On rational relations with multiplicity

A classical concept revisited

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Automata theory has been first developped for (formal) languages, that is for *subsets* of *free monoids*. It has then been generalized in two directions which proved both to be fruitful and enlightening. On one hand, from languages to subsets of general monoids, and in particular subsets of direct products of free monoids; this gives the suitable framework for the study of *rational relations*. On the other hand, from languages to languages with multiplicity in a semiring $\mathbb{K}$, called *(formal power) series* over A^*. In particular, rational relations may be seen as well as series over A^* with multiplicity in $\operatorname{Rat} B^*$. And the two generalizations can possibly be combined in the definition of $\mathbb{K}$-series over M — provided certain restrictions on the pair $\mathbb{K}, M$.

The notions of rationality, recognizability, the intersection theorem can well be transported to these more general objets; their nature, their meaning appear more clearly when applied to subsets of non free monoids or to series. Somehow, this was the underlying philosophy in Eilenberg's traetise [1].

Rational relations between two free monoids A^* and B^* or even between two general monoids M and N can be considered in two ways. First, as subsets of $M \times N$, or, second, as *additive* mappings from $\mathfrak{P}(M)$ into $\mathfrak{P}(N)$. Relations with multiplicity should be at the cross-road of the generalization of the two faces of relations. First, as series over $M \times N$, second as additive, *i.e. linear*, mapping from $\mathbb{K}\langle\langle M\rangle\rangle$ into $\mathbb{K}\langle\langle N\rangle\rangle$.

Two difficulties arise. First, it seems that it is necessary to assume that $\mathbb{K}$ is commutative in order to achieve linearity. Second, it may well happen that the image of a series s by a $\mathbb{K}$-relation τ is *not definied.* A classical answer is found in the work of Jacob ([2]), and exposed in [4, 3]. It consists in assuming that $\mathbb{K}$ is commutative, and considering only the so-called *regulated* rational $\mathbb{K}$-relations from A^* into B^*, a class of $\mathbb{K}$-relations that are defined via their matrix representation.

We shall present a theory of rational $\mathbb{K}$-relations which gives a perfect parallel with the theory without multiplicity, provided it is accepted that a $\mathbb{K}$-relation from M into N may well not be defined on the whole set $\mathbb{K}\langle\langle M\rangle\rangle$. The assumption that $\mathbb{K}$ is commutative is replaced by the one that *the coefficients* of the relation are in the *center* of $\mathbb{K}$.

In particular, Elgot-Mezei's theorem extends to rational $\mathbb{K}$-relations: *Let θ be a rational $\mathbb{K}$-relation from M into N and σ a rational $\mathbb{K}$-relation from N into P such that the composition product $\sigma \circ \theta$ is defined. If N is a free monoid, then $\sigma \circ \theta$ is rational.*

References

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