Lecture Script

— Dynamical Systems and Autonomous Agents —

Part I: Theory of Dynamical Systems[1]

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[1]This is a version of the script where the quality if the figures is reduced in order to restrict the size of the file. For a version with high-resolution pictures please contact the authors.

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nonlinear dynamics and autonomous agents

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Table 1: Schedule of the course. Entries after May 31, 2002 are subject to change.

Notes and comments

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Part I

Theory of Dynamical Systems

Chapter 1

Introduction

1.1 Agents in Artificial Intelligence and Cognitive Science

In the seventies the notion of an agent has been introduced into computer science. An agent in the general sense is an entity which receives information from its environment and may execute various actions in response to the inputs received. In the context of artificial intelligence, agents are pieces of software acting on data structures. In contrast to this, we will be interested in agents acting in a material, physical environment via material actuators. Before we are going to discuss problems connected with agents in the real world, some more on the characteristics of software agents are to be mentioned.

A software agent is "licensed to act". It carries out tasks autonomously, makes decisions and reacts to changes in the environment according to its task. Agency refers to the degree of autonomy or independence (www.ifs.tuwien.ac.at/oegai/). *Autonomy* (Eigengesetzlichkeit) and *agency* (Urheberschaft) are to be distinguished. Often, the term agency sufficiently describes the property of the agent to actively behave in its environment. Autonomy goes one step further in allowing the agent to find itself ways of evaluating it present success or at least to establish by itself and when acting subgoals of its prescribed task.

An *intelligent (software) agent* is a program that processes questions in an autonomous, flexible and user-specific fashion and that returns to the user a structured result. An agent tries to analyze or even to modify the task specified by the user in order to obtain answers that are relevant to the user and optimized with respect to the user's intentions. An agent can be characterized by the following properties: goal-directedness, flexibility, cooperativity, the ability to start by itself, communicability, adaptability, security, and mobility (adapted from Frank Puhl, Saarland University 1999). By main stream Artificial Intelligence (AI) agents are thus considered as rational beings in an complex though artificial environment.

A related, but somewhat different view is taken in cognitive science. Cognitive science focuses on specific aspects of human intelligence which are assumed to be implementable without direct reference to specific properties of the underlying physical matter. The assumption that this is a meaningful approach has been put forward by Newell and Simon (1976) in form of the *physical symbol system hypothesis.* We cannot go into the details of debate on this strong claim. Critics from the point of view of connectionism has been formulated by Smolensky (1988) and has provoked the anti-critics by Fodor and Pylyshyn (1988). We should at least briefly note two fundamental problems which cannot be conveniently solved by physical symbol systems (Dreyfus and Dreyfus 1990)

1. Frame problem. How to model change? What is relevant in a given context? What is changing and what is constant? (D. C. Dennett, 1984)

2. Symbol grounding problem. How symbols relate to the real world? (symbol grounding problem, Harnad, 1991; syntax grounding, Searle, 1990)

The formulation of a symbolic description is (after its prospective completion) supposed to allow for efficient (or intelligent) reasoning concerning those aspects of the world, which are captured by the symbolic description. The very formulation, however, conveys already the essential part of the intelligence which we are then tempted to attribute to the symbolic reasoning system. An agent that is (in place of ourselves) acting in the real world is thus confronted with the task to construct symbolic systems on its own. The thus obtained symbolic descriptions are to be efficient with respect to the aspects of the world which are important *to the agent*. What is relevant cannot be decided by the designer of the agent, because this is unknown before the agent actually interacts with its environment and must therefore be decided upon autonomously by the agent when it is autonomously interacting with the world.

It cannot be excluded (actually it will indeed be assumed in the following) that the agent for many tasks will not need to produce a symbolic description. The systems, that the agent is equipped with in order to construct, calibrate, apply and verify its internal systems, are referring to the real world or to what is known about the real world to the agent. And already these systems, if of any use at all to the agent, would necessarily allow the agent to do without symbolic descriptions in many instances. The fact that e.g. humans have developed symbolic world models seems to imply that at some stage of intelligence symbolic representations indeed may become essential. But we may not be able understand why they are essential and from what level of intelligence they indeed are. Further, considering the level of intelligence presently implementable in agents, it is questionable whether the large loop via symbolic representations might not be easily become short-cut by the agents built-in drive for efficiency unless it is suggested by a purely symbolic environment as considered in artificial intelligence.

1.2 Causality

Central to the argument given below is the notion of causality, i.e. the postulate that all phenomena require a cause in order to come into existence. In its strong sense, causality states that similar causes produce similar effects. Strong causality is indeed obeyed by many (though by no means all) natural phenomena. Also it seems that to reveal strongly causal relations is the main goal when humans strive for knowledge. Nature, however, cannot be reduced to strongly causal relations, which has been exemplified with slight exaggeration by the famous butterfly effect. Such 'exceptions' to strongly causal relations are generally related to non-linear behavior. In these cases of weak causality, effects can be still related to causes, but tiny deviations in the causes may give rise to enormous variations in the effects.

Weakly causal systems can be very efficient in carefully controlled environments. For example, a program code lives essentially only in computers rather than in natural environments. Here weakly causal codes do well (though an the expense of heavy debugging and fundamental restrictions on testability). In order to obey strong causality, e.g. to allow for random distortions, the code must be strongly redundant. Generally, in controlled environments, weakly causal systems can be successful, but in open, unknown, natural environments successful performance should imply strong causality and in this way insensitivity to noise, distortions, unforeseen changes etc. In the following, we will try to substantiate the claim that, if there are possibilities for control, such as exerted by an agent, weakly causal relations can be beneficially exploited even in natural environments and even if the environment is not fully controllable. The claim is that a well-balanced combination of weak and strong causality has the potential of providing a basis for intelligence.

1.3 Embedded agents

The challenges which became unavoidable in control of autonomous robots have lead to fields of study which are called embodied cognitive science or situated artificial intelligence. Central to these approaches is the notion of situatedness meaning a situation in which an agent is in maximum contact (by its sensors) with the environment so that it can plan its further actions in an informed way even without a world model. Material constraints and properties are to be exploited rather than being ignored. Or in the words of Rodney Brooks: “Nature is your friend, not your enemy”.

1.4 Autonomous robots

Mobile robots are instantiations of autonomous agents, which most clearly represent the approach followed here. In order to behave successfully, a traditionally programmed robot would need to apply a program of a complexity comparable to the complexity of the environment which is even in controllable environments (such as a robot soccer ground or an office environment) a difficult, but in general a hopeless task. Further the designer of the robot will be tempted to overly exploit specificities of the environment to optimize the robot with respect to a given task. We will instead focus on what is done under a given paradigm and evaluate the behavior in order to find tasks for which it might be useful, rather than to try to tailor the behavior regarding a specific task.

1.5 Living systems

What we expect situated autonomous agents to be able to do, has long been achieved by living beings. We can therefore only gain when considering parallels of our approach known from the animal kingdom. However, animals have not only adapted to meet the challenges of their respective environment, but have also contributed to the very properties of these environments. For various reasons we should not assume our agents to modify the environment for the purpose of their well-functioning unless specifically required by the designer of the agent’s tasks. Yet, the agent is not opposed to an absolute world, but maintains interactions with its surroundings that give rise to various relative world views, which indeed are subject to optimization in order to allow for more efficient functioning of the agent. We do not want the agent, say, to cut down any plants in the environment in order to be able to move faster and more reliably, but instead to develop patterns of locomotion which are adapted to a given state of the environment. The intelligent agent has control over its relations to the environment, but not over the environment.

1.6 Dynamical systems

Dynamical systems provide a general framework to describe change, development, evolution, relations, and interactions. In continuous domains such as the physical world differential equations provide the language that expresses these processes. When taking into account that controllers for agents are implemented based on digital processors there is a discretization of time and state spaces imposed. Also, sometimes, a discrete formulation of the problems under consideration is conceptually easier in an initial phase of study. Still differential formulations do allow for the application of a highly developed apparatus of analysis and thus for understanding of the relation the agent is engaged in with its environment.

Differential equations do actually describe *space* itself – by specifying the structure of space. Knowing the structure of spaces allows to behave efficiently in these spaces including physical processes in the environment, manifolds of behaviors, perceptions as well as internal representation of all of these.

Chapter 2

Conceptual framework

2.1 The model of an agent

In the formal sense an agent is an input-output system which at instances $n = 0, 1, 2, ...$ receives inputs $x_n \in \mathbf{I}$ and produces an output which we call $y_n \in \mathbf{O}$ where the input space $\mathbf{I}$ is our space of input values and the output space $\mathbf{O}$ corresponds to the actions. In the most simple case the agent can be modeled by a function

$$K: \quad \mathbf{I} \to \mathbf{O}$$

or

$$y = K(x)$$

In many cases however the action is not a function of the present sensor values alone but also of previous ones and also of other influences. Whatever the conditions are, at each instant of time our agent embedded into the world produces a pair (x_n, y_n). It is a well known result of system theory (early references are Arbib, Kalman) that under very general conditions any sequence of pairs

$$\Xi = \{(x_n, y_n) \mid n\} = 0, 1, 2, \ldots$$

can be modeled by introducing a state system, i.e. assume the agent depends on a state vector $z \in \mathbf{Z}$ which is updated each time a new vector x of sensor value comes in so that the action the agent proposes at time step n is

$$\begin{array}{rcl} y_n & = & K(x_n, z_n) \\ z_{n+1} & = & I(x_n, z_n) \end{array} .$$

This comprises the case that the output of the agent depends on previous values of the sensors since we always may introduce part of the state variables to store the previous sensor values, examples below. For the same reason we might also write the output as a function of the internal state alone, i.e. $y_n = K(z_n)$ since the latter may also include the present sensor values. We will however not use the latter notation since in fact we will use the shorthand notation

$$y_n = K(x_n)$$

hiding the dependence of the agent on its internal state.

2.2 The agent in the world

Up to now we have the sensor values received by the robot treated as being given. However, usually the agent is interacting with the world so that the sensor values are influenced by its own actions. This will lead us to an extended update rule by including the dynamics of the world. In the present section we consider the generic example of a robot moving in an external world which may be either static or dynamic (with moving objects or so). We assume for the moment that the agent receives sensor values $x \in \mathbf{R}^n$ at discrete instances of time which we number by $n = 0, 1, 2, \ldots$ as before. The agent outputs an action

$$y_n = K(x_n, z_n) \tag{2.1}$$

which in general will contribute to the change of the state of the world.

The physical state of the world is to be characterized completely by a vector $q \in \mathbf{R}^m$ and the change of the state q is put as

$$q_{n+1} = q_n + W(q_n, y_n) \tag{2.2}$$

where the sensor values as a function of the physical state of the system are

$$x_n = S(q_n) \tag{2.3}$$

S being the sensor characteristics,

$$y_n = K(x_n, z_n) \tag{2.4}$$

and the internal state of the agent is updated as

$$z_n = I(x_{n-1}, z_{n-1}) \tag{2.5}$$

Equation (2.2) together with eqs. (2.3), (2.4), and (2.5) formally describe the behavior of the agent in the world.

We note in passing that in realistic cases the underlying assumption of deterministic update rules is not justified so that the updates will be noisy as will be discussed later.

2.3 The principle of adaptivity

In practical applications the behavior of the agent should be dependent on a set of parameters $c \in \mathbf{R}^p$ so that the behavior can be adapted to the external conditions and/or the needs of the user. In the case of the robot c is a set of parameters for the controller. In particular, if the controller is realized by a neural network, c would contain the synaptic strengths and threshold values and the like of the neurons. We write the model of the adaptive agent now as

$$\begin{aligned} y_n &= K(x_n; z_n, c) \\ z_{n+1} &= I(x_n; z_n, c) \end{aligned}$$

The parameters can be tuned by hand form outside until the desired behavior of the robot is achieved. In an adaptive system the parameters are tuned by the agent itself according to some update rule given to the agent by the designer. Usually there is an error function measuring the distance between the actual behavior and the target one. Then the update rule for the parameters can be obtained by gradient descent

$$c_{n+1} = c_n - \eta \frac{\partial}{\partial c_n} E(x_n; c_n)$$

on the error function E which depends on both the actual state $x_n \in \mathbf{R}^n$ and the controller parameters $c_n \in \mathbf{R}^p$. In principle we can explode our vector of internal states z to include the parameters c so that the learning dynamics can be included into the internal state dynamics. However, in the theory of dynamical systems the dependence of the behavior on the parameters c is of central interest (bifurcation theory) so that we will not do this here. Adaptive agents will be considered in more detail below, in particular in section **??**.

2.4 Dynamical systems

All of us know dynamical systems as will be clear from considering the following program fragment

```
n=0;x=x0;
while (n<n_max) {
          x += F(x);
          n += 1;
}
```

where $F(x)$ is an arbitrary function given from outside.

This update rule changes iteratively in each cycle of the loop the value of the state variable x. Thus we may consider x as a state variable which is a function of the discrete time n, i.e we write the loop as

$$x_{n+1} = x_n + F(x_n)$$

and we immediately see that the evolution equations for the state of our agent in the world is a dynamical system in the sense introduced here.

2.4.1 Time-discrete dynamical systems as iterated maps

Mathematically the above iterative update rule is an iterated map. A function

$$G: \quad \mathbf{R}^n \to \mathbf{R}^n$$

maps $x \in \mathbf{R}^n$ to a new value $G(x) \in \mathbf{R}^n$. Writing $x + F(x) = G(x)$ we see that the value x_n is obtained as the $n-$fold application of the map G to itself

$$x_n = G(G(\ldots G(x_0)\ldots))$$

Of course for many time steps the result will be very complicated in general.

The flavor of the theory of dynamical systems is that one can in many cases give systematic rules for the behavior of the state x under these iterated mappings. This is of course of interest for the many dynamical systems which describe the behavior of systems in nature, techniques or economy. On the other hand the computer scientist may also benefit from this theory in that it gives him a better understanding of what may happen in loops of the above kind.

Examples

We also know that such systems can have quite different behavior and we will give a few prototypes.

The most simple ones are the linear type where the update $F(x)$ is just a constant, i.e. $F(x) = a$ for all x so that

$$x_n = x_0 + an$$

Others are of the exponential type where the update is proportional to the state x itself, i.e. $F(x) = gx$, i.e.

$$x_{n+1} = x_n + gx_n$$

so that

$$x_n = (1+g)^n x_0$$

and so on.

Most of us probably also know that depending on the nature of the function F the behavior may be very complicated. An extreme example of the dynamical complexity engendered by simple dynamical systems is given by the pseudo random number generators used in the computer. In many cases these are based on the iterated mapping defined by a convenient function. Examples are

$$x_{n+1} = (ax_n + c) \bmod m \tag{2.6}$$

where $x_0, a, c \in \mathbf{N}$, x_0 being the starting value, and $0 \le x_0, a, c < m$. Typical values are $m = 2^{32}$, $a = 1664525$, and $c = 1013904223$. For $n = 0, 1, 2, ...$ the dynamical system generates pseudo random numbers which are more or less identically distributed in the interval $[0, m-1]$. These numbers are not truly random since after a (very long) time they repeat periodically.

The pseudo random dynamics generated by the dynamical system of equation 2.6 is not so very surprising in view of the properties of the mod-Operation on very large numbers. However we get a very complex, as we will find chaotic, behavior also from extremely simple mapping functions. A well known case is the so called logistic map so that

$$x_{n+1} = \alpha x_n (x_n - 1)$$

For certain values of the parameter α the dynamics is known to be chaotic.

2.5 Time continuous dynamical systems

In many applications the update in any instant of time is only very small and it is only the number of iterations which lead to large changes in the state of the system. This can be accounted for in the above formulation by using F as the rate of change of the state, i. we introduce the true (physical) time lag θ and write

$$x_{t+\theta} = x_t + \theta F(x_t)$$

instead of equation **??**, where $t = n\theta$ is the physical time. The advantage is that the rate of change F is more or less independent of the length of the time lag.

As a consequence we may consider time steps of width θ rather than of unity and take the limit $\theta \to 0$, i.e. we consider

$$\frac{x_{t+\theta} - x_t}{\theta} = F(x) \tag{2.7}$$

and write $x(t)$ for x as a function of the continuous time variable t. Writing

$$\frac{d}{dt} x(t) = \dot{x}(t)$$

for the time derivative we obtain

$$\dot{x} = F(x) \tag{2.8}$$

which is a differential equation. If we have more than one dimension, i.e. $x \in \mathbf{R}^n$ and $F: \mathbf{R}^n \to \mathbf{R}^n$ we understand equation (2.8) as a system of differential equations, i.e. we write

$$\begin{array}{rcl} \dot{x}_1 & = & F_1(x_1, \ldots, x_n) \\ \vdots & & \\ \dot{x}_n & = & F_n(x_1, \ldots, x_n) \end{array} \tag{2.9}$$

where $x = (x_1, \ldots, x_n)$ and $F = (F_1, \ldots, F_n)$, each F_i being an $n-$point function ($n-$stellige Funktion).

The formulation of eq. (2.9) includes also higher order differential equations, cf. section 3.2 and thus many classical physical systems. Our aim however is not to investigate specific physical systems but instead to understand the characteristic properties of certain classes of dynamical systems. Our specific aim in this context is to describe the behavior of autonomous agents in the world in terms of the language of dynamical systems.

2.6 Perspectives

These few examples seem to show that the behavior is very diverse and it is not clear what kind of rules are to be expected. In order to give some hints we mention for instance the universality of scaling laws in chaotic systems, the understanding of phase transitions, the emergence of new modes (behaviors) due to spontaneous symmetry breaking driven by the noise, the slaving principle of the theory of self-organization, the qualitative rules for the classification of systems (Morse-Conley) and so on. Moreover there is also a lot of simple models displaying new dynamic phenomena like self-organized criticality.

So what the theory of dynamic systems and above all the chaos theory has given us is a set of tools for understanding and dealing with these new phenomena, in particular

- classification of system behaviors
- analytical understanding
- approaches to dynamic complexity

Autonomous agents as complex beings, their evolution, and dynamics of ensembles of agents seem to belong to this class of dynamic complexity so that it is these tools which can help us in understanding the complex world of biological agents and also help us in designing the principles for artificial agents which are of the same dynamic complexity as the biological ones.

Chapter 3

Linear differential equations

3.1 Phase portraits

Consider the one-dimensional differential equation

$$\dot{x} = f(x) \tag{3.1}$$

at any point x where $f(x) > 0$ the function $x(t)$ will tend to increase.

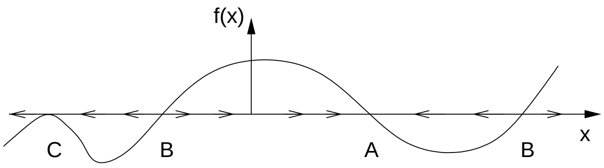

Figure 3.1: Phase portrait. To each state x the one-dimensional system (3.1) uniquely assigns a rate of change $\dot{x}$. If $\dot{x} = f(x)$ is larger than zero then x experiences an increase, if $\dot{x} < 0$ then x is bound to decrease. For $\dot{x} = 0$, x remains constant: The system is said to have a fixed point. If at both sides of a fixed point x tends towards this fixed point, the fixed point is stable (B), otherwise it is unstable (B). The arrows on the x axis indicate the *flow* of the system. Fixed points may also be neither fully stable nor unstable, cf. the case of the a turning point (C) an the left. Further it becomes obvious that one-dimensional systems on a line cannot have cycles: To each point x there is only one value of $\dot{x}$, therefore the state x cannot go up once and go down at a later time.

Zeros of $f(x)$ are of particular interest. They are called fixed points, because at values x with $f(x) = 0$ the function $x(t)$ will not change in time. (A) If left of the zero x tends to increase and right x tends to decrease, then this points is in a sense *attractive* to the time course of the function $x(t)$, if (B) the situation is vice versa, the point seems to repel the future x-values. Thus, starting near an repulsive fixed point, the function will tend towards the next attractive (or stable) fixed point.

It is also obvious that no cyclic behavior is possible in one-dimension. Each value of x leads to a well-defined (when the restrictions given in the next section are satisfied) behavior, and may not be different when $x(t)$ passes this place another time.

3.2 Systems of differential equations

If a higher order (here: second order) differential equation can be written as

$$\ddot{x}(t) = F(x, \dot{x}, t)$$

then by introducing a derived auxiliary function y, we can consider instead

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= F(x, y, t)\end{aligned}$$

In particular, in the case of an additive function F, i.e. if $F(x, y, t) = F_1(x(t)) + F_2(y(t))$, we have

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= F_1(x) + F_2(y)\end{aligned}$$

such that is usually suffices to consider systems of first-order differential equations. We will restrict ourselves to these cases.

Using an n dimensional vector x we write

$$\dot{x} = F(x)$$

Considered as a function of time t and of the initial condition $x(0) = x_0$ the right hand side F is called the *flow* of an autonomous differential equation.

Before we consider more general equations we will take a look on linear ones, here one can write (F_1 and F_2 are now constant coefficients):

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & y \\ F_1(x) & F_2(y) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ F_1 & F_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

3.3 Linear systems

The equation

$$\dot{x}(t) = \lambda\, x(t),\ \ x(t) \in \mathcal{R},\, \sqcup \in [\prime, \infty) \tag{3.2}$$

describes (for fixed λ) a set of functions, which can be parameterized by specifying initial values $x(0) = x_0$. To find out what functions satisfy (3.2), we divide 3.2 by $x(t)$ (assuming $x(t) \neq 0$) and integrate over t

$$\int_0^s x(t)^{-1} \frac{dx(t)}{dt} dt = \int_0^{x(s)} \frac{dx}{x} = \lambda \int_0^s dt$$

and find

$$\log\left(x(s)\right) - \log\left(x(0)\right) = \lambda s$$

or

$$x(s) = x(0) \exp\left(\lambda s\right) \tag{3.3}$$

i.e. in dependence on whether λ is greater or less than zero the magnitude of the initial value will rise or decay. The solution (3.3) suggests a similar approach for the corresponding multidimensional equation:

$$\dot{x}(t) = \Lambda x(t) \tag{3.4}$$

where now $x(t) = (x_1(t), \ldots, x_n(t))$ and Λ is a real $(n \times n)$ $-$matrix. The ansatz

$$x(t) = \exp\left(\lambda t\right) y \tag{3.5}$$

reduces (3.4) to an eigenvalue problem: Namely, inserting (3.5) into 3.4 and dividing by $\exp(\lambda t)$ leads to

$$\lambda y = \Lambda y$$

or

$$(\Lambda - \lambda I)\, y = 0.$$

If λ is not an eigenvalue of Λ then all components of the vector y must be zero, and, hence, $x(t)$ will be identical zero (which is always a solution of (3.4)). Otherwise, for each eigenvalue λ_k, $k \leq n$, the corresponding eigenvector y_k determines via 3.5 a solution of 3.4, a.k.a. *fundamental* solutions. Are other solutions possible? Consider to solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ then also $\alpha x^{(1)}(t) + \beta x^{(2)}(t)$ is a solution of (3.4), which becomes obvious when inserting $x(t) = \alpha x^{(1)}(t) + \beta x^{(2)}(t)$ into (3.4). This simple fact is known as the *superposition principle,* which turns out to be of great use, although it is essentially restricted to linear equations. (Remains the question whether all solutions of the linear system are given by superpositions of the fundamental solutions. But this is actually no problem, because there cannot be more than n independent solutions, which have been obtained already, and any other solution is thus already included.

3.3.1 Strogatz' love affairs

EXAMPLE

Strogatz (1994) considers the wide variability in the course of love affairs as an interesting field of application of the theory of dynamical systems. After having defined the functions

$$\begin{aligned} R(t) &: \text{ Romeo's love/hate for Juliet at time } t \\ J(t) &: \text{ Juliet's love/hate for Romeo at time } t \end{aligned}$$

and accepted that these functions obey differential equations as these ones

$$\begin{aligned} \dot{R} &= aR + bJ \\ \dot{J} &= cR + dJ \end{aligned}$$

all kinds of love affairs are theoretically treatable.

As an example, consider the case, where Romeo is clearly excited by the love he experiences from Juliet, whereas Juliet on the other hand, is more busy with her own feeling and reacts disapproving to the advances of Romeo:

$$\begin{aligned} \dot{R} &= J \\ \dot{J} &= J - R \end{aligned}$$

If the two never noticed each other, i.e. if $R = 0$ and $J = 0$ initially, this will never change from itself. But assume either of the two lovers has had a spontaneous empathy to the other one, be it love at first sight, be it a sudden emotional flicker. If it is Romeo who started to show some affection (while $J = 0$), J will decrease and stay zero. But there is hope: As soon as Romeo senses the aversion of Juliet, he will treat her badly as well. This macho behavior is what Juliet likes about Romeo. If it becomes sufficiently strong, it compensates her own negative feelings, and turns her on. It is visible that there will be a never ending change of their attitude towards each other and that the various emotions become stronger and stronger as time passes. Clearly, this is not a boring relationship, although only during one quarter of the time they will experience mutual love. In real life, there will occur some exhaustion when the emotional waves are too strong for an extended period, also one may guess that the romantic style, i.e. the parameters of the system may change in the process of the relationship. What remains from the example is perhaps nothing but a nice example of the “unstable spiral” behavior, just one case in the classification of linear dynamical systems, cf. section 3.4.

With regards to her “romantic style”, each lover comes in four variants

- $a, b > 0$: eager beaver
- $a, b < 0$: misanthrope
- $a < 0$, $b > 0$: cautious lover
- $a > 0$, $b < 0$: tragic romantic

In the example above Juliet is clearly a tragic romantic, whereas Romeo is something in between an eager beaver and an cautious lover, not really the macho Juliet is being fond of.

But also the absolute values of a and b are of importance. Namely, when $a^2 > b^2$ the relationship fizzles out to mutual indifference. In contrast, if $a^2 < b^2$, the feelings will eventually be mutual, however, not necessarily as indented initially. We will explain (cf. section 3.4.1) why we can expect this behavior and have a look at a few more examples when we have presented the qualitative theory of linear systems in the following section 3.4.

3.4 Qualitative analysis

The individual solutions of (3.4) are determined by n dimensional vectors of initial values, but we are interested in properties that are common to all solutions for a given matrix Λ. It will turn out, that important statements can be made (for linear systems) solely based on the eigenvalues of Λ. Consider, to keep things easy, the case $n = 2$. The two eigenvalues λ_1 and λ_2 of Λ are given as solutions of

$$\left(\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\lambda_{1/2} = \frac{1}{2}(\Lambda_{11} + \Lambda_{22}) \pm \frac{1}{2}\sqrt{(\Lambda_{11} + \Lambda_{22})^2 - 4(\Lambda_{11}\Lambda_{22} - \Lambda_{12}\Lambda_{21})}.$$

λ_1 and λ_2 are either both real or complex conjugate, and not necessarily different. Table 3.1 list the various possibilities for the two eigenvalues of the system's matrix. W.l.o.g we assume the real part of λ_1 larger than that of λ_2. The corresponding behaviors can be found in figure 3.2.

	eigenvalues	eigenspace	type of fixed point
λ_1, λ_2 real	$\lambda_1 < 0$, $\lambda_2 < 0$	2d	stable node
	$\lambda_1 = \lambda_2 < 0$	1d	stable improper (Jordan) node
	$\lambda_1 > 0$, $\lambda_2 > 0$	2d	unstable node
	$\lambda_1 = \lambda_2 > 0$	1d	unstable improper (Jordan) node
	$\lambda_1 > 0$, $\lambda_2 < 0$	2d	saddle (hyperbolic)
	$\lambda_1 = \lambda_2 < 0$	2d	stable star
	$\lambda_1 = \lambda_2 > 0$	2d	unstable star
	$\lambda_1 = 0$, $\lambda_2 < 0$	1d	line of stable fixed points
	$\lambda_1 > 0$, $\lambda_2 = 0$	1d	line of unstable fixed points
$\lambda_1 = \overline{\lambda_2}$	$\lambda_1 + \lambda_2 < 0$	2d	stable spiral
	$\lambda_1 + \lambda_2 > 0$	2d	unstable spiral
	$\lambda_1 + \lambda_2 = 0$	2d	elliptic fixed point (center)
	$\lambda_1 = \lambda_1 = 0$	0d	superstable or nilpotent fixed point

Table 3.1: *Table of flow structures in two dimensional linear systems.*

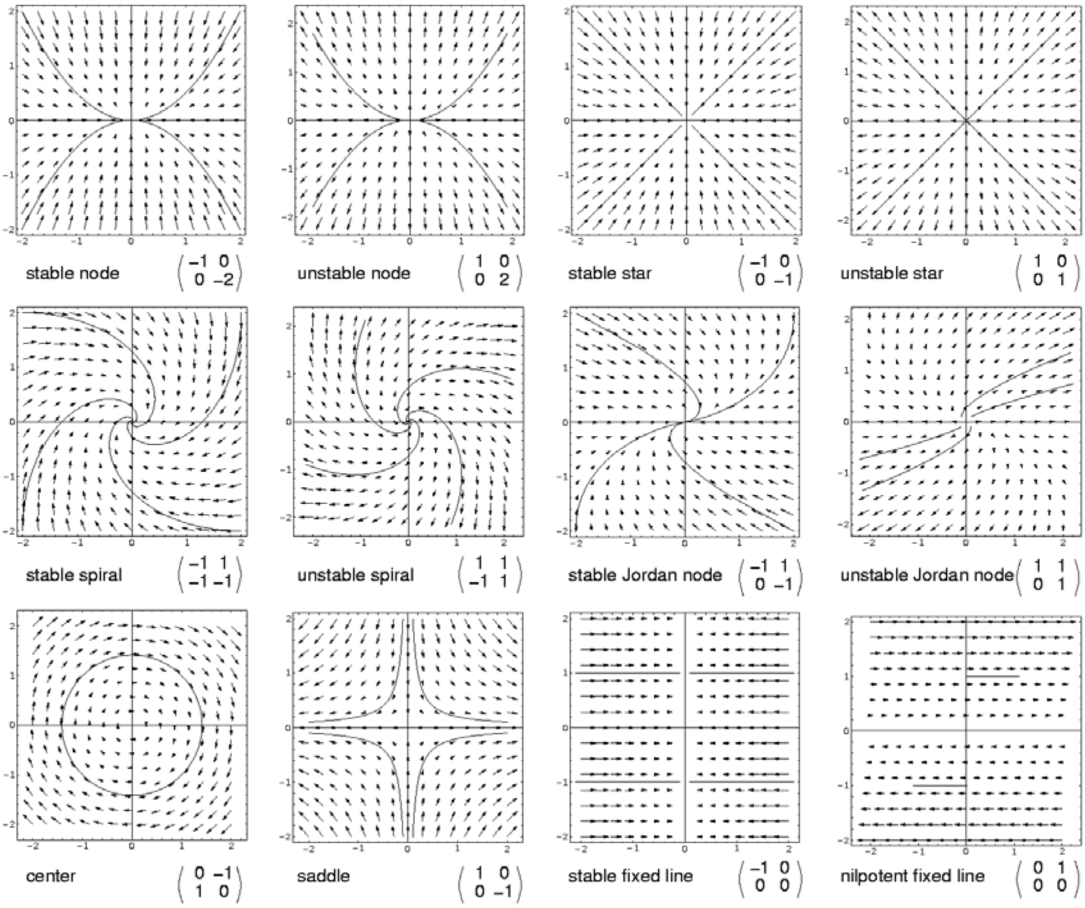

Figure 3.2: *Examples of flow structures in two dimensional linear systems and sample trajectories as well as typical coefficient matrices.*

3.4.1 More on love affairs

Strogatz (cf. [9, 10]) asks his readers to take a look on a few more examples on what may happen in the realm of love affairs. Suppose Romeo and Juliet react to each other, but not to themselves: $\dot{R} = aJ$, $\dot{J} = bR$. What happens? Let us write the system in matrix form:

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

The eigenvalues of $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ are given by $\lambda_{1/2} = \pm\sqrt{ab}$. If a and b have the same sign, there are two real eigenvalues, one positive and one negative: The qualitative behavior is that of a saddle, i.e. the state of the affair is bound to diverge: If a and b are both positive, Romeo and Juliet will end up in excessive bliss or in deadly mutual hatred. What actually will happen depends on what feeling was initially stronger. The lesson to be learned is: try a little love.

If a and b have opposite signs the feelings will remain opposite if they are in the beginning, or if both had initially the same attitude, the one with the stronger attitude will force the other one to change the feeling.

Do opposites attract? The system $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$ has the eigenvalues $\lambda_{1/2} = \pm\sqrt{-b^2}$, i.e. purely imaginary ones. The behavior of the system is that of a center: The total amount of emotion is conserved ($R^2 + J^2 = \text{const}$) but the emotions vary cyclically. A stable and continuously "interesting" affair.

If Romeo and Juliet are romantic clones ($\dot{R} = aR+bJ$, $\dot{J} = bR+aJ$) should they expect boredom or bliss? Now, this depends. It depends actually on the magnitudes of a and b, namely, as mentioned above, on whether $a^2 < b^2$ or not. This is seen from the eigenvalues of the system in question: $\lambda_{1/2} = a \pm \sqrt{a^2 - b^2}$, i.e. the sign of $a^2 - b^2$ determines whether we have complex or real eigenvalues. Further the sign of a determines stability, because for $a^2 - b^2 > 0$ the root cannot override the sign of a. But also for $a^2 - b^2 < 0$ the stability is determined only by a. Since the root yields only the imaginary part, $a < 0$ is sufficient for stability.

Finally consider "Romeo the robot": Nothing could ever change the way Romeo feels about Juliet: $\dot{R} = 0$, $\dot{J} = aR + bJ$ Does Juliet end up loving him or hating him? R is constant for all times, but its sign gives a bias (that might be inverted for $a < 0$) to the emotional state of Juliet. We may consider instead of $\dot{J} = aR + bJ$ the equation $\dot{J} = b\left(J - \frac{aR}{b}\right) = b\tilde{J}$. Note that $\dot{\tilde{J}} = \dot{J}$, i.e. we can directly infer the behavior of J from $\tilde{J}$. Thus if $b < 0$ then $\tilde{J}$ tends to zero and J tends to $\frac{aR}{b}$ which can be either positive or negative. If $b > 0$ then $\tilde{J}$ explodes to the same side which is J initially relative to $\frac{aR}{b}$.

After having solved all the love issues we will turn back to a set of mathematical problems which require as much care as the ones connected with love.

3.5 Remarks

3.5.1 First integrals for nodes and saddles

In many cases, e.g. for saddles or proper nodes, we can rotate the system to two independent equations, (cf. sect. 3.5.2):

$$\begin{array}{rcl} \dot{x} &=& \lambda_1 x \\ \dot{y} &=& \lambda_2 y \end{array}, \tag{3.6}$$

if both eigenvalues have the same sign ($\lambda_1 \cdot \lambda_2 > 0$, node) than it holds that

$$\left(\frac{y}{y_0}\right)^{|\lambda_2|} - \left(\frac{x}{x_0}\right)^{|\lambda_1|} = 0. \tag{3.7}$$

If they are of opposite sign (saddle), analogously

$$\left(\frac{y}{y_0}\right)^{-|\lambda_2|} - \left(\frac{x}{x_0}\right)^{|\lambda_1|} = 0. \tag{3.8}$$

Equations (3.7) and (3.8) can be viewed as conservation laws a.k.a. first integrals. The dynamics of the respective linear system is thus that (3.7) or (3.8), resp., is satisfied. If the system is not already in the form (3.6) slightly more complex relations hold. Let's briefly summarize how a given systems is transferred to the form(3.6).

3.5.2 Coordinate transformations

Consider the system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We assume that the matrix Λ is diagonalizable which is equivalent to the existence of n independent eigenvalue. Note however that this condition is not satisfied in all of the cases listed in table 3.1.

Diagonalizability means that there is a matrix R that transforms the matrix Λ to diagonal form in the following way:

$$R\Lambda R^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The variables of the diagonal system (e.g. (3.7) and (3.8)) are obtained by

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} u \\ v \end{pmatrix}$$

If Λ is symmetric, the matrix R is easily constructed once the eigenvectors of Λ are known. R contains the eigenvector as rows. For more on diagonalizations cf. any standard linear algebra book.

3.5.3 Spirals in polar coordinates

A typical spiral is present in the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.9}$$

The eigenvalues of Λ are obtained from $(\alpha - \lambda)^2 + \omega^2 = 0$, i.e. $\lambda_{1/2} = \alpha \pm \omega$. In order to see that there are actually spiraling solutions to this system, we introduce polar coordinates by

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{x}{y}\right) \end{aligned} .$$

The system (3.9) is expressed in terms of ρ and θ by calculating the derivatives of ρ and θ.

$$\begin{aligned} \dot{\rho} &= \frac{1}{\rho}\left(x\dot{x} + y\dot{y}\right) \\ \dot{\theta} &= \frac{1}{\rho^2}\left(x\dot{y} - y\dot{x}\right) \end{aligned}$$

inserting $\dot{x} = \alpha x + \omega y$ and $\dot{y} = \alpha y - \omega x$ we find

$$\begin{aligned} \dot{\rho} &= \frac{1}{\rho}\left(\alpha\left(x^2 + y^2\right)\right) \\ \dot{\theta} &= \frac{1}{\rho^2}\left(-\omega\left(x^2 + y^2\right)\right) \end{aligned}$$

or

$$\begin{aligned} \dot{\rho} &= \alpha\rho \\ \dot{\theta} &= -\omega \end{aligned} \tag{3.10}$$

The later form of the system (3.9) shows easily that the angular coordinate moves with constant speed, where the orientation of the spiral is dependent on the sign of ω. From the second equation we find $\theta = \omega t$ such that we have the relation

$$\rho = \rho_0 \exp\left(\frac{\alpha}{\omega}\theta\right)$$

The distance from the origin increases (or decreases) exponentially if $\alpha > 0$ ($\alpha < 0$), because for $t > 0$ the signs of θ and ω cancel.

3.5.4 Time inversion

What about $t < 0$ in the previous example? Then we would have obtained for the initial value problem a solution in terms of $t - t_0$, which is positive. On the other hand we can (formally) allow the time to run in opposite direction. This results in an reversion of the stability properties as well. Under the transformation $t \to -t$, i.e. when solving a differential equation from some starting value backwards in time, stable fixed points turn into unstable ones and vice versa. Time inversion, although practically meaningless, becomes thus a powerful tool for the analysis of dynamical systems. A problem is that unstable behavior rarely occurs and unstable fixed points are rarely ever visited by the system. If one wants to get information about the unstable points, manifold etc., one consider the stable points in the time-reversed system, which are essentially identical to their unstable counterparts one is interested in.

3.5.5 Center

Consider (3.9) in the case $\alpha = 0$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.11}$$

Now, (3.10) simplified to

$$\begin{array}{rcl} \dot{\rho} & = & 0 \\ \dot{\theta} & = & -\omega \end{array} , \tag{3.12}$$

i.e. the radius does not shrink or expand. Solutions from circles (or ellipses) in the phase plane. If the initial amplitude is at a value ρ_0 the system will remain at $\rho(t) = \rho_0$ for all times. A corresponding physical system is a low-amplitude pendulum without friction or damping.

How ever weak influences such as by noise or small nonlinearities which have been considered to be irrelevant in the formulation of (3.11) may gain importance here, because at least in the radial direction the system shows no dominant dynamics. We will continue here later, when dealing with small nonlinearities.

3.5.6 Higher dimensions

For higher dimensional there are of course more combination possible. Generally, there are proper nodes and spirals, and a lot of degenerated cases. Later, however, when instead of linear systems linearizations of general systems are considered, we will see that the degeneracies are not merely a reduction to the lower dimensional cases, but may reveal new structure which is brought about by the nonlinearity of the system.

3.6 Light-seeking robot

We consider as a first and rather trivial example a robot in front of a light source. The robot faces the light source under an angle ϕ and is able to turn, i.e. to change this angle ϕ. The robot has one sensor which output a value x as a function of the angle ϕ

$$x = b(\phi)$$

Our task is to equip the robot with a controller which turns the robot so that $x = \max$.

Let (to simplify the problem)

$$x = \cos(\phi)$$

The controller output affects the angle ϕ as follows

$$\Delta\phi = \Delta t\, \alpha y$$

The fact that the robot should turn to the direction of maximal light intensity is expressed by a cost function which is to be optimized by the robot's actions. Choosing $E = \cos\phi$, we have posed the task to the robot to maximize the light intensity w.r.t. y. It turns out, however, that E does not directly depend on y. Only when considering the difference

$$E(t - \Delta t) - E(t) = \cos(\phi + \Delta\phi) - \cos(\phi)$$

the $y-$dependency becomes visible and can be used to change y such that E is further increased:

$$\Delta y = \frac{\partial}{\partial y} E = \frac{\partial}{\partial y} \cos(\phi + \Delta t\, \alpha y) = \Delta t\, \alpha \sin(\phi + \Delta t\, \alpha y)$$

This procedure, called gradient ascent (or analogously: gradient descent if the task is to minimize E), establishes a convenient and simple way of parameter adaptation (or if you like: learning) and will reoccur later more often.

Taking the limit $\Delta t \to 0$ we find

$$\dot{y} = \alpha \sin(\phi).$$

We can for small deviations from the light source (which is assumed here to be visible at an angle $\phi = 0$) linearize the sin function and obtain the following linearized system

$$\begin{pmatrix} \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} y \\ \phi \end{pmatrix}$$

The eigenvalues are $\lambda_{1/2} = \pm\alpha i$, which means that the behavior is always oscillatory. What can be done in order to have an actual stabilization towards the point of maximal light intensity? Actually, the present algorithm does not distinguish between the point of maximal and the point of minimal light intensity. Also this should be taken into account: the point of maximal intensity should be a stable fixed point, whereas the point of minimal intensity should be unstable.

Of course we should have asked how y is actually to be controlled from the information accessible by the robot rather than expressing in term of the objective coordinates, which are not directly available to the robot.

What went wrong in the above example? We wanted to stabilize the fixed point at $\phi = 0$ but obtained merely a center around the fixed point.

We have used (unlike in the computer program controlling a robot for this task) a few differential relations. Differential relations refer to two infinitely (or at least nearby) points, say, in time. What happened is that in this way we have failed to take into account causality o the events: The control action cannot be based on later measurements x or ϕ cannot react to later control actions y.

Let's be a bit more careful now. We indicate the direction of causality by arrows and shall use the relations only in the causal temporal order. When introducing a wrong causality by some relation we can indeed obtain any stability behavior. Wrong causality means time inversion, which as we have seen, changes stability properties.

We want the objective dynamics, i.e. that of ϕ, to have a stable fixed at $\phi = 0\,(+2k\pi)$ and unstable one elsewhere, in order to approach $\phi = 0$ directly it should be located at $\phi = \pi\,(+2k\pi)$. This means, $\dot{\phi}$

should cross zero decreasing at 0 and increasing at π. Similar to the error function in the previous trail approach, we choose $\dot{\phi} = -\sin\phi$, which is qualitatively the same as maximizing $\cos\phi$, but now the error function is not really accessible to the robot because is based on non-measurable quantities.

$$\dot{\phi} = \alpha y$$

$$y = -\frac{1}{\alpha}\sin\phi$$

$$\dot{y} = -\frac{1}{\alpha}\cos\phi\,\dot{\phi} = -\frac{1}{\alpha}\cos\phi\,(ay)$$

$$\begin{pmatrix} \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} -\cos\phi & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ \phi \end{pmatrix}$$

This system is stable near the fixed point $\begin{pmatrix} y^* \\ \phi^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, although one observes that the system is stable for any ϕ if only $y = 0$: Trivially, if the robot does not move, the system is at a fixed point. It is thus interesting to note, that for

and unstable near $\phi = \pi$.

If we however reformulated the system such that it contains closed expressions in terms of the information available to the robot

$$x = \cos\phi$$

$$\dot{y} = -xy \tag{3.13}$$

$$\dot{x} = -\sin\phi\,\dot{\phi} = -y^2 \tag{3.14}$$

we find for the linearized system with $f(y,x) = -xy$, and $g(y,x) = -y^2$:

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -y & -x \\ 0 & -2y \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}.$$

Thus, near the fixed point $\begin{pmatrix} y^* \\ x^* \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ approximately holds

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ x - x^* \end{pmatrix}.$$

This is the nilpotent case of he classification. It presents a non-trivial dynamics with two zero eigenvalues. The phase plane contains a line of fixed points, i.e. any y is OK if only $x = x^*$, if, however, x deviated from x^* then y is changed. Since, however, x^* is the maximal possible value for x, we always have $x - x^* > 0$ such that $\dot{y} < 0$ and y decreases. The nonlinear system (3.13), (3.14) than takes care of the correct control. Here, although having identified a theoretically interesting special case of a linear system, the limited information available to the robot does not allow for a linearization, but must take into account the full non-linear system. The mechanism is simply, that if x is at the fixed point one cannot know from a single short deviation the robot cannot guess to which side (in terms of ϕ) it actually went.

3.7 Stability

Stability will be the subject of the beginning of the next section. Here we merely give some simplified definitions.

Lyapunov stability means that trajectories that are close together will stay close together. Asymptotic stability implies that nearby trajectories will become arbitrarily close for sufficiently long times. Both notions are clearly different. It may happen, that close trajectories stay close be never approach, such as near an elliptic fix points. On the other hand (Consider the system $\dot{\phi} = 1 - \cos(\phi)$, where a starting value slightly above zero will run for a full circle before settling to the (one-sided) stable fixed point at $\phi = 0$.) asymptotic stability does not imply Lyapunov stability. If both conditions a fulfilled we will call a fixed point stable and if neither is satisfied, we will call it unstable.

There is also the notion of orbital stability, which is a weaker formulation of Lyapunov stability. Here trajectories stay close although individual points may deviate as time passes. E.g. two planets may follow nearby orbits while having different orbital speeds.

Later another aspect of stability, namely with regards to changes in the parameters is discussed. The notions of stability mentioned above relate to dynamical stability, whereas when stability with respect to the parameters is considered we speak of structural stability.

Chapter 4

Nonlinear systems

4.1 Linearization

4.1.1 Stability analysis in the one dimensional case

Let x^* be a fixed point of the one dimensional system $\dot{x} = f(x)$. We perturb the system by a small-amplitude function $\eta(t) = x(t) - x^*$. The perturbation happens actually at $t = 0$ and we are interested in the fate of the deviation $\eta(t)$ for $t > 0$. We consider the derivative

$$\dot{\eta}(t) = \frac{d}{dt}\left(x(t) - x^*\right) = \dot{x} = f(x) = f(x^* + \eta) \tag{4.1}$$

Since η is small we can Taylor-expand the r.h.s. of the previous equation (4.1):

$$f\left(x^* + \eta\right) = f\left(x^*\right) + \eta f'\left(x\right) + O\left(\eta^2\right)$$

f' denotes the derivative of f with respect to x, and the symbol O tell something about the error we have made by omitting the remaining terms of the Taylor expansion, here the error is at most of the order of η^2 which is very small if η is already small. From (4.1) we have thus

$$\dot{\eta} = f'\left(x^*\right)\eta. \tag{4.2}$$

Note that we can neglect the $O\left(\eta^2\right)$ terms only if we are close to x^* and if $f'\left(x^*\right)$ is nonzero (otherwise the evolution of η is determined by the higher-order terms). (4.2) is (since $f'\left(x^*\right)$ is a constant) actually a linear differential equation. Obviously, if $f'\left(x^*\right) < 0$ the initial deviation $\eta(0)$ decays to zero for $t \to \infty$, whereas for $f'\left(x^*\right) > 0$ the perturbation grows exponentially, i.e. the fixed point x^* is unstable in the latter case (and stable in the former case). We see that it made sense to exclude the case $f'\left(x^*\right) = 0$, because here we would expect from the linear system that the perturbation would remain unchanged by the dynamics. Whereas for $f'\left(x^*\right) \neq 0$ the exponential behavior dominates the higher-order terms, for $f'\left(x^*\right) = 0$ the higher-order terms will determine the dynamics. We will present several interesting examples later when we are dealing with center manifolds.

4.1.2 Stability analysis for dimensions larger than one

Assume that the system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

has a fixed point at (x^*, y^*), i.e. $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. As in the previous example we study the problem in terms of deviations from the fixed point $u = x - x^*$ and $v = y - y^*$. Consider at first u :

$$\begin{aligned}
\dot{u} &= \dot{x} \\
&= f(x^* + u, y^* + v) \\
&\approx f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} \\
&= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}
\end{aligned}$$

where we have omitted terms quadratic in u and v (cf. previous section) and have used that $f(x^*, y^*) = 0$. Analogously, we find

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}$$

or, taken together,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.3}$$

The matrix $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$ is called Jacobian matrix and is the multi dimensional analog of f'. Since (4.3) is a linear differential equation we can exploit the relations we have derived in the section about linear systems (Sect. 3.3) and have found now a possibility to carry over the analytic tools from the linear case to the nonlinear case, however, only in a local sense, i.e. in the vicinity of a fixed point.

4.2 The moving one-eyed robot

Let us consider first a simple two-wheel robot which is assumed to have but one sensor which measures the distance to the closest obstacle. In physical space the state of the robot is defined by the two coordinates

1. $a =$ distance to the obstacle and
2. $\phi =$ angle between the direction of the forward motion of the robot and the connecting line between the robot and the obstacle.

The robot is to be equipped with a controller which outputs the new turn velocity $\dot{\phi} = \frac{d}{dt}\phi$ of the robot as a function of the distance a and the angle ϕ. The forward velocity v is assumed to be fixed. The dynamical system describing the motion of the robot in **physical** space is

$$\begin{aligned}
\dot{a} &= v \sin \phi \\
\dot{\phi} &= K(a, \phi)
\end{aligned} \tag{4.4}$$

where

$$K(a, \phi) = \alpha \frac{v_r - v_l}{D} \tag{4.5}$$

D being the distance between the wheels, v_r and v_l are the velocities of the right and left wheels, respectively, and α is a hardware constant.

Equation (4.4) describes the motion in physical space. The robot however receives information about the world and hence about its state in physical space only by its sensors. This information can be incomplete or unreliable or both. In the present model the sensor output

$$x = s(a)$$

is to be a function of the distance a alone so that the angle ϕ is a hidden variable.

In sensor space we obtain by means of

$$\dot{x} = s'(a)\,\dot{a} = s'\left(s^{-1}(x)\right)\dot{a} = q(x)\,\dot{a}$$

the equations of motion as (absorbing v into q)

$$\begin{aligned}\dot{x} &= q(x)\sin\phi \\ \dot{\phi} &= K(x)\end{aligned}$$

For a first analysis we note that the system has a fixed point $\phi = 0$, x^* where $K(x^*) = 0$. Near $\phi = 0$ we can approximate the sin linearly: sin$\phi \approx \phi$. Further we can assume the $q(x)$ is more or less constant for $x \approx 0$, i.e. $q(x) \approx u$. Finally, we assume a linear controller $K(x) = kx$ or we approximate the controller linearly. This linearization near the fixed point yields the dynamical system

$$\begin{aligned}\dot{x} &= u\,\phi \\ \dot{\phi} &= k\,x\end{aligned}$$

or

$$\begin{pmatrix}\dot{x}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}0 & u\\ k & 0\end{pmatrix}\begin{pmatrix}x\\ \phi\end{pmatrix}$$

The fixed points are obtained from the characteristic equation $\lambda_{1/2} = \pm\sqrt{ku}$. We immediately see that the system is unstable if $ku > 0$ and is oscillatory if $ku < 0$. Hence, with the information the robot obtains from its sensor it never can move in a fixed distance to some wall but instead the best it can do is to follow an oscillatory path at a distance given by the sensor value x^*.

4.3 Two-eyed robot

Let us now assume that the robot has two sensors at one side of its body, the sensors measuring the distance to the wall. Looking to the right, the sensor signals are x_1 (front) and x_2 (end) the distance between the sensors being A. If the robot moves parallel to a wall we consequently have $x_1 = x_2$. The controller which is to yield the target turn velocity of the robot as before may now depend on both sensor values which on their hand depend on both $y = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ and ϕ.

$$\begin{aligned}\dot{y} &= v\sin\phi \\ \dot{\phi} &= K(x_1, x_2)\end{aligned}$$

4.3.1 Dynamics in sensor space.

The above equations are formulated in terms of the distance y and angle ϕ of the robot relative to the wall. These must be extracted from the primary sensor data in a preprocessing procedure. In order to formulate the dynamics in terms of the primary sensor values x_1 and x_2 itself, we use a linear sensor characteristics and absorb hardware constants into D and the time scale so that

$$y = \frac{x_1 + x_2}{2}$$

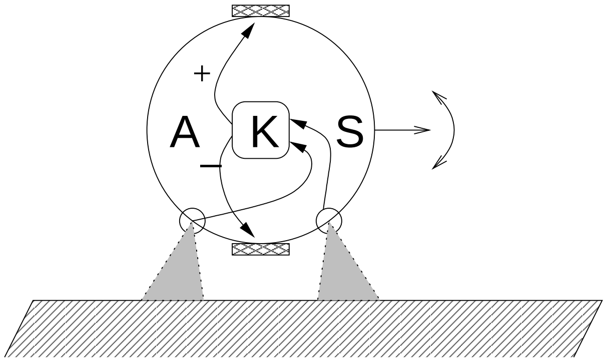

Figure 4.1: Schematic drawing of a two-eyed robot. It senses walls or obstacles on its sides by sensors in the front and in the back. We consider here only sensors on the right side of the robot. The sensor readings are fed to the robot's controller which determines the speed of the two wheels (or rather the steering angle, cf. text).

and

$$\sin\phi = \frac{x_1 - x_2}{D} \tag{4.6}$$

In the case of small ϕ we may replace $\sin\phi$ by ϕ and find

$$\begin{aligned} \dot{x}_1 &= vD^{-1}(x_1 - x_2) + \tfrac{D}{2}K(x_1, x_2) \\ \dot{x}_2 &= vD^{-1}(x_1 - x_2) - \tfrac{D}{2}K(x_1, x_2) \end{aligned} \tag{4.7}$$

Depending on the controller we may observe all kinds of motions which are possible in a two dimensional system. In particular we observe all attractor types to be discussed below.

In order to discuss the general case we introduce

$$z = \frac{x_1 - x_2}{D}$$

we find $\dot{z} = \dot{\phi}\sqrt{1 - z^2}$ so that from (4.4)

$$\begin{aligned} \dot{y} &= vz \\ \dot{z} &= \sqrt{1 - z^2}K(y, z) \end{aligned} \tag{4.8}$$

This equation describes the motion in sensor space in terms of the generalized sensor coordinates y and z. It can easily be reformulated in terms of x_1 and x_2.

The root term may be eliminated by the following trick. We introduce an auxiliary variable $u = \cos\phi$ so that $\dot{u} = -z$. Then we have the dynamical system

$$\begin{aligned} \dot{u} &= -z \\ \dot{y} &= vz \\ \dot{z} &= u\,K(y, z) \end{aligned} \tag{4.9}$$

However this equation is more general than equation (4.8). It boils down to the latter when observing the initial condition

$$u(t_0) = \sqrt{1 - z^2(t_0)}$$

Another form of writing equation (4.9) is obtained from eliminating z

$$\begin{aligned} \ddot{u} &= -u\,K\,(y, -\dot{u}) \\ \ddot{y} &= v\,u\,K\,(y, -\dot{u}) \end{aligned} \tag{4.10}$$

The above equations define the dynamics of a robot encountering a wall for an arbitrary controller given by K.

4.3.2 The linear control regime

For small z and using a linear controller we have

$$\begin{aligned} \dot{y} &= v\,z \\ \dot{z} &= \alpha y + \beta z + \gamma \end{aligned} \tag{4.11}$$

so that obviously for small deviations of the robot from the ideal line we have the dynamics

$$\dot{x} = Hx \tag{4.12}$$

where $x = (x_1, x_2)$ and the matrix H is chosen appropriately. Equation (4.11) is equivalent to

$$\ddot{z} = \beta \dot{z} + \alpha v z \tag{4.13}$$

which is the equation of the damped harmonic oscillator with friction term. If the friction constant β is < 0 than we have a damping so that the robot now will move in a stable motion along the wall. This is possible because there are no hidden variables in the present case.

Adding noise

Noise can appear in various forms in the dynamical system depending on the origin of the stochastic influences. Measurement noise makes both y and z noisy which leads to an **additive** noise term in eqs. (4.12), (4.13), e.g. Controller noise may influence the variables α, β, γ and v which leads to a **multiplicative** noise term in these equations.

4.4 A sample limit cycle

EXAMPLE

We consider a system in polar coordinates

$$\begin{aligned} \dot{r} &= r\left(1 - r^2\right) \\ \dot{\theta} &= 1 \end{aligned} \tag{4.14}$$

Since θ moves with constant speed independent of r, we can study the temporal evolution of r separately. The one dimensional system

$$\dot{r} = f(r) = r\left(1 - r^2\right)$$

has exactly two fixed points (well, there is another one at $r = -1$, but in polar coordinates r is restricted to nonnegative values). Setting $\dot{r} = 0$, we find $r_1^* = 0$ and $r_2^* = 1$. The derivative of the right hand side is

$$f'(r) = \frac{df}{dr} = 1 + r - 3r^2$$

such that

$$f'\left(r_1^*\right) = 1 \,\text{and}\, f'\left(r_2^*\right) = -1,$$

i.e. r_1^* is an unstable fixed point and r_2^* is a stable one. Starting exactly at $r(0) = r_1^*$ the trajectory will stay at $r(t) = r_1^* = 0$. For any other value of $r(0)$ the trajectory $r(t)$ will tend towards $r_2^* = 1$, because for $r > r_2^*$ there is a tendency for r to decrease, whereas r will increase for $r_1^* < r < r_2^*$.

For the system (4.14) this means that any the trajectory $\begin{pmatrix} r \\ \theta \end{pmatrix}$ different from $\begin{pmatrix} r_1^* \\ \theta \end{pmatrix}$ will approach the circle $\begin{pmatrix} r_2^* \\ \theta \end{pmatrix}$, $\theta \in \mathbf{R}$, i.e. the system (4.14) has a limit cycle with radius r_2^*. Without having solved the equations explicitly, we have thus identified the qualitative behavior of the system (4.14).

4.5 The Poincare-Bendixon Theorem

Suppose the following conditions are satisfied

1. A domain Q is a closed and bounded subset of the two dimensional plane.
2. There is a open domain $P \supset Q$ on which the function f is everywhere differentiable.
3. The system $\dot{x} = f(x)$ has no fixed point on Q.
4. There exists a trajectory $x(t)$ (with $\dot{x} = f(x)$) that once having entered Q it never leaves Q again.

Then there exist a closed orbit in Q.

Remark: We have already seen, that in one dimension there are no limit cycles. Now we find that in two dimensions there are no more wild behaviors possible than limit cycles. Of course there are also fixed points in two dimensions. Other dynamical behaviors, e.g. the nilpotent fix point line in the above classification, may contain lines or curve consisting of fixed points, but apart from fixed points and limit cycles nothing is to be expected.

The proof of the Poincare-Bendixon theorem is lengthy (cf. e.g. Alligood et al., 1997), and contains in addition to the uniqueness property that we have referred to when showing that there are no cyclic behaviors in one dimension, a trick that captures the trajectory iteratively within its own previous points.

In order to apply the theorem the first three conditions are mostly easy. If there are only finitely many fixed points, one has to cut out a small open region around the fixed points to obtain a closed region R. More difficult is the fourth condition: One may instead of looking for special trajectories consider a so called “trapping region” to be used for the region R. One then has to make sure that at all the boundaries trajectories can only enter R and can nowhere leave. If it can be proven that trajectories exist for infinite times (cf., however, the counter example above), then condition 4 is satisfied.

4.6 Three dimensional systems

4.6.1 Two incommensurate frequencies form a torus

Consider the equation

$$\ddot{x} + c^2 x = 0, \quad x \in \mathbf{R} \tag{4.15}$$

of equivalently

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -c^2 x \end{aligned}$$

Inserting $x(t) = a\sin(ct+b)$, shows that we have found a solution. Since the solution depends on two parameters (which can be determined from the initial conditions) we can be sure that we found already all solutions.

In the following examples we will assume that $a = 1$, $b = 0$, and $c = 2$. The two-dimensional system (4.15) is of center type with ellipses as solutions which are traveled through twice when t runs through $[0, 2\pi)$.

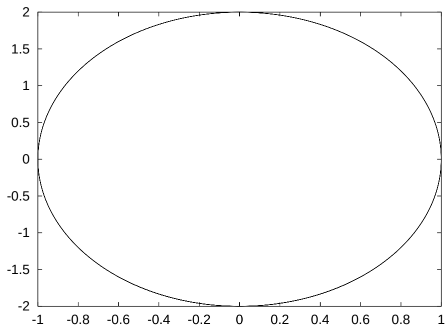

Figure 4.2: Typical trajectory of the unperturbed system (4.15). Horizontally x and vertically $\dot{x}$ is represented. Corresponding to the periodic behavior of the system the trajectory is closed and is ran through by the system twice every clock cycle.

Now we apply a second frequency to the system.

$$\ddot{x} + c^2 x = \left(c^2 - 1\right)\sin t \tag{4.16}$$

Actually we are now considering a two-dimensional system with driving. The Poincare-Bendixon theorem does not apply to such systems, because of the explicite time dependency. We can however express (4.16) as an autonomous three-dimensional system.

$$\begin{aligned}\dot{x} &= v\\ \dot{v} &= -c^2 x + (c^2-1)\sin\theta\\ \dot{\theta} &= 1\end{aligned}$$

where the explicite time dependency is covered by a third trivial equation. A typical solution (cf. figure 4.3) of (4.16) is

$$x(t) = \sin t + \sin ct. \tag{4.17}$$

Note, that the behavior of (4.16) for long times does not depend on the initial conditions. In order to get right to the interesting solutions we choose the initial conditions such that the trajectory will must not go through a transient behavior, namely $x(0) = 3c$???

Using (4.17) we find also

$$v(t) = \cos t + c\cos ct$$

Taking into account that

$$\theta(t) = t$$

we have the following picture, cf. figure 4.4 (left)

In the right picture (above), we have used the fact that θ is actually a cyclic variable, that we have rather plotted $\sin\theta$ along the z-axis (unfortunately it is turned by 90^o).

The closed orbit can be embedded into a torus, cf. figure 4.5.

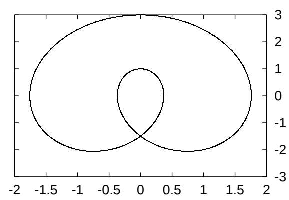

Figure 4.3: In this example the frequency of the external driving is half the natural frequency of the unperturbed system. Both frequencies are superimposed forming thus a cycle of period two. In contrast to the unperturbed system the amplitude of the system is now fixed, i.e. the system sooner or later approaches a limit cycle, i.e. the trajectory displayed here.

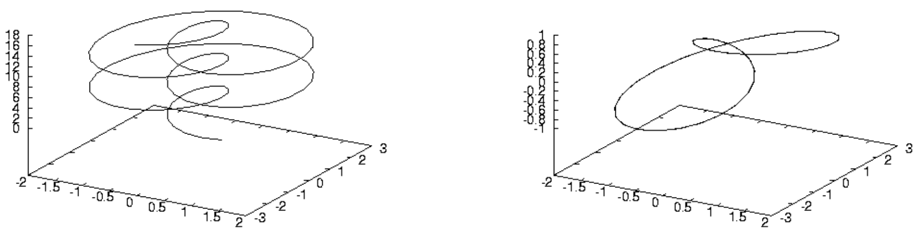

Figure 4.4: Embedding the trajectory of the periodically perturbed system as an autonomous system into a three dimensional space. On the left time is an unbounded continuous variable, whereas on the right the periodicity of the external drive is appropriately taken into account.

So far in the pictures only the case $c = 2$ has been displayed. What happens at other values of c? Below in figure 4.6 we illustrate the cases $c = 3$, $c = 1.05$, and $c = 3.05$.

Interestingly, for rational $c = \frac{p}{q}$ the trajectory closes after q cycles, and it winds pq times around the torus. For irrational values of c, however, the trajectory never closes, and fills thus the whole torus. In this case we speak of quasiperiodic behavior. Quasiperiodicity is different from chaotic behavior (to be considered next) in that nearby trajectories will not depart from each other, i.e. quasiperiodic systems are stable in the Lyapunov sense.

In nature related phenomena occur, if periodic behavior is disturbed periodically. A particle in the ring of the planet Saturn has a natural frequency which is determined by its distance to the planet. The large moon Titan (?) periodically attracts these particles with a different frequency. If the two frequencies have a rational ratio, the particle's trajectory is not stable as it takes up energy resonantly. For irrational ratios energy uptake will average out and the trajectory is stable. If $c = \frac{p}{q}$ is rational, but p and q are large, the particle will be able to stay for (even astronomically) long times on its orbit. At this scales also the effect of other moons etc. cannot be neglected.

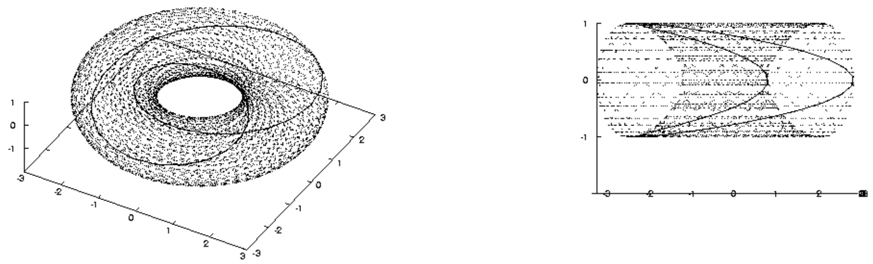

Figure 4.5: The trajectory of the double-periodic system embedded in a torus. For other initial conditions the trajectory is shifted around the torus. On the right a side view of the torus is given.

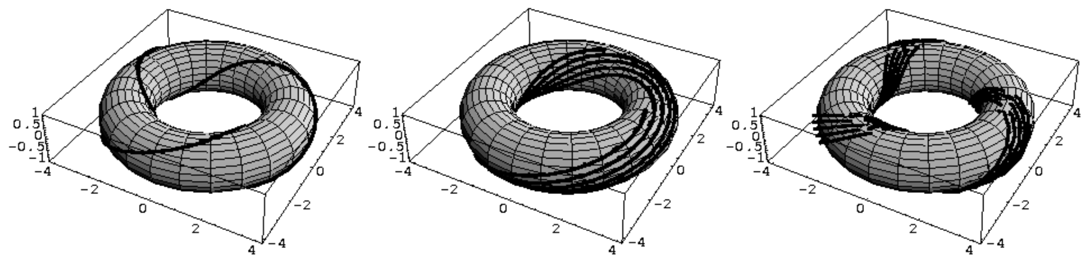

Figure 4.6: Trajectories with different period. Intrinsic frequencies for $c = 1.05$ and $c = 3.05$ close at, resp., twenty and sixty cycles (only a part of the trajectory is presented in these cases).

4.7 Braitenberg vehicles

4.7.1 Vehicle 1

Vehicle 1 is equipped with a single sensor that increases (or decreases) the speed of the vehicle in dependence of a certain stimulus. In order to obtain interesting behavior we have to assume that the vehicle deviates randomly from its straight path. On a large length scale the vehicle can be described by a random walk in a potential which is formed by the stimulus quantity. If x denotes a position and $V(x)$ is the value of the stimulus at this location, over large times the probability of the vehicle to be a certain place will be high at low V and low at high V (or vice versa for inhibitory sensor-motor coupling). This can be made more explicitely.

4.7.2 Vehicle 2

Consider a vehicle with two light sensors the output of which is fed into the wheel motors. We study in the present section the case that the left (right) sensor is connected to the right (left) wheel with strength factor $1 + w$ $(1 - w)$. Let (x, y) be the position of the center of the axis A of the robot and

$$\vec{u} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

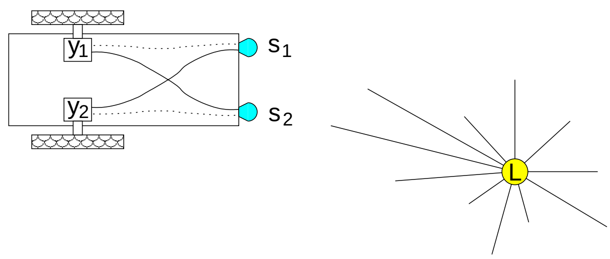

Figure 4.7: Braitenberg vehicle near a light source. Either the solid lines, the dashed ones or both a present in, resp., Braitenberg's vehicles 2a, 2b, and 2c. Each vehicle exhibits a specific behavior in relation to the light source.

be the unit vector pointing from A to the nose of the robot. The sensors are to be located along the the axis at a distance $-b/2$ and $b/2$ to the left and right of A, respectively. The distance of A from light source i is

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$

We put the intensity of the light source as a function of the distance as

$$I_i = \frac{\alpha_i}{r_i^2 + R^2}$$

where R provides a cutoff of the intensity close to the light source. It may be interpreted as a kind of radius of the source. The power supply received by the motors is (we put the source to the origin for the moment)

$$P_{l/r} = \frac{1 \mp w}{\left(x \pm \frac{1}{2} b \sin\phi\right)^2 + \left(y \mp \frac{1}{2} b \cos\phi\right)^2 + R^2}$$

where the upper (lower) sign is for the left (right) motor, respectively.

We consider the case that the sensors are close together which means $b << R$. In leading order of b, the forward velocity of the robot is

$$\begin{aligned} v &= \tfrac{s}{2}\left(P_l + P_r\right) \\ &= sI\left(x, y\right) \end{aligned} \tag{4.18}$$

the difference velocity of the wheels is obtained as

$$\begin{aligned} v_l - v_r &= s\left(P_l - P_r\right) \\ &= -2swI + 2sbQ \end{aligned}$$

where s is a hardware quantity and

$$Q\left(x, y, \phi\right) = \alpha \frac{-x\sin\phi + y\cos\phi}{\left(x^2 + R^2 + y^2\right)^2} = I\left(x, y\right) \frac{-x\sin\phi + y\cos\phi}{\left(x^2 + R^2 + y^2\right)}$$

If there are more than one sources both I and Q are simply the sum of the corresponding quantities.

The equations of motion of the robot are

$$\begin{aligned} \dot{x} &= s\,I\left(x, y\right)\cos\left(\phi\right) \\ \dot{y} &= s\,I\left(x, y\right)\sin\left(\phi\right) \\ \dot{\phi} &= 2sI\left(x, y\right)\left(-w + b\frac{-x\sin\phi + y\cos\phi}{x^2 + y^2 + R^2}\right) \end{aligned}$$

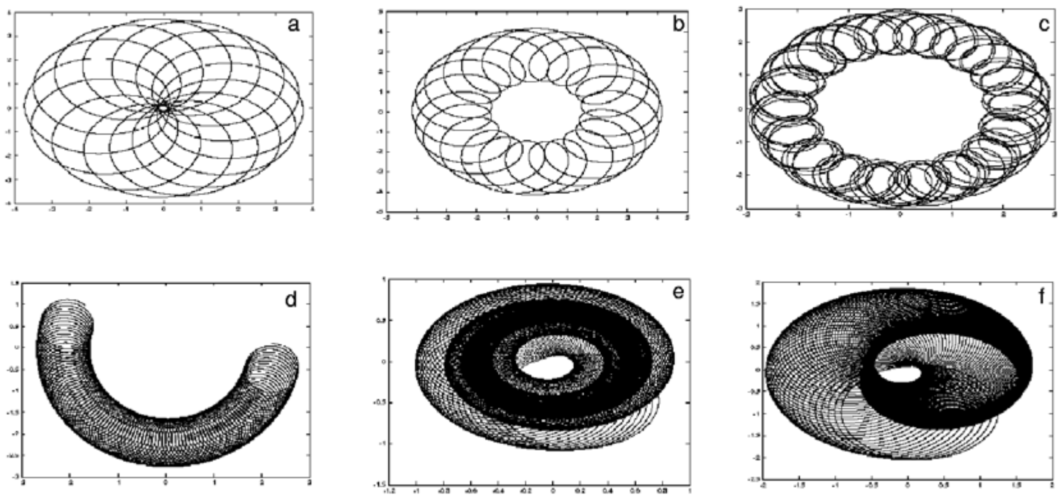

Figure 4.8: Trajectories of Braitenberg vehicles. (a) $x(0) = 0.2$, $b = 1$, $w = 0.5$ (b) $x(0) = 2.2$, $b = 1$, $w = 0.5$ (c) $x(0) = 2.2$, $b = 1$, $w = 0.9$ (d) $x(0) = 2.2$, $b = 0.1$, $w = 0.9$ (e) $x(0) = 0.2$, $b = 0.1$, $w = 0.9$ (f) $x(0) = 0.2$, $b = 0.1$, $w = 0.5$. Further variables: $y\,(0) = 0.1$, $\phi\,(0) = 0$, $R = 1$.The behavior depends on the value of the parameter w in an essential way. (but also on b)

Chapter 5

Bifurcation theory

5.1 Introduction

How do fixed points behave when the system changes? A dynamical system usually depends on a number of parameters, such as the coefficients of the Jacobian matrix in a linearized system. Changes in these parameters can induce changes of the number of fixed points, their stability, multiplicity or change the properties of limit cycles, tori or chaotic attractors. Such parameter values are called bifurcation values. Bifurcations denote thus the qualitative change of the dynamics of a system.

Bifurcations always involve changes in the signs of eigenvalues of the linearized dynamics caused by small changes in the control parameters. Vice versa, dynamical systems are called topologically equivalent if the numbers of eigenvalues of the linearization with positive and negative sign, resp., are the same (and if the eigenspaces of purely imaginary eigenvalues are linearly equivalent).

For dynamical systems describing agents in an environment, bifurcations are particularly important because they correspond to behavioral changes. The parameters of the dynamical system are thus the key to the control of the agent. (Actually the parameters are often called control parameters even in abstract theory of dynamical systems.)

5.2 General framework

The general dynamical system

$$\dot{x} = f(x, c) \tag{5.1}$$

is governed by a set of parameters (for simplicity we concentrate on the case of a single scalar parameter c). Qualitative changes in the behavior of the system in dependence of the parameter are revealed by the effect of varying c on the fixed points of the system. Near any parameter value c_{crit} where the behavior is expected to change, we expand the right hand side of 5.1 with respect to both x and c at a fixed point x^*. If the resulting power series can be transformed to one of the forms discussed in the following sections, the system is said to undergo a bifurcation.

$$\begin{aligned} \dot{x} &= f(x^*, c_{\text{crit}}) + (x - x^*) \left.\frac{\partial f}{\partial x}\right|_{(x^*, c_{\text{crit}})} + (c - c_{\text{crit}}) \left.\frac{\partial f}{\partial c}\right|_{(x^*, c_{\text{crit}})} && (5.2) \\ &+ (x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, c_{\text{crit}})} + \left(x - x^*\right)(c - c_{\text{crit}}) \left.\frac{\partial^2 f}{\partial x \partial c}\right|_{(x^*, c_{\text{crit}})} + (c - c_{\text{crit}})^2 \left.\frac{\partial^2 f}{\partial c^2}\right|_{(x^*, c_{\text{crit}})} + \ldots && (5.3) \end{aligned}$$

The constant term obeys $f(x^*, c_{\text{crit}}) = 0$ because x^* is a fixed point. The further terms in the expansions determine the types of the fixed points of (5.1) in the vicinity of the critical parameter value c_{crit}. By a simple reparametrization we can achieve $c_{\text{crit}} = 0$ which will be assumed in the following. (5.3) now becomes

$$\begin{aligned} \dot{x} &= (x - x^*) f_x(x^*, 0) + c f_c(x^*, 0) \\ &+ (x - x^*)^2 f_{xx}(x^*, 0) + \left(x - x^*\right) c f_{xc}(x^*, 0) + c^2 f_{cc}(x^*, 0) + \ldots \end{aligned}$$

Whether or not the various derivatives of f are zero determines the type of the bifurcation point at c_{crit}. The following sections discuss the main cases.

Supercritical and subcritical

5.3 Saddle-node bifurcation

The system

$$\dot{x} = c + x^2$$

has fixed point at $x^* = \pm\sqrt{-c}$ if $c \leq 0$, cf. figure 5.1. In order to understand the configuration of the fixed point of this and other systems we combine the flows, i.e. the vector fields along the x-axes in figure 5.1, for different values of c and arrive at a plot as in the left display of figure 5.2. Usually, the fixed point configuration is displayed as a function of the parameter c, as presented on the right side of figure 5.2, i.e. the axes are interchanged.

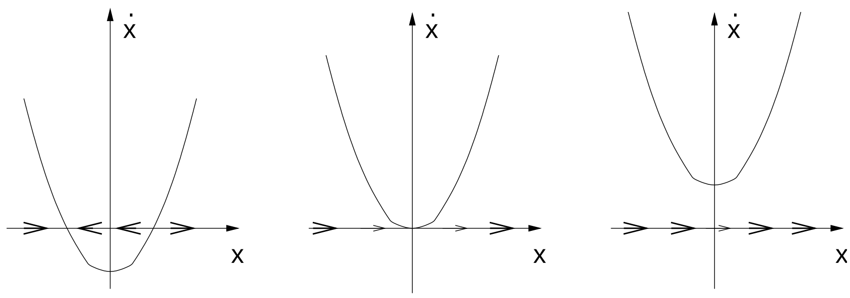

Figure 5.1: Parameter dependent fixed point configuration: Saddle-node bifurcation.

It is obvious from figure 5.1 and even more from figure 5.2 that an unstable fixed point and a stable one collide and annihilate, leaving a system without fixed points. There are other situations, cf. 5.3, where for increasing parameter the fixed points appear out of the blue or where the roles of the stable and the unstable fixed point are interchanged.

Even if no fixed point is present, but c is close to its critical value, the dynamical properties of the system are still influenced by the neighborhood (in the direction of c not in x !) of the fixed points: The trajectory is slowed down for some time but gains speed later. The time spent near (w.r.t. x) the close (w.r.t. c) fixed point scales as $\frac{1}{\sqrt{c - c_{\text{crit}}}}$, i.e. as soon as the fixed point is there, i.e. at $c = c_{\text{crit}}$, the trajectory simply stops at x^* forever.

The naming of this bifurcation type is derived from the two-dimensional case: there instead of an unstable and a stable fixed point a stable node and a saddle collide along the unstable manifold of the saddle, cf. below in sect. 5.6.1.

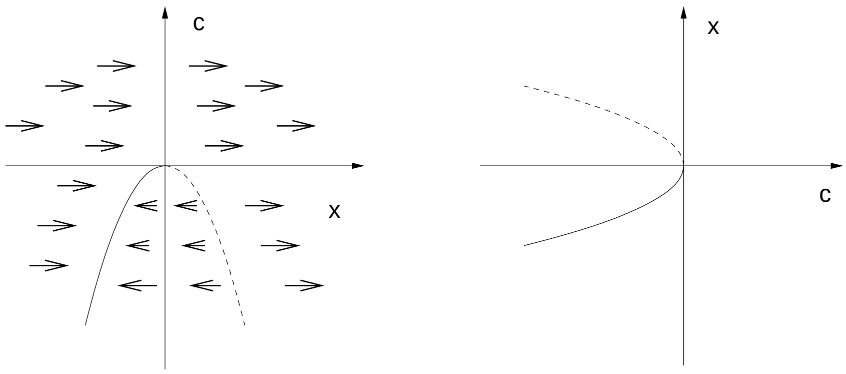

Figure 5.2: From fixed point configurations to standard bifurcation diagrams: horizontal lines in the left picture correspond to the x-axes in figure 5.1. The subfigure on the right is simply a representation of the right one with interchanged axes.

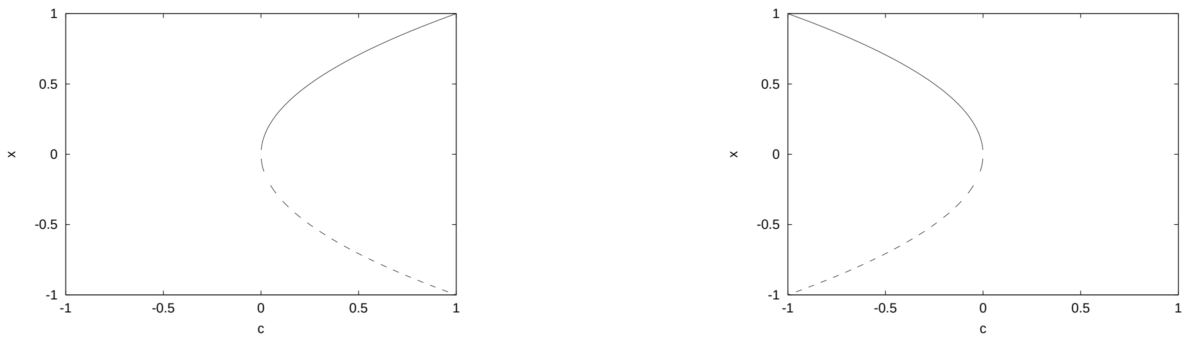

Figure 5.3: Saddle-node bifurcation.

5.4 Transcritical bifurcation

For transcritical bifurcation to occur in a dynamical system the first non-zero terms in the normal form are of second order in x and there is a mixed term. The normal form reads

$$\dot{x} = cx - x^2,$$

i.e. $x^* = 0$ is a fixed point for any c, and there is a fixed point at $x^* = c$. Thus, if c is negative, the second fixed point is unstable, but it becomes stable as soon as c crosses zero. It appears as if the fixed points change their roles when c changes sign, cf. figure 5.4.

5.5 Pitchfork bifurcation

5.5.1 Standard case

The normal form of a pitchfork bifurcation is characterized by the absence of linear terms and of quadratic terms in x and any higher order term in the bifurcation parameter c. Thus only the mixed term and a cubic term is relevant.

$$\dot{x} = cx - x^3$$

When c traverses zero from below the single stable fixed point at $x = 0$ becomes unstable while two new stable fixed points appear at the non-trivial roots $x^* = \pm\sqrt{c}$ of the equation $0 = x\big(c - x^2\big)$. The situation

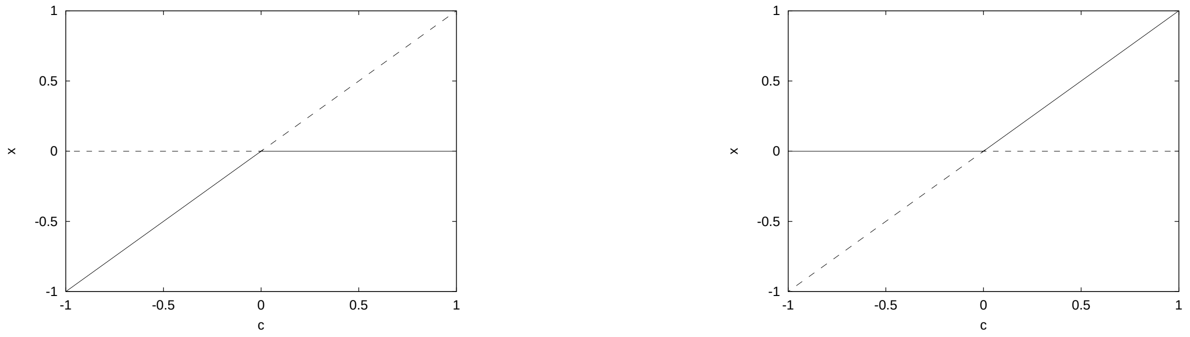

Figure 5.4: Transcritical bifurcation.

can be described such that although the system is no longer stable at $c > 0$ in the linear sense the non-linear terms induce stability to the system. The new fixed points are affected by the non-linearity and a moving away from zero if the relative strength of the non-linearity increases. Actually, for larger c the linear instability of the system becomes stronger such that only at larger x the stabilizing non-linearity can compensate the linear instability.

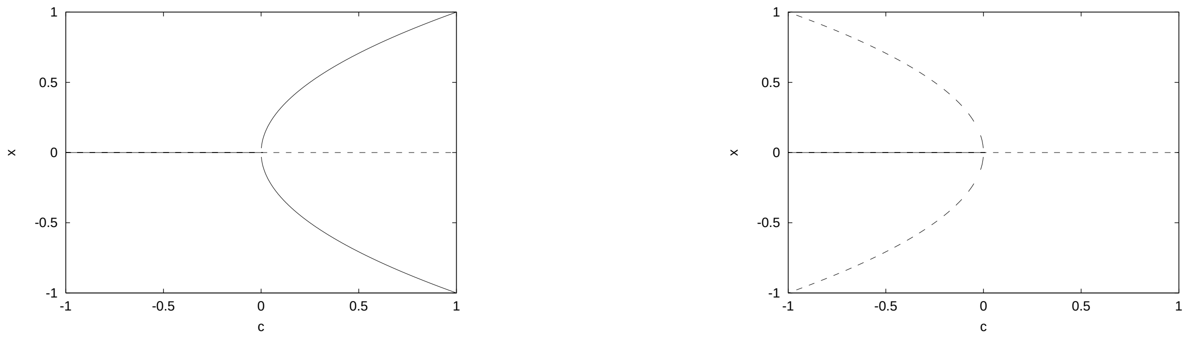

Figure 5.5: Pitchfork bifurcation.

5.5.2 Subcritical pitchfork bifurcation

Subcritical is the pitchfork bifurcation occurring in the system

$$\dot{x} = cx + x^3$$

An interesting case arises if also higher powers are of importance such as in the system (cf. figure 5.6)

$$\dot{x} = cx + x^3 - x^5 \tag{5.4}$$

In addition to the fixed point at zero, which is stable for $c < 0$, further fixed points are found from $0 = c + x^2 - x^4$. We set $z = x^2$ and find $z_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + c}$. Thus, for $0 > c > c_s = -\frac{1}{4}$there are five fixed points, two of which are unstable. The transition from a single fixed point two the five-fixed-point configuration happens via two saddle-node bifurcations at $x_c = \pm\frac{1}{\sqrt{2}}$ and $c = c_s$. At $c = 0$ the two unstable fixed points merge as a subcritical pitchfork into an unstable fixed point, while the outer stable fixed points remain.

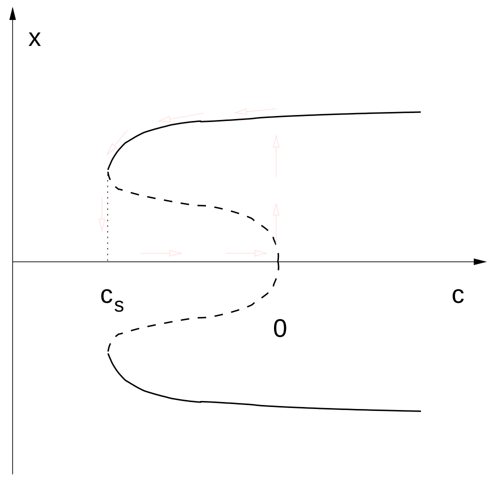

Figure 5.6: Subcritical pitchfork bifurcation embedded into a fixed point configuration of a system involving effects to fifth power. The faint grey arrows indicate the hysteretic behavior of the stable fixed points of x when c is moving up and down. The fixed points are match those of (5.4) only qualitatively.

5.5.3 Imperfect bifurcation and catastrophes

$$\dot{x} = h + cx - x^3 \tag{5.5}$$

For $h = 0$ we have a supercritical pitchfork bifurcation, cf. fig. 5.5. How is the behavior changed when $h \neq 0$? First, consider the images in figure 5.7. The fixed points of (5.5) can be determined graphically, i.e. they can be identified qualitatively by the intersection of the graph of the r.h.s. of the homogeneous system $\dot{x} = cx - x^3$ and the horizontal lines at $-h$. The left image represents the case of negative c, here only a single fixed point exists. For $c > 0$ three fixed points are present if $|h| < |h_c|$. h_c is defined for $c > 0$ and is equal to the value of $f(x)$ at the local maximum at $x_{\max} = \sqrt{\frac{c}{3}}$ hence $h_c = cx_{\max} - x_{\max}^3 = \frac{2c}{3}\sqrt{\frac{c}{3}}$. h_c and $-h_c$ as functions of c are displayed in figure 5.9. The dependence of the fixed point configuration for fixed $h > 0$ on c is depicted by figure 5.8. Finally, the surface in figure 5.9 represents the dependence of the fixed point configuration of (5.5) on both parameters c and h. It is visible that for sufficiently large c hysteretic behavior occurs when h changes. Namely, if h increases starting from a small value a pair of a stable fixed point and an unstable one annihilates such that x jumps to the upper part of the sheet. When h decreases the state x will stay for some time at the upper region, but will jump down later. At fixed $c > c_{\text{crit}}$ we may thus observe large effects from small changes of the parameter h. Actually, here a combination of two saddle-node bifurcations is present, which is usually referred to as an *elementary catastrophe* of cusp type.

5.5.4 Decision making in robots

The system

$$\dot{x} = cx - x^3 + h$$

can be used by robot to decide, e.g., whether it should pass an obstacle left or right. Here x is the steering angle, c is proportional to the inverse distance to the obstacle, and by $-h$ the deviation of the obstacle from the center is expressed. When the robot approaches the obstacle, c increases more and more and the dynamics becomes more and more biased to either side. If h is nonzero the direction of the robot is steered to the opposite side. If h should remained close to 0 it does not matter anyway which side the obstacle is to be passed: the central direction becomes so unstable that ultimately a decision is taken due to some small fluctuation. Hopefully, this happens before the robot bumps into the obstacle. (Steinhage et al.)

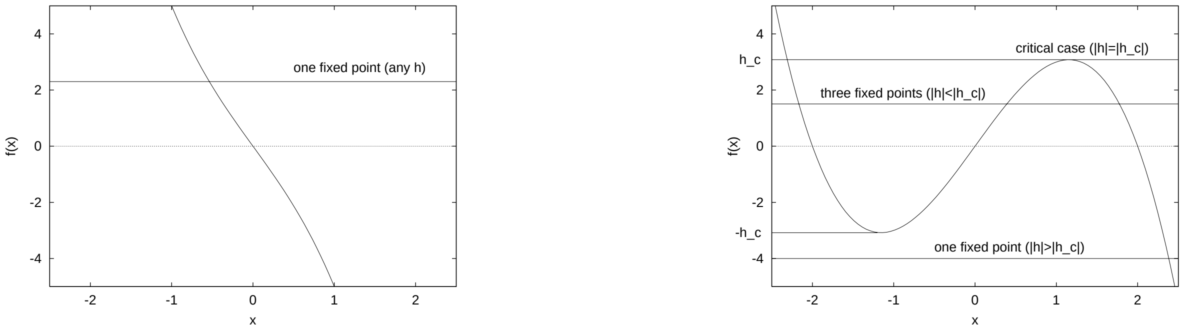

Figure 5.7: Phase portrait of the system $\dot{x} = f(x) = cx - x^3$ for $c < 0$ (left) and $c > 0$ (right). Fixed points of the system $\dot{x} = h + cx - x^3$ are obtained by the intersection of the horizontal lines at negative h value.

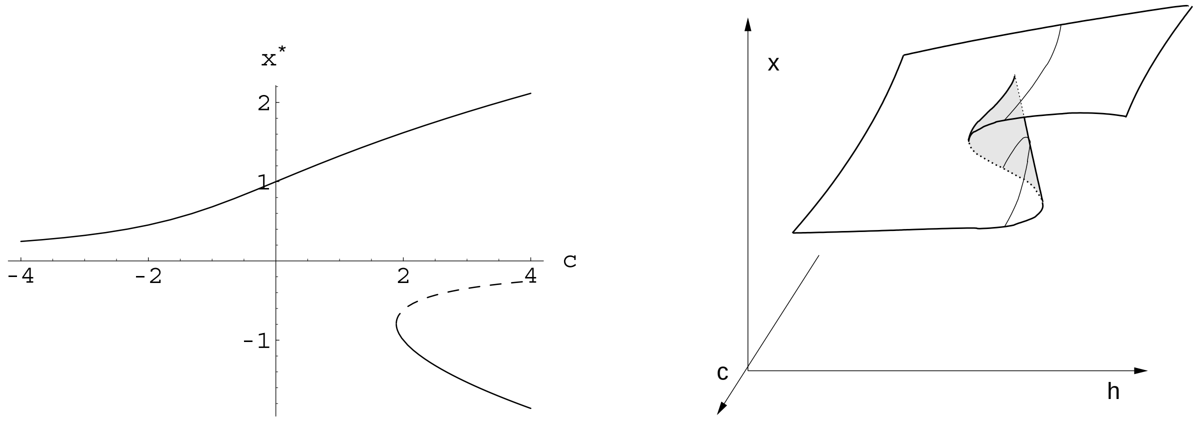

Figure 5.8: At fixed h, increasing c leads from a single fixed point (left side of the previous figure) to the occurrence of three fixed points as soon as $|h| > |h_{\max}|$. Dashed lines represent instable fixed points. Here we have $h > 0$. For $h < 0$ the picture is up-side-down.

5.6 Hopf bifurcation

5.6.1 Higher-dimensional systems

For the previous cases is has been sufficient to consider one-dimensional systems. For saddle-node and transcritical bifurcations two fixed points collide, such that the situation can be described on the one dimensional line connecting the two fixed points. The pitchfork bifurcation is in the vicinity of the critical point symmetrical such that it can be treated in the same way.

Therefore these case carry over to higher dimensional systems without much complications. Naturally, transitions among other attractor types are possible as well, e.g. a limit cycle may become a torus etc. Some examples will be discussed in the following chapter. A simple and common type, namely the Hopf bifurcation, is the transition of a fixed point into a limit cycle. Hopf bifurcations occur in systems of dimension larger than one and are related to pitchfork bifurcation.

It may be good to come back to the saddle-node bifurcation introduced in section 5.3. Most important applications involve a pair of fixed points on a circle, an unstable and a stable one. The circle itself is stable, i.e. the unstable fixed point is actually a saddle point: along the circle states tend away from this point, while across the circle states are attracted. If in the course of the bifurcation the fixed points meet and annihilate, the circle becomes a stable limit cycle with a possible reduced speed near the region

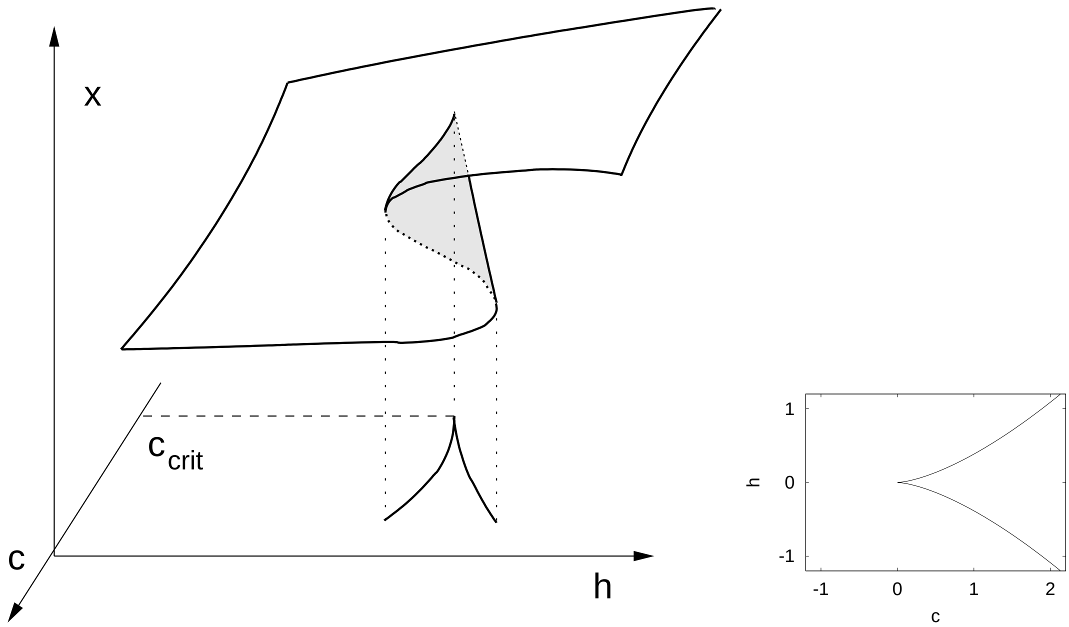

Figure 5.9: Elementary catastrophe: Cusp. The region of instable fixed points is shaded.

where the fixed points used to be. If the parameter can be controlled externally, the system can switch between rotary motion along the limit cycle and waiting behavior near they stable fixed point. This type of a dynamical system provides also a nice model for biological neurons: In the absence of input, the neuron is near its resting state, i.e. near a stable fixed point, if the fixed points become closer the neural state is likely to jump over the unstable fixed point and will follow the whole circle before it reaches the stable fixed point again. The run along the circle is interpreted as the activity of the neuron, the neuron sends a spike or is firing. If the fixed points are annihilated the neuron necessarily will fire a sequence of spikes until the control parameter is set back to values the reintroduce the fixed points. Surely enough, the saddle-node bifurcation can also be present, if the unstable manifold of the saddle does not connect back to the stable node, i.e. if there is only a one-way connection between the fixed points.

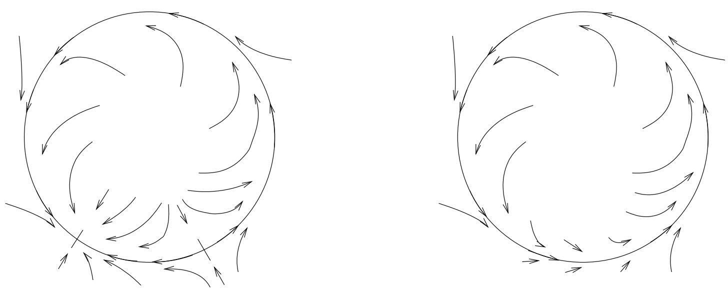

Figure 5.10: Saddle-node bifurcation on a stable cycle. The size of the arrows gives an impression (in particular in the subfigure on the right) of the flow velocity which is low near the position where the fixed points were located.

5.6.2 Standard case

The two dimensional system in polar coordinates

$$\begin{aligned}\dot{r} &= cr - r^3 \\ \dot{\theta} &= \omega\end{aligned}$$

has a stable fixed point for $c \leq 0$ at $r = 0$, whereas for $c > 0$ is shows a limit cycle (cf. above). Formally, the first equation $\dot{r} = cr - r^3$ represents the normal form of a pitchfork bifurcation, but the difference is that r is restricted to positive values, such that only the upper branch of the pitchfork is relevant. Together with the angular equation, which is easily solved by $\theta = \omega t$, the stable fixed point changes above the critical point $c_{\text{crit}} = 0$ into an unstable one surrounded by a stable fixed point.

The situation become perhaps more transparent when displayed in Cartesian coordinates. Using $\dot{x} = r\cos\theta$ and $\dot{y} = r\sin\theta$ we find

$$\begin{aligned}\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ &= \left(cr - r^3\right)\cos\theta - r\omega\sin\theta \\ &= \left(c - \left(x^2 + y^2\right)\right)x - \omega y \\ &\approx cx - \omega y\end{aligned}$$

Analogously we obtain

$$\dot{y} = \omega x + cy.$$

In order to find out about the fate of the fixed point of the system at $c \leq 0$ it remains thus to determine the eigenvalues of $\begin{pmatrix} c & -\omega \\ \omega & c \end{pmatrix}$. These are easily evaluated to $\lambda_{1/2} = c \pm i\omega$. If c passes zero the two eigenvalues move from the half plane of negative real part to the positive one.

If $\dot{r} = cr + r^3$ (cf. the center image in figure 5.11) the situation is vice versa: At $c = c_{\text{crit}} = 0$ an unstable limit cycle is shrunk to an unstable fixed point thereby annihilating a stable fixed point which was present in the center of the limit cycle. Here we have an example of an subcritical bifurcation. Considering the eigenvalues as above we find, however, that the two cases, the supercritical and the subcritical one, cannot be distinguished: in both cases a pair of complex conjugate eigenvalues moves from the left half-plane to the right half-plane, i.e. fixed point of spiral type changes from stability to instability. The behavior of the limit cycle is simply not captured by the linear analysis.

There exists a formula (Guggenheimer and Holmes, 1983, pp. 152-165)

$$\tilde{a} = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{\omega}\left(f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right) \tag{5.6}$$

that allows to capture the nature of the Hopf bifurcation occurring in the system

$$\begin{aligned}\dot{x} &= -\omega y + f(x,y) \\ \dot{y} &= \omega x + g(x,y)\end{aligned}.$$

The derivatives f_{xxx} etc. are evaluated at the origin, where also the bifurcation point is located. Now, it can be shown that if $\tilde{a} < 0$ the bifurcation is supercritical, and it is subcritical for positive $\tilde{a}$. In practical cases, however, it should be more convenient to decide the properties if this bifurcation numerically rather than evaluating (5.6).

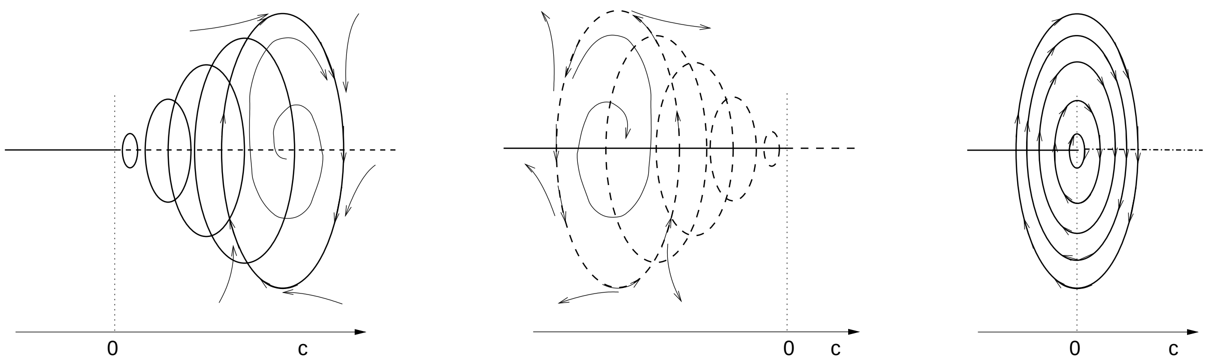

Figure 5.11: Hopf bifurcation: supercritical, subcritical and degenerate.

5.6.3 Degenerate case

In some cases of a supercritical bifurcation the fixed point does not ed up in a limit cycle but in a center. Whereas for the standard Hopf bifurcation the effect is only local (large r are attracted to an initially small limit cycle, i.e. except for the small region within the cycle the origin is attractive), here a transition beyond the critical point changes the behavior in the whole (r, θ)-plane. A typical example of the case would be

$$\begin{aligned} \dot{r} &= cr^3 \\ \dot{\theta} &= \omega \end{aligned}$$

Here in the vicinity of $c = 0$ the derivative of r is essentially zero such that the behavior is of center type, although for slightly larger $c > 0$ the system will show unstable spirals.

As in subsection 5.5.3 also imperfections due to a constant term can occur for systems with a Hopf bifurcation. Try out some computer experiments on such systems and check out the Lorenz equation below.

5.6.4 Hopf bifurcation in a Braitenberg vehicle

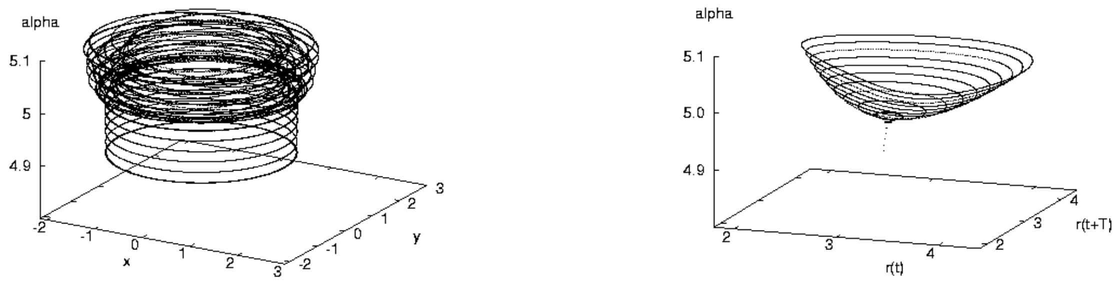

Figure 5.12: Hopf bifurcation in a Braitenberg vehicle. On the left the trajectory of the vehicle around a light source at (0,0) is shown. For sufficiently high light intensities the circular motion changes into a quasiperiodic one. On the left the return map of the distance from the light source is shown. If the motion is circular the distance remains constant for fixed α, and increases slightly with increasing α until a critical value α_{crit} is reached at which the behavior becomes quasiperiodic. Then the two distances at time t and at $t + T$ (displayed at the x and y axes) move around a circular contour: the distance to the center is no long constant but is oscillating at a period different from the period of the robot surrounding the light source. The Hopf bifurcation is here with respect to the radius. With respect to the trajectory a limit cycle is changed into a torus (cf. also the upper three image in figure 4.8).

Chapter 6

Chaos

Effects of chaotic dynamics seemed strange to the scientists of the previous century. It took two decades before substantial experimental evidence was accepted by the scientific community (and sufficiently understood by the experimenters, perhaps) and one more decade before they made their way to the public. Theoretical investigation have been performed without much attention of the scientific community already from the times of Poincaré, i.e. since the end of the nineteenth century. Presently there is some common knowledge that systems that involve nonlinear interactions have parameter ranges where they behave irregular and very complex, show a sensitive dependency on initial conditions, but is still predictable on short time scales, i.e. is deterministic.

The sensitivity to initial conditions is popularly expressed as the butterfly effect by a quotation, which is attributed to Lorenz himself (although it was formulated originally slightly different): A butterfly in China flutters its wings, which triggers a huge, complex series of events that results in a tornado in Texas (or a snowstorm in New York, whatever is worse).

The prototypical chaotic dynamics is present in the baker transformation: Just as the dough is handled by a baker typical chaotic systems combine stretching and folding: nearby trajectories are teared apart and brought close to other trajectories. The expansion part increases also any imprecision in the knowledge about the present state such that the prediction is eventually completely unrelated to the actual course of the trajectory. If we would have started near an unstable fixed point of a non-chaotic system the expansion would be similar. The folding of the phase space takes care that trajectories do not simply go away from the fixed point but are returning to nearby points and the stretching starts anew.

In order to be chaotic, the irregularity of the behavior persist for arbitrary long times. The interactions in the system will, however, suppress some degrees of freedom, such that the system eventually will be describable be a certain number of variables. These variables which are functions of the original variables form the embedding space of the system (Takens theorem). Interestingly, the trajectories of the systems (usually) do not fill out any dense fraction of the embedding space, even if no embedding with a smaller number of variables is possible. The “region” which is filled by the attractor is usually of a very complex shape with rough non-differentiable boundaries. This is expressed by the mathematical term of a *fractal*: A region in a space which fills a substantial part, but has holes and cavities everywhere, such that its volume is actually zero. The extend to which the fractal fills the space is expressed by the fractal dimension, a number which is (unlike a dimension in elementary mathematics) a non-integer number. The fractal dimension can be used to characterize the dynamical behavior of a system.

A chaotic attractor can be considered of being made up by an infinite number of unstable limit cycles or alternatively as being formed by an infinitely long limit cycle which returns to each of its point arbitrarily closely but never exactly. This means chaoticity is instability everywhere in the (*strange* or chaotic) *attractor*. Inside the attractor the distance between two initial conditions is further and further increased and mixed with trajectories from other parts in the attractor. Still this is a continuous process and is thus predictable, but only for a few time units: Any imprecision in the knowledge about the present state increases until the prediction is completely unrelated to the actual course of the trajectory.

Predictability is essential to distinguish chaotic behavior from merely noisy ones. Note, however, that noise must be produced by some noise source, which usually can be assumed to be chaotic as well. Determining fractal dimensions is prohibitively difficult if the minimal embedding space is high dimensional. Therefore we term low-dimensional irregular, sensitive systems chaotic if they are low-dimensional and deterministic, and noisy otherwise.

6.1 Sample systems

6.1.1 Rössler system

The Rössler system [6] has been introduced to describe phenomena related to chemical waves. It seem to be the most simple continuous system that shows chaotic behavior: Remember that the Poincaré-Bendixon theorem implies that three equations are needed to produce chaos. Here we have two linear equations and a single multiplicative non-linearity in the third equation.

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + (x - c)z\end{aligned}$$

Typical parameter values are $a = 0.2$, $b = 0.2$ and $c = 5.7$, a different set with similar results is $a = 0.1$, $b = 0.1$ and $c = 14.0$. The bifurcation scenario is best view

Consider for a moments the system with the condition $z = 0$:

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x + ay\end{aligned},$$

which is a linear system, since the nonlinearity affects only the $z-$coordinate. It is governed by the dynamic matrix

$$\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$$

which has the eigenvalues $\lambda_{1/2} = a \pm \frac{1}{2}\sqrt{a^2 - 4}$, i.e. at $0 < a < 2$ the eigenvalues are complex, but with a positive real part, i.e. the system forms an unstable spiral. Returning now to the full system we realize that (for small b) z is stable only if x is slightly smaller than c. When the spiral of the x-y subsystem increases the value of x sufficiently strongly, the dynamics of z becomes unstable. Increasing z values, however, tend to reduce the value of x until it becomes negative and reduces y to negative values and drives z back near to zero. Then the cycle starts again. In dependence of the parameters it may take several runs through the spiral before x exceeds c or excursions of z of various heights. The more outwards the systems is in the x-y spiral the larger becomes z and the more is the radius of the x-y spiral reduced. Also for small radii of the spiral a not much larger radius is obtained after one cycle. Only at intermediate radii, when z is not strongly turned on, large radii result, cf. figure Thus, with respect to the radii in the x-y plane a stretching and folding is present at suitable parameter values, rendering the dynamics chaotic.

6.1.2 The Lorenz system

The Lorenz system[5]

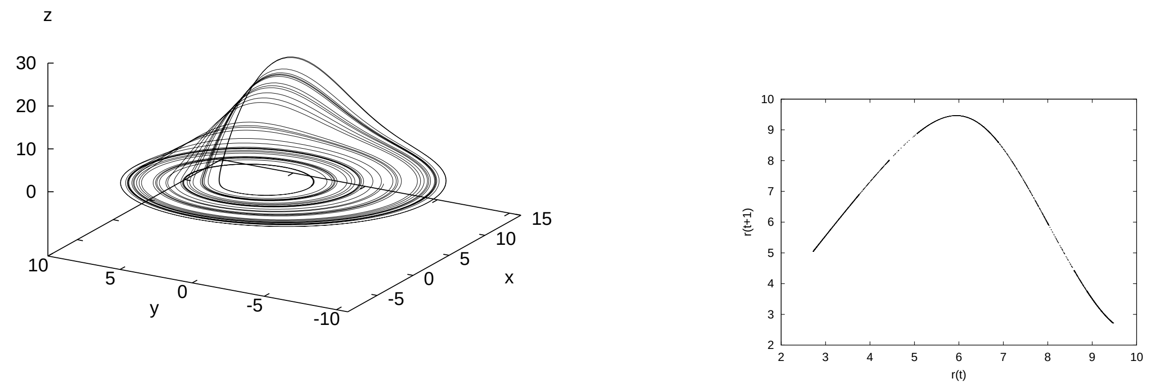

Figure 6.1: Sample trajectory of the Rössler attractor and return map of $\sqrt{x^2+y^2}$ at the diagonal section $x = y$, $x, y < 0$.

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz
\end{aligned} \tag{6.1}$$

looks similar as the Rössler system, spiraling outwards from an unstable fixed point until a non-linearity send the trajectory back. Here we have, however, two unstable points and the trajectory either remains in its spiraling behavior around one fixed or if it to much outward is is thrown near the other fixed point, the more outward it be came, when the nonlinearity become essential, the close the trajectory is then at the other fixed point. Also the return map looks (except for the stretching and folding behavior) different to the one of Rössler's system.

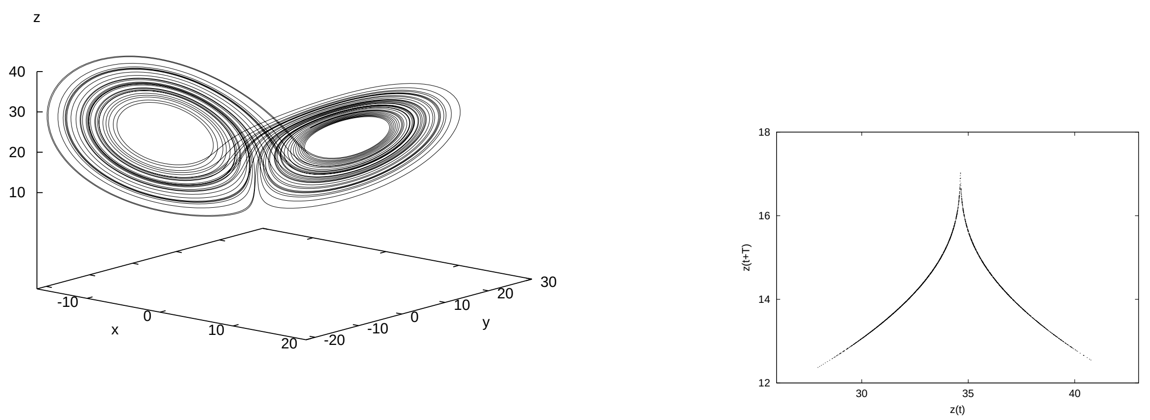

Figure 6.2: Sample trajectory of the Lorenz attractor and return map of the peaks of the z-coordinate $z_{\max}$.

Typical parameter values (the ones which have been used by Lorenz) are $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. The parameters have a physical meaning, since the system describes the Rayleigh-Benard convection (originally Lorenz studied a twelve dimensional system from meteorology, but he found the interesting dynamics to be due to effects produced by equations (6.1). σ is the Prandtl number, r is the Rayleigh number, and b is related to the thickness of the liquid layer which is heated from below and where the convection happens.

Increasing r leads from a stable fixed point at the origin ($r \leq 1$) by a pitchfork bifurcation to two stable fixed points ($1 < r < 24.74$), later these two fixed points become unstable and the system run through a series of chaotic orbits and cycles of various periods. But already while the fixed points are stable there exist complex behavior, called transient chaos, i.e. trajectories with positive (local) Lyapunov exponents which after long transients run into the fixed points ($13.926 < r < 24.06$). For $r > 24.06$ there exist a chaotic attractor in an area outside the (still stable) fixed points, which is globally attracting for $r > 24.74$.

6.1.3 Chaotic Braitenberg vehicle

The are other systems like Chua's circuit, Duffing equation, the Toda Oscillator, the Van der Pol Oscillator, and the pendulum. In the context of this work we are interested whether robot dynamics can show chaos. Figure 6.3 shows the behavior a Braitenberg vehicle surround an oscillating light source. Displayed is the return map of the distance from the light source, compare figure 5.12.

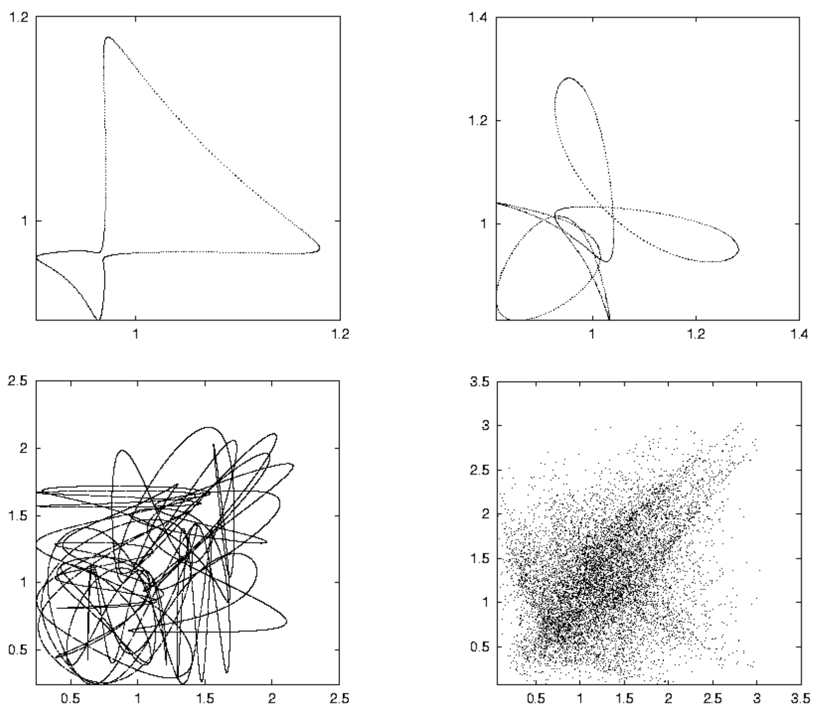

Figure 6.3: Braitenberg vehicle near an oscillating light source. Plotted is the return map ($\Delta t = 20$) of the distance from the light source at intervals of the light source change. Plots are for various values of the light amplitude ($\alpha = 0.4$, 0.5, 2.1, and 2.3. Below $\alpha \approx 0.37$ the plot would appear as a circle (very slow convergence) indicating a nearly incommensurable but regular torus for the behavior of the robot. For larger values of α the simple torus breaks down and at a smaller distance from the light source an increasingly complex toroid behavior sets in. Note that at $\alpha = 2.1$ a perfect though very complex torus is still maintained. From $\alpha \approx 2.2$ the numerics suggests chaotic behavior. At even higher values of α other types of complex tori reappear.

6.2 Attractors

We have learned about fixed points, limit cycles, and tori, and now we will see a more strange type of limit set, strange attractors. Attractors generally represent the features of the long term behavior of a dynamical system, which can be trivial as in a fixed point, periodic as in limit cycles or rational tori, or quasiperiodic as in tori. It can as well be irregular in some sense, as in strange attractors. For the latter case we need to make the notion of an attractor more precise, namely, a closed set A of states of a dynamical system is called an attractor if it shows the following properties.

1. A is an invariant set. A trajectory that starts in A will never leave A.
2. A attracts an open set of initial conditions. The set of initial conditions outside A that are attracted towards A are called the basin of attraction of A.
3. A is minimal. A does not contain any other set the satisfies the previous two conditions.

For a stable fixed point of a linear system, e.g. the whole $\mathbf{R}^n$ forms the basin of attraction. The attractor is due to the third property only the fixed point itself. A chaotic attractor is an attractor which is everywhere unstable in the sense that trajectories leave any point and in different directions as any other trajectory leaving this point. Small deviations have a major effect in chaotic systems. We will make this more precise in the following section 6.3.

6.3 Lyapunov exponents

In the examples considered above, the period a certain cyclic behavior became larger and larger when tuning a appropriate control parameter. We have reasons to assume that cycle length eventually becomes infinite, then it would be justified to speak of chaotic behavior. But can we really be sure? The usual definition of chaoticity of a system is thus not based on the cycle length, but rather on a measure of the sensitivity of the state of the system to the initial conditions. If two trajectories on average tend to depart quickly from each other, we will speak of a chaotic system. This is formalized by the notion of the Lyapunov exponent.

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{dl_i(t)}{dr}\right) \tag{6.2}$$

The situation behind this definition is the following: We take an initial state x_0 and consider a small ball of radius r around x_0. Then the system is allowed to evolve for some time t. By this time all the initial conditions inside the ball (if the ball was sufficiently small) form an ellipsoid around $x(t)$. This is because r was small, so only linear effects are of importance. In some directions of the phase space the ball may have been compressed, in others is was stretched. The principal axes l_i of the ellipsoid are measured and may be used to characterize the dynamics. By (6.2) we obtain n numbers if the dynamics has n degrees of freedom. Only λ_n, the largest of these numbers, is of interest.

Because of the log, the sign of (6.2) depends on whether the derivative is smaller or larger than unity. If for all i the diameter l_i is smaller than r then the initial ball contracts and the dynamics is stable. If x_0 is on or near a stable limit cycle then the largest diameter l_n remains of the same size as r. l_n will be in the direction of the limit cycle, because in any direction orthogonal to the limit cycle, the dynamics is stable, and the corresponding derivatives are smaller than unity. In the direction of the limit cycle the derivative is about one and, hence, λ_n is about zero. For unstable systems at least l_n increases beyond r such that $\log\left(\frac{dl_n}{dr}\right)$ is positive. Now, if l_n increases exponentially with time, i.e. $l_n \sim \exp -\lambda_n(t)t$, then we have $\lambda_n = \lim_{t\to\infty} \lambda_n(t)$ if the limit exists.

So far we have considered only a single initial position x_0. If e.g. x_0 is an unstable fixed point, a locally defined Lyapunov exponent would be larger than zero. For chaotic behavior we further require that for the average over initial conditions λ_n is still larger than zero, i.e. that the system is on average more unstable than it is stable. It may occur that the dynamics is in some regions of the phase space contractive, even if is chaotic, we require only that the dynamics is dominated by the unstable regions.

6.3.1 Prediction horizon

The impossibility of predicting the fate of a specific trajectory is to be taken quite seriously. Using the largest Lyapunov exponent λ the dynamics of a perturbation $\eta(t)$ can be approximated by

$$\eta(t) \sim \eta(0) \exp \lambda t$$

Allowing for a tolerance of a, i.e. requiring that $\eta(t) < a$, allows to estimate

$$t_{\text{horizon}} \sim \frac{1}{\lambda} \log \frac{a}{\eta(0)}$$

Naturally $a > \eta(0)$ such that by decreasing $\eta(0)$ the length of the time interval where a prediction better than a is possible can be extended. Concerning the effectiveness of measuring the initial conditions more precisely, consider the following example.

Suppose a prediction with a tolerance of 10^l is to be made. and the initial condition are known to k decimal places ($a > \eta(0)$ implies that $k > l$). Then

$$t_{\text{horizon}} = \frac{1}{\lambda} \log \frac{10^{-l}}{10^{-k}} = (k - l)\frac{\log 10}{\lambda}.$$

E.g. let $a = \frac{1}{1000}$when improving $\eta = 0$ from 10^7 to 10^{13} i.e. by a factor of 1.000.000, then $k - l$ becomes 10 instead of 4. This means that the usually completely unrealistically precise measurement increase the prediction range in time merely about a factor of $\frac{5}{2}$. This makes clear that prediction horizon and the impossibility of long term predictions is not merely a practical issue, but will soon touch upon fundamental physical limits. On the other hand, e.g. in order to extend the weather forecast from three to four days, it might be still worth the effort.

6.3.2 Numerical calculation of the largest Lyapunov exponent

J.C. Sprott describes in some detail numerical methods (and effects of numerical errors) in nonlinear systems. We summarize here his notes on the numerical calculation of the largest Lyapunov exponent (cf. sprott.physics.wisc.edu/chaos/lyapexp.htm) The algorithm consists of six steps:

1. start with any initial condition in the basin of attraction
2. iterate until the orbit is in the attractor: usually this takes a few hundred time steps, but one should be aware, that the dynamics can be very slow (cf. figure 4.8) or that transient chaos might be present as in the Lorenz map. Ideally one would start with a point inside the attractor, but on the other hand even trajectories slightly off the attractor give good estimates of the Lyapunov exponent.
3. Select a nearby point separated by d_0
4. Iterate both orbits one iteration and calculate d_1

5. Evaluate $\log \left| \frac{d_1}{d_0} \right|$. Note that the Lyapunov exponent depends on the base of the logarithm. Although the natural choice is the natural logarithm it is often interesting to use dual logarithm because then the information loss about the initial conditions can be expressed in bits.
6. The crucial point is (in order to avoid to calculate "all" initial conditions starting in a ball of size d_0 and than to analyze the ellipsoid the ball is mapped to) to choose good starting points, i.e. choose the initial perturbation such that the trajectory is mapped to the maximal extend of the ellipsoid. This can be done iteratively (i.e. without explicitely solving the system) by readjusting the initial perturbation $\tilde{x}_0$:
$$\tilde{x}_0 = x_1 + \frac{d_0}{d_1} (\tilde{x}_1 - x_1)$$
7. The steps 4 to 6 are to be repeated a few thousands of times while averaging step 5 in order to estimate about two significant digits of the Lyapunov exponent. It is advisable to plot the estimated Lyapunov exponent as a function of the length of the trajectory and to start several times at other initial conditions in order to get sufficiently reliable results.

6.4 Poincaré section and Poincaré map

(from: monet.physik.unibas.ch/~elmer/pendulum/bterm.htm)

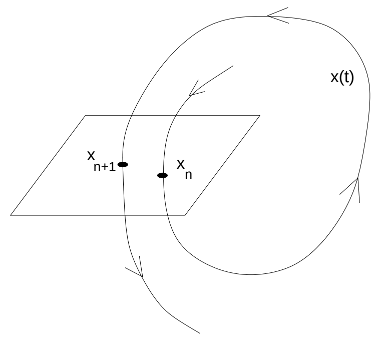

Figure 6.4: Poincaré section.

A carefully chosen (curved) plane in the phase space that is crossed by almost all orbits. It is a tool developed by Henri Poincaré (1854-1912) for a visualization of the flow in a phase space of more than two dimensions. The Poincaré section has one dimension less than the phase space. The Poincaré map maps the points of the Poincaré section onto itself (not individually, but some to others). It relates two consecutive intersection points. Note, that only those intersection points count which come from the same side of the plane. A Poincaré map turns a continuous dynamical system into a discrete one. If the Poincaré section is carefully chosen no information is lost concerning the qualitative behavior of the dynamics. Poincaré maps are invertible maps because one gets x_n from x_{n+1} by following the orbit backwards.

In periodically driven systems it is often convenient to use a stroboscopic map of the system by choosing a phase ϕ_s of the driving and setting $x_n = x(t_n)$ if $\phi(t_n) = \phi_s$. Here, snapshots of the system are taken at equidistant time intervals, where as in the Poincaré section method above a spatial criterion was used instead and time intervals are not necessarily equidistant.

Poincaré maps of the stationary behaviors considered so far are as follows:

- A limit cycle becomes a fixed point of the map. The Poincaré section should be positioned orthogonal to the trajectory and stretch from the unstable fixed point in the center of the cycle across the peripheral trajectory.

- Tori become sets of periodic points. If the ratio of the frequencies is $\frac{p}{q}$ than the period is q. If the frequencies are incommensurable any point in the section of the torus will occur such that a (possibly deformed) circle is mapped onto itself. Tori are in some sense mapped to limit cycles.
- Stability properties for directions that lie in the section carry over from the original system to the mapped system (orbital stability cannot be studied within Poincaré sections).

6.5 Maps

The Poincaré map introduced in section 6.4 allows to relate time-continuous to time-discrete dynamical systems (provided that the continuous involves some periodicity). This allows to restrict ourselves (to begin with) to the study of discrete systems. Whereas in continuous systems the Poincare-Bendixon theorem excludes chaotic behavior from dimensions less or equal to two, chaos can be observed already in one dimensional maps.

For example the Roessler attractor which is produced by a three dimensional continuous system can be mapped to the sequence of points in a suitably chosen Poincaré section, cf. figure 6.5. We may consider at first discrete maps the most prominent of which is the logistic map

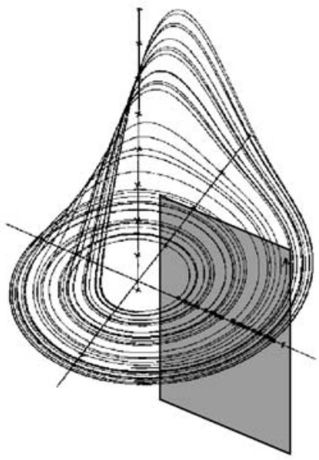

Figure 6.5: Roessler attractor. The intersections of the trajectories with the Poincaré section are essentially contained in a one dimensional subset of the section. Therefore, the Poincaré map can be expressed as a one dimensional map.

6.6 Definition of a chaotic map

An equivalent definition is obtained in several steps starting from the notion of a *forward set.*

Let f be a map and let x_0 be an initial condition. The forward limit set of the orbit $\{f^n(x_0)\}$ (the set of all points x_0 is mapped to by the iterated map) is the set: $\omega(x_0) = \{\, x \mid \forall N \forall \varepsilon > 0 \exists n > N : |f^n(x_0 - x)| < \varepsilon \}$.

If $\omega(x_0) \supseteq \omega(x_1)$ we say that x_1 is attracted to $\omega(x_0)$.

A bounded orbit $\{f^n(x_0)\}$ is called chaotic if it is not (asymptotically) periodic and if the largest Lyapunov exponent is greater than zero.

$\omega(x_0)$ is called a *chaotic set*.

An *attractor* is a chaotic set which attracts a set initial values of non-zero measure.

If a chaotic set is also an attractor it is called a *chaotic attractor*.

6.6.1 Two simple sample maps

The tent map

$$x_{n+1} = \begin{cases} 2x_n & \text{if } x < \frac{1}{2} \\ 1 - 2x_n & \text{otherwise} \end{cases}$$

and the "$2x \mod 1$" map

$$x_{n+1} = 2x_n \mod 1 = \begin{cases} 2x_n & \text{if } x < \frac{1}{2} \\ 2x_n - 1 & \text{otherwise} \end{cases}$$

map the unit interval onto itself, cf. figure 6.6. Both have similar properties. The tent map is rather similar to the Poincaré map of the Lorenz system and in some sense also to the logistic map (cf. section 6.7). The "$2x \mod 1$" map allows directly for an interpretation in terms of binary numbers.

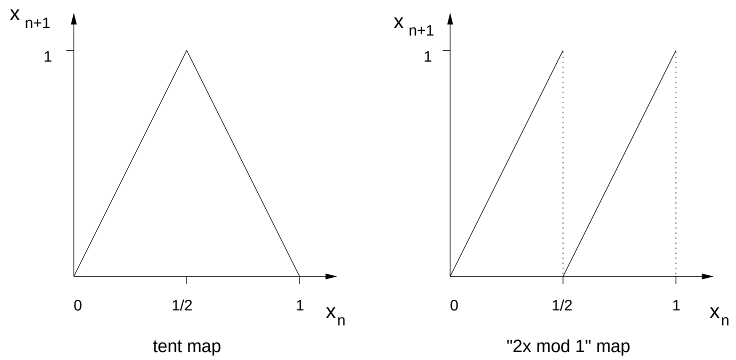

Figure 6.6: Two simple maps that possess chaotic orbits.

6.6.2 Lyapunov exponent of the $2x \bmod 1$ map

The points of the unit interval that are eventually mapped to $\frac{1}{2}$ by the $2x \bmod 1$ map form a countable set, since they are mapped to $\frac{1}{2}$ by finitely many interactions. Excluding these points excludes only a set of measure zero out of the unit interval. For the points that never reach $\frac{1}{2}$ we can calculated the derivative of the map as well as the derivative of any iterate of the map, which is needed in order to determine the Lyapunov exponent:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(x_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2 = \ln 2$$

The Lyapunov exponent is defined as an average over initial conditions, but since the excluded points are of measure zero they do not contribute to the average (the generalized derivative at $\frac{1}{2}$ bounded between ± 2). We have thus found a positive Lyapunov exponent, indicating that there exist a chaotic attractor.

This does not imply that there is a single chaotic orbit, actually there is an infinite number of chaotic orbits. The Lyapunov exponent of the tent map amounts to the same value. Interestingly, also the logistic map (at $\lambda = 4$) has the same Lyapunov exponent. In order to find out more about the structure of the attractors we consider the so-called symbolic dynamics of the maps.

6.6.3 Symbolic dynamics in the $2x \mod 1$ map

The dynamics of the map $x_{n+1} = f(x_n) = 2x_n \mod 1$ has a very suggestive interpretation in terms of binary numbers. We define the intervals $L = \left[0, \frac{1}{2}\right)$ and $R = \left[\frac{1}{2}, 1\right)$. Starting from the initial value x_0 we keep track of whether x_n is in the interval L or in R. E.g. $x_0 = \frac{1}{5}$ has the binary representation $x_0 = 0.0011\overline{0011}$. The digit just right of the decimal point determines whether $x \in L$ (in case of a "0") or $x \in R$ (in case of a "1"). Applying f to this binary expression results in shifting all the digits one place to the left. If in this way a "1" passes the decimal point it is cut away by the $\mod$ function. Let us consider the iterates of $x_0 = \frac{1}{5}$:

n	x_n	binary representation	left/right
0	$\frac{1}{5}$	$0.0011\overline{0011}$	L
1	$\frac{2}{5}$	$0.011\overline{0011}$	L
2	$\frac{4}{5}$	$0.11\overline{0011}$	R
3	$\frac{3}{5}$	$0.1\overline{0011}$	R
4	$\frac{1}{5}$	$0.0011\overline{0011}$	L
5	$\frac{2}{5}$	$0.011\overline{0011}$	L

The special example suggest, that a starting value x_0 the binary of which eventually become periodic, the sequence of L's and R's becomes periodic as well after some iterations of the map f, i.e. the orbit $f^n(x_n)$ created by x_0 is periodic if x is rational.

In the special case that the period of x_0 consists of only 0's the map necessarily ends up in the point $0.\overline{0}$ (zero) when the a-periodic part of the sequence of binary digits is shifted beyond the decimal point and deleted by the $\mod$ operation. Before $0.\overline{0}$ is reached, say this happen at step k the orbit passes $x_{k-1} = \frac{1}{2} = 0.1$ (binary) unless it is not already at $0.\overline{0}$, in other words, the orbits which are excluded in the above calculation of the Lyapunov exponent are the ones which run into the fixed point at $0.\overline{0}$ after passing $\frac{1}{2}$. The periodic binary sequences, i.e. the rational numbers which do not have a denominator of 2^k, will stay away from the fixed point and follow periodic orbits forever. The derivative of f is, however, as we have seen, equal to 2, i.e. the orbits are unstable: If two initial conditions are different only after the k binary place it takes k iterations of the map until the orbits end up in different intervals L and R, i.e. until the two orbits are separated.

The number rational points in the unit interval is "only" countably infinite, i.e. it is small in comparison to the number of irrational points. Irrational numbers produce an infinite, a-periodic sequence of L's and R's, which corresponds uniquely to the number (to be mathematically exact we have to exclude sequences of only 1's (or R's) after a certain binary digital place). Any aperiodic sequence is a chaotic orbit. And since there are uncountably many aperiodic orbits there are also uncountably many chaotic behaviors possible. Each of the orbits contains all the left-shifted variants of a prototypical sequence, and any right-shifted variants (with arbitrary insertions of new digits) are mapped eventually to the orbit.

6.7 Logistic map

6.7.1 Phenomenology

The sequence of points defined by the logistic map

$$x_{n+1} = F(x_n) = \lambda x_n (1 - x_n)$$

is approaching an attractor which is independent of the initial value x_0. For different values of $\lambda \in [0,4]$ the attractor is different. For small values of λ the limit set is a single fixed point, the position of which is however dependent on λ. For large λ the fixed point changes into a period-two cycle by a pitchfork bifurcation. One of the two alternating points can be considered as a fixed point of the two-step map $x_{n+2} = F(F(x_n))$ for even n, the other one for odd n. Further increasing of λ lead to more pitchfork bifurcations and each time to cycles of double lengths. Eventually, the distance between bifurcations goes to zero the cycle length tend to infinity and a chaotic range is reached. The chaotic regime is interspersed with periodic windows where now cycle lengths occur which are not powers of two, cf. figure 6.7.

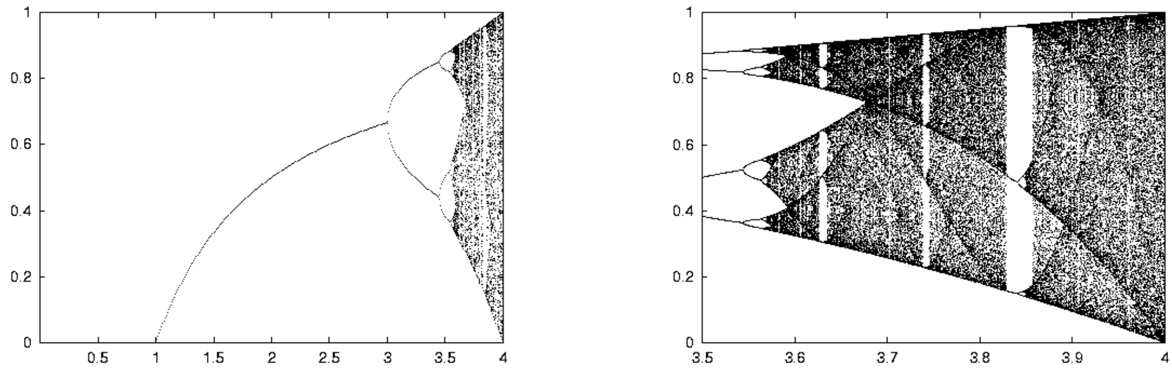

Figure 6.7: Bifurcation scenario of the logistic map. (horizontal axis: bifurcation parameter λ, vertical axis: state x). (left) full range of parameter λ. (right) region of large λ magnified form the left figure.

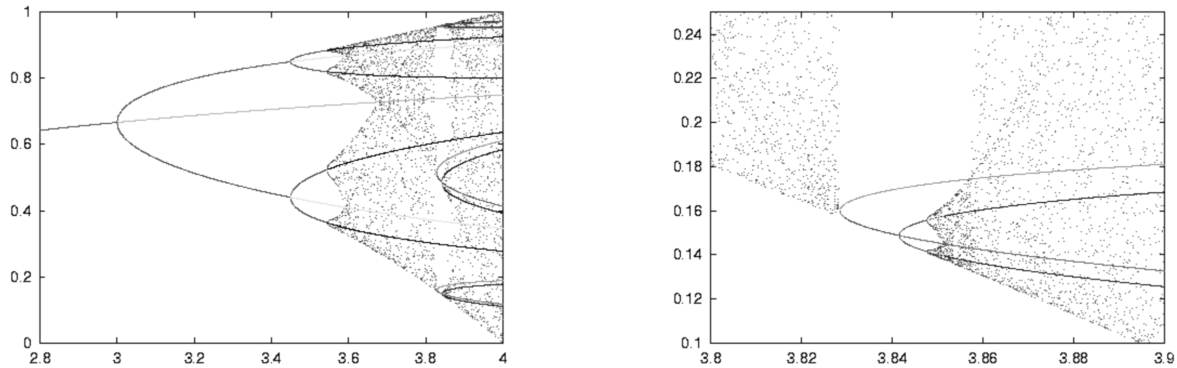

Figure 6.8: Bifurcation scenario of the logistic map (cf. figure 6.7). Here also the unstable fixed point and unstable cycles with small periods are plotted (left). In the chaotic regions the unstable cycles cover the attractor densely. Note that the period-three cycle is created by a saddle-node bifurcation when the chaotic region is left, rather than by a pitchfork bifurcation. Later, the stable branch of the 3-cycle undergoes a pitchfork bifurcation into a 6-cycle and an unstable 3-cycle, as visible in the magnified image on the right.

6.7.2 Controlling the logistic map

At a pitchfork bifurcation (cf. figure 6.8) the single stable fixed point does not simply cease to exist, but is continues as an unstable fixed point. That is, when the control parameter is increased quickly, the state will remain for a short time at the now unstable fixed point, because the previously stable dynamics has moved the state sufficiently exactly to that position. Before the state is eventually driven away from the unstable fixed point, an control algorithm may change the parameter such that the system remains stable. Actually, the combined system of original dynamics and the control algorithm is supposed to have a stable fixed point where the original dynamics has an unstable one. If the control algorithm has failed to stabilize the state it will get another chance, since because of chaoticity the state will return close to the fixed point again (Here we have assumed, that the system has the additional property of being *mixing*. A merely chaotic system can consist of several components such that the return property holds only in each component separately.)

How such a control algorithm shall look like?

Chaotic attractors cover a multitude of unstable limit cycles and tori. In order to exploit this richness of behavior one should have a means to select and stabilize behaviors, and to switch to different behaviors. The advantage is that control of unstable behaviors requires only tiny control actions, although these have to be chosen 'intelligently'. The complexity of the behavior is thus to be stored in the algorithm.

6.8 Screen creatures

Let us now consider the case that the state x defines the pixel position of a "creature" moving on the n-dimensional screen ($n = 2$). There is a function $\pi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ which maps an arbitrary position to a pixel position. We write

$$X = \pi(z)$$

where $z \in \mathbf{R}^2$ and $X = (i, j)$ is a pixel position. The mapping of an arbitrary position z to a pixel position ("pixelization") can be done according to one of the following scenarios

1. In each dimension: Map $x_{new,i}$ to the nearest neighbor pixel coordinate.
2. In each dimension: Map $x_{new,i}$ to a pixel coordinate which is chosen randomly (with equal probability) from a given neighborhood.
3. Quite generally we use periodic boundary conditions, i.e. actually we use

   $$X = \pi(\hat{z})$$

   where

   $$\hat{z}_l = z_l \bmod N_l$$

   and N_l is the number of pixels in the $l = i, j$ direction of the screen.

The controller $K(x)$ of the creature defines the new target position in terms of the current one. With a linear controller we may write the time discrete dynamical system governing the motion of the screen creature as

$$\begin{aligned}\Delta x_1 &= c_{11}x_1 + c_{12}x_2 + \xi_1 \\ \Delta x_2 &= c_{21}x_1 + c_{22}x_2 + \xi_2\end{aligned}$$

or

$$\Delta x = cx + \xi$$

where $\Delta x = x_{t+1} - x_t$, $x, \xi \in \mathbf{R}^2$ and the noise ξ results from the pixelization

$$\xi = \pi(x + \Delta x) - (x + \Delta x)$$

The dynamics maps between pixel position if only the starting state is a pixel position.

We note that the noise can be of different kind depending on whether we use the random pixelization scheme or the nearest neighbor one. In the latter case the noise depends in a sensitive way on the state of the system itself. The screen creatures as introduced will display an very interesting behavior only for larger values of c_{ij} due to the periodic boundary conditions. However this changes drastically if we use a nonlinear controller and even more so if we introduce ensembles of interacting creatures which may model ecologies.

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MORE STUFF TO COME.

Appendix A

Main ideas of this part

Here a collection of the main fact, ideas, results etc. is (to be) presented. This list is not intended to be understandable by itself, refer to the main text in case of any problems.

- situated AI vs. GOFAI
- agent architectures (general architecture, SPA-architecture, subsumption architecture, homeokinetic systems)
- linear systems
- linearization of non-linear systems
- bifurcation theory (pitchfork bifurcation, Hopf bifurcation, transcritical bifurcation; subcritical and supercritical versions of these bifurcation types)
- chaos in nonlinear continuous systems and in simple maps