Homeokinesis and the moderation of complexity in neural systems
– DRAFT VERSION –

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Contents

Abstract

We consider brain like systems in the sensor-motor loop which may be considered as hierarchies of feed-back loops. With the synaptic dynamics included these systems are of an overwhelming complexity. Our question is how systems may self regulate into operational conditions of moderate complexity as they seem to be realized in biological systems. A simple stability argument like the homestatic principle is not sufficient since in the sensory-motor loop the agent changes its internal state due to its own actions. We propose to impose the requirement for homeostasis only after transforming away the state changes induced by the agent itself. We call this paradigm homeokinesis since quite generally the principle supports the spontaneous creation of behavioral modes instead of stationarity. The new principle is applied to drive the synaptic dynamics of a neuron in a simple sensory-motor loop. Formally under the condition of slow learning the synaptic dynamics corresponds to the maximization of the local Lyapunov exponent of the system. Effects found so far for the combinded dynamics are the self-regulation to the stochastic bifurcation point, self-induced search oscillations via a Hopf bifurcation, the adaptation to the regime of highest sensitivity and the like. These mathematical findings suggest that learning under the principle of homeokinesis drives the system towards working conditions characterized by the optimum payoff between creativity and stability.

1 Introduction

In the time since the early neuron model of McCulloch and Pitts we have more and more understood that neural systems are of an overwhelming complexity. From a system theoretic point of view neurons occurr embedded into recurrent NNs which on their hand are integrated into sensor motor loops integrating the reality. We hence may view neural systems as hierarchies of intertwined feed-back loops with the SM loop as the top level of the hierarchy. The fast synaptic plasiticity discovered recently adds a new stage in this scenario. The reason for this is in the observation that the neuron may change its synaptic efficacy on the fly depending on the spike timing engendered by the neuronal dynamics. In this sense we may say that the inclusion of the fast synaptic processes opens a new level at the bottom of the hierarchy of feed-back loops.

The wealth of effects which systems of such complexity may display is potentially infinite. It is by far not clear how nature makes use of such an offer. The concern of the present paper is in the search for general principles which make these systems operational. One root of our approach can be found in the principle of homeostasis as formulated in 1939 [2]. His claim was that the overall goal oriented functionality of the body is achieved in a more or less self-organized way by the "desire" of each of the individual control circuits to stay in a stationary state. So the ideal state of the system is one of overall stationarity which can be upheld only if the system manages to embed itself successfully into its environment.

The main objective of these processes is to upheld the functionality of the system under external perturbations. This stability argument has proven helpful in understanding many of the functions of the body. The idea was extended by Ross Ashby [10] with its attempt to create self-regulating ultrastable systems. In recent times a number of homeostatic regulation mechanisms have been identified which help neurons and circuits to maintain stability in their firing properties in the face of the dramatic synaptic reconfiguration occurring during learning and development of mammals, cf. [11, 12]. Of particular interest is activity dependent synaptic scaling which adjustes the strength of all synapses in response to changes in average postsynaptic activity which has been observed in vivo by [3].

However stability is only one half of the story. Living beings have to be able of adapting their functionality to changing external conditions and actively search for new functionalities. There are hints that such systems operate in a regime which is somewhere between the fully deployed complexity and a more or less ordered state. This is argued to be the edge of chaos by some authors [7, 6, 4, 5] or the border of criticality [1] or instability [8, 9]. On the behavioral level one the intriguing properties of such systems is a maximum capability of the agent to discriminate while still being able to rapidly switch between alternatives.

Our aim is to find general principles for the self-regulation of systems into this regime of moderate complexity. The main result of the present paper is that the desired self-regulation porperties can be achieved from a simple but essential modification of the principle of homeostasis based on the following considerations. We consider brains as controllers of the behavior in the SM loop. From a system theoretic point the brain is described by a state z which is the mediator between sensor values and actions. If an agent is to become active in the external world it produces changes in its sensor values all over the time. The actions are found by processing the sensor values and this can be done only if the internal state z of the agent is in some correspondence to the sensor values. As a consequence if internal equilibrium is the ultimate goal then the agent's most favorite behavior is the "do nothing" one.

The above naive analysis of the HS principle suggests a generalization in the following way. We must transform away the motions in internal state space which are produced by the agent istself and apply the HS principle to this transformed state. Technically this is done by using the interaction representation $\hat{z}$ of the state z which is known from mechanics. We view the changes in the internal state values as motion in state space and introduce a co-moving coordinate system obtained by describing the system in terms of a model dynamics. The dynamics in the co-moving coordinate system is driven by the interaction between the model and the true world alone. The **generalized HS principle is the requirement that the system tries to stay stationary with respect to the state in interaction representation**. As explained in some detail below this paradigm leads to a destabilization of the "do nothing" behavior. As a result the system does not go into a stationary state internally but instead develops elementary behavior modes by way of spontaneous symmetry breaking. We call our principle therefore the homeokinetic one.

We investigate this principle in the present paper for a leaky integrator neuron embedded into a simple sensor-motor loop under the rate coding paradigm. The state variable is the membrane potential z. We include the synaptic dynamics so that we have a dynamics in the product space. The synaptic dynamics is driven by the generalized HS principle which aims at keeping the transformed state $\hat{z}$ stationary. As it turns out this amounts to maximizing the local Lyapunov exponent of the noisy system.

As a result in the most simple case the neuron under this adpatation dynamics is found to self-regulate towards the stochastic bifurcation point which is characterized by latent bistability. In this regime the neuron already realizes state alternatives which are however still sufficiently close to each other so that the noise switches the state between the altenatives. Moreover with a further control parameter under its adaptation dynamics the neuron is driven by the noise into self-sustaining search modes which make the neuron to implore more actively its possibilities of behavior (self-tuning Hopf bifurcation).

The present results are for a single neuron. We consider this as the generic self-regulating circuit which is both creating a dynamics of its own but still remains sensitive to the influences of its environment. Our expectation is that (hierarchical) networks of such circuits may self-organize into a common mode of behavior which is in agreement with the environment.

This brings us back to the complexity issue. We argue that the homeokinetic principle is able of moderating the dynamical complexity of´ systems involving hierarchies of feed-back loops the main mechanism being the self-tuning towards latent decision making. This means that the neuron induces the investigation of the options which are present in its host SM loop favoring the decisions which are more safe in the sense of homeokinesis. The complexity of this behavior is higher than the one of the ordered state where the neuron has chosen one of the pertinent options. On the other hand the system still is not chaotic as it might be with a different synaptic dynamics.

2 The leaky integrator neuron in the sensor-motor loop

Consider a neuron under the rate coding paradigm with membrane potential z and

$$\dot{z} = -\gamma z + H + \sum_{i=1}^{n} c_i x_i \tag{1}$$

where x_i is the input into the i-th synapse with strength c_i, H is the bias (threshold). We consider the case that the x_i are essentially the responses of the environment of the neuron to its output

$$y = \tanh(z) \tag{2}$$

coding the current rate of firing. The world is modelled as

$$x_i = a_i y + I_i + \xi_i \tag{3}$$

We may interprete the a_i as the response strength of channel i to the output y of the neuron and ξ_i as a noise which comprises all effects not contained in the simple linear response expresssion.

We will in the present paper consider the most simple case that there is only one channel so that our model boils down to

$$\dot{z} = -\gamma z + H + cx \tag{4}$$

where

$$x = ag(z) + I + \xi$$

In the calculations we will assume that ξ is a white Gaussian noise hence

$$\overline{\xi(t)\,\xi(t')} = D\delta(t - t')$$

where D is the diffusion constant.

We will consider below a concrete application where y is the target velocity of a robot and x is the true velocity as read back from the wheel counters. We will use the gain function eq. 2 in the present paper in order to cover this case but everything in the derivations below is easily translated to the case of a gain function like

$$g(z) = \frac{1}{1+e^{-z}}$$

Moreover we may rescale both time and the synaptic stregth so that γ and a are both equal to 1. This is what we assume in the following.

3 The principle of homeokinesis

It is clear from the above considerations that the neuronal dynamics depends in an essential way on the values of the parameters c and H. In particular bifurcations and switching between fixed points can be induced by small changes of these parameters. Therefore the inclusion of a concomitant synaptic dynamics may have interesting effects on the dynamics in the product space formed by the state and the parameter space. When using the homeostatic principle we do not get an interesting dynamics since this would mean to stabilize the internal state z and hence the output of the neuron at a given value. Things change very much if we apply the homeostatic principle to the state only after transforming away the effects which are caused by the actions (output) of the neuron itself. This means the transition to the interaction representation which will be introduced in the following.

3.1 Interaction representation

Quite generally assume that we have a system

$$\dot{z} = F(z) \tag{5}$$

where $F: \mathbf{R}^n \to \mathbf{R}^n$ is a function which is generally stochastic and not known explicitely. The interaction representation is the transition to a co-moving coordinate sytem which can be defined by a reference (model) dynamics given by the ODE

$$\dot{q} = M(q) \tag{6}$$

where M defines the model which is assumed to capture the essential features of the full dynamics. Both F and M define a motion in state space. Writing $F = M + \delta M$ we want to find a representation $\hat{z}(t)$ of the state $z(t)$ which is driven by δM alone. In order to do so we define a fixed reference instant of time t_0 and use eq. 6 to transform the state $z(t)$ back to time t_0, i.e. we solve eq. 6 down to t_0 starting at t with $q(t) = z(t)$. The result is the desired state $\hat{z}(t)$.[1] Formally we write

$$\partial_t \hat{z} = \delta\hat{M}(t)$$

where $\delta\hat{M}(t) \to 0$ if $\delta M \to 0$. We may interprete $\delta\hat{M}(t)$ as the interaction between the model and the true dynamics hence the name interaction representation.

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[1] The state depends also on t_0 which we do omit in the notation.

In the case of our sensor-motor loop we use the full deterministic part as the model. We need the dynamics of $\hat{z}$ only over a short interval of time so that we may approximate eq. 4 as

$$\dot{z} = F\left(z_0\right) + \Gamma\left(z_0\right)\left(z - z_0\right) + c\xi\left(t\right) \tag{7}$$

where

$$F\left(z\right) = -z + c\tanh\left(z\right) + cI + H$$

$$\Gamma\left(z\right) = -1 + c\tanh'\left(z\right)$$

The dynamics in the interaction represention reads (see Appendix and Sec. 3.4)

$$\partial_t \hat{z} = c\hat{\xi}\left(t\right) \tag{8}$$

where $\hat{z}$ is the state in interaction representation and

$$\hat{\xi}\left(t\right) = e^{-(t-t_0)\Gamma}\xi\left(t\right)$$

is the interaction of the system with its environment in this representation.

3.2 Formulation of the principle of homeokinesis

The idea of our generalization of the homeostatic idea is to keep $\hat{z}$ and not z itself stationary as good as possible. We write the solution of eq. 8 as

$$\hat{z}\left(t\right) - \hat{z}\left(t_0\right) = cu\left(t\right)$$

where $\hat{z}\left(t_0\right) = z\left(t_0\right)$ and the driving force for the deviation $u\left(t\right)$

$$u\left(t\right) = \int_{t_0}^{t} \hat{\xi}\left(t'\right) dt'$$

is seen to be the noise in interaction representation. We will consider the average of u^2 as a measure of the strength of this driving force. We note that in a strictly mathematical sense the stochastic differential equation for the state of the system is written as

$$dz = f\left(z\right) dt + \sqrt{D} dW\left(t\right)$$

where $f\left(z\right) = \left(-z + cg\left(z\right) + H + I\right)$ and $W\left(t\right)$ is the Wiener process. We will consider therefore $\tau^{-1}\overline{u^2}$, i.e. we divide the trivial dependence on the time interval $\tau = t - t_0$ out so that

$$\begin{aligned} E &= \frac{1}{\tau}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \overline{\hat{\xi}\left(t'\right)\hat{\xi}\left(t''\right)} \\ &= \frac{D}{\tau\Gamma}\left(1 - e^{-2\tau\Gamma}\right) \end{aligned}$$

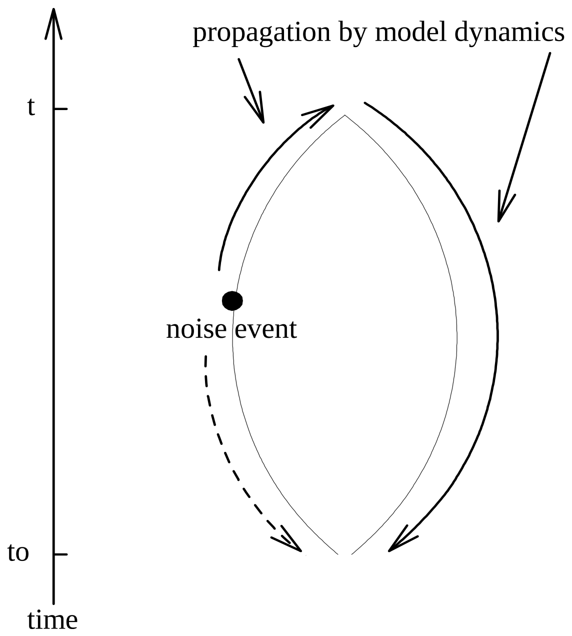


Figure 1: The time loop effect. In the linear approximation the full dynamics (left branch of the time loop) can be viewed as the superpositon of noise events which are propagated from the time t_n of the event to the current time t by the model dynamics. Then the result is propagted back from t to the reference time t_0 by the model dynamics again. Since the model dynamics forward and backward in time cancel the net effect is a propagation backward in time from t_n to t_0 (dashed arrow).

is our measure.

For small τ we obtain

$$E = 2D\left(1 - \tau\Gamma\left(z\right)\right) \tag{9}$$

where z is the current state $z\left(t\right)$ of the system. (Actually z is $z\left(t_0\right)$ but we can replace this by $z\left(t\right)$ in lowest order of τ). Please note that Γ depends on the parameters both directly and indirectly via the dependence of z on these parameters. The latter dependence can be viewed in different ways. We invoke in the present paper the quasi-equilibirium hypothesis. We assume that the time scales are well separated so that the learning is much slower than the state dynamics. Then we may assume that the state dynamics is in a state of quasiequilibrium and we may replace the current $\Gamma\left(z\right)$ with its quasi-equilibrium average $\overline{\Gamma\left(z\right)}$.

3.2.1 The quasi-equilibrium probability distribution

In order to find the probability distribution for z we note that the state dynamics can be written as

$$\dot{z} = -\frac{\partial}{\partial z} V(z) + c\xi(t)$$

where the potential

$$V(z) = \frac{z^2}{2} - (cI + H)\, z - c \ln\cosh z$$

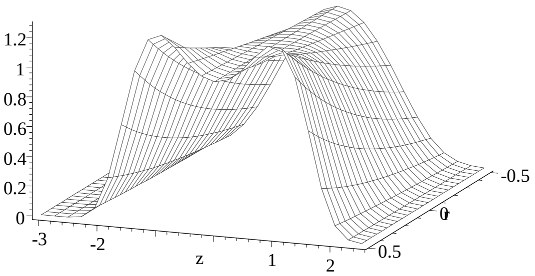
since

$$\frac{\partial}{\partial z} \ln\cosh z = \tanh z$$

Hence the equilibrium probability that the state z is observed is given by the Gibbs distribution

$$P(z) = \frac{1}{N} \exp\left(-V(z) / c^2 D\right) \tag{10}$$

where $D = 2\overline{\xi^2}$ (???) is the diffusion constant and the factor c^2 is due to the fact that z is driven by $c\xi(t)$.



Relative probability distribution for the state for values of $r = c - 1$ in the region -0.5 *to* 0.5 . The diffusion constant is $D = 0.5$

The aim now is to find a self-regulation mechanism which makes the neuron to comply to the homeokinetic principle, i.e. to behave such that E is minimized meaning that the state $\hat{z}$ is as stationary as possible.

3.3 The learning rule

The error eq. 9 is a measure of the degree to which the state $\hat{z}$ is in equilibrium. The aim of the adaptation of the neuron in the sensor-motor loop is to keep $\hat{z}$ in equilibrium as good as possible which can be achieved by gradient descending the error. Together with the state dynamics the full equations including learning are

$$\dot{z} = -z + cg(z) + cI + H + c\xi(t)$$

$$\dot{c} = \varepsilon \frac{\partial}{\partial c} \overline{\Gamma(z)} \tag{11}$$

$$\dot{H} = \varepsilon \frac{\partial}{\partial H} \overline{\Gamma(z)}$$

where factors D and τ were absorbed into ε.

The learning rate ε in the learning rule eq. 11 is to be chosen such that quasi-equilibrium condition is always fulfilled which means that ε must be sufficiently small. If the noise is not too large and $c \neq 1$ we may assume that the state z is always close to the FP and $\overline{\Gamma(z)}$ may be considered as the average local LE of the system. The learning then aims at driving the system to the values of c, H which correspond to the largest negative LE.

3.4 The time loop effect

We are now going to give an additional interpretation of the measure E in order to better understand the effect of the homeokinetic paradigm. Looking from time t_0 the state $\hat{z}(t)$ may be viewed as the result of propagating $z(t_0)$ forward in time until t by means of the exact dynamics and then back to time t_0 by means of the model dynamics. Thus we may say that the state $\hat{z}$ is the result of running through a time loop.

This is illustrated by considering the fate of a noise event at time t' in this time loop. In the full dynamics the noise is propagated from t' to t as $S(t-t')\,\xi(t')$ where

$$S(\kappa) = e^{\kappa\Gamma}$$

is the propagator of the state dynamics over a time κ. The transformation from time t to t_0 is done with the same propagator so that

$$\hat{\xi}(t') = S^{-1}(t-t_0)\,S(t-t')\,\xi(t') = S^{-1}(t'-t_0)\,\xi(t')$$

since $S^{-1}(\kappa) = S(-\kappa)$ and $S(a)\,S(b) = S(a+b)$. Overall we are left with a propagation backward in time from t' to t_0.

This is the ultimate reason why the "do nothing" behavior is destabilized if the behavior is adapted by minimizing the time loop error E. The point is that forward in time a noise is damped if the deterministic dynamics is stabilizing a FP. Time reversal inverts attractors into repellers and vice versa so that minimizing E generates an instable dynamics.

In the present case we may see this explicitly if we consider the FP at $z = 0$ which is stable for $c < 1$. Ignoring the dependence of Γ on z we find the learning dynamics for c as

$$\dot{c} = \varepsilon_0 \tau D$$

which increases c unconditionally so that after some time $c = 1$ is exceeded and the FP at $z = 0$ is destabilized. With c growing further the state will settle in one of the stable branches of the bifurcation diagram. The state dynamics is symmetric with respect to the operation $z \to -z$. However when driven by the noise the system chooses one of the branches this symmetry is broken since with c growing unrestrictedly the state eventually will not return. However this is only half of the story since Γ is independent of z only if z is sufficiently small. Hence the symmetry breaking brings the nonlinerity effects into play which counteract the unrestricted increase of c. Learning becomes stationary if the two effects balance each other. It will be discussed below that this happens just when the system is in the regime where the noise switches the state between the branches of the bifurcation diagram in an effective way.

We note in passing that these considerations are not special for the present very simple model. Instead this general scenario which is well known from the theory of self-organization comes up here from the principle of HS in a natural way.

4 Properties of the homeokinetic neuron

We are now going to study the properties of our learning neuron and start with the limiting case of vanishingly small noise.

4.1 The deterministic case

Let us start from the update

$$\dot{p} = \varepsilon \frac{\partial}{\partial p} \overline{\Gamma(z)}$$

where $\varepsilon = \varepsilon_0 D \tau$ and $p \in \{c, H\}$ and consider the case that $D \to 0$ (and $\tau \to 0$) with $\varepsilon_0 \to \infty$ such that $\varepsilon = const.$ With $D << 1$ the distribution is sharply peaked so that in the limit we may write

$$\overline{\Gamma} = \Gamma(z^*) = -1 + cg'(z^*) \tag{12}$$

where z^* is the FP. This formula is appropriate whenever $c \neq 1$.

4.1.1 No bias – Learning drives the system to the bifurcation point

We are now going to study some special cases and will start with the case that both $H = 0$ and $I = 0$. In order to find $\partial_c \Gamma(z^*)$ we need the derivatives of z^* with respect to c. Let us consider the FP equation $z = cg(z)$ at $z = z^*(c) + \Delta z$. In order that the equation is correct we must replace c with $c + \Delta c$ so that

$$\Delta z = g(z^*) \Delta c + \Delta z c g'(z^*)$$

The equation is fulfilled if $\Delta z = \left(1 - g'\left(z^*\right)\right)^{-1} g\left(z^*\right) \Delta c$ hence

$$\frac{\partial z^*}{\partial c} = \frac{g\left(z^*\right)}{\left(1 - cg'\left(z^*\right)\right)}$$

The complete dynamics can now be written as

$$\begin{aligned} \dot{z} &= -z + cg\left(z\right) + c\xi\left(t\right) \\ \dot{c} &= \varepsilon g'\left(z\right)\left(1 - 2c\frac{g^2\left(z\right)}{1 - cg'\left(z\right)}\right) \end{aligned}$$

where

$$g\left(z\right) = \tanh z$$

We used $g' = 1 - g^2$ and replaced z^* with the current value of z which is appropriate under the assumed validity of the quasi-equilibrium assumption. We note that $1 - cg'\left(z\right) = -\Gamma\left(z\right) > 0$ for $c \neq 1$ since we are always at the stable FP. It is easily seen (Figure???) that the update is always negative for $c > 1$. On the other hand we find that for $c < 1$ the gradient $\partial_c\overline{\Gamma\left(z\right)} = 1$ since z and hence $g\left(z\right) = 0$ in this case.

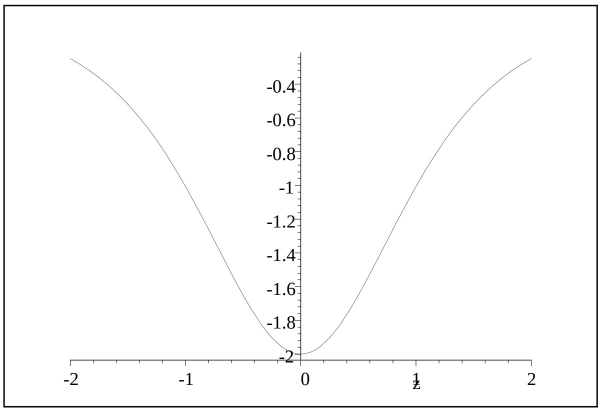


Figure 2: The gradient $\partial_c\overline{\Gamma\left(z\right)}$ as a function of the fixed point z for values of $c > 1$ and the case of $H = 0$ and $I = 0$.

The findings can be made more explicite if we consider the case of $c = 1 + \rho$ where $0 < \rho << 1$ so that we may approximate

$$g\left(z\right) = z - \frac{1}{3}z^3$$

The FP is

$$z = \sqrt{3\rho}$$

hence

$$\frac{\partial z}{\partial c} = \begin{cases} 0 \text{ for } \; c < 1 \\ \\ \frac{3}{2z} \text{ for } \; c > 1 \end{cases}$$

and

$$\dot{c} = \begin{cases} \varepsilon \text{ for } c < 1 \\ \\ -2\varepsilon \text{ for } c > 1 \end{cases} \tag{13}$$

Obviously the learning dynamics drives c unconditionally towards the bifurcation point at $c = 1$.

4.1.2 Interpretation in terms of the local LE

In view of the interpretation of Γ as the local Lyapunov exponent we consider the latter as a function of c which in the vicinity of $c = 1$ can be obtained as

$$L(\rho) = \begin{cases} \rho - (1+\rho)\tanh^2\left(\sqrt{3\rho}\right) & \text{if} \quad \rho \geq 0 \\ \rho & \text{if} \quad \rho < 0 \end{cases}$$

so that the maximization of the LE drives the system towards $c = 1$.

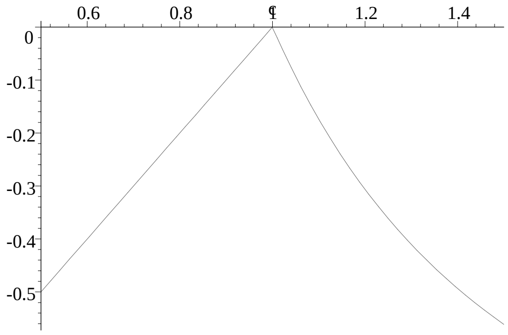


Figure 3: The local Lyapunov exponent as a function of c. The maximum is at $c = 1$ which is the bifurcation point of the dynamical system.

4.1.3 The sensitization effect

The above findings refer to the case that $I = H = 0$ where the global dynamics settles at the pitchfork bifurcation point (BP) which is singled out in that case. However with I different from zero (but still $H = 0$) the BP diagram displays one globally stable branch and a BP where a stable and an unstable branch arise. It is interesting where the learning settles the system in this case. Let us

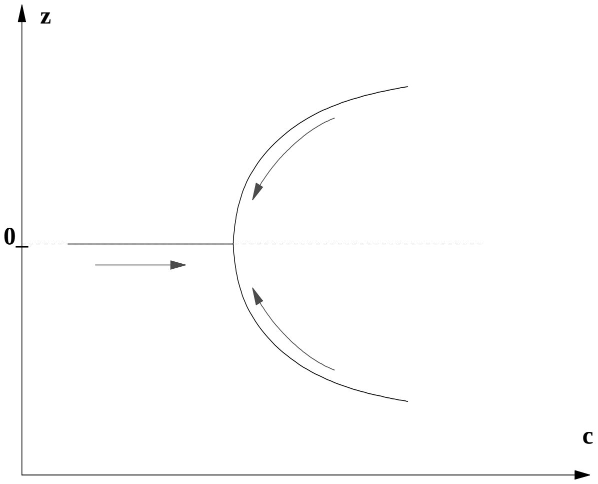


Figure 4: The flow of the fixed points due to the learning by gradient ascending the local Lyapunov exponent in the very low noise case.

define the response strength μ which measures the response of the membrane potential z to a small variation of the input I as

$$\delta z = c\mu\delta I$$

the response being mediated by the coupling strength c. With z at the FP we have

$$\mu = \frac{1}{c}\frac{\partial z}{\partial I} = \frac{1}{1 - cg'(z)} = -\frac{1}{\Gamma(z)}$$

so that

$$\partial_c \mu = \mu^2 \partial_c \Gamma \tag{14}$$

Learning stops at $\partial_c \Gamma = 0$ hence eq. 14 implies that learning drives the system to that parameter region where μ has an extremum with respect to c, see also Figure 5.

The response factor measures the shift mediated by c of the FP due to a change δI of the input. Since the membrane potential determines the output we may also say that μ measures the sensitivity of the neuron. The result shows that with each input I the **neuron adapts such that its sensitivity is at maximum**. Moreover, μ measures also the reaction of the membrane potential to a change in H, i.e.

$$\mu = \frac{\partial z}{\partial H}$$

Interpreting H as a bias in the membrane potential we see that the learning drives the system towards the maximum sensitivity to a change in H.

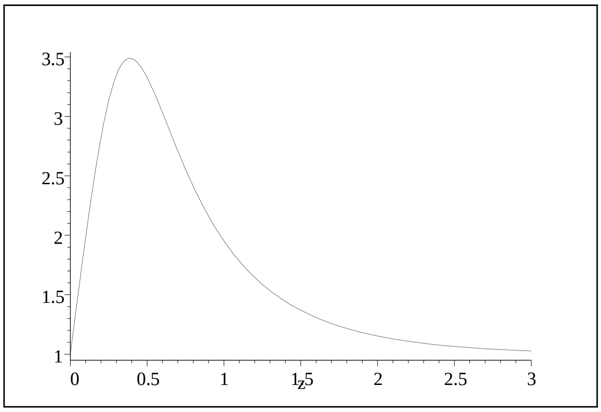


Figure 5: The response strength μ over the fixed point z for varying values of c and $I = 0.1$.

4.2 The moderate noise case

The noise modifies the results found in the preceding Sec. 4.1 essentially close to the bifurcation point.

4.2.1 Low noise approximation

Let us start the discussion with a few results for weak noise. We use the small z expansion of $\tanh(z)$ everywhere in the following so that

$$V(z) = -(c-1)\frac{z^2}{2} + c\frac{z^4}{12}$$

When in one of the stable regions of the bifurcation diagram we put $z = z^* + a$ and use the approximate distribution as (Gardiner Chap. 9.1)

$$P(a) = \sqrt{\frac{V''(z^*)}{2\pi c^2 D}} e^{-\frac{V''(z^*)}{2c^2 D}a^2} = \sqrt{\frac{1}{2\pi\chi^2}} e^{-\frac{a^2}{2\chi^2}}$$

where

$$\chi = \sqrt{\frac{c^2 D}{V''(z^*)}}$$

and in leading order of $\rho = c - 1$

$$V''(z^*) = \begin{cases} 2(c-1) \text{ for } c > 1 \\ \\ 1 - c \text{ for } c < 1 \end{cases}$$

so that

$$\overline{\Gamma(z)} = \rho - c\overline{z^2} = \rho - c\chi^2 - c(z^*)^2$$

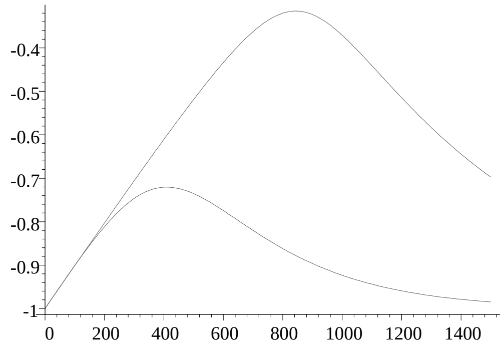


Figure 6: The local Lyapunov exponent over c (in units of 10^{-3}) for the dynamics $z' = -z + c\tanh z + cI$ with $I = 0.1$ (upper) and $I = 1$ (lower curve). The maximum for the $I = 0.1$ case is at $c = 0.83$ where $z = 0.39$ in agrreement with the maximum of the response curve.

We have $z^* = 0$ and $\chi^2 = D/(1-c)$ for $c < 1$ and $3\rho = (z^*)^2$, $\chi^2 = D/2(c-1)$ for $c > 1$ in leading order of ρ. Eventually we obtain

$$\frac{\partial}{\partial c}\overline{\Gamma(z)} = \begin{cases} 1 - \frac{D}{(c-1)^2} \text{ for } c < 1 \\ \\ -2 + \frac{D}{2(c-1)^2} \text{ for } c > 1 \end{cases} \tag{15}$$

The result is appropriate only if $D/(c-1)^2 << 1$. At $D = 0$ the result obtained agrees with the direct approach in the deterministic case where we take the derivative of $\Gamma(z)$ with z at the fixed point, cf. eqs. 13.

At $c = 1$ we have

$$P(z) = \frac{1}{N}\exp\left(-\frac{z^4}{12D}\right)$$

and find exactly (Appendix)

$$\partial_c\overline{\Gamma} = 1 - \sqrt{4D/3}$$

Hence the update is independent of c and positive. Together with eq. 15 we find that $\partial_c\overline{\Gamma}$ smoothly decreases form 1 to -2 when increasing c from $c << 1$ to $c > 1$, the slope at $c = 1$ being the larger the smaller D. This is corroborated very well by numerical results.

4.2.2 The Arrhenius point – the optimum of probing into alternatives

Like in the deterministic case we are now going to study the stationary state of the learning dynamics under noise. Let us consider first the case that $H = 0$

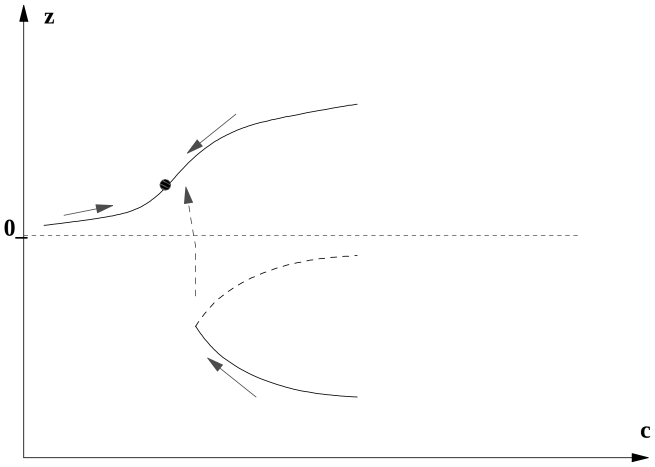


Figure 7: The flow of the fixed points due to the learning by gradient ascending the local Lyapunov exponent in the very low noise case if there is an additional constant input I. The learning converges where the sensitivity of the system to a change in I is at maximum.

and $I = 0$. The first point to be noted is that the noise obviously drives the system towards the regime where $c > 1$. This is obvious from the fact that we still have $\partial_c \overline{\Gamma} > 0$ at $c = 1$. When looking form the other side, i.e. from large values of c we find that $\partial_c \overline{\Gamma} < 0$ so that c is decreased, the decrease being diminished by the noise. Obviously the noise determines the point c_{cross} where the c–dynamics becomes stationary.

Interestingly the stationary state has a clear interpretation which can be found from considering the rate of barrier crossings as a function of c with $c > 1$ where the potential $V(z)$ is split into two wells. The probability for the crossing of the barrier between the wells is given by Arrhenius' law

$$P_{cross} = e^{-\Delta V/D}$$

where ΔV is the barrier height. Let us study the low noise case where z is small in the transition region. The value of c_{cross} in this case is according to eq. 15 somewhere in the region of the solutions of $(c-1)^2 = D$ and $(c-1)^2 = D/4$. Let us put ´therefore

$$c_{cross} = 1 + \alpha\sqrt{D}$$

with an empirical factor α. Numerically we found that $\alpha = 0.45$.

The potential barrier in the small z approximation of the potential is (in leading order of $c-1$)

$$\Delta V = \frac{3}{4}(c-1)^2$$

so that

$$P_{cross} = e^{-\frac{3}{4}\alpha^2}$$

Hence the crossing rate is of the order of 1 independent of the value of D.

The result shows that the learning drives the system towards that region of parameter space where driven by the noise the state switches between its two alternatives with a moderate rate. This is a region of optimality of noise induced decision making since the amplitude of the stochastic oscillations of z (which exponentially increases with z) is large whereas the crossing rate (which exponentially decreases with c) is still substantial. We can also say that the system probes cautiously into its alternatives.

4.2.3 The emergence of search

So far we have only considered the learning dynamics for c. We found that the learning converges towards a value of c so that the neuron is on the one hand most sensitive to changes in the input and on the other hand is driven to the edge of noise induced decision making. With respect to the learning behavior we do have the usual situation that the learning converges the parameters towards fixed values with the behavior of the system specified as soon as the parameters are converged. The situation changes if we have additional inputs into the neuron which are not integrated into the SM loop. We will see that with the learning the system realizes a limit cycle attractor.

Let us interprete our parameter H in the state dynamics eq. 4 either as a threshold or as a tonic input of unit strength coupled into the system via a synapsis of strength H. From eq. **??** we have the update of H as

$$\dot{H} = \varepsilon \partial_H \overline{\Gamma(z)}$$

In the low noise limit this is

$$\dot{H} = -\varepsilon \frac{cg'(z)}{1 - cg'(z)} g(z)$$

where z is at the FP. Under the quasi-equilibrium hypothesis we may replace z with the current state. Let us assume for the moment that $c = 1 + \rho$ where $0 < \rho << 1$ is chosen and kept fixed. The combined dynamics is for small z and ρ approximated as (ignoring the noise term in the state dynamics)

$$\begin{aligned} \dot{z} &= \rho z - \frac{z^3}{3} + H \\ \dot{H} &= -\varepsilon z \end{aligned}$$

where the positive factors $g'(z)$ and $(1 - cg')^{-1}$ were absorbed into ε since these factors only modulate the speed of learning. The eigenvalues of the linearized system are $\lambda_\pm = \frac{1}{2}\rho \pm \frac{1}{2}\sqrt{\rho^2 - 4\varepsilon}$ where $\rho = c - 1 > 1$.

It is easily seen that this dynamical system supports a stable limit cycle behavior, the frequency being defined by ε and the amplitude a is roughly estimated by the zeros of $\rho z - \frac{z^3}{3}$ so that $a = \sqrt{3\rho}$. This is very well born out by computer simulations (with a different overall factor) as can be seen from Fig.

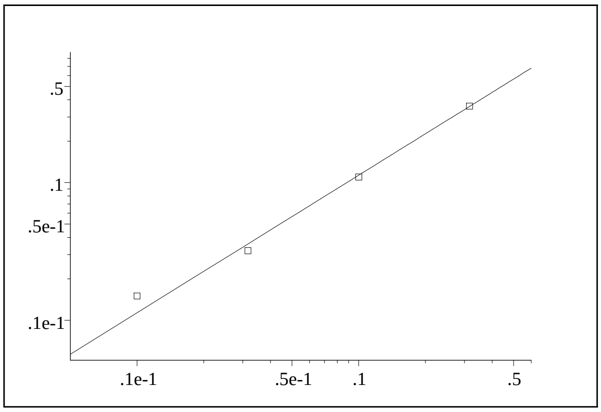


Figure 8: Plot of the amplitude a of the state oscillations over $\sqrt{\rho}$ for the combinded dynamics with c kept fixed at $c = 1 + \rho$. We considered ρ in the range 0.0001 to 0.5. The learning rate was $\varepsilon = 0.1$. The straight line is a fitting curve with $a = 1.13\sqrt{\rho}$.

8. Obviously the interaction of the H and the state dynamics creates a system which actively searches the state space.

These results have been obtained by assuming c to be fixed by hand. However with H oscillating the inclusion of the H dynamics influences the average value of c not essentially so that our assumption of keeping c fixed is essentially self-consistent. This is well corroborated by computer simulations which clearly demonstrate that the noise strength determines essentially the value of c and this one the amplitude of the self-sustaining oscillations insinutated by the H dynamics. In this way we find an active searching of the state space by the combined dynamics. More experiments and results are given below (see robot behavior???).

We emphasize once again that this behavior is not a property of the learned system but instead is a result of the permanent change of the learning parameter H. Hence we may say that the behavior in state space is largely produced by the concomitant learning dynamics. Technically the behavior in phase space is a projection of the limit cycle dynamics in the product space. We may understand the learning dynamics again as the maximization of the local Lyapunov exponent, cf. Fig. 9

5 A simple explicite learning rule

The learning rules derived and discussed in the preceding sections are still rather complicated and do not lend themselves to a ready biological interpretation. We are now going to derive an approximate learning dynamics which still displays the essential properties of the full dynamics and yet is much more simple and

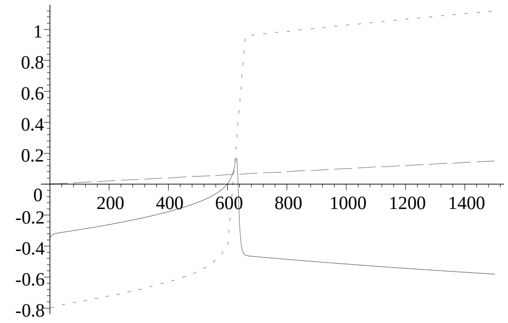


Figure 9: The local Lyapunov exponent $\Gamma(z)$ (solid line) as a function of H (dashed) for $c = 1.2$ fixed. H starts at $H = 0$ and is gradually increased. Positive values occurring in the transition region of the state z (dotted) are to be considered as an artifact connected with the jump to the other branch of the bifurcation diagram. During the jump the system is not at a stable fixed point so that the conditions of the quasi-equilibrium hypothesis is violated.

biologically more plausible. In order to do so we first introduce another formulation of the exact learning dynamics.

5.1 Effective state approximation

The investigation of the update is simplified by introducing the notion of an effective state which is defined by the solution of

$$\overline{\Gamma(z)} = \Gamma\left(z^{(*)}\right)$$

which is equivalent to

$$\overline{g^2(z)} = g^2\left(z^{(*)}\right)$$

so that

$$z^{(*)} = \operatorname{arctanh}(\sigma)$$

where $\sigma = \sqrt{\overline{g^2(z)}}$.

In terms of this notion we may now write the update in the learning rule as

$$\frac{\partial}{\partial c}\overline{\Gamma(z)} = \frac{\partial}{\partial c}\Gamma\left(z^{(*)}\right) = g'\left(z^{(*)}\right) + g''\left(z^{(*)}\right)\frac{\partial z^{(*)}}{\partial c}$$

Using $g' = 1 - g^2$ we find also

$$\frac{\partial}{\partial c}\overline{\Gamma(z)} = g'\left(z^{(*)}\right)\left(1 - 2cg\left(z^{(*)}\right)\frac{\partial z^{(*)}}{\partial c}\right)$$

The essential effect of the noise is that $\partial_c z^{(*)}$ does not any longer diverge at $c = 1$ as is the case with

$$\partial_c z^* = \frac{g\left(z^*\right) + I}{1 - cg'\left(z^*\right)}$$

This is clearly born out by the Figure 10.

5.2 The simplified learning rule

The interesting point now is that looking from large c the deviation between $z^{(*)}$ and z^* starts in the region where the barrier hoppings occurr with a substantial rate. It is then that the curvature of z^*as a function of c is washed out. The smooth behavior of $z^{(*)}$ is clearly seen from Fig. 10. Let us therefore write

$$\partial_c z^{(*)} = \phi\left(z^{(*)}\right)\left(g\left(z^{(*)}\right) + I\right)$$

where from the behavior of $z^{(*)}$ we conclude that $\phi\left(z^{(*)}\right) = \left(1 - cg'\left(z^{(*)}\right)\right)^{-1}$ is a smooth positive function so that we may write the learning dynamics as

$$\frac{1}{\varepsilon}\dot{c} = \left(\phi^{-1} - 2cg\left(g + I\right)\right)$$

Moreover we may replace ϕ with its value at just c_{cross} without changing the stationary state of c qualitatively. However we do not know explicitly the critical value of ϕ. On the other hand we know that the learning is to stop at c_{cross}. Looking from $c > c_{cross}$ we may replace $z^{(*)}$ with z (at the FP) so that the critical value of ϕ^{-1} is obtained from the solution of ($I = 0$, small z)

$$\phi^{-1} = 2z^2$$

where z is at the FP corresponding to c_{cross} which was obtained above as

$$c_{cross} = 1 + \alpha\sqrt{D}$$

The FP is at $z^2 = 3\left(c_{cross} - 1\right)$ so that $\phi^{-1} = 6\alpha\sqrt{D}$.

The full dynamics with the simplified learning rules is consequently

$$\begin{aligned}
\dot{z} &= -z + H + cx \\
\dot{c} &= \varepsilon D\left(\alpha'\sqrt{D} - cxy\right) \\
\dot{H} &= -\varepsilon Dcy
\end{aligned} \tag{16}$$

where $\overline{\xi\left(t\right)} = 0$ was assumed

$$x = g\left(z\right) + I + \xi\left(t\right)$$

$$y = g\left(z\right)$$

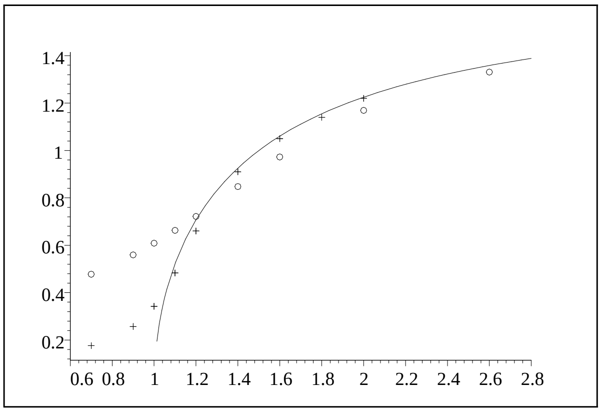


Figure 10: The effective state $z^{(*)} = \sqrt{< z^2 >}$ over c for the case of $D = 0.01$ (crosses) and $D = 0.1$ (circles). The full line depicts the fixed point as a function of c.

and α' is taken as a new empirical constant of order 1 which essentially influences the rate of the noise driven barrier crossing. Decreasing (increasing) this factor means the increase (decrease) of the hopping rate since then c is driven towards smaller (larger) values.

We note moreover that in practice the diffusion constant D actually measures the average noise strength in the sense of a time averaging. Therefore D may change in the course of time which produces interesting effects as will be seen below, see Sec. 7. The time averaging can be achieved on-line by considering D as an auxiliary variable with dynamics

$$\dot{D} = \varepsilon \left(\xi^2 (t) - D \right)$$

The noise ξ can be evaluated from the observable variables as

$$\xi = x - I - y$$

where $\dot{z}$ is to be evaluated empirically from the state. Moreover we may let the prefactor D multiplying the learning rate be generated in the sense of stochastic approximation so that we obtain the explicite formula

$$\begin{aligned} \dot{z} &= -z + H + cx \\ \dot{c} &= \varepsilon \xi^2 \left(\alpha' \sqrt{D} - cxy \right) \\ \dot{H} &= -\varepsilon \xi^2 cy \\ \dot{D} &= \varepsilon' \left(\xi^2 - D \right) \end{aligned} \tag{17}$$

which is of interest if the strength of the noise changes rapidly as a function of the state itself. This will be of interest in the robotics application in Sec. 7. The learning rate ε' typically is $\varepsilon' = 0.1\varepsilon$.

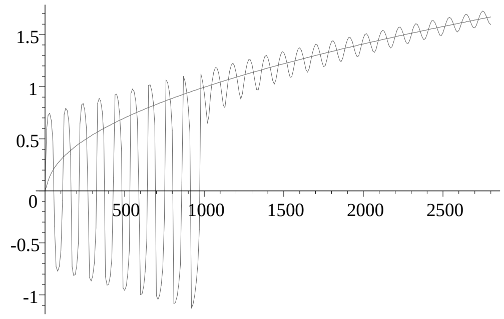


Figure 11: The stochastic bifurcation scenario illustrated. The state z in the dynamics $\dot{z} = -z + c \tanh z + h(t)$ where h is a periodic driving force. The coupling parameter c is slowly increased starting at $c = 1$. The small driving force oscillates the state between the brances of the bifurcation diagram until a critical value of c called the Arrhenius point is exceeded.

Equations 16 and 17 have been derived by the homeokinetic principle. Although an approximation they are seen in computer simulations to behave essentially in the same way as the exact ones. The interesting point here is that the principle obviously finds a very simple expression in terms of an overall driving term for the synapsis together with a simple (anti-) Hebbian learning rule. This enhances the probability that nature realizes this principle because on the one hand both terms are observed in real synaptic processes. On the other hand systems with this kind of learning dynamics should be clearly favored in evolution because of their useful properties as will be more clear from the application in the robot domain.

5.3 Biological interpretation

The learning dynamics for c involves the constant driving term $\sqrt{3D}$ which increases c so that the the system is driven towards the bifurcated regime. This effect is counterbalanced by the second term which may be interpreted as an anti-Hebbian learning rule since it involves the product of input times output of the neuron. Analogously the update for H can be interpreted as the product of the tonic input 1 times the output of the neuron. Note that we can remove the c factor in these terms since c is close to 1 in the region of interest.

It would be interesting to study the realization of these effects in terms of the spike timing synaptic plasticity. This seems to be realistic since when working with integrate and fire neurons the constant driving term can be produced by a lag in time between the presynaptic input and the firing of the neuron. On the other hand the anti-Hebbian terms can be produced in the usual ways.

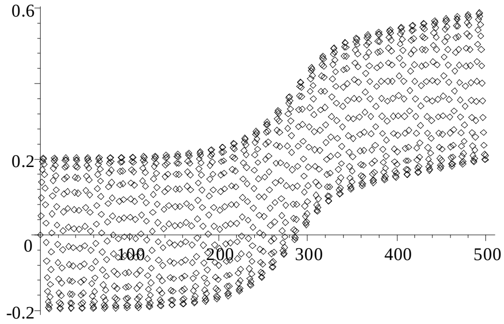


Figure 12: The stochastic bifurcation scenario illustrated. The state z in the dynamics $\dot{z} = -z + c \tanh z + h(t)$ where $h = +.1 \cos(.5t)$. The coupling parameter c is slowly increased as $c(t) = (1.02 + .0001t)$. Plotted are snapshots of $z(t)$.

However as discussed above the effects produced by the H dynamics depend on the permanent change of the value of H so that it is also here that fast synaptic processes might be of interest for the realization of this dynamics. Note that the change of H is only rather small despite the rather large effects produced. This is a consequence of the critical (for small z) dependence of z on H close to the BP.

6 Entrainment and the perspective of self-organization

An interesting potentiality of the homeokinetic neuron is that the self-induced search generated by the H dynamics is modified as soon as there is a signal from the outside world. This is clearly seen from Fig. **??** where the neuron is receiving a weak periodic input with an amplitude much smaller than that of its self-generated limit cycle oscillations. After a very short time the membrane potential is synchronized with the perturbation although the frequencies are largely different. This entrainment effect shows that besides of the search the neuron is still able to "listen" to the outside world and to harmonize with the latter.

Viewing the neuron as a unit which sends signals into the outside world and receives the echo of its own signals together with the sigals produced by other units and/or the outside world we would say that the neuron is able of concerting its actions with those of the other neurons. Thus we might expect that a system of many such units is able of finding a concerted action which can be sustained in the interaction with the outside world. Thus we argue that such a system is able of self-organization. We hope to consider this point in a

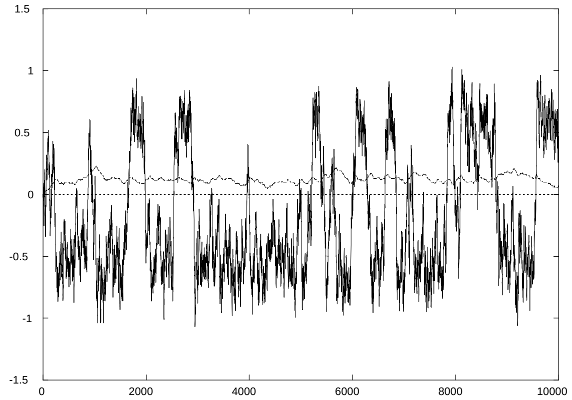


Figure 13: Noise driven transitions of the state z with parameters $D = 0.1$, $\varepsilon = 0.001$. The state is seen to jump occasionally from one side to the other, the value of $c - 1$ (dashed) fluctuating around 0.12. The empirical factor in front of the $\sqrt{D}$ term in the update rule for c was chosen as 0.6.

later paper.

7 Application in the robot domain

The properties of the homeokinetic neuron are best seen in a real SM loop. We considered the case that the output $y = g(z)$ of the neuron is the target wheel velocity of a Khepera robot whereas the input x is the true velocity of the wheels as read back from the wheel counters. If the robot is moving with a constant velocity in free space the average of x and y should coincide which we will assume in the following.

7.1 Time discrete SM loop

We have a discretization of time since the wheel counters are read back every 50 mS or so. Let x_t, y_t, z_t be the sensor reading, the output of the neuron, and its internal state, respectively. Then

$$x_t = y_{t-1} + \xi_t$$

where

$$y_t = g(z_t)$$

and the noise ξ_t comprises the effects due to the discretization (ten steps in the experiments) of the velocities and from slip and friction effects of the wheels in the real environment.

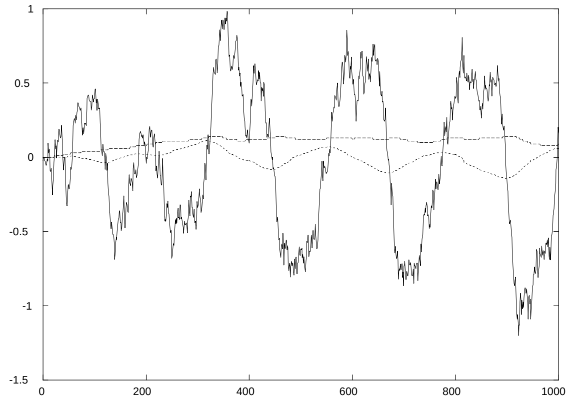


Figure 14: Same as Fig. 13 with the H dynamics included. The time is scaled by a factor of 0.1 so that the oscillations of the state z are much faster than in the former case. The behavior of H (dashed) is seen to be phase shifted by $\pi/2$.

The dynamics of the membrane potential in discrete time reads now

$$z_{t+1} = (1-\eta)\, z_t + \eta c x_t + \eta H$$

where η with $0 < \eta < 1$ is a hardware constant.

We apply the learning rule according to eq. 17 which in the discrete time domain now reads

$$\Delta c = \varepsilon\, (x_t - y_{t-1})^2 \left(\sqrt{3D} - c x_t y_t\right)$$

$$\Delta H = -\varepsilon\, (x_t - y_{t-1})^2\, c y_t \tag{18}$$

$$\Delta D = \varepsilon' \left((x_t - y_{t-1})^2 - D\right)$$

With a physical Khepera robot we must take some additional precautions due to the discretization in the realization of the wheel velocities. In fact once $x_t = 0$ and $|z|$ below a certain threshold the discretized target velocity is equal to zero so that the robot stalls at $x = 0$. Then there is no learning and the full dynamics stops as well. In practice this case must be accounted for though it is seldom because of the H dynamics which creates an active search.

7.2 Experiments

We have used the learning procedure of eq. 18 to control the forward velocity of both a Khepera and a Pioneer robot instead of giving it the fixed velocity.

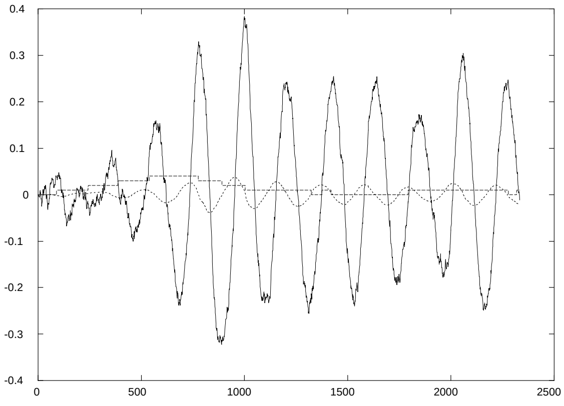


Figure 15: Same as Fig. 14 with $D = 0.001$.

A robot controlled by the learning procedure of eq. 18 will move forward for some time and then reverse its velocity and so on, the distances covered by this erratic motion depending on the strength of the noise and the learning rate in a systematic way. The most interesting property of this controller paradigm however is observed if the robot collides with some obstacle. In this case the noise (difference between true and target velocity) is largely increased which leads to a very rapid relearning of the parameters c and in particular H such that the velocity of the robot is reversed almost immediately. In this way our learning dynamics may be said to generate an explorative behavior of the robot with a sensitive reaction to perturbations from the environment. Due to the sensitive reaction of the neuron to "slip and friction noise" our robot survives in nearly arbitrary environments without getting stuck in corners or at other obstacles. Moreover, learning is found to be extremely fast and permanently alert while under stationary environmental conditions it is convergent and reproducible.

8 Summary and outlook

In the present paper we considered brain like systems acting as controllers in the SM loop. We considered these systems from a system theoretic point of view as hierarchies of feed-back loops. With the synaptic dynamics included these systems are of an overwhelming complexity. Our question is how systems may self regulate into operational conditions of moderate complexity as they seem to be the appropriate for biological systems. A simple stability argument like the homestatic principle is not sufficient since in the sensory-motor loop the agent changes its internal state due to its own actions. We introduced a new paradigm – homeokinesis – which essentially corresponds to the application of the homeostatic principle to the internal state of the system in interaction representation which is obtained by transforming away state changes caused by

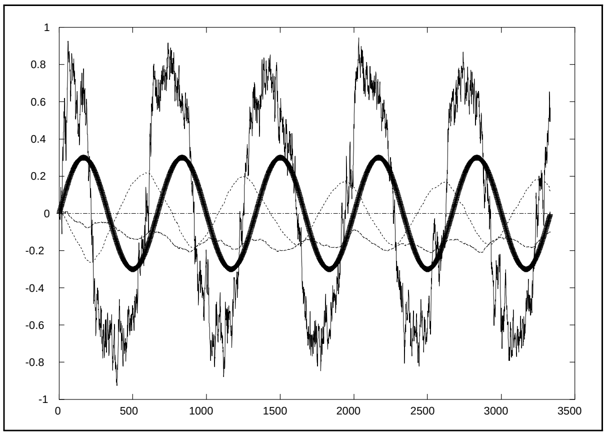


Figure 16: Same as Fig. 14 with an additional input $a \sin(2\pi\omega t)$ with $a = 0.3$ (crosses). The frequency of the oscillations is five times smaller than that of the unperturbed system. The state is seen to be entrained by the external perurbation. The phase lag arises from the time scale of the learning dynamics.

the agent itself. Quite generally the principle supports the spontaneous creation of behavioral modes instead of stationarity.

The new principle is applied to drive the synaptic dynamics of a neuron in a simple sensory-motor loop and was shown to produce a number of intersting behaviors like the self-regulation to the stochastic bifurcation point, self-induced search oscillations via a Hopf bifurcation, the adaptation to the regime of highest sensitivity and the like. These findings suggest that learning under the principle of homeokinesis drives the system towards working conditions characterized by the optimum payoff between creativity (probing into alternatives) and stability (realizing alternatives).

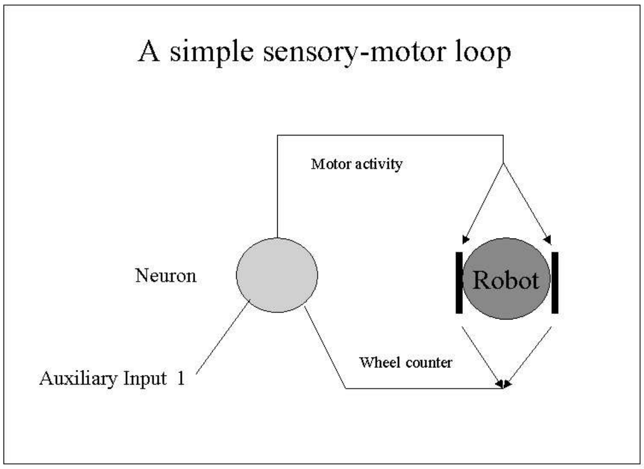


Figure 17: A neuron controlling the velocity of a robot. The output of the neuron is the target velocity of both wheels. The true wheel velcity is the input into the neuron.

One interesting issue was a new functionality of the neuron seen when including the adpatation of the parameter H. By the learning concomitant to the neuron dynamics the combined system realizes a limit cycle which may be interpreted as a self-induced **searching mode** of the state emerging in the state space. This was exemplified in the robot case where the robot was seen to react in this mode sensitively to the influences of the environment.

From a mathematical point of view the effect of homeokinesis as a paradimg for driving adaptation is qualified in the following general sense. Asssume there is a parameterized (stochastic) dynamical system

$$\dot{z} = F\left(z, c\right) \tag{19}$$

where $z \in \mathbf{R}^1$ and $c \in \mathbf{R}^p$. Then a parameter dynamics is added as

$$\dot{c} = \varepsilon \xi^2 \frac{\partial}{\partial c} \Gamma - \varepsilon' \frac{\partial}{\partial c} \xi^2 \tag{20}$$

where Γ is the local Lyapunov exponent of the system and ξ is the "noise" which is essentially the deviation between the true dynamics of the system and a model dynamics defined by a dynamical system

$$\dot{q} = M\left(q, c\right)$$

In the present paper ξ was a true noise independent on the state so that the second term in eq. 20 was equal to zero. We have seen in the present paper that combined dynamical systems given by eqs. 19,20 show very interesting property and might be an object of independent mathematical study. The generalization to the case $z \in \mathbf{R}^n$ is easily possible and will be considered in a later paper.

9 Appendix A: Interaction representation

Consider the linear dynamical system

$$\dot{x} = Cx + M \tag{21}$$

where $x \in \mathbf{R}^n$ and the matrix C may be split

$$C = A + B$$

where A is defining that part of the dynamics we want to take into account explicitely. In a similar way we split the inhomogeneity as

$$M = P + Q$$

The idea of the interaction representation is a transformation of the state such that in the new representation the dynamics is driven only by the (modified) B and Q terms. In order to find this dynamics let us consider first the explicite solution

$$x(t) = e^{(t-t_0)C} x(t_0) + \int_{t_0}^{t} e^{\left(t-t'\right)C} M(t') \tag{22}$$

of the full equation starting from some reference instant of time t_0 with state $x(t_0)$ given. Then we transform this solution back in time by using the unperturbed equation of motion

$$\dot{s} = As + P$$

down to time t_0 starting at time t with $s(t) = x(t)$ as given by eq. 22. We obtain

$$s(t_0) = e^{(t-t_0)A} x(t) + \int_{t}^{t_0} e^{\left(t_0-t'\right)A} P(t')$$

We may consider this as a function of t and call this $\hat{x}(t)$ (omitting the reference point in the notation). The state $\hat{x}(t)$ which is now in the so called interaction representation is seen to obey the differential equation

$$\partial_t \hat{x}(t) = B(t-t_0)\,\hat{x}(t) + B(t-t_0) \int_{t_0}^{t} \hat{P}(t')\,dt' + \hat{Q}(t)$$

where

$$B(\tau) = e^{-\tau A} B e^{\tau A}$$

and

$$\hat{P}(t) = e^{-(t-t_0)A} P(t) \ , \ \hat{Q}(t) = e^{-(t-t_0)A} Q(t)$$

We note that $\hat{x}(t)$ is constant if $B = Q = 0$ which is interpreted by saying that the dynamics of $\hat{x}$ is driven by the effect of B and Q alone.

The equations are simplified if we omit the splitting of M, i.e. put $Q = M$ so that

$$\partial_t \hat{x}(t) = B(t - t_0)\hat{x}(t) + \hat{M}(t)$$

10 Appendix B: Stability of the stationary point

The point of interest now is the behavior in the vicinity of the stationary state, i.e. assume that for I given $c = c_{cross} + \delta c$ is not yet converged. We may further assume the state z to be at the FP $z = z^*(c, I)$ for fixed values of c and I. The question is what is the behavior of $\partial_c \Gamma$ as a function of δc. The dependence may not be simple because in the case $I = 0$ we have found above that the value of Q is -2 or 1 below and above the critical value $c = 1$.The situation where $\partial_c \Gamma = 0$ so to say is sandwiched between these two situations.

We consider the low noise limit where we approximate $\overline{\Gamma(z)} = \Gamma(z)$ with z at the FP.

$$\dot{c} = \frac{\partial}{\partial c}\Gamma(z) = \Gamma'(z)\frac{\partial z}{\partial c}$$

The behavior of c around the stationary point $c_0(I)$ is found from

$$\delta\dot{c} = W\delta c$$

where

$$W = \left.\frac{\partial^2}{\partial c^2}\Gamma(z)\right|_{c=c_{cross}} = \Gamma''(z)\left(\frac{\partial z}{\partial c}\right)^2 + \left.\Gamma'(z)\frac{\partial^2 z}{\partial c^2}\right|_{c=c_{cross}}$$

Using

$$\frac{\partial z}{\partial c} = \frac{(\tanh z + I)^2}{\tanh z + I - z\tanh' z}$$

and $c = z/(\tanh z + I)$ we find

$$\frac{\partial^2}{\partial c^2}\Gamma(z) = -\frac{(\tanh z + I)^2}{\tanh z + I - z\tanh' z}\frac{\partial}{\partial z}\frac{z\tanh z(\tanh z + I)}{I + \tanh z - z\tanh' z}$$

The Figure shows that W is negative so that the learning dynamics is converging to the stationary point.

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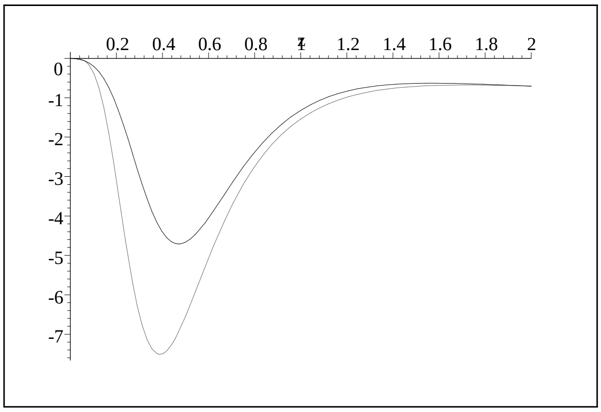

Figure 18: The slope W of the gradient in the learning rule as a function of the FP z for $I = 0.1$ and $I = 0.01$ (dotted). The slope is always negative so that the stationary point is stable.

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