On Ashby's homeostat:
A formal model of adaptive regulation

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Abstract

Ashby's homeostat performs a random search in the parameter space of a linear system until a stable behavior is retrieved following e.g. structural perturbations. We consider a continuous adaptive version of the homeostat where the parameters are determined by a gradient dynamics with respect to a given goal state. In this way much shorter transients are achieved by a parameter dynamics that runs on the same time scale as the state evolution. The model allows for analytical solution and forms in so far an interesting toy model for nonlinear adaptive systems with implications for control mechanisms in natural or artificial embedded agents.

1. Introduction

Ashby's homeostat (Ashby 1948, 1952) was invented and actually built half a century ago. It was intended as a working illustration of the principle of homeostasis, a term coined by Cannon (1932) in the 1920s. Presently, we are witnessing a revived interest in the concept of homeostasis in various fields ranging from synaptic plasticity (e.g. Turrigiano, 1998) to the control of autonomous robots (Di Paolo, 2000).

Originally homeostasis was conceived as an approach to complex control mechanisms in living beings. Blood pressure, e.g., depends on many local variables which are in turn affected by many hormonal, nervous and chemical processes, but it is stabilized globally without a central control unit. Ashby studied properties that underly such mechanisms in living beings from a theoretical point of view and tried to transfer this principle to the design of control architectures in machines. He found it natural in this context to consider dynamical systems involving switching processes, intermittency, memory and dynamic relations with the environment. In particular the notion of ultrastability appeared to him of central importance for a theory of the mechanisms underlying life.

Ashby claimed that in complex dynamical systems the notion of linear stability is not sufficient. Such systems should be able to cope with large deviations from a target state or even to produce them in order to achieve structural reorganization in their course. Ultrastability refers to systems which in addition to their states also possess dynamical parameters which are supposed to modulate the system dynamics such that eventually a target state is approached even in the presence of structural deformations.

In the cybernetical era the notion of homeostasis has served as a kind of metaphorical explanation of various phenomena ranging from physiological control to social relations. Relevant work has been performed by Zemanek (1958) and his student Hauenschild (1956), who constructed a replica of Ashby's homeostat, including some display devices such as a face with movable parts driven by the output of the homeostat and a glass screen where two spots of light were controlled each by two units of the homeostat. According to Wilkins (1968) they considered also coupling to an external environment, but perhaps this refers only to these output devices.

In the early 1960s Haroules and Haire (1960) constructed a homeostat with 16 units (cf. also Williams, 1961) called Jenny, it is however not clear whether any results have been obtained. Finally, Wilkins (1968) has started an even larger project on homeostats, which unfortunately could not be continued. Only recently (Di Paolo, 2000, 2002, Balaam, 2001, Ducker, 2001) the idea of homeostasis experienced a revived interest. Schwefel (1994) has noted that the function of Ashby's original homeostat bears some similarity with evolutionary algorithms.

The present contribution tries to relate more directly to the early works for several reasons, while leaving aside the peculiarities of the hardware implementation of the homeostat. We will see that the main idea of the homeostat can be formulated in terms of continuous adaptive dynamical systems and treated analytically, where explicit solutions can be obtained at least in a number of special cases. Practical applications of the homeostatic principle to robot control (Der et. al., 1999, 2002) are beyond the scope of the present paper, but have given a starting point for the present work. Further, the learning homeostat may turn out to provide a standard model with implications for various problems in the theory of adaptive behavior, autonomous robotics, neural learning dynamics, and nonlinear control. We will first present a formal account of homeostatic dynamical systems. Later we will introduce an adaptive dynamics which adjusts the parameters of the system, provide some analytical considerations and numerical results. Finally, we will discuss presently relevant implications of homeostatic dynamics.

2. Ashby's homeostat

Ashby has built a machine consisting of four rotatable magnets whose deviations from their target orientation gave rise to currents in a number of coils that in turn influenced the orientation of the magnets. The strengths of the interactions among the magnets are subject to switching processes realized by pseudo-random resistor values produced by multi-selectors. The dynamics of the system can be described mathematically (Ashby, 1948, 1952, cf. also the detailed presentation in Hauenschild, 1956) by a seemingly linear system in $n = 4$ dimensions.

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j \qquad (1)$$

The self-couplings a_{ii} are adjusted by hand to negative values initially, such that the system has a bias towards stability. The coefficients a_{ij} for $i \neq j$ are subject to a switching dynamics and are automatically chosen from a set of 9 possible values, which are fractions of 0.48, 0.73, 0.89 and 1.0 of some maximal value, the corresponding negative values or zero. Switching occurs if a variable x_j reaches a threshold $|x_j| \geq \theta = \frac{\pi}{4}$ with the target being at the origin. For each of the four vectors (a_{21}, a_{31}, a_{41}), (a_{12}, a_{32}, a_{42}), (a_{13}, a_{23}, a_{42}), and (a_{14}, a_{24}, a_{34}) there are 25 combinations of values stored out of the $9^3 = 729$ possible ones such that the system (1) includes up to $25^4 = 390625$ different dynamical behaviors.

The system state x is allowed to evolve for some time. The system will either approach some attractor inside the critical surface or eventually reach a critical surface. If $|x_i| \geq \frac{\pi}{4}$ a new set of random variables a_{ij} for all off-diagonal couplings towards unit i is drawn such that the system is changed each time a potential instability is encountered. Since in the hardware implementation the critical surface is simply given by a mechanical constraint, the state is assumed not to evolve further once the critical surface is reached even if the newly chosen couplings do not reintroduce stability into the system. The switching process stops if the coefficients of (1) are such that all eigenvalues have negative real parts.

In the stable region of the combined parameter and state space, the system tolerates small noise. But by an external impact that is sufficiently strong, e.g. the loss of a physical connection between node i and j or the reverse of the sign of an a_{ij}, the system returns via a large deviation in state space to a possibly different parameter setting with the desired dynamical properties. For random couplings, negativity of all eigenvalues is realized only with probability 2^{-n}, i.e. homeostats with large n will converge only after very long transients.

Further, oscillatory behavior or slow dynamics may occur inside the critical region such that the time between switching events may be long. If the eigenvalues are of small negative real part the system will be prone to noise. The fact that the system dynamics stops when x_i reaches the critical surfaces implies a separation of time scales which is not clear for the envisaged applications of homeostasis. When the critical surface is specified by soft thresholds, state may be controlled to avoid the boundaries and to stay near the target state by the exploitation of gradients of the threshold functions.

3. The learning homeostat

While in Ashby's approach stabilizing weights are found by random search, we suggest an incremental weight update, because it is reminiscent to the modification of synaptic efficacies in neural networks and will turn out to show a directed convergence.

The parameters a_{ij} in (1) are to be selected such that the system is stable at the fixed point $x = 0$. We will now discuss a learning scheme for the parameters a_{ij} which in a sense realizes Ashby's original ideas in a supposedly straightforward way. We consider the more general case that the system is to be stabilized at an arbitrarily given target state $\tilde{x}$. Because $x^* = 0$ is the only fixed point of (1), an adaptable inhomogeneity is required which can shift the fixed point towards $\tilde{x}$.

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + b_i \tag{2}$$

The aim is realized if the matrix $A = (a_{ij})$ is invertible and has eigenvalues with negative real parts, such that the inhomogeneity $b = (b_i)$ can be obtained by $b = A^{-1}\tilde{x}$.

In order to derive an adaptation rule for the parameter matrix A we introduce the target state $\tilde{x}$ which is to be approached by the system state x. Hence, the goal is to minimize

$$E = \frac{1}{2} \|x - \tilde{x}\|^2 . \tag{3}$$

Note that only future states $x(t+\tau)$, $\tau > 0$, can be influenced by present parameter changes. More precisely, we have

$$E = \frac{1}{2} \|x(t+\tau) - \tilde{x}\|^2 , \tag{4}$$

instead of (3), where $x(t+\tau)$ can be approximated for small τ by $x(t+\tau) \approx x(t) + \tau \dot{x}(t)$. Specifically for the system (2) we have

$$x(t+\tau) \approx x(t) + \tau \left(A(t)x(t) + b(t) \right) .$$

The objective function (4) becomes

$$\begin{aligned} E &= \tfrac{1}{2} \|x + \tau (Ax + b) - \tilde{x}\|^2 \\ &\approx \textstyle\sum_i (x_i - \tilde{x}_i)^2 + \tau (x_i - \tilde{x}_i) \left(\sum_j a_{ij} x_j - b_i \right) + O(\tau^2) \end{aligned}$$

and suitable parameter changes may be determined by gradient descent

$$\begin{aligned} \Delta a_{ij} &= -\varepsilon \tfrac{\partial E}{\partial a_{ij}} = -\varepsilon \tau (x_i - \tilde{x}_i) x_j \\ \Delta b_i &= -\varepsilon \tfrac{\partial E}{\partial b_i} = -\varepsilon \tau (x_i - \tilde{x}_i) \end{aligned} \tag{5}$$

In matrix form this reads

$$\begin{aligned} \Delta A &= -\varepsilon \tau (x - \tilde{x}) x^T \\ \Delta b &= -\varepsilon \tau (x - \tilde{x}) \end{aligned} \tag{6}$$

We assume that the parameter update is effective after a time delay τ_1 such that $A(t+\tau_1) = A(t) + \Delta A$ with $0 < \tau_1 \ll \tau$. For $\tau \to 0$, $\tau/\tau_1 = \text{const}$, we consider

$$\frac{A(t+\tau_1) - A(t)}{\tau_1} = -\varepsilon \frac{\tau}{\tau_1} (x - \tilde{x}) x^T$$

and obtain from the analogous relation for b and (2) the dynamical system (with ε rescaled by τ/τ_1)

$$\begin{aligned} \dot{x} &= Ax + b \\ \dot{A} &= -\varepsilon (x - \tilde{x}) x^T \\ \dot{b} &= \varepsilon (x - \tilde{x}) \end{aligned} \tag{7}$$

The same learning dynamics is obtained when aiming directly at a decrease of the distance from the target state $\tilde{x}$. This task can be expressed as the minimization of the function

$$\begin{aligned} E &= \frac{1}{2} \frac{d}{dt} \|x - \tilde{x}\|^2 = \frac{1}{2} \sum_i \frac{d}{dt} (x_i - \tilde{x}_i)^2 \\ &= \sum_i (x_i - \tilde{x}_i) \dot{x}_i = \dot{x}^T (x - \tilde{x}) \end{aligned}$$

A differential error function, namely,

$$E = \frac{1}{2} \|\dot{x}\|^2$$

gives rise to the dynamical system

$$\begin{aligned} \dot{x} &= Ax + b \\ \dot{A} &= -\varepsilon (Ax + b) x^T \\ \dot{b} &= -\varepsilon (Ax + b) \end{aligned} \tag{8}$$

Here stationary states are given by $Ax + b = 0$, such that $x^* = -A^{-1}b$. Since no explicit information on the target state x^* is present in the adaptation rules there is a multitude of solutions.

4. Analysis of simple homeostats

We will first consider a homeostat with a one dimensional state space and with $b = 0$. The system (7) reduces to

$$\begin{aligned} \dot{x} &= ax \\ \dot{a} &= -\varepsilon x^2 \end{aligned} \tag{9}$$

which describes the dynamics in the two dimensional (x, a)-space with target $x_0 = 0$. The dynamics of (9) can be easily understood qualitatively. If $a(0) < 0$, the state x will quickly run into the fixed point $x^* = 0$, while $a(t)$ tends to become more negative. a is strictly decreasing also for $a(0) > 0$, but now the magnitude of x grows as long as a is above zero. At $a = 0$ the state x reaches its maximal deviation from x_0 such that a will further decrease below zero, such that now x returns towards x_0. Eventually a saturates, and x further converges exponentially to zero with a nearly fixed rate, cf. Fig. 1.

The final value of a can be identified by exploiting the existence of a first integral of (9). By calculating the derivative $\frac{d}{dt}F(x,a)$ it is easily checked that the quantity

$$F(x,a) = \varepsilon x^2 + a^2 = \text{const} \tag{10}$$

is conserved by the dynamics (9). This means that due to $x(\infty) = 0$ the value $a(\infty)$ is given by

$$a(\infty) = -\sqrt{a^2(0) + \varepsilon x^2(0)},$$

where the sign is determined by the above qualitative consideration. Thus, for $a(0) > 0$ and $\varepsilon x^2(0) \ll a(0)$ the parameter a simply perform a sign switch, i.e. the parameter dynamics turns out to perform a directed, smooth analog of Ashby's original switching mechanism.

The conservation law (10) can be used for an explicite solution of (9), because it allows to eliminate one dimension. Introducing polar coordinates

$$\begin{aligned} x &= \tfrac{1}{\sqrt{\varepsilon}} r \sin\phi \\ a &= r\cos\phi \end{aligned} \tag{11}$$

we have $r^2 = \text{const}$ instead of (10). Inserting (11) into (9) we find a closed equation for ϕ

$$\dot{\phi} = r\sin\phi \tag{12}$$

which can be solved exactly as

$$\phi(t) = \pm 2\arccos\left(\pm\left(1 \pm \exp(2rt)\tan^2\left(\frac{\phi(0)}{2}\right)\right)^{-\frac{1}{2}}\right)$$

where the various signs reflect symmetry properties of (12). $\phi^* = 0$ is an unstable fixed point of (12) such that for $\phi(0) > 0$ ($\phi(0) < 0$) the dynamics converges towards $\phi = \pi$ ($\phi = -\pi$) i.e. to $x = 0$ and $a = -r$. A special solution of (9) is shown in Figure 1.

In case of a non-zero bias b we consider

$$\begin{aligned} \dot{x} &= ax + b \\ \dot{a} &= -\varepsilon x^2 \\ \dot{b} &= -\varepsilon x \end{aligned} \tag{13}$$

If x is close to zero, its evolution will be essentially determined by b and assume the same sign as b, which results eventually in a decay of b towards zero. If x and b have the same sign, the dynamics is similar to the case without a bias. Analogously to (10), for (13) it can be shown that

$$\frac{d}{dt}\left(\varepsilon x^2 + a^2 + b^2\right) = 0, \tag{14}$$

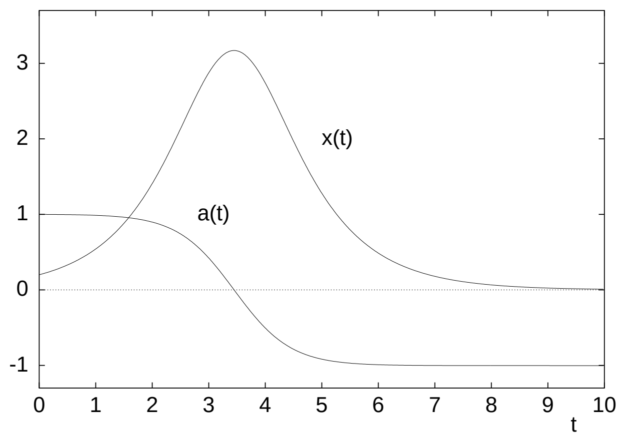

Figure 1: Time course of state x and parameter a for the one dimensional homeostat. Initial values are $a(0) = 1$ and $x(0) = 0.2$ and the learning rate is $\varepsilon = 0.1$. The excursion of x is reversed as soon as a changes sign. When x finally tends to zero, a settles near $-a(0) = -1$.

In order to find an explicite solution, equation (13) can be transformed into spheric coordinates

$$\begin{aligned} \sqrt{\varepsilon}x &= r\cos\phi\sin\theta \\ a &= r\sin\phi\sin\theta \\ b &= r\cos\theta \end{aligned}$$

such that r is invariant with respect to the dynamics and we are left with the reduced system

$$\begin{aligned} \dot{\phi} &= -r\cos\phi\sin\theta + \sqrt{\varepsilon}\sin\phi\cot\theta \\ \dot{\theta} &= -\sqrt{\varepsilon}\cos\phi \end{aligned} \tag{15}$$

The second equation has fixed points at $\phi = \frac{\pi}{2} + k\pi$, such that the first equation implies $\theta = \frac{\pi}{2} + l\pi$. The linearization of (15) yields eigenvalues $\frac{1}{2}\left(\pm r - \sqrt{r^2 - 4\varepsilon}\right)$ and $\frac{1}{2}\left(\pm r + \sqrt{r^2 - 4\varepsilon}\right)$ with negative real part if $k + l$ is an odd number. The eigenvalues are real if $4\varepsilon < r^2$, i.e. for large ε the solution is fluctuating before it settles to a stable fixed point.

5. Multidimensional homeostat

5.1 Homogenous case

Consider now the multidimensional case of (9)

$$\begin{aligned} \dot{x} &= Ax \\ \dot{A} &= -\varepsilon x x^T \end{aligned} \tag{16}$$

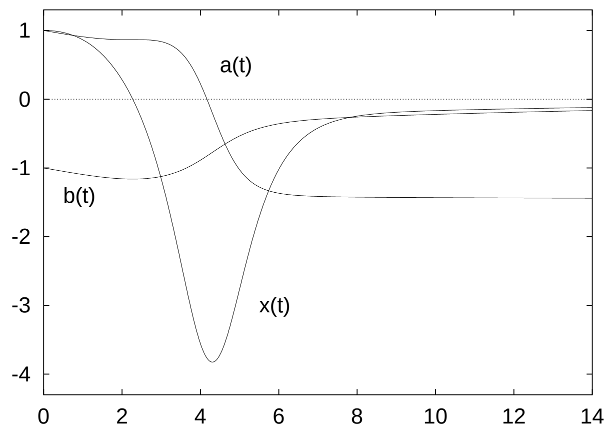

Figure 2: Time course of state x and parameters a and b for the one dimensional homeostat with bias. Initial values are $x(0) = 1$, $a(0) = 1$ $b(0) = -1$, and $\varepsilon = 0.1$. For this set of initial conditions the excursion of x undergoes now a sign change which is due to the initially large negative bias. As soon as b and x have the same sign the bias is no longer interfering the stabilizing dynamics and the further evolution is similar as shown in Figure 1, but with a smaller terminal value of a.

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$. $n = 4$ corresponds to Ashby's homeostat. The expression

$$F(x, A) = \varepsilon x x^T + AA^T \tag{17}$$

is a first integral of (16), i.e. the dynamics of the full system is restricted to the set

$$\{(x, A)|F(A, x) = V = \text{const}\}$$

with V being a semidefinite symmetric matrix.

For a symmetric parameter matrix the convergence of the state dynamics can be shown easily, thus we will not assume that A is symmetric. We consider the dynamics of $r^2 = x^T x$ which obeys

$$\dot{r^2} = \dot{x}^T x + x^T \dot{x} = x^T \left(A^T + A\right) x$$

for (16). The norm of x is thus governed only by the symmetric part of A. r^2 tends to zero if $B = A^T + A$ has only strictly negative eigenvalues. Let b_i an eigenvector of B and β_i the corresponding eigenvalue and assume that A has a full set of n eigenvectors. Then, taking the time derivative of the eigenvalue equation we obtain

$$\dot{\beta}_i b_i + \beta_i \dot{b}_i = \dot{B} b_i + B \dot{b}_i$$

or, using $\dot{B} = -2\varepsilon x x^T$ and multiplying by b^T from the left,

$$\dot{\beta}_i b_i^T b_i = -2\varepsilon b_i^T x x^T b_i + b_i^T \left(B - \beta_i\right) \dot{b}_i$$

where the last term vanishes, because b_i^T is an eigenvector of $B^T = B$. Thus, we have

$$\dot{\beta}_i = -2\varepsilon \left\|x^T b_i\right\|^2 / \left\|b_i\right\|^2 \leq 0, \tag{18}$$

i.e. all eigenvalues λ_i with $x^T b_i \neq 0$ are strictly decreasing. Because we can assume generic initial conditions, i.e. $x^T(0)\, b_i(0) > \eta > 0$, and that $x^T b_i$ even grows for positive β_i all eigenvalues will eventually become negative.

Assume $A(0)$ and $A(\infty)$ being symmetric with real entries and of full rank. If $\|x(0)\| \ll 1$ and $x(\infty) = 0$, then both $A(0)$ and $A(\infty)$ are generalized roots of V and we can write approximately

$$A(\infty) = QA(0)$$

where $Q^{-1} = Q^T$ is an orthogonal matrix. We further have that

$$A(\infty) = A^T(\infty) \Rightarrow QA(0) = A(0)Q^T.$$

Thus

$$Q = A(0)Q^T A(0)^{-1}$$

which implies that Q and Q^T have the same eigenvalues, i.e. $\lambda = \overline{\lambda}$, thus the eigenvalues are real and, because Q is orthogonal, they are ± 1. In this way we have obtained an analogon of the sign-switch property from the one dimensional case.

5.2 Global analysis

Eq. (17) implies that

$$\varepsilon x x^T + AA^T = V$$

with $V = V^T$. By diagonalizing V via $V = TDT^T$, where T is an orthogonal matrix and D is a diagonal one with non-negative entries, we define

$$y = \sqrt{\varepsilon} T x, \quad G = TAT^T. \tag{19}$$

Using (19) the system (16) transforms into

$$\begin{array}{rcl} \dot{y} & = & Gy \\ \dot{G} & = & -yy^T \end{array} \tag{20}$$

where the first integral (17) becomes now

$$D = yy^T + GG^T = \tilde{F}(G, y) = \text{const.} \tag{21}$$

which can be written as

$$\tilde{C}\tilde{C}^T = \tilde{D} \tag{22}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 \\ y & G \end{pmatrix} \in \mathbf{R}^{(n+1)\times(n+1)}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \in \mathbf{R}^{(n+1)\times(n+1)}$$

The matrix $\tilde{C}$ in Eq. (22) can by termed a generalized root of $\tilde{D}$. It can be proven that the set

$$M_D = \{(G, y) \in \mathbf{R}^{n\times n} \times \mathbf{R}^n : F(G, y) = D\} \tag{23}$$

is an $\frac{n(n+1)}{2}$-dimensional compact submanifold of $\mathbf{R}^{n\times n} \times \mathbf{R}^n$, if the matrix D has full rank, which is the case for almost every initial condition.

This allow to characterize the manifold of fixed points of (16). (21) allows to eliminate the y equation in (20) and to consider the reduced dynamics

$$\dot{G} = GG^T - D. \tag{24}$$

The set of fixed points of (24) is determined by $GG^T = D$, hence any stationary G can be written as $G = \sqrt{D}O$, where $O^T = O^{-1}$ is an arbitrary orthogonal matrix. We can express the set of solutions by $\sqrt{D}O(n)$ where $O(n)$ denotes the set of orthogonal matrices. Because every solution G has full rank, at the fixed point of the joint dynamics we have $y = 0$. Consequently, the fixed-point manifold $\sqrt{D}\mathcal{O}(n) \times \{0\} \subset M_D$ is of the same dimension as $O(n)$, i.e. $\frac{n(n-1)}{2}$. Recalling that D is fixed by the initial values, we arrive at the reduced form

$$\dot{y} = \sqrt{D}$$

The manifold which combines the dynamics of the full set of variables of (16) can also be characterized using (21). Since $\det D \neq 0$, the set M_D (23) is diffeomorphous with M_I, where I is the $n \times n$ unit matrix. The corresponding diffeomorphism is given by $G \to \sqrt{D}G \quad y \to \sqrt{D}y$. Instead of (21) we can restrict ourself thus to the special case

$$yy^T + GG^T = I$$

For any y we define an orthogonal $n \times n$ matrix R_y with $\det R_y = 1$ and a positive semi-definite diagonal matrix $D_{|y|}$, such that

$$\begin{aligned} R_y GG^T R_y^T &= R_y(I - yy^T)R_y^T \\ &= \begin{pmatrix} 1-|y|^2 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix} = D_{|y|}^2 \end{aligned}$$

Further, we define an orthogonal $n \times n$ matrix V_y with $\det V_y = -1$ which satisfies

$$V_y = R_y \begin{pmatrix} -1 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix} R_y^T.$$

Then, each matrix in the set $U_1 = M_I \cap \{(G, y) : \det G > 0, |y| < 1\}$ can be represented by the form $(R_y D_{|y|} R_y^T T, y)$ and each one from $U_2 = M_I \cap \{(G, y) : \det G < 0, |y| < 1\}$ by $(R_y D_{|y|} R_y^T V_y T, y)$, resp., where T is orthogonal with $\det T = 1$. Both U_1 and U_2 are diffeomorphous to $\mathcal{SO}(n) \times E^n$, where $SO(n)$ is the set (group) of orthogonal matrices with positive determinant and E^n is the open unit sphere $M_I = \bar{U}_1 \cup \bar{U}_2$

$$\lim_{|y|\to 1} R_y D_{|y|} R_y^T T = \lim_{|y|\to 1} R_y D_{|y|} V_y R_y^T$$

$$= T_y \begin{pmatrix} 0 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix} T_y^T T = g(T, y)$$

$$g(T_1, y) = g(T_2, y),\ T_1, T_2 \in \mathcal{SO}(n) \ \Leftrightarrow T_1 = T_2$$

The two parts of the set of solutions can be joined by identifying the margins of $\bar{E}^n$ (E.g. for $n = 2$ the margins are one dimensional circles surrounding two dimensional disks, which are glued together yielding an object which is topologically equivalent to a sphere). The joint set of solutions M_I is thus homeomorphous to $SO(n) \times S^n$, with S^n being the n dimensional sphere. Formally this process is denoted as

$$\begin{aligned} &(\mathcal{SO}(n) \times \bar{E}^n) \cup_{\mathrm{id}_{\mathcal{SO}(n)\times S^{n-1}}} (\mathcal{SO}(n) \times \bar{E}^n) \\ = \ & \mathcal{SO}(n) \times S^n \end{aligned}$$

Since M_D is homeomorphous to M_I if $\det D \neq 0$, we have also that the set M_D is homeomorphous to $SO(n) \times S^n$. Remember that D is given by the initial conditions of (16).

By construction it is clear that with respect to the above homeomorphism the fixed-point manifold is

$$\mathcal{SO}(n) \times \{''\text{north pole}''\} \cup \mathcal{SO}(n) \times \{''\text{south pole}''\},$$

although here the somewhat artificial nature of this representation becomes obvious: the state vector x always converges to the same fixed point $x = 0$, but the parameter matrix develops in two qualitatively different ways. In order to prove the stability of a fixed point of (16) we need to know the tangents of the manifold M_D in that fixed point. Expansion of the equation $GG^T + yy^T = D$ around the fixed point $(\sqrt{D}T, 0)$ yields

$$\delta G\sqrt{D}T + T^T\sqrt{D}\delta G^T = 0.$$

δG is thus tangential with respect to the fixed point manifold $\sqrt{D}O(n)$ or is zero, while δy remains arbitrary. The eigenvalues in the direction of the tangents of the fixed-point manifold are all zero. Perturbations with $\delta G = 0$ behave within a linear approximation as

$$\dot{\delta y} = \sqrt{D} T \delta y,$$

such that the stability of a fixed point is determined by the eigenvalues of $\sqrt{D}T$.

Consider for example $n = 2$ and $D = \mathrm{diag}(\alpha^2, \beta^2)$. An two-dimensional orthogonal matrix X has the form

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

if $\det X = 1$. In this case the eigenvalues

$$\frac{1}{2}\left(\cos(\phi)(\alpha+\beta) \pm \sqrt{\cos(\phi)^2(\alpha+\beta)^2 - 4\alpha\beta}\right)$$

are obtained. If $\det X = -1$, i.e. for matrices of the form

$$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

we find analogously

$$\frac{1}{2}\left(\cos(\phi)(\alpha-\beta) \pm \sqrt{\cos(\phi)^2(\alpha-\beta)^2 + 4\alpha\beta}\right)$$

5.3 Multidimensional with bias

Here the simple homeostat with a bias term (7) is formulated for an n dimensional state space, i.e. we consider the system

$$\dot{x} = Ax + b.$$

The parameters A and b are modified such that x approaches the target state $\tilde{x}$, which is expressed as the gradient dynamics (6)

$$\begin{aligned} \dot{A} &= -\varepsilon\,(x - \tilde{x})\,x^T \\ \dot{b} &= -\varepsilon\,(x - \tilde{x}) \end{aligned}$$

which as derived from (4). We find a conservation law

$$0 = \frac{d}{dt}\left(\varepsilon(x-\tilde{x})(x-\tilde{x})^T + AA^T + bb^T\right).$$

as a multidimensional analog of (14). The considerations on dynamics and stability are basically similar as in the homogeneous case, but more involved due to the additional dynamical bias vector as will be seen in the following section.

5.4 "Concomitant" learning

In the dynamical systems like (7) or (8) the parameter and the state dynamics are coupled via the learning rate ϵ. If ϵ is not infinitely small, parameter and state dynamics severely influence each other. In particular there is no seperation into a learning and performance phase so that we call this scenario concomitant learning. Tis interaction is particularly strong if the bias is included. This is seen most promoinently when comparing the dynamics $\dot{x} = Ax + b$ with the similar dynamics $\dot{x} = A(x+b)$, the difference being only the different definitions of the bias. Using the error function $E = \|x(t) + \tau\dot{x}(t) - \tilde{x}\|^2$ the combined system dynamics is

$$\begin{aligned} \dot{x} &= A(x+b) \\ \dot{A} &= -\varepsilon\tau(x-\tilde{x})(x+b)^T \\ \dot{b} &= -\varepsilon A^T(x-\tilde{x}) \end{aligned}$$

There is again a conservation law

$$\frac{d}{dt}\left(\varepsilon(x-\tilde{x})(x-\tilde{x})^T + AA^T\right) = 0$$

For a one dimensional state space the above system can be formulated in spheric coordinates (cf. (15)) which leads to the reduced dynamics

$$\begin{aligned} \dot{\phi} &= \frac{2\sqrt{\varepsilon}\cos\theta\left(\sqrt{\varepsilon}\cos\theta + \cos\phi\sin\theta\right)(\tilde{x}\cos\phi - r\cos(2\phi)\sin\theta)}{\cos(2\phi)+\cos(2\theta)} \\ \dot{\theta} &= \frac{2\sqrt{\varepsilon}\sin\phi\left(\sqrt{\varepsilon}\cos\theta + \cos\phi\sin\theta\right)(\tilde{x}\sin\theta + r\cos(2\theta)\cos\phi)}{\cos(2\phi)+\cos(2\theta)} \end{aligned} \tag{25}$$

MATHEMATICA produces a rather complex solution of (25) for the case $\tilde{x} = 0$. The system (25) has many fixed points. Of particular interest are the ones given by $\sqrt{\varepsilon}\cos\theta + \cos\phi\sin\theta = 0$, which form a one dimensional manifold of fixed points in (ϕ, θ)-space. The stability properties along the manifold vary due to sign switches of the other terms. Further fixed points of (25) are at (for $\tilde{x} = 0$), $\phi = \pm\frac{\pi}{2}$, $\theta = \pm\frac{\pi}{2}$ with all combinations of signs and zero eigenvalues of the Jacobian. The stationary states will thus be on the appropriate parts of the $\sqrt{\varepsilon}\cos\theta + \cos\phi\sin\theta = 0$ manifold.

6. Dynamics with noise

We consider both the case of the states being subject to Gaussian white noise

$$\begin{aligned} \dot{x} &= Ax + \sigma\xi \\ \dot{A} &= -\varepsilon x x^T \end{aligned} \tag{26}$$

as well as also the parameters are affected by noise

$$\begin{aligned} \dot{x} &= Ax + \sigma\xi \\ \dot{A} &= -\varepsilon x x^T + \rho\zeta \end{aligned} \tag{27}$$

both ξ and ζ being of zero mean and unit variance. It is obvious that for state noise (26) the general behavior of the homeostat is not much affected, expect in the sense that the eigenvalues of A diverge to $-\infty$ (with a rate proportional to $\sqrt{t}$) because $\dot{A}$ is continuously receiving negative contributions by the noisy fluctuations of x. From the numerical studies we observe also that the off-diagonal elements of A decay to zero if the noise is uncorrelated among different dimensions. Therefore the negative contributions to A directly change the eigenvalues.

For the noise affecting both states and variables the off-diagonal elements of A do not vanish, but rather perform a random walk with increasing amplitude. The real parts of the eigenvalues of A may thus return to the zero, but are pushed back as soon as a some deviation of x has been caused by the momentary instability, cf. Figure 3.

In particular if $\sigma \ll \varepsilon$ the state dynamics will stay very close to the fixed point of the deterministic dynamics, such that A performs essentially an unbiased random walk which is, however, reflected from the x-axis if an eigenvalue of A reaches zero from below. The dynamics of x looks spiky, i.e. every time when the system becomes unstable x performs a substantial deviation from zero which, however, quickly is reset by the response of the A dynamics, cf. Figure 4.

For weak parameter noise temporary instabilities are not excluded, and for larger noise they continue to occur, although at a lower rate.

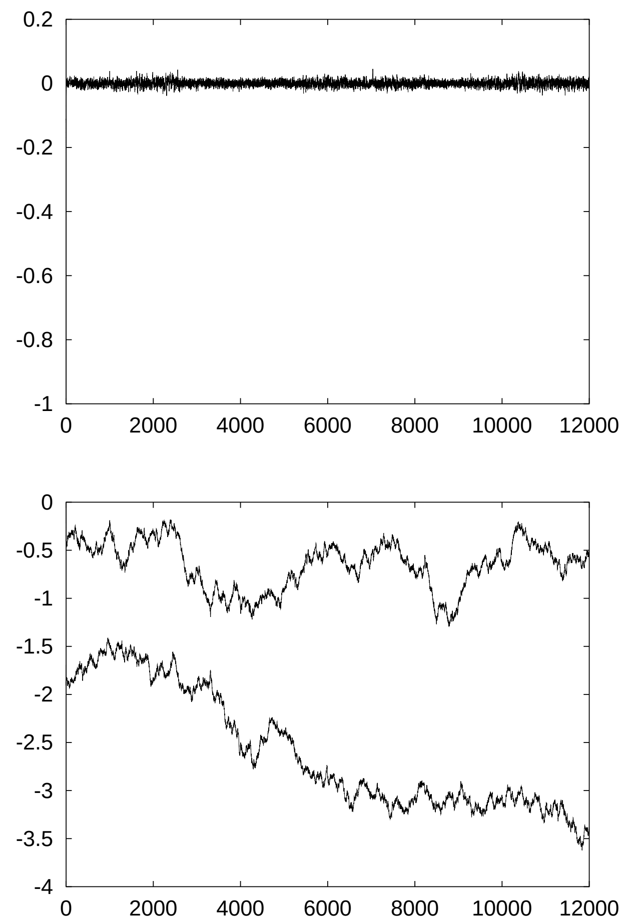

Figure 3: Dynamics with state noise in a two-unit homeostat. $\sigma = \rho = 0.1$. The top image presents one coordinate of the state, the bottom one the two eigenvalues of the parameter matrix.

7. Discussion

Parameter changes in Ashby's homeostat are performed by switching, i.e. whenever a state variable reaches a certain threshold, a subset of the parameters is replaced by a random configuration. Disregarding the specific architecture of the homeostat we may consider the parameter changes in the homeostat to occur on a much faster time scale than the state dynamics. In this way it is not possible to guide the parameters into suitable directions.

Adaptive systems, on the other hand, are mostly based on the opposite assumption, that parameter changes are much slower than the state dynamics. If the states are in a simple attractor, the information available from the states is limited and will not be extendable to other regions of the state space. The consequence is usually an increase of stability at the cost of an insensitivity to changes. This is in other words an instance of the stability-plasticity dilemma. Therefore, the state dynamics has to be controlled in some way in order to enable learning. If, on the other hand, the state dynamics is unstable, a slow parameter adaptation may be unable to prevent the system from divergence. Although there are ways to cope with such problems, such as resetting, shuffling or weighting the states as inputs to the learning algorithm, on-line learning in living systems or in machines as envisioned by Ashby requires parameter

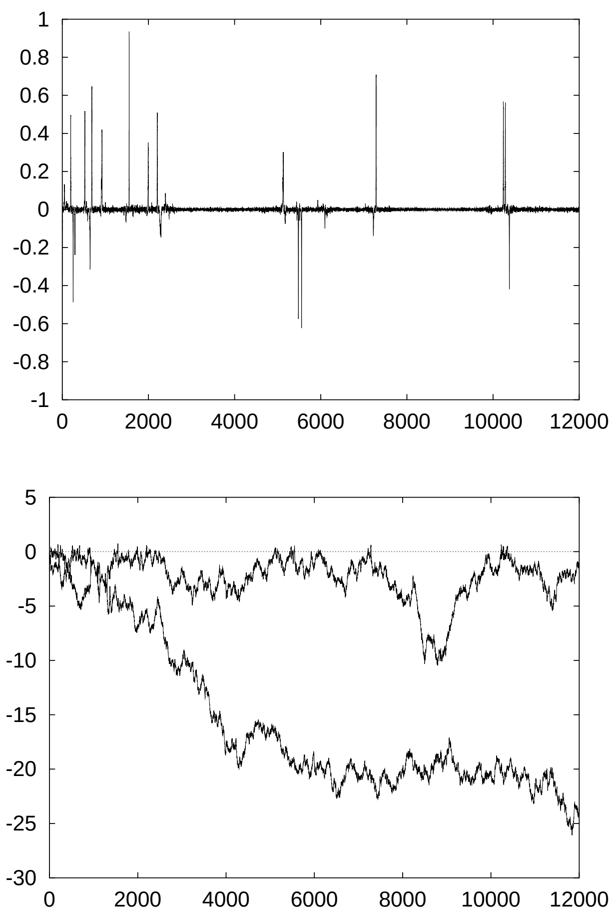

Figure 4: Dynamics with parameter noise in a two-unit homeostat. $\sigma = 0.1$ and $\rho = 1.0$.. The top image presents one coordinate of the state, the bottom one the two eigenvalues of the parameter matrix.

dynamics on a time scale which is comparable with that of the state dynamics. In this way states and parameters are dynamically equivalent and form a joint dynamical system with inseparable time scales, although generally different dynamical behaviors. Because then all variables are subject to intrinsic control, the agent can be improved only by structural modifications, such as performed by genetic algorithms.

If under certain conditions the parameter dynamics is converged, the corresponding behavior is governed by the state dynamics, i.e. on small scales ultrastability implies stability. If the external conditions change or some internal failure occurs, then the stability of the state dynamics may be lost and the parameter dynamics is required to reinstall stability of the target region. For larger excursions from the locally stable state, ultrastability requires thus the existence of possibly complex returning trajectories. Globally stable systems such as stable linear systems are in this sense not ultrastable, because ultrastability does not refer just to the dynamical variables (states) of the system, but also to the parameters, which can no longer be considered as fixed. The joint dynamics of parameters and states will thus be nonlinear and give rise to all kinds of nonlinear phenomena such as bifurcations, self-organization etc.

Although ultrastable system can cope with large scale changes, more complex systems than the one considered here should possess capabilities beyond thus, such as active learning and memory, e.g. in order to be able to predict long excursion or to initiate them in order to approach other goal states without an essential reorganization of the system properties. One step in this direction relates to active destabilization or, ideally, criticalization of the state dynamics. In such adaptation schemes the parameter dynamics tends to create center manifolds of the state dynamics (instead of stable fixed points of limit cycles) which allows for low-risk explorative behavior near the center and fast stabilization in the presence of unexpected external perturbations.

Hauenschild (1956) claims that the long transients in large homeostats can be avoided if the connectivity matrix is a block matrix, i.e. the described function separates into subfunctions which are controlled by low-dimensional servo-systems. More interesting is the possibility of considering blocks of different dynamical behaviors to subspaces of the state space. This allow to model systems with partial feedback arising in incomplete observable problems or in models that include an autonomous external or internal dynamics. Similarly, different learning rates or objective function may be applied to different blocks of the connectivity matrix, such as to achieve optimal exploration in certain directions while stabilizing other ones.

Ashby's homeostat is an attempt to mimic natural systems which are both resistive and reactive to the external world and are able to survive even structural disruptions. Prominent to Ashby's ideas is the switching of system parameters subject to some internal criteria (like the state reaching the boundary). In the present paper we introduce instead a parameter dynamics which rests on gradient descending an objective function. We have shown that in simple cases the adaptive system created in the way shares prominent features with Ashby's homeostat. In particular we observe a parameter switching dynamics as an emergent feature of the system to maintain stability.

To conclude, we have considered a particularly simple example of a nonlinear system which allows for analytical considerations some of which have been presented here. Applications of the present formal model will pose further questions such as the behavior in the presence of colored or correlated noise, the effect of time delays, the capabilities for tracking control and explorative dynamics.

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