unrestricted direct product of $\aleph_0$ copies of $\mathbb{Z}$ and a single translation g on the copies.

The main initial example of a chain X to which the theorem may be applied is given by taking $\mathbb{Q}$ for X, and $\mathbb{Z}^\alpha$ for Y for any countable ordinal α, giving strong uncountable cofinality of $A(Z)$ for 'half' the structures Z in Morel's list, namely those of the form $\mathbb{Q}.\mathbb{Z}^\alpha$. There are however many other examples, one for instance being obtained by letting X be the smallest 'long rational line', which is the lexicographic product $\omega_1 \times \mathbb{Q}$ (with the same range of possible choices for Y).

Now we remark on the situation for chains of the form $\mathbb{Z}^\alpha$ for countable ordinals α, the members of Morel's classification not covered so far.

Theorem 2.4. *(i) For any countable successor ordinal α, $A(\mathbb{Z}^\alpha)$ does not satisfy the Bergman property.*

(ii) For any countable limit ordinal λ, $A(\mathbb{Z}^\lambda)$ is expressible as $\bigcup_{i\in\omega} G_i$ where $G_0 \subset G_1 \subset G_2 \subset \ldots$ and each G_i is a proper subgroup.

Proof: (i) Let $\alpha = \beta + 1$ say. Thus $\mathbb{Z}^\alpha = \mathbb{Z}.\mathbb{Z}^\beta$, so it is the union of $\mathbb{Z}$ copies of $\mathbb{Z}^\beta$, X_n say for $n \in \mathbb{Z}$. Let E be the set of elements of $A(\mathbb{Z}^\alpha)$ which map X_0 into $\bigcup_{|i|\leq 1} X_i$. Then each $g \in A(\mathbb{Z}^\alpha)$ maps X_0 into $\bigcup_{|i|\leq n} X_i$ for some n, so $g \in E^n$, showing that $A(\mathbb{Z}^\alpha) = \langle E \rangle$. But here since n cannot be bounded, we have $A(\mathbb{Z}^\alpha) \neq E^n$ for any n.

(ii) Let α_n for $n \in \omega$ be an increasing sequence with limit λ. Then $\mathbb{Z}^\lambda = \bigcup_{n\in\omega} \mathbb{Z}^{\alpha_n}$ under the natural identification of $\mathbb{Z}^{\alpha_n}$ as a subset of $\mathbb{Z}^{\alpha_{n+1}}$ for each n. Let G_n be the group of members of $A(\mathbb{Z}^\lambda)$ which fix $\mathbb{Z}^{\alpha_n}$ setwise. Then this provides a strictly increasing chain of subgroups, and any member g of $A(\mathbb{Z}^\lambda)$ lies in some G_n since 0 must be mapped to a member of some $\mathbb{Z}^{\alpha_n}$ and it follows that g fixes $\mathbb{Z}^{\alpha_n}$ setwise so lies in G_n. $\Box$

This shows that strong uncountable cofinality fails for all $\mathbb{Z}^\alpha$ for $\alpha > 0$, but we remark that we do not know whether or not $A(\mathbb{Z}^\alpha)$ has the Bergman property when α is a limit ordinal.

3 Uncountable cofinalities for weakly 2-transitive trees

We begin this section by recalling more of the notation and results from [7, 8]. We need the language of lattice-ordered groups, or 'ℓ-groups', which are groups endowed with a lattice structure which is compatible with the group operation. A subgroup which is also a sublattice is called an *ℓ-subgroup*. It was shown by Holland [14] that the automorphism group of any chain is a lattice-ordered group under the pointwise operations, and conversely, any lattice-ordered group can be (lattice- and group-) embedded in some such $A(\Omega)$. A subgroup H of the group $A(\Omega)$ is said to be closed under *piecewise patching* if for any convex subchain S of Ω and $\{a_i; i \in \mathbb{Z}\}$ and $\{b_i; i \in \mathbb{Z}\}$ in S which are coterminal in S and with $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for each i, if h_i are members of H taking $[a_i, a_{i+1}]$ to $[b_i, b_{i+1}]$ for each i, then there is $h \in H$ which agrees with h_i on $[a_i, a_{i+1}]$ and fixes all points outside S. Now viewing $A(\Omega)$ as a lattice-ordered group, we say that the subgroup H is closed under *disjoint suprema* if whenever $\{h_i : i \in I\} \subseteq H$ with $h_i \wedge h_j = 1$ for $i \neq j$, then the supremum h of the h_i in $A(\Omega)$ belongs to H. A 2-transitive ℓ-subgroup H of $A(\Omega)$ closed under piecewise patching and disjoint suprema is called *large* in $A(\Omega)$. By [6], if H is large in $A(\Omega)$, then H has strong uncountable cofinality.

In the analysis of the normal subgroup structure of $A(T)$ given in [7] a particular normal subgroup $S(T)$ is introduced, and it is shown that it is the unique minimal normal subgroup. It is defined to be the group of all those $g \in A(T)$ such that for some $x \in T$, $x < supp(g)$. In other words, if $gy \neq y$ then $x < y$. The only result we need about this subgroup is Lemma 4.1 of [7] I, which says that if C is a maximal chain, $c \in T$, and f is any automorphism, then there is $g \in S(T)$ such that gf fixes C setwise, and also every point $\geq c$.

We say that T has *countable coinitiality*, if T contains a countable descending sequence with no lower bound in T.

Theorem 3.1. *Let $(T, \leq)$ be a weakly 2-transitive tree with countable coinitiality. Then $A(T)$ has strong uncountable cofinality.*

Proof: Let $(U_i)_{i \in \omega}$ be an ascending sequence of subsets of $A(T)$ with union $A(T)$ such that for each $i \in \omega$, $U_i = U_i^{-1}$ and $\forall i \exists j (U_i U_i \subseteq U_j)$. We shall show that $A(T) = U_m$ for some $m \in \omega$, giving a contradiction.

Let C be a convex subchain of T which is unbounded below in T and has countable coterminality. Such chains exist by the assumption that T has countable coinitiality. We let $A(T)_C$ be the group consisting of all $g \in A(T)$ which fix C setwise and which fix every element of T which lies above all of C. (In the case that C is maximal, there are no such elements, but we do not know whether there are maximal subchains in T having countable coterminality.) We let $A(T)_C^C$ be the group induced on C by $A(T)_C$ (that is, the family of restrictions to C of members of $A(T)_C$). Then by the proof of [7] I, Proposition 4.2, it follows that $A(T)_C^C$ is large in $A(C)$. We break the argument into six steps.

(1) There is $m_1 \in \omega$ such that for any $g \in A(T)_C$ there is $h \in U_{m_1} \cap A(T)_C$ agreeing with g on C.

For intersecting with $A(T)_C^C$ we have $A(T)_C^C = \bigcup_{i\in\omega}(U_i \cap A(T)_C)^C$, so by applying the main result of [6] mentioned above, there is m such that $A(T)_C^C = (U_m \cap A(T)_C)^C$.

(2) There are a moiety M of T and $m_2 \in \omega$ such that every $f \in A(T)_C$ whose support is contained in M lies in U_{m_2}.

For this we explain what a 'moiety' is this time. We use the notation $\langle a, b\rangle$ as in [7] I, where $a < b$ in T, to stand for the set of those $x \in T$ such that $a \le x \le b$, or $a < x \parallel b$. We modify this slightly here to give $\langle a, b\rangle^o = (a, b) \cup \{x : a < a' < x$ for some $a' < b$ and $b \not\le x\}$. Intuitively, these are the points of T which branch off in T at a point (strictly) between a and b, together with the open interval from a to b as well. We say that a subset M of T is a *moiety* if there is a coterminal $\mathbb{Z}$-sequence (a_i) in $\overline{C}$ such that $a_i < a_{i+1}$ for each i, such that all a_{2i} lie in the same $A(T)$-orbit, and so do all the a_{2i+1}, each (a_{2i}, a_{2i+1}) has countable coterminality, and $M = \bigcup_{i\in\omega}\langle a_{2i}, a_{2i+1}\rangle^o$. (Compare [7] page 464 section 3.)

Now we remark that there is an infinite set $\{M_j : j \in \omega\}$ of pairwise disjoint moieties, and we use a similar 'diagonalization' argument as given in section 2 to show that for some j, every member of $A(T)_C$ fixing M_j agrees on M_j with a member of U_j (and we once again say that M_j is *full* for U_j). Let us write $M = M_j$ for this j. This now gives us a choice of a_i for $i \in \mathbb{Z}$.

Choose $h \in A(T)_C$ having each (a_{2i}, a_{2i+1}) as a single positive orbital and with support contained in M. Then $h \in U_m$ for some m. By [7] I Lemma 3.5, for every f in $A(T)_C$ whose support is contained in M, there are k_1 and k_2 in $A(T)_C$ with support contained in M such that $f = h^{k_1} \cdot (h^{-1})^{k_2}$. Then there are $u_1, u_2 \in U_j$ whose restrictions to M are k_1, k_2, and this implies that $f = h^{u_1} \cdot (h^{-1})^{u_2} \in (U_j \cdot U_m \cdot U_j)^2$, which provides m_2 as desired.

(3) There is $m_3 \in \omega$ such that for every $f \in A(T)_C$ which fixes each a_i, $f \in U_{m_3}$.

For this we first choose $k \in A(T)_C$ such that $a_{2i-2} < ka_{2i-1} < ka_{2i} < a_{2i-1}$ for each $i \in \mathbb{Z}$. Now any $f \in A(T)_C$ fixing each a_i may be written in the form $f_1f_2f_3$ where $f_1, f_2, f_3 \in A(T)_C$, $supp(f_1) \subseteq M$, $supp(f_2) \subseteq \bigcup_{i\in\mathbb{Z}}\langle a_{2i-1}, a_{2i}\rangle^o$, and f_3 is the restriction of f to the set of all points branching off at a_i not lying in C. It follows from this that $supp(f_2^k) \subseteq M$. Furthermore, k also maps each a_i into M, and hence f_3^k has support contained in M. By (2), it follows that f_1, f_2^k, and f_3^k all lie in U_{m_2}. If m' is such that $k \in U_{m'}$, we deduce that $f \in U_{m_3}$, where m_3 is independent of f.

(4) There is $m_4 \in \omega$ such that $A(T)_C \subseteq U_{m_4}$.

Let $g \in A(T)_C$. By (1), there is $h \in U_{m_1} \cap A(T)_C$ which coincides with g on C. Hence $h^{-1}g \in A(T)_C$ fixes C pointwise. By (3), we have $h^{-1}g \in U_{m_3}$ giving the desired statement.

(5) If S_x stands for the set of those elements g of $A(T)$ for which $x < supp(g)$, where x is any element of T, then there is $m_5 \in \omega$ such that $S_{a_0} \subseteq U_{m_5}$.

For this we choose a convex subchain C' of T with countable coterminality which is unbounded below in T and contains an element incomparable with a_0. By (4) we may choose $r \in \omega$ such that $A(T)_{C'} \subseteq U_r$, and this implies that $S_{a_0} \subseteq U_r$.

(6) There is $m_6 \in \omega$ such that, in the notation of (5), every S_{a_i} is contained in U_{m_6}.

For any $i \in \mathbb{Z}$ there is $k \in A(T)_C$ such that $a_0 < ka_i$. Hence $S_{a_i}^k \subseteq S_{a_0} \subseteq U_{m_5}$ by (5). But by (4), $k \in U_{m_4}$ for some m_4 independent of k, and hence $S_{a_i} \subseteq U_{m_4} U_{m_5} U_{m_4}$.

Now we can conclude the proof. Choose any $f \in A(T)$. By [7] I, Lemma 4.1 there is $g \in S(T)$ such that $gf \in A(T)_C$. Find i such that $g \in S_{a_i}$. Then from the above it follows that f lies in $U_{m_6} U_{m_4} \subseteq U_m$ for some m. $\square$

4 Cycle-free partial orders

In [10] the automorphism groups G of the countable 3-CS-transitive cycle-free partial orders ('$CFPO$s') were studied, and it was determined precisely which of them are simple groups. Moreover, those simple groups for which a 'bounded number of conjugates' sufficed for the simplicity proof were characterized. By this is meant that there is some fixed number m such that for any non-identity elements g and h of G, it is possible to write h as the product of at most m conjugates of g or g^{-1}. It is immediate that the ones in the list for which G is simple, but there is no bound on the number of conjugates, fail the Bergman property. For we may fix $g \neq 1$, and let E be the union of $\{1\}$ and the set of conjugates of g or g^{-1}. As G is simple, E generates G, but as there is no bound on the number of conjugates required, there is no m such that $G = E^m$. This means that such G cannot have the Bergman property. It turns out however that *all* the ones which have a simple group fail to have the Bergman property, for a related reason (to do with the fact that there is a notion of 'distance' in the structure which can take arbitrarily large values), and indeed a much wider class, in fact all the 1-transitive $CFPO$s which embed ALT. In this section we establish however that all the simple groups in our list have uncountable cofinality (except possibly one sporadic).

The classification that we refer to is summarized in [10], and we recall the main points here. A particular $CFPO$ which is needed to describe the classification is the 'infinite alternating chain' ALT, which has vertices x_i for $i \in \mathbb{Z}$, ordered by $x_{2i} < x_{2i\pm 1}$ (with no other relations). As shown in [21], any countable 3-CS-transitive $CFPO$ M which embeds ALT either has all maximal chains finite (and then they have length 2), or all maximal chains infinite. The former fall into two types, the so-called 'skeletal' ones, in which the maximal chains of the completion are infinite, and the 'sporadics', in which the maximal chains are finite (of length at most 4 in fact). Each of these gives rise to a number of cases, which are parametrized by cardinals κ and λ between 2 and $\aleph_0$ (which are upward and downward ramification orders) and in some cases, also a countable 1-transitive linear order Z (which either is, or easily describes, the order-type of a maximal chain in M^+). The finite chain ones are written using script letters from $\mathcal{A}^Z_{\kappa\lambda}$ to $\mathcal{K}$ for skeletals, and $\mathcal{M}_{\kappa\lambda}$, $\mathcal{N}_{\kappa\lambda}$, $\mathcal{P}_{\kappa\lambda}$, $\mathcal{P}'_{\kappa\lambda}$ for sporadics, and the infinite chain ones are written using Gothic letters from $\mathfrak{A}_{\kappa\lambda}$ to $\mathfrak{V}$. An important feature of the classification is that it is shown that any of the $CFPO$s under consideration can be constructed from (any) one of the maximal chains of M^+, given information about how the points ramify, and indeed the classification is found by analyzing which possible maximal chains can arise. If C is a maximal chain in M^+, then we may colour the orbits of the points of C under its setwise stabilizer in $A(M)$ by distinct 'colours', and so view C as a coloured chain.

Using this notation as in [10], the ones which have simple automorphism groups are $\mathcal{A}^{\mathbb{Q}}_{\kappa\lambda}$, $\mathcal{B}_{\kappa\lambda}$, $\mathcal{C}_{\kappa\lambda}$, $\mathcal{D}_{\lambda}$, $\mathcal{D}'_{\kappa}$, $\mathcal{E}_{\lambda}$, $\mathcal{E}'_{\kappa}$, $\mathcal{F}^{\mathbb{Q}}_{\lambda}$, ${\mathcal{F}'_{\kappa}}^{\mathbb{Q}}$, $\mathcal{G}^{\mathbb{Q}}$, $\mathcal{H}^{\mathbb{Q}}$, ${\mathcal{H}'}^{\mathbb{Q}}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{J}'$, $\mathcal{K}$, all those in the infinite chain case classification, and the sporadic $\mathcal{M}_{\kappa\lambda}$ for $\kappa, \lambda \geq 3$.

A key property which arose in the analysis of which of them have simple automorphism groups in [10] was whether a maximal chain of M^+ has a doubly transitive orbit, and the way in which this condition enters into the picture here is shown in the following lemma. In the proof the notion of 'extended cone' from [21] needs to be referred to, so we briefly describe what this is. In a tree, an (upper) *cone* at a ramification point x is a set C of points of the tree lying strictly above x maximal subject to the property that any two points have a common lower bound in C. In a $CFPO$ M the structure may branch downwards as well as upwards, so we immediately also get a notion of a 'lower cone' at a downward ramification point. For each such upper or lower cone C, we also get a corresponding larger subset called the corresponding *extended cone*, which consists of all points y of M such that the path from y to x intersects C.

Lemma 4.1. *Let M be any (finite chain) skeletal or infinite chain countable 3-CS-transitive cycle-free partial order for which any maximal chain of M^+ has a doubly transitive orbit, and let C_1 and C_2 be disjoint maximal chains of M^+ such that for some points x of C_1 and y of C_2, $x < y$ and (x, y) contains a dense set of points of a doubly transitive orbit of its setwise stabilizer. Then $A(M)$ is generated by the union of the setwise stabilizers of C_1 and C_2.*

Proof: Let H be the group generated by $A(M)_{C_1}$ and $A(M)_{C_2}$, and let C be any other maximal chain of M^+. We shall show that $A(M)_C \leq H$.

Since M is connected, there is a path from a member of C to a member of C_1, and also to a member of C_2. We shall use induction of the least length of a path from a member of C to a member of $C_1 \cup C_2 \cup [x, y]$.

In the first case, the length is 0, which means that C shares a point with $C_1 \cup C_2 \cup [x, y]$. One possibility is that $[x, y] \subseteq C$. Then any order-automorphism of C may be written as a product of two elements fixing either $C \cap (-\infty, x]$ or $C \cap [y, \infty)$ pointwise. Extending suitably to the extended cones at ramification points, it follows that any member of $A(M)_C$ may be written as the product of two elements of $A(M)_{C_1} \cup A(M)_{C_2}$ (even the pointwise stabilizers in this case), and hence $A(M)_C \leq \langle A(M)_{C_1} \cup A(M)_{C_2} \rangle = H$. If $[x, y] \not\subseteq C$ then there is some $z \in (x, y)$ such that $C \cap (C_1 \cup C_2 \cup [x, y])$ is a subset of either $C_1 \cup [x, z]$ or $C_2 \cup [z, y]$. In the first case, taking the conjugate by an automorphism fixing C_2 (pointwise) we may assume that $C \cap (C_1 \cup C_2 \cup [x, y]) \subseteq C_1$, and similarly in the second case for C_2.

To give more detail in the first case for instance, if already $C \cap (C_1 \cup C_2 \cup [x, y]) \subseteq C_1$ we do not need to conjugate at all, so suppose that $C \cap (x, y) \neq$

$\emptyset$. Then as C is a chain, $C \cap (C_1 \cup C_2 \cup [x, y]) \subseteq (C_1 \cap (-\infty, x]) \cup [x, z]$. Let $g \in A(M)_{C_2}$ fix $(C_1 \cap (-\infty, x]) \cup [x, y]$ setwise and map z to a point of $(-\infty, x]$. Then $g(C \cap (C_1 \cup C_2 \cup [x, y])) \subseteq C_1 \cap (-\infty, x]$ from which it follows that $g(C) \cap (C_1 \cup C_2 \cup [x, y]) \subseteq C_1 \cap (-\infty, x]$ and we note that $A(M)_C \subseteq g^{-1} A(M)_{gC} g$.

Let us now therefore concentrate on the case where $C \cap (C_1 \cup C_2 \cup [x, y]) \subseteq C_1$.

If $C = C_1$ then the result is immediate. Otherwise, $C \cap C_1$ is bounded either above or below in C_1, and so by multiplying by a member of $A(M)_{C_1}$ we may suppose that $C \cap C_1$ is a subset of either $(-\infty, x) \cap C_1$ or $(x, \infty) \cap C_1$.

First suppose that $C \cap C_1 \subseteq (-\infty, x) \cap C_1$. By multiplying by a member of $A(M)_{C_2}$ we may suppose that for some $z < x$, $C \cap C_1 \supseteq (-\infty, z)$. Pick $u < z$ and $v > x$ in C_1. Then any order-automorphism of C may be written as a product of two automorphisms which fix either $(-\infty, u)$ or (z, ∞) pointwise. Hence by extending suitably on the extended cones, any member of $A(M)_C$ may be written as a product of members of $A(M)$ which fix either all members of extended cones from points of $(-\infty, u)$ or all members of extended cones from points of (z, ∞) pointwise. The latter all lie in $A(M)_{C_2}$. Let $g \in A(M)$ fix all members of the extended cones from points of $(-\infty, u)$, and let $h \in A(M)_{C_1}$ take u to v. Then hgh^{-1} fixes all members of extended cones from points of $C_1 \cap (-\infty, v)$ so lies in $A(M)_{C_2}$. Hence $g \in h^{-1} A(M)_{C_2} h \leq \langle A(M)_{C_1} \cup A(M)_{C_2} \rangle$.

Next if $C \cap C_1 \subseteq (x, \infty) \cap C_1$ we may similarly suppose first that $C \cap C_1 \supseteq (z, \infty)$ for some $z > x$, and then choosing $u < x$ and $v > z$ in C_1 write any member of $A(M)_C$ as a product of members of $A(M)$ which fix either all members of extended cones from points of $(-\infty, z) \cap C$ or all members of extended cones from points of (v, ∞) pointwise. The former all lie in $A(M)_{C_2}$, and using the same trick of conjugating by a member of $A(M)_{C_1}$ the action of automorphisms of the latter kind below u, these also lie in $A(M)_{C_2}$.

If the length n is greater than zero, by cycle-freeness we may identify the following possible ways in which a path from a member of C can meet $C_1 \cup C_2 \cup [x, y]$: (i) there is a path from a point of C to a point z of C_1 which is disjoint from $C_2 \cup (x, y]$, (ii) there is a path from a point of C to a point of C_2 which is disjoint from $C_1 \cup [x, y)$, (iii) there is a path from a point of C to some $z \in (x, y)$ which is disjoint from $C_1 \cup C_2 \cup (x, z) \cup (z, y)$. The first two cases are essentially the same, so we just do (i) and (iii). In each case let t be the point on the path next to z (which exists since $n > 0$).

For (i), by multiplying by a member of $A(M)_{C_1}$ we may suppose that $z < x$. Let C_3 be a maximal chain containing t and z, and if $t < z$ let $C_3 \supseteq C_1 \cap [x, \infty)$ and if $t > z$ let $C_3 \supseteq C_1 \cap (-\infty, z)$. Then C_3, C_2, $[x, y]$ if $t < z$ or C_3, C_2, $[z, y]$ if $t > z$ is a configuration of the same type as given in the statement of the lemma, and the least distance from a member of C to a member of $C_3 \cup C_2 \cup [x, y]$ is $n - 1$, since the same path, but with $(t, z]$ deleted will serve. Hence by induction hypothesis, $A(M)_C \leq \langle A(M)_{C_3} \cup A(M)_{C_2} \rangle$. By the basis case, $A(M)_{C_3} \leq \langle A(M)_{C_1} \cup A(M)_{C_2} \rangle$, so $A(M)_C \leq \langle A(M)_{C_1} \cup A(M)_{C_2} \rangle$ as desired.

For (iii), once more let C_3 be a maximal chain containing z and t. Then either (x, z) or (z, y) contains a dense set of points of a doubly homogeneous orbit of the setwise stabilizer of (x, y), suppose the former for example. We apply the induction hypothesis to $C_1 \cup C_3 \cup [x, z]$ and obtain the same conclusion as before.

Since we have shown that $A(M)_C \leq H$ for every maximal chain, and the family of maximal chains is preserved by the action of $A(M)$, it follows that H is a normal subgroup of $A(M)$. But by the results of [10], $A(M)$ is simple, and it follows that $H = A(M)$ as required. $\square$

Theorem 4.2. *The automorphism groups of the following cycle-free partial orders have uncountable cofinality but not the Bergman property:* $\mathcal{A}^{\mathbb{Q}}_{\kappa\lambda}$, $\mathcal{B}_{\kappa\lambda}$, $\mathcal{C}_{\kappa\lambda}$, $\mathcal{D}_{\lambda}$, $\mathcal{D}'_{\kappa}$, $\mathcal{E}_{\lambda}$, $\mathcal{E}'_{\kappa}$, $\mathcal{F}^{\mathbb{Q}}_{\lambda}$, $\mathcal{F}'^{\mathbb{Q}}_{\kappa}$, $\mathcal{G}^{\mathbb{Q}}$, $\mathcal{H}^{\mathbb{Q}}$, $\mathcal{H}'^{\mathbb{Q}}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{J}'$, $\mathcal{K}$; *and all those in the infinite chain case classification.*

Proof: As remarked in [10], for all the $CFPO$s listed, any maximal chain C of M^+ has a doubly transitive orbit. Let M be one of these $CFPO$s, and let $(U_i)_{i\in\omega}$ be an ascending sequence of subgroups of $A(M)$. We begin by following the first few steps in the proof of Theorem 3.1. Let C be a maximal chain of M. Then in M^+ there is a doubly transitive orbit of $A(M)$ (and that was how the list was compiled), so as before, using [7] I, Proposition 4.2, extended to $CFPO$s, the group induced on C by its setwise stabilizer in $A(M)$ is large, here meaning that it acts doubly transitively on some orbit (as well as the other conditions).

(1) As before, the fact that $A(M)^C_C$ is large implies that there is m_1 such that any automorphism of M fixing C setwise agrees on C with a member of U_{m_1}.

(2) This step carries straight over provided we define 'moiety' suitably. Consider coterminal points a_i for $i \in \mathbb{Z}$ so that $a_i < a_{i+1}$ lying in a doubly transitive orbit. We say that a subset X of M is a *moiety* if there is a choice of such points for which a point x lies in X if and only if the path from x to C first meets C between a_{2i} and a_{2i+1} for some i. This requires a little explanation. The *definition* of cycle-free partial order is that between any two points x and y of M, there is in M^D a unique path from x to y. So if $x \in M$ is given, and we choose any point y of C, this notion makes sense for x and y. We may say that the path from x to C *first meets* C *between* a_{2i} *and* a_{2i+1} if there is some y between these points such that the path from x to y contains no points of C other than y. The diagonalization is exactly as before. Hence there is $m_2 \in \omega$ such that every $f \in A(M)_C$ whose support is contained in M lies in U_{m_2}.

(3) There is m_3 such that for every $f \in A(M)_C$ which fixes each a_i, $f \in U_{m_3}$.

(4) There is $m_4 \in \omega$ such that $A(M)_C \subseteq U_{m_4}$.

These two steps are proved just as in 3.1.

We conclude the proof as follows. Choose any maximal chain C_1, and upward ramification point $x \in \overline{C_1}$, and downward ramification point $y > x$ such that (x, y) contains a dense set of points of a 2-transitive orbit and such that for some $a \in C_1$, x is the infimum of a and y. Now choose $b \in M$ such that $b < y$ and y is the supremum of x and b, and let C_2 be a maximal chain containing b such that $y \in \overline{C_2}$.

By (4), there is some $q \in \omega$ such that $A(M)_{C_1}, A(M)_{C_2} \subseteq U_q$. By Lemma 4.1,

$A(M) = \langle A(M)_{C_1} \cup A(M)_{C_2} \rangle$, so as U_q is a subgroup it follows that $A(M) = U_q$, giving the desired conclusion.

Now we observe that none of the automorphism groups has strong uncountable cofinality. Pick a maximal chain C, and let U_n be the set of all elements of $A(M)$ which move no member of C to a point at distance greater than n from a member of C (where 'distance' was defined towards the end of section 1). Then for any automorphism g of M, there is n such that all points of gC are at distance at most n from all points of C, and so $g \in U_n$. Thus $A(M) = \bigcup_{n \in \omega} U_n$. But the distance that C can be moved is clearly unbounded, and so $U_n \subset A(M)$ for every n. $\square$

We conclude by remarking that the last part of the preceding proof actually applies in much greater generality; namely to all the $CFPO$s classified in [21, 4, 13] and many others (for instance, the hypothesis of countability is not required).

Theorem 4.3. *Let M be a 1-transitive $CFPO$ embedding ALT. Then $A(M)$ does not satisfy the Bergman property.*

Proof: The proof is read off from what we have just given. The fact that ALT embeds ensures that the distances from C are unbounded, and 1-transitivity is sufficient to ensure that C can be moved to arbitrarily great distances. $\square$

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